MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: TRANSFORM TECHNIQUES

SEMESTER: IV

PREPARED BY: PROF.T.N.REKHA

UNIVERSITY OF MADRAS

B.Sc. DEGREE COURSE IN MATHEMATICS

SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC07

CORE-VII: TRANSFORM TECHNIQUES

(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 4 **YEAR: II**
Credits : 4 **SEMESTER: IV**

Learning outcomes:

Students will acquire knowledge

- About Laplace Transforms and its inverse
- To apply Laplace transform in solving Ordinary Differential Equations with constant coefficients, simultaneous Ordinary Differential Equations.
- To solve problems in Fourier series and Fourier transforms.

UNIT I: The Laplace Transforms-Definitions-Sufficient conditions for the existence of the Laplace transform(without proof)-Laplace transform of periodic functions-some general theorems-evaluation of integrals using Laplace transform-Problems.

Chapter 5: Section-1 to 5.

UNIT II:The inverse Laplace Transforms- Applications of Laplace Transforms to ordinary differential equations with constant co-efficients and variable co-efficients, simultaneous equations and equations involving integrals-Problems.

Chapter 5: Section-6 to 12.

UNIT III: Fourier series- Expansion of periodic functions of period 2π- Expansion of even and odd functions, Half range Fourier series-Change of intervals –Problems.

Chapter 6: Section-1 to 6.

UNIT IV: Fourier Transform- Infinite Fourier Transform(Complex form) – Properties of Fourier Transform – Fourier cosine and Fourier sine Tranform – Properties – Parseval's identity – Convolution theorem - Problems.

Chapter 6: Section-8 to 15.

UNIT V: Z Transforms: Definitionof Z-Transform and its properties - Z-Transforms of some basic functions- Examples and simple problems

Chapter 7: Sections -7.1 to 7.3.

Contents and treatment as in

1. "Calculus-Volume III" – S.Narayananand T.K.ManicavachagamPillai. (Ananda Book Depot)(**for Units I to IV**)
2. "Engineering Mathematics for Semester III- Third Edition – T.Veerarajan (Tata McGraw-Hill Publishing Company Ltd, New Delhi) **(for Unit-V**)

Reference Books

1. Engineering Mathematics Volume III – P.Kandasamy and others (S.Chand and Co.)
2. Advanced Engineering Mathematics- Stanley Grossman and William R.Devit.

Engineering Mathematics III-A.Singaravelu, Meenakshi Agency, Chenani, 2008

e-Resources:

1. http://mathworld.wolfram.com.
2. http://www.sosmath.com.

UNIT I
LAPLACE TRANSFORM

- **Definition of Laplace transform**
- **Properties of Laplace transform**
- **Laplace transforms of derivatives and integrals**
- **Inverse Laplace transform**
- **Properties of Inverse Laplace transform**
- **Convolution theorem and applications**

Introduction

In mathematics the Laplace transform is an integral transform named after its discoverer Pierre-Simon Laplace . It takes a function of a positive real variable t (often time) to a function of a complex variable s (frequency).The Laplace transform is very similar to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with t > 0. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Also techniques of complex variables can be used directly to study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. This perspective has applications in probability theory.

Introduction

Let f(t) be a given function which is defined for all positive values of t, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called Laplace transform of f(t) and is denoted by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The inverse transform, or inverse of L{f(t)} or F(s), is

$$f(t) = L^{-1}\{F(s)\}$$

where s is real or complex value.

Laplace Transform of Basic Functions

1. $L\ [1] = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$

2. $L\ [t^a] = \int_0^\infty t^a e^{-st} dt = \int_0^\infty (\frac{u}{s})^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$

3. $L\ [e^{at}] = \int_0^\infty e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = \frac{1}{s-a}$

4. $L\ [e^{iat}] = \frac{1}{s-ia} \Rightarrow L\ [\cos at + i \sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

   $\therefore L\ [\cos at] = \frac{s}{s^2+a^2}$, and $L\ [\sin at] = \frac{a}{s^2+a^2}$

5. $L\ [\sinh at] = L\ [\frac{e^{at}-e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} - \frac{1}{s+a}) = \frac{a}{s^2-a^2}$

   $L\ [\cosh at] = L\ [\frac{e^{at}+e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} + \frac{1}{s+a}) = \frac{s}{s^2-a^2}$

1. Linearity

$$\mathcal{L}\,[af(t)+bg(t)]=\int_0^\infty [af(t)+bg(t)]e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s)+bG(s)$$

EX: Find the Laplace transform of $\cos^2 t$.

$$\text{Solution : } \mathcal{L}[\cos^2 t] = \mathcal{L}[\frac{1+\cos 2t}{2}] = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right) = \frac{s^2+2}{s(s^2+4)}$$

2. Shifting

$$(a)\ \mathcal{L}\ [f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

Let $\tau = t-a$, then

$$\mathcal{L}\ [f(t-a)u(t-a)] = \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sa}F(s)$$

$$(b)\ F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty [e^{at}f(t)]e^{-st}dt = \mathcal{L}\ [e^{at}f(t)]$$

EX: What is the Laplace transform of the function $f(t) = \begin{cases} 0, & t<4 \\ 2t^3, & t\ge 4 \end{cases}$

Solution: $f(t)=2t^3u(t-4)$

$\mathcal{L}\,[f(t)]=\mathcal{L}\,\{2[(t-4)^3+12(t-4)^2+48(t-4)+64]u(t-4)\}$

$$= 2e^{-4s}\left(\frac{3!}{s^4}+12\times\frac{2!}{s^3}+48\times\frac{1}{s^2}+\frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4}+\frac{12}{s^3}+\frac{24}{s^2}+\frac{32}{s}\right)$$

3. Scaling

$$\mathcal{L}\ [f(at)] = \int_0^\infty f(at)e^{-st}dt$$

Let $\tau = at$, then

$$\mathcal{L}\ [f(at)] = \int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}d\frac{\tau}{a} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F(\frac{s}{a})$$

EX: Find the Laplace transform of $\cos 2t$.

$$\text{Solution :} \because \mathcal{L}\ [\cos t] = \frac{s}{s^2+1}$$

$$\therefore \mathcal{L}\ [\cos 2t] = \frac{1}{2}\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

4. Derivative

(*a*) Derivative of original function

$$\mathcal{L}\,[f'(t)]=\int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt$$

(1) If $f(t)$ is continuous, equation (2.1) reduces to

$\mathcal{L}\,[f'(t)]=-f(0)+sF(s)=sF(s)-f(0)$

(2) If $f(t)$ is not continuous at $t=a$, equation reduces to

$\mathcal{L}\,[f'(t)]= f(t)e^{-st}\Big|_0^{a^-} + f(t)e^{-st}\Big|_{a^+}^\infty + sF(s) = [f(a^-)e^{-sa}-f(0)]+[0-f(a^+)e^{-sa}]+sF(s)$

$=sF(s)-f(0)-e^{-sa}[f(a^+)-f(a^-)]$

(3) Similarly, if $f(t)$ is not continuous at $t=a_1, a_2, \cdots,\cdots,a_n$, equation reduces to

$$\mathcal{L}\,[f'(t)]= sF(s)-f(0)-\sum_{i=1}^n e^{-sa_i}[f(a_i^+)-f(a_i^-)]$$

If $f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

(b) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}[f(t)e^{-st}]dt = \int_0^\infty (-t)f(t)e^{-st}dt = \mathcal{L}[(-t)f(t)]$$

$$\text{[Deduction]}\ \frac{d^n F(s)}{ds^n} = \mathcal{L}[(-t)^n f(t)]$$

EX: Find the Laplace transform of te^t.

$$\text{Solution}: \mathcal{L}(e^t) = \frac{1}{s-1} \Rightarrow \mathcal{L}(te^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

EX: $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, find $\mathcal{L}[f'(t)]$.

$$\text{Solution}: f(t) = t^2[u(t) - u(t-1)]$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2u(t)] - \mathcal{L}[t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L}\{[(t-1)+1]^2 u(t-1)\}$$

$$= \frac{2}{s^3} - \mathcal{L}\{[(t-1)^2 + 2(t-1) + 1]u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right)$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)]$$

$$= \left[\frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s} + 1\right)\right] - 0 - e^{-s}(0-1) = \frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s}\right)$$

5. Integration

(a) Integral of original function

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt$$

$$= \frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty f(t)e^{-st}dt\right] = \frac{1}{s}F(s)$$

$$\Rightarrow \mathcal{L}\left[\int_0^t\int_0^t \ldots \int_0^t f(t)dtdt\cdots dt\right] = \frac{1}{s^n}F(s)$$

(b) Integration of Laplace transform

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty f(t)e^{-st}dtds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{-t}\Bigg|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

$$\Rightarrow \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)dsds\cdots ds = \mathcal{L}\left[\frac{1}{t^n}f(t)\right]$$

EX: Find $(a)\ \mathcal{L}\ [\frac{1-e^{-t}}{t}]\quad (b)\ \mathcal{L}\ [\frac{1-e^{-t}}{t^2}]$.

$$\text{Solution}: (a)\mathcal{L}\ [1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$$

$$\mathcal{L}\ [\frac{1-e^{-t}}{t}] = \int_s^\infty (\frac{1}{s} - \frac{1}{s+1})ds = \ln s - \ln(s+1)\Big|_s^\infty = \ln\frac{s}{s+1}\Big|_s^\infty$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

$$(b)\mathcal{L}\ [\frac{1-e^{-t}}{t^2}] = \int_s^\infty \ln\frac{s+1}{s}ds = s\ln\frac{s+1}{s}\Big|_s^\infty - \int_s^\infty s(\frac{1}{s+1} - \frac{1}{s})ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^\infty + \int_s^\infty \frac{1}{s+1}ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^\infty$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^\infty = s\ln s - (s+1)\ln(s+1)$$

EX: Find $(a)\ \int_0^\infty \frac{\sin kt\, e^{-st}}{t}dt \qquad (b)\ \int_{-\infty}^\infty \frac{\sin x}{x}dx$.

$$\text{Solution}: (a)\int_0^\infty \frac{\sin kt e^{-st}}{t}dt = \mathcal{L}\ [\frac{\sin kt}{t}]$$

$$\because \mathcal{L}\ [\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\ [\frac{\sin kt}{t}] = \int_s^\infty \frac{k}{s^2+k^2}ds = \frac{1}{k}\int_s^\infty \frac{1}{(\frac{s}{k})^2+1}ds$$

$$= \tan^{-1}\frac{s}{k}\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

$$(b)\int_{-\infty}^\infty \frac{\sin x}{x}dx = 2\int_0^\infty \frac{\sin x}{x}dx$$

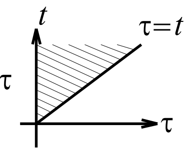
$$= 2\lim_{\substack{k\to1\\ s\to0}} \int_0^\infty \frac{\sin kte^{-st}}{t}dt$$

$$= 2\lim_{\substack{k\to1\\ s\to0}} (\frac{\pi}{2} - \tan^{-1}\frac{s}{k}) = \pi$$

6. Convolution theorem

$$\mathcal{L}\ [\int_0^t f(\tau)g(t-\tau)d\tau] = \int_0^\infty\int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^\infty\int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}dtd\tau$$



Let $u = t-\tau$, $du = dt$, then

$$\mathcal{L}\ [\int_0^t f(\tau)g(t-\tau)d\tau] = \int_0^\infty f(\tau)\int_0^\infty g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau\int_0^\infty g(u)e^{-su}du = F(s)G(s)$$

EX: Find the Laplace transform of $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$.

$$\text{Solution}: \because L\ [e^t] = \frac{1}{s-1},\ L\ [\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore L\ [\int_0^t e^{t-\tau}\sin 2t\, d\tau] = L\ [e^t * \sin 2t] = L\ [e^t]\cdot L\ [\sin 2t]$$

$$= \frac{1}{s-1}\cdot\frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: $f(t+T) = f(t)$

$$L\ [f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$\text{and } \int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore L\ [f(t)] = (1 + e^{-sT} + e^{-2sT} + \cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

EX: Find the Laplace transform of $f(t) = \frac{k}{p}t,\ 0 < t < p,\ f(t+p) = f(t)$.

$$\text{Solution}: L\ [f(t)] = \frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$$

$$= \frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p - \int_0^p e^{-st}dt)]$$

$$= \frac{-k}{ps(1-e^{-ps})}(te^{-st} + \frac{1}{s}e^{-st})\Big|_0^p$$

$$= \frac{-k}{ps(1-e^{-ps})}(pe^{-sp} + \frac{e^{-sp}}{s} - \frac{1}{s})$$

8. Initial Value Theorem:

$$\because L\ [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$$

we get initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Deduce general initial value theorem: $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$

9. Final Value Theorem:

$$\mathcal{L}\,[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0} sF(s) - f(0) \Rightarrow$$

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0) \Rightarrow \text{final value theorem}: \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\text{General final value theorem}: \lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{s\to 0}\frac{F(s)}{G(s)}$$

EX: Find $\mathcal{L}\,[\int_0^t \frac{\sin x}{x}dx]$.

$$\text{Solution: Let } f(t) = \int_0^t \frac{\sin x}{x}dx \Rightarrow f'(t) = \frac{\sin t}{t},\ f(0) = 0$$

$$\mathcal{L}\,[tf'(t)] = \mathcal{L}\,[\sin t] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}\,[f'(t)] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = \frac{1}{s^2+1} \Rightarrow \frac{d}{ds}[sF(s)] = -\frac{1}{s^2+1}$$

$sF(s) = -\tan^{-1}s + C$

From the initial value theorem, we get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$0 = -\frac{\pi}{2} + C \quad \therefore C = \frac{\pi}{2}$$

$$sF(s) = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$$

$$F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

EX: Find $\mathcal{L}\left[\int_t^\infty \frac{e^{-x}}{x}dx\right]$.

$$\text{Solution: Let } f(t) = \int_x^\infty \frac{e^{-x}}{x}dx \Rightarrow f'(t) = -\frac{e^{-t}}{t},\ \lim_{t\to\infty} f(t) = 0$$

$$\mathcal{L}\,[tf'(t)] = \mathcal{L}\,[-e^{-t}] = -\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = -\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)] = \frac{1}{s+1}$$

$$sF(s) = \ln(s+1) + C$$

$$\text{From the final value theorem}: \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$0 = 0 + C \Rightarrow C = 0, \text{ and } F(s) = \frac{\ln(s+1)}{s}$$

Note: $\int_0^t \frac{\sin x}{x}dx$, and $\int_t^\infty \frac{e^{-x}}{x}dx$ are called sine, and exponential integral function, respectively.

UNIT II
INVERSE LAPLACE TRANSFORM

I. Inversion from Basic Properties

1. Linearity

Ex. 1.

$(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]$ $\qquad$ $(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]$.

Solution : $(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}] = \mathcal{L}^{-1}[2\frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}] = 2\cos 2t + \frac{1}{2}\sin 2t$

$(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}] = \mathcal{L}^{-1}[4\frac{s}{s^2-4^2} + \frac{4}{s^2-4^2}] = 4\cosh 4t + \sinh 4t$

2. Shifting

Ex. 2.

$(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $\qquad$ $(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

Solution : $(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$

$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}] = e^{-t}\sin t$

and $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$

$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}] = e^{-(t-\pi)}\sin(t-\pi)u(t-\pi) = -e^{-(t-\pi)}\sin t u(t-\pi)$

$(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}] = \mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}] = 2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$

3. Scaling

Ex. 3.

$\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]$.

Solution : $\mathcal{L}^{-1}[\frac{4s}{16s^2-4}] = \mathcal{L}^{-1}[\frac{4s}{(4s)^2-2^2}] = \frac{1}{4}\cosh 2\cdot\frac{1}{4}t = \frac{1}{4}\cosh\frac{t}{2}$

4. Derivative

Ex. 4.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $\qquad$ $(b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

$$\text{solution}: (a) \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}[t\sin\omega t] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\text{Let } F(t) = t\sin\omega t \Rightarrow \mathcal{L}[F'(t)] = s\cdot\frac{2\omega s}{(s^2+\omega^2)^2} - F(0)$$

$$\mathcal{L}[F'(t)] = 2\omega\frac{s^2}{(s^2+\omega^2)^2} = 2\omega\left[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}\right] = 2\omega\left[\frac{1}{s^2+\omega^2} - \frac{\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= 2\mathcal{L}[\sin\omega t] - \frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2} = \frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t - F'(t)]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] = \frac{1}{2\omega^3}\cdot[2\sin\omega t - F'(t)] = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$$

$$(b) \text{ Let } \mathcal{L}[f(t)] = \ln\frac{s+a}{s+b} = \ln(s+a) - \ln(s+b)$$

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}[\ln(s+a) - \ln(s+b)] = \frac{1}{s+b} - \frac{1}{s+a} = \mathcal{L}[e^{-bt} - e^{-at}]$$

$$\therefore f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

5. Integration

Ex. 5.

$$(a)\mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right] \qquad (b)\mathcal{L}^{-1}\left[\ln\frac{s+a}{s+b}\right].$$

$$\text{Solution}: (a)\mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}\right] = \int_0^t e^{-t}dt - \int_0^t\int_0^t e^{-t}dtdt$$

$$= -(e^{-t}-1) + \int_0^t (e^{-t}-1)dt = -(e^{-t}-1) - (e^{-t}-1) - t = 2 - 2e^{-t} - t$$

$$(b)\mathcal{L}[e^{-bt} - e^{-at}] = \frac{1}{s+b} - \frac{1}{s+a}$$

$$\mathcal{L}\left[\frac{e^{-bt}-e^{-at}}{t}\right] = \int_s^\infty \left(\frac{1}{s+b} - \frac{1}{s+a}\right)ds = \ln\frac{s+b}{s+a}\Bigg|_s^\infty = \ln\frac{s+a}{s+b}$$

$$\therefore \mathcal{L}^{-1}\left[\ln\frac{s+a}{s+b}\right] = \frac{e^{-bt}-e^{-at}}{t}$$

6. Convolution

Ex. 6.

$$(a)\mathcal{L}^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] \qquad (b)\mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right].$$

Solution : $(a)\mathcal{L}[\sin\omega t]=\dfrac{\omega}{s^2+\omega^2}\Rightarrow \mathcal{L}[\dfrac{1}{\omega}\sin\omega t]=\dfrac{1}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega^2}\int_0^t\frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]d\tau$$

$$=\frac{1}{2\omega^2}\int_0^t[\cos(2\omega\tau-\omega t)-\cos\omega t]d\tau=\frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau-\omega t)-\tau\cos\omega t\right]_0^t$$

$$=\frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t-\sin(-\omega t))]-t\cos\omega t\}=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$(b)\mathcal{L}[\dfrac{1}{\omega}\sin\omega t]=\dfrac{1}{s^2+\omega^2}\qquad \mathcal{L}[\cos\omega t]=\dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}]=\frac{1}{\omega}\int_0^t\sin\omega\tau\cos\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega}\int_0^t\frac{1}{2}[\sin(\omega\tau+\omega t-\omega\tau)+\sin(\omega\tau-\omega t+\omega\tau)]d\tau$$

$$=\frac{1}{2\omega}\int_0^t[\sin\omega t+\sin(2\omega\tau-\omega t)]d\tau=\frac{1}{2\omega}\left[\tau\sin\omega t+\frac{-1}{2\omega}\cos(2\omega\tau-\omega t)\right]_0^t$$

$$=\frac{1}{2\omega}\{t\sin\omega t-\frac{1}{2\omega}[\cos\omega t-\cos(-\omega t)]\}=\frac{t}{2\omega}\sin\omega t$$

II. Partial Fraction

If $F(s)=\dfrac{P(s)}{Q(s)}$, where deg[$P(s)$]<deg[$Q(s)$]

1.$Q(s)$=0 with unrepeated factors $s-a_i$

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}$$

$$A_k=\lim_{S\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)]=P(a_k)\lim_{S\to a_k}\frac{s-a_k}{Q(s)}$$

$$=P(a_k)\lim_{s\to a_k}\frac{1}{Q'(s)}=\frac{P(a_k)}{Q'(a_k)}$$

$$\frac{P(s)}{Q(s)}=\frac{P(a_1)/Q'(a_1)}{s-a_1}+\frac{P(a_2)/Q'(a_2)}{s-a_2}+\cdots\cdots+\frac{P(a_n)/Q'(a_n)}{s-a_n}$$

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=\frac{P(a_1)}{Q'(a_1)}e^{a_1t}+\frac{P(a_2)}{Q'(a_2)}e^{a_2t}+\cdots\cdots+\frac{P(a_n)}{Q'(a_n)}e^{a_nt}$$

Ex. 7.

$\mathcal{L}^{-1}[\dfrac{s+1}{s^3+s^2-6s}]$.

$$\text{Solution}: \frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$A_1 = \lim_{s \to 0} \frac{s+1}{(s-2)(s+3)} = -\frac{1}{6}$$

$$A_2 = \lim_{s \to 2} \frac{s+1}{s(s+3)} = \frac{3}{10}$$

$$A_3 = \lim_{s \to -3} \frac{s+1}{s(s-2)} = \frac{-2}{15}$$

$$L^{-1}[\frac{s+1}{s^3+s^2-6s}] = \frac{-\frac{1}{6}}{s} + \frac{\frac{3}{10}}{s-2} + \frac{\frac{-2}{15}}{s+3} = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

2. $Q(s)=0$ with repeated factors $(s-a_k)^m$

$$\frac{P(s)}{Q(s)} = \frac{C_m}{(s-a_k)^m} + \frac{C_{m-1}}{(s-a_k)^{m-1}} + \cdots\cdots + \frac{C_1}{s-a_k}$$

$$\frac{P(s)}{Q(s)}(s-a_k)^m = C_m + C_{m-1}(s-a_k) + C_{m-2}(s-a_k)^2 + \cdots + C_1(s-a_k)^{m-1}$$

$$C_m = \lim_{s \to a_k}[\frac{P(s)}{Q(s)}(s-a_k)^m]$$

$$C_{m-1} = \lim_{s \to a_k}\{\frac{d}{ds}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}$$

$$C_{m-2} = \lim_{s \to a_k}\{\frac{d^2}{ds^2}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{2!}$$

…………………………………………………………………………

$$C_1 = \lim_{s \to a_k}\{\frac{d^{m-1}}{ds^{m-1}}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{(m-1)!}$$

$$L^{-1}[\frac{P(s)}{Q(s)}] = e^{a_k t}[C_m \frac{t^{m-1}}{(m-1)!} + C_{m-1}\frac{t_{m-2}}{(m-2)!} + \cdots + C_2 t + C_1]$$

Ex. 8.

$$L^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}].$$

Solution : $\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}=\frac{C_2}{s^2}+\frac{C_1}{s}+\frac{A_1}{s-1}+\frac{A_2}{s-2}+\frac{A_3}{s-3}$

$$C_2=\lim_{s\to 0}\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}=\frac{-12}{-6}=2$$

$$C_1=\lim_{s\to 0}\frac{d}{ds}[\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}]$$

$$=\frac{4(-1)(-2)(-3)-(-12)[(-2)(-3)+(-1)(-3)+(-1)(-2)]}{[(-1)(-2)(-3)]^2}=\frac{-24+12\times 11}{6^2}=3$$

$$A_1=\lim_{s\to 1}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-2)(s-3)}=\frac{-1}{2}$$

$$A_2=\lim_{s\to 2}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-3)}=\frac{8}{-4}=-2$$

$$A_3=\lim_{s\to 3}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)}=\frac{9}{18}=\frac{1}{2}$$

$$L^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}]=2t+3-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

3. $Q(s)=0$ with unrepeated factor $(s-\alpha)^2+\beta$, where $\beta>0$

$$\frac{P(s)}{Q(s)}=\frac{As+B}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]=As+B$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}=A(\alpha+i\beta)+B$$

$$R+iI=(A\alpha+\beta)+iA\beta$$

where R and I are the real and imaginary parts of $\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}$, respectively

then, $\begin{cases}A\alpha+B=R\\ A\beta=I\end{cases}$, where we can get A and B, and

$$L^{-1}[\frac{P(s)}{Q(s)}]=L^{-1}[\frac{A(s-\alpha)+(A\alpha+B)}{(s-\alpha)^2+\beta^2}]=e^{\alpha t}\left(A\cos\beta t+\frac{A\alpha+B}{\beta}\sin\beta t\right)$$

Ex. 9.

$$L^{-1}[\frac{s^2}{s^4+4}].$$

$$\text{Solution}: \frac{s^2}{s^4+4} = \frac{s^2}{(s^2)^2 + 2\cdot s^2\cdot 2 + 2^2 - 2\cdot s^2 \cdot 2} = \frac{s^2}{(s^2+2)^2 - (2s)^2}$$

$$= \frac{s^2}{(s^2+2s+2)(s^2-2s+2)} = \frac{A_1 s + B_1}{(s+1)^2+1} + \frac{A_2 s + B_2}{(s-1)^2+1}$$

$$\lim_{s\to -1+i} \frac{s^2}{(s-1)^2+1} = A_1(-1+i) + B_1 \Rightarrow \frac{-2i}{4-4i} = (-A_1 + B_1) + iA_1$$

$$\frac{8-8i}{32} = (-A_1+B_1) + iA_1 \Rightarrow A_1 = -\frac{1}{4}, B_1 = 0$$

$$\lim_{s\to 1+i} \frac{s^2}{(s+1)^2+1} = A_2(1+i) + B_2 \Rightarrow \frac{2i}{4+4i} = (A_2 + B_2) + iA_2$$

$$\frac{8+8i}{32} = (A_2+B_2) + iA_2 \Rightarrow A_2 = \frac{1}{4}, B_2 = 0$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{s^4+4}\right] = \mathcal{L}^{-1}\left[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1} + \frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}\right]$$

$$= \frac{e^{-t}}{4}(-\cos t + \sin t) + \frac{e^{t}}{4}(\cos t + \sin t)$$

4. Q(s)=0 with repeated complex factor $[(s-\alpha)^2+\beta]^2$, where $\beta>0$

$$\frac{P(s)}{Q(s)} = \frac{As+B}{[(s-\alpha)^2+\beta^2]^2} + \frac{Cs+D}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2 = As + B + (Cs+D)[(s-\alpha)^2+\beta^2]$$

$$\lim_{s\to\alpha+i\beta}\left\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\right\} = A(\alpha+i\beta)+B$$

$$R_1 + iI_1 = (A\alpha+B) + iA\beta \Rightarrow \begin{cases} A\alpha + B = R_1 \\ A\beta = I_1 \end{cases}, \text{ where } A \text{ and } B \text{ can be obtained}$$

$$\lim_{s\to\alpha+i\beta}\frac{d}{ds}\left\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\right\} = A + [C(\alpha+i\beta)+D]\lim_{s\to\alpha+i\beta}\frac{d}{ds}[(s-\alpha)^2+\beta^2]$$

$$R_2 + iI_2 = A + [C(\alpha+i\beta)+D]2i\beta = (A - 2C\beta^2) + i(2\alpha\beta C + 2\beta D)$$

$$\Rightarrow \begin{cases} A - 2C\beta^2 = R_2 \\ 2\alpha\beta C + 2\beta D = I_2 \end{cases}, \text{ where we get } C \text{ and } D, \text{ hence}$$

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \mathcal{L}^{-1}\left\{\frac{A(s-\alpha)+(A\alpha+B)}{[(s-\alpha)^2+\beta^2]^2}\right\} + \mathcal{L}^{-1}\left[\frac{C(s-\alpha)+(C+D)}{(s-\alpha)^2+\beta^2}\right]$$

$$= e^{\alpha t}\left\{\left[\frac{At}{2\beta}\sin\beta t + (A\alpha+B)\frac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)\right] + \left[C\cos\beta t + (C\alpha+D)\frac{1}{\beta}\sin\beta t\right]\right\}$$

Ex. 10.

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}].$$

$$\text{Solution}: \frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}=\frac{As+B}{[(s-1)^2+1]^2}+\frac{cs+D}{(s-1)^2+1}$$

$$\lim_{s\to 1+i}(s^3-3s^2+6s+4)=A(1+i)+B$$

$$2i=(A+B)+iA \Rightarrow A=2, B=-2$$

$$\lim_{s\to 1+i}\frac{d}{ds}(s^3-3s^2+6s+4)=A+[c(1+i)+D]\lim_{s\to 1+i}\frac{d}{ds}[(s-1)^2+1]$$

$$0=A+(c+ic+D)2i=(A-2c)+2i(c+D)$$

$$c=1, D=-1$$

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}]=\mathcal{L}^{-1}\{\frac{2(s-1)}{[(s-1)^2+1]^2}\}+\mathcal{L}^{-1}[\frac{s-1}{(s-1)^2+1}]$$

$$=e^t(2\cdot\frac{t}{2}\sin t+\cos t)=e^t(t\sin t+\cos t)$$

IV. Differentiation with Respect to a Number

Ex. 11.

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}].$$

$$\text{Solution}: \frac{d}{d\omega}(\frac{1}{s^2+\omega^2})=\frac{-2\omega}{(s^2+\omega^2)^2}\Rightarrow \mathcal{L}^{-1}[\frac{d}{d\omega}(\frac{1}{s^2+\omega^2})]=\mathcal{L}^{-1}[\frac{-2\omega}{(s^2+\omega^2)^2}]$$

$$-2\omega\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{d}{d\omega}\mathcal{L}^{-1}[\frac{1}{s^2+\omega^2}]=\frac{d}{d\omega}(\frac{1}{\omega}\sin\omega t)=-\frac{1}{\omega^2}\sin\omega t+\frac{t}{\omega}\cos\omega t$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

V. Method of Differential Equation

Ex. 12.

$$\mathcal{L}^{-1}[e^{-\sqrt{s}}].$$

Solution : $\bar{y} = e^{-\sqrt{s}} \Rightarrow \bar{y}' = -\dfrac{e^{-\sqrt{s}}}{2\sqrt{s}}, \ \bar{y}'' = \dfrac{e^{-\sqrt{s}}}{4s} + \dfrac{e^{-\sqrt{s}}}{4\sqrt{s^3}}$

we get the equation $4s\bar{y}'' + 2\bar{y}' - \bar{y} = 0 \Rightarrow 4\mathcal{L}[\dfrac{d}{dt}(t^2 y)] + 2\mathcal{L}[-ty] - \mathcal{L}[y] = 0$

$$4\frac{d}{dt}(t^2 y) - 2ty - y = 0 \Rightarrow 4t^2 y' + (6t-1)y = 0 \Rightarrow \frac{dy}{y} + \frac{6t-1}{4t^2}dt = 0$$

$$\ln y + \frac{3}{2}\ln t + \frac{1}{4t} = c_1 \Rightarrow y = ct^{-\frac{3}{2}}e^{-\frac{1}{4t}}$$

$\because \mathcal{L}[t^{-\frac{1}{2}}] = \dfrac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$, and $\mathcal{L}[ty] = \mathcal{L}[ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}]$

while $\mathcal{L}[ty] = -\bar{y}' = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}} \Rightarrow \mathcal{L}[ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}] = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}}$

Apply general final value theorem $\lim\limits_{t\to\infty} \dfrac{ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}}{t^{-\frac{1}{2}}} = \lim\limits_{s\to 0} \dfrac{\frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{\frac{\sqrt{\pi}}{\sqrt{s}}} \Rightarrow c = \dfrac{1}{2\sqrt{\pi}}$

$$\therefore y = \frac{1}{2\sqrt{\pi}t^{3/2}}e^{-\frac{1}{4t}}$$

Applied to Solve Differential Equations

I. Ordinary Differential Equations with Constant Coefficients

Ex. 1.

$y'' + y' + y = g(x),\ y(0) = 1,\ y'(0) = 0$, where $g(x) = \begin{cases} 1 & 0 < x < 3 \\ 3 & x > 3 \end{cases}$.

Solution : $g(x) = u(x) + 2u(x-3)$

$$[s^2Y - sy(0) - y'(0)] + [sY - y(0)] + Y = \frac{1}{s} + 2\frac{e^{-3s}}{s}$$

$$(s^2 + s + 1)Y = s + 1 + \frac{1}{s} + 2\frac{e^{-3s}}{s}$$

$$Y = \frac{s+1}{s^2+s+1} + \frac{1}{s(s^2+s+1)} + \frac{2e^{-3s}}{s(s^2+s+1)}$$

$$= \frac{s+1}{s^2+s+1} + (\frac{1}{s} - \frac{s+1}{s^2+s+1}) + 2e^{-3s}(\frac{1}{s} - \frac{s+1}{s^2+s+1})$$

$$\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow \mathcal{L}^{-1}[\frac{s+1}{s^2+s+1}] = e^{-\frac{x}{2}}(\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x)$$

$$y(x) = u(x) + 2u(x-3)\{1 - e^{-\frac{x-3}{2}}[\cos\frac{\sqrt{3}}{2}(x-3) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(x-3)]\}$$

Ex. 2.

$$y'''(t)-2y''(t)+5y'(t)=0,\ y(0)=0,\ y'(0)=1,\ y(\frac{\pi}{8})=1.$$

$$\text{Solution}: [s^3Y-s^2y(0)-sy'(0)-y''(0)]-2[s^2Y-sy(0)-y'(0)]+5[sY-y(0)]=0$$

$$y''(0)=c$$

$$Y=\frac{s+c-2}{s(s^2-2s+5)}=\frac{A}{s}+\frac{Ps+Q}{(s-1)^2+2^2}$$

$$A=\lim_{s\to 0}\frac{s+c-2}{s^2-2s+5}=\frac{c-2}{5}$$

$$P(1+2i)+Q=\lim_{s\to 1+2i}\frac{s+c-2}{s}=\frac{-1+c+2i}{1+2i}=\frac{c+3}{5}+\frac{4-2c}{5}i$$

$$P=\frac{2-c}{5},\quad Q=\frac{2c+1}{5}$$

$$y(t)=\frac{c-2}{5}+e^t(\frac{2-c}{5}\cos 2t+\frac{c+3}{10}\sin 2t)$$

$$y(\frac{\pi}{8})=1\Rightarrow 1=\frac{c-2}{5}+e^{\frac{\pi}{8}}(\frac{2-c}{5}\frac{1}{\sqrt{2}}+\frac{c+3}{10}\frac{1}{\sqrt{2}})\Rightarrow c=7$$

$$\therefore y(t)=1+e^t(-\cos 2t+\sin 2t)$$

II. Ordinary Differential Equations with Variable Coefficients

Ex. 3.

$ty''+(1-2t)y'-2y=0,\ y(0)=1,\ y'(0)=2.$

$$\text{Solution}: -\frac{d}{ds}[s^2Y-sy(0)-y'(0)]+\{[sY-y(0)]+2\frac{d}{ds}[sY-y(0)]\}-2Y=0$$

$$(-s^2Y'-2sY+1)+[(sY-1)+2(sY'+Y)]-2Y=0$$

$$(-s^2+2s)Y'+(-2s+s+2-2)Y=0$$

$$-(s-2)Y'=Y\Rightarrow\frac{dY}{Y}=-\frac{ds}{s-2}\Rightarrow\ln Y=-\ln(s-2)+c_1$$

$$Y=\frac{c}{s-2}\Rightarrow y(t)=ce^{2t}$$

$$y(0)=1,\ \therefore 1=c,\ y(t)=e^{2t}$$

III. Simultaneous Ordinary Differential Equations

Ex. 4.

$$\begin{cases}\dfrac{dx}{dt}=2x+y+2e^{5t}\\[2mm]\dfrac{dy}{dt}=x+2y+3e^{2t}\end{cases},\quad x(0)=y(0)=0.$$

$$\text{Solution : }\begin{cases} sX - x(0) = 2X + Y + \dfrac{2}{s-5} \\ sY - y(0) = X + 2Y + \dfrac{3}{s-2} \end{cases} \Rightarrow \begin{cases} (s-2)X - Y = \dfrac{2}{s-5} \\ -X + (s-2)Y = \dfrac{3}{s-2} \end{cases}$$

$$X = \frac{(s-2)\dfrac{2}{s-5} + \dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{2s^2 - 5s - 7}{(s-1)(s-2)(s-3)(s-5)}$$

$$Y = \frac{\dfrac{2}{s-5} + (s-2)\dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{3s - 13}{(s-1)(s-3)(s-5)}$$

$$X = \frac{5/4}{s-1} + \frac{-3}{s-2} + \frac{1}{s-3} + \frac{3/4}{s-5} \Rightarrow x(t) = \frac{5}{4}e^{t} - 3e^{2t} + e^{3t} + \frac{3}{4}e^{5t}$$

$$Y = \frac{-5/4}{s-1} + \frac{1}{s-3} + \frac{1/4}{s-5} \Rightarrow y(t) = -\frac{5}{4}e^{t} + e^{3t} + \frac{1}{4}e^{5t}$$

UNIT III
FOURIER SERIES

- **Definition of periodic function**
- **Determination of Fourier coefficients**
- **Fourier expansion of periodic function in a given interval of length 2π**
- **Fourier series of even and odd functions**
- **Fourier series in an arbitrary interval**
- **Half- range Fourier sine and cosine expansions**

INTRODUCTION:

Fourier series which was named after the French mathematician "Jean-Baptise Joseph Fourier" (1768-1830).Fourier series is an infinite series representation of periodic function in terms of trigonometric sine and cosine functions. It is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms. We know that Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives because of the periodic nature Fourier series constructed for one period is valid for all. Fourier series has been an important tool in solving problems in many fields like current and voltage in alternating circuit, conduction of heat in solids, electrodynamics etc.

Periodic function

A function f:R→R is said to be periodic if there exists a positive number t such that f(x+T)=f(x) for all x belongs to R .

- T is called the **period** of f(x).
- If a function f(x) has a smallest period T(>0) then this is called **fundamental period of f(x) or primitive period of f(x)**

EXAMPLE

- Sin x, cos x are periodic functions with primitive period 2π
- Sinnx,cosnx are periodic functions with primitive period $\frac{2\pi}{n}$
- Tanx are periodic functions with primitive period π
- Tannx are periodic functions with primitive period $\frac{\pi}{n}$
- If f(x)= constant is a periodic function but it has no primitive period

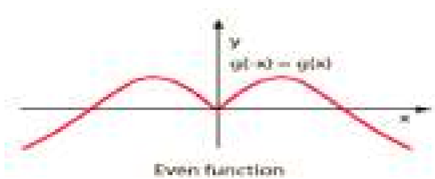
NOTE

- Any integral multiple of T is also a period i.e. if f(x) is a periodic then f(x+nT)=f(x).where T is a period and n ϵ Z
- If f_1 and f_2 are periodic functions having same period T then $f(x)=c_1f_1(x)+c_2f_2(x)$,[c_1,c_2 are constants] is also the periodic function of period T
- If T is the period of f then $f(c_1x+c_2)$also has the period T [c_1,c_2 are constants]
- If f(x) is a periodic function of x of period T
  - (1) f(ax),a≠0 is periodic function of x of period T/a
  - (2) f(x/b), b≠0 is periodic function of x of period Tb

EVEN FUNCTION:

A function f(x) is even function if f(-x)=f(x)

Ex: f(x)= cos x, x^2

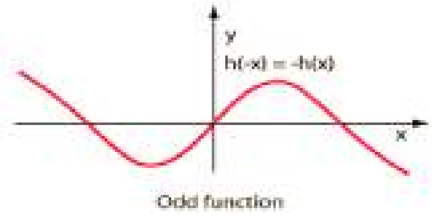


- ➢ The graph of even function y=f(x) is symmetric about Y-axis
- ➢ If f(x) is even function $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

ODD FUNCTION:

A function f(x) is odd function if f(-x)= -f(x)

Ex: f(x)=sin x, x^3



- The graph of odd function y=f(x) is symmetric about the origin
- If f(x) is odd function $\int_{-a}^{a} f(x)dx = 0$

NOTE

- There may be some functions which are neither even nor odd

  Ex: f(x) =4sinx +3tanx-e^x
- The product of two even functions is even
- The product of two odd functions is even
- The product of an even and odd function is odd

TRIGONOMETRIC SERIES: A series of the form

$f(x)=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\ldots\ldots a_n\cos x+\ldots b_1\sin x+b_2\sin 2x+b_3\sin 3x+\ldots\ldots b_n\sin x+\ldots$

$f(x)= a_0+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0,a_1,a_2,\ldots a_n$ and $b_1,b_2,\ldots b_n$ are coefficient of the series.Since each term of the trigonometric series is a function of period 2π it can be observed that if the series is convergent then its sum is also a function of period 2π

CONDITIONS FOR FOURIER EXPANSION (DIRICHLET CONDITIONS)

A function f(x) defined in [0,2π] has a valid Fourier series expansion of the form

$$\mathbf{f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

Where a_0, a_n, b_n are constants, provided

- f(x) is well defined and single-valued, except possibly at a finite number of point in the interval [0,2π] .
- f(x) has finite number of finite discontinuities in the interval in [0,2π] .
- f(x) has finite number of finite maxima and minima.

Note: The above conditions are valid for the function defined in the Intervals [-π,π],[0,2l], [-l,l]

- {1,cos 1x,cos 2x,….,cosnx,…..,sin 1x,sin 2x,….,sin nx,….}

  Consider any two, All these have a common period 2π . Here 1=cos 0x
- $\{1,\cos\frac{\pi x}{l},\cos\frac{2\pi x}{l},\ldots,\cos\frac{n\pi x}{l},\ldots,\sin\frac{\pi x}{l},\sin\frac{2\pi x}{l},\ldots,\sin\frac{n\pi x}{l},\ldots\}$

  All these have a common period 2l .

These are called complete set of orthogonal functions.

Then the Fourier series converges to f(x) at all points where f(x) is continuous. Also the series converges to the average of the left limit and right limit of f(x) at each point of discontinuity of f(x).

Example

- $\sin^{-1} x$ cannot be expanded as fourier series since it is not single valued
- Tan x cannot be expanded as Fourier series in $(0,2\pi)$ since tan x is infinite at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $c \le x \le c+2\pi$ is given by

$$f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx dx$$

These values are known as Euler's Formulae.

Proof:consider $f(x)=\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$----------(1)

Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_c^{c+2\pi} f(x)dx = \int_c^{c+2\pi} \frac{a_0}{2}dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty} a_n \cos nxdx + \int_c^{c+2\pi}\sum_{n=1}^{\infty} b_n \sin nxdx$$

$$\int_c^{c+2\pi} f(x)dx = \int_c^{c+2\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nxdx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nxdx$$

$$\int_c^{c+2\pi} f(x)dx = \frac{a_0}{2}[c+2\pi - c]$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

Multiplying cosnx and Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_c^{c+2\pi} f(x)\cos nx\, dx = \int_c^{c+2\pi} \frac{a_0}{2}\cos nxdx + \int_c^{c+2\pi}\sum_{n=1}^{\infty} a_n \cos nx \cos nxdx + \int_c^{c+2\pi}\sum_{n=1}^{\infty} b_n \sin nx \cos nxdx$$

$$\int_c^{c+2\pi} f(x)\cos nx\, dx = \int_c^{c+2\pi} \frac{a_0}{2}\cos nxdx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos^2 nxdx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx \cos nxdx$$

$$\int_c^{c+2\pi} f(x)\cos nx\, dx = a_n \pi$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\, dx$$

Multiplying sinnx and Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_{c}^{c+2\pi} f(x)\sin nx\,dx = \int_{c}^{c+2\pi} \frac{a_0}{2}\sin nx dx + \int_{c}^{c+2\pi}\sum_{n=1}^{\infty} a_n \cos nx \sin nx dx + \int_{c}^{c+2\pi}\sum_{n=1}^{\infty} b_n \sin nx \sin nx dx$$

$$\int_{c}^{c+2\pi} f(x)\sin nx\,dx = \int_{c}^{c+2\pi} \frac{a_0}{2}\sin nx dx + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2\pi} \cos nx \sin nx dx + \sum_{n=1}^{\infty} b_n \int_{c}^{c+2\pi} \sin^2 nx dx$$

$$\int_{c}^{c+2\pi} f(x)\sin nx\,dx = b_n \pi$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

DEFINITION OF FOURIER SERIES

- Let f(x) be a function defined in $[0,2\pi]$. Let $f(x+2\pi)=f(x)$ then the Fourier Series of is given by

  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

  Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

  $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$

  $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

  These values a_0, a_n, b_n are called as Fourier coefficients of f(x)in $[0,2\pi]$.

- Let f(x) be a function defined in $[-\pi,\pi]$. Let $f(x+2\pi)=f(x)$ then the Fourier Series of is given by

  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

  Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

  $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

  $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x)in $[-\pi,\pi]$

- Let f(x) is a function defined in $[0,2l]$. Let $f(x+2l)=f(x)$ then the Fourier Series of is given by

  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$

  Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

  $a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\, dx$

  $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l} dx$

  These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in $[0,2l]$

- Let f(x) be a function defined in $[-l,l]$. Let $f(x+2l)=f(x)$ then the Fourier Series of is given by

  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in [-*l*,*l*]

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that if *f(x)* be a function defined in [-π, π]. Let f(x+2π) =f(x), then the Fourier series *f(x)* of is given by

$$f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in [-π, π]

<u>Case (i):</u> When f(x) is an even function

- $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$

Where $a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

- $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$

Where $a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$

$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

<u>Case (ii):</u> When f(x) is an Odd Function

- $f(x)= \sum_{n=1}^{\infty} b_n\sin nx$

Where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

- $f(x)= \sum_{n=1}^{\infty} b_n\sin nx$

Where $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

Let f(x) be defined by f(x) = $\begin{cases} f_1(x), c < x < x_0 \\ f_2(x), x_0 < x < c + 2\pi \end{cases}$

Where x_0 is the point of discontinuity in (c, c+2π)
Then the Fourier coefficient is given by

$$\boldsymbol{a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\;dx\right]}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]}$$

The Fourier series converges to $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity of *f(x)*

PROBLEMS

1 **Find the Fourier series expansion of f(x) = x², 0 < x < 2π. Hence deduce that**

$(i)\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.............=\frac{\pi^2}{6}$

$(ii)\quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-.............=\frac{\pi^2}{12}$

$(iii)\quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.............=\frac{\pi^2}{8}$

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{0 + \frac{(4\pi)(1)}{n^2} - 0\right\} - \{0 + 0 - 0\}\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3}\right\} - \left\{0 + 0 + \frac{2}{n^3}\right\}\right]$$

$$= -\frac{4\pi}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$$

$$f(x) = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + ...\right] - 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]$$

Put $x = 0$ in the above series we get

$$f(0) = \frac{4\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right] - 4\pi(0) \quad \text{--------------- (1)}$$

But $x = 0$ is the point of discontinuity. So we have

$$f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{(0) + (4\pi^2)}{2} = 2\pi^2$$

Hence equation (1) becomes

$$2\pi^2 = \frac{4\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right]$$

$$2\pi^2 - \frac{4\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \quad --------(2)$$

Now, put $x = \pi$ (which is point of continuity) in the above series we get

$$\pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\right] - 4\pi(0)$$

$$\pi^2 - \frac{4\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots \quad --------(3)$$

Adding (2) and (3), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots$$

2 **Expand in Fourier series of f(x) = $x \sin x$ for $0 < x < 2\pi$ and deduce the result**

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots = \frac{\pi - 2}{4}$$

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi + 0) - (0 + 0)\right]$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx \sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx, \quad n \neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x \sin(n-1)x\,dx$$

$$= \frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n+2}}{n+1}+0\right\}-\{0+0\}\right]-\frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n-2}}{n-1}+0\right\}-\{0+0\}\right]$$

$$= \frac{-1}{n+1}+\frac{1}{n-1}$$

$$a_n = \frac{-(n-1)+(n+1)}{(n+1)(n-1)}$$

$$a_n = \frac{2}{n^2-1}\ , \quad n \neq 1$$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi\left(\frac{-1}{2}\right)+0\right\}-(0+0)\right]$$

$$= -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x-\cos(n+1)x\right]dx\ , \quad n \neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\ dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\ dx$$

$$= \frac{1}{2\pi}\left[(x)\left(\frac{\sin(n-1)x}{n-1}\right)-(1)\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{\sin(n+1)x}{n+1}\right)-(1)\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n-2}}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n+2}}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\left\{0+\frac{1}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{1}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$b_n = 0\ , \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \left\{x\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{4}\right)\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi^2 - 0 - \left(\frac{1}{2}\right)\right\} - \left\{0 - 0 - \frac{1}{2}\right\}\right]$$

$$= \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$$

$$= \frac{-2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{(n-1)(n+1)}\cos nx + \pi\sin x + 0$$

$$x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + 2\left[\frac{\cos 2x}{1.3} + \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} + \frac{\cos 5x}{4.6} + \ldots\ldots\ldots\ldots\right]$$

Put $x = \frac{\pi}{2}$ in the above series we get

$$\frac{\pi}{2}(1) = -1 - 0 + \pi(1) + 2\left[\frac{-1}{1.3} + 0 + \frac{1}{3.5} + 0 + \frac{-1}{5.7} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi}{2} - \pi + 1 = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi - 2\pi + 2}{2} = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{-\pi + 2}{2} = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots$$

3 **Obtain the Fourier expansion of f(x)=e^{-ax} in the interval (-π, π).**

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

$$f(0)=1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$1 = \frac{2\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

4 **Find the Fourier series for the function $f(x) = 1 + x + x^2$ in $(-\pi, \pi)$. Deduce**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$$

Sol The given function is neither an even nor an odd function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1 + x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi + \frac{\pi^2}{2} + \frac{\pi^3}{3}\right\} - \left\{-\pi + \frac{\pi^2}{2} - \frac{\pi^3}{3}\right\}\right]$$

$$= \frac{1}{\pi}\left[2\pi + \frac{2\pi^3}{3}\right] = 2 + \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(1+x+x^2)\left(\frac{\sin nx}{n}\right)-(1+2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{0+\frac{(1+2\pi)(-1)^n}{n^2}-0\right\}-\left\{0+\frac{(1-2\pi)(-1)^n}{n^2}-0\right\}\right]$$

$$= \frac{(-1)^n}{\pi n^2}\left[1+2\pi-1+2\pi\right]$$

$$= \frac{(-1)^n}{\pi n^2}(4\pi) = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(1+x+x^2)\left(\frac{-\cos nx}{n}\right)-(1+2x)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-(1+\pi+\pi^2)\frac{(-1)^n}{n}+0+\frac{2(-1)^n}{n^3}\right\}-\left\{-(1-\pi+\pi^2)\frac{(-1)^n}{n}+0+\frac{2(-1)^n}{n^3}\right\}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[-1-\pi-\pi^2+1-\pi+\pi^2\right]$$

$$= \frac{(-1)^n}{n\pi}(-2\pi) = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$= \frac{1}{2}\left(2+\frac{2\pi^2}{3}\right)+\sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx+\frac{2(-1)^{n+1}}{n}\sin nx\right]$$

$$= 1+\frac{\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+..........\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-..........\right]$$

$$f(x) = 1+\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-...........\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-...........\right]$$

Put $x = \pi$ in the above series we get

$$f(\pi) = 1+\frac{\pi^2}{3}-4\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-...........\right]+2(0) \quad \text{---------------} \ (1)$$

But $x=\pi$ is the point of discontinuity. So we have

$$f(\pi) = \frac{f(-\pi)+f(\pi)}{2} = \frac{(1-\pi+\pi^2)+(1+\pi+\pi^2)}{2} = \frac{2+2\pi^2}{2} = 1+\pi^2$$

Hence equation (1) becomes

$$1+\pi^2 = 1+\frac{\pi^2}{3}-4\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\ldots\ldots\ldots\ldots\right]$$

$$\pi^2-\frac{\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots$$

5 **Find the Fourier series expansion of $(\pi - x)^2$ in $-\pi < x < \pi$.**

Sol Fourier series is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-3\pi}\left[0-8\pi^3\right]$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right)-[2(\pi - x)(-1)]\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\{0+0-0\}-\left\{0-\frac{(4\pi)(-1)^n}{n^2}-0\right\}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right)-[2(\pi - x)(-1)]\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{0+0+\frac{2(-1)^n}{n^3}\right\}-\left\{-(4\pi^2)\frac{(-1)^n}{n}+0+\frac{2(-1)^n}{n^3}\right\}\right]$$

$$= \frac{4\pi(-1)^n}{n}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{1}{2}\left(\frac{8\pi^2}{3}\right)+\sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx+\frac{4\pi(-1)^n}{n}\sin nx\right]$$

$$f(x)=\frac{4\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\ldots\ldots\right]+4\pi\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\ldots\ldots\right]$$

$$(i.e.)\ f(x)=\frac{4\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\ldots\ldots\right]-4\pi\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\ldots\ldots\right]$$

6 **Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.**

Sol Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3}+b_n\sin\frac{2n\pi x}{3}\right)$$

$$a_0=\frac{1}{(3/2)}\int_0^3 f(x)\,dx=\frac{2}{3}\int_0^3(2x-x^2)\,dx$$

$$=\frac{2}{3}\left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$=\frac{2}{3}\left[\left(9-\frac{27}{3}\right)-(0-0)\right]$$

$$=0$$

$$a_n=\frac{1}{(3/2)}\int_0^3 f(x)\cos nx\,dx=\frac{2}{3}\int_0^3(2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]$$

$$=\frac{2}{3}\left[\left\{0-(-4)\left(\frac{-9}{4n^2\pi^2}\right)+0\right\}-\left\{0-(2)\left(\frac{-9}{4n^2\pi^2}\right)+0\right\}\right]$$

$$=\frac{2}{3}\left[\frac{-54}{4n^2\pi^2}\right]$$

$$=\frac{-9}{n^2\pi^2}$$

$$b_n=\frac{1}{(3/2)}\int_0^3 f(x)\sin nx\,dx=\frac{2}{3}\int_0^3(2x-x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{2}{3}\left[\left\{(-3)\left(\frac{-3}{2n\pi}\right)+0-2\left(\frac{27}{8n^3\pi^3}\right)\right\}-\left\{0+0-2\left(\frac{27}{8n^3\pi^3}\right)\right\}\right]$$

$$=\frac{3}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$$

$$= 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{3}{n\pi}\sin\frac{2n\pi x}{3}\right)$$

$$(i.e.)\ f(x) = -\frac{9}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} + \frac{1}{3^2}\cos\frac{6\pi x}{3} + \dots\dots\right]$$

$$+ \frac{3}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \dots\dots\right]$$

7 **Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$ and using this series find the root square mean value of $f(x)$ in the interval.**

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{l^2}{2} - \frac{l^3}{3}\right\} - \left\{\frac{l^2}{2} + \frac{l^3}{3}\right\}\right]$$

$$= \frac{1}{l}\left(\frac{-2l^3}{3}\right) = \frac{-2l^2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(x - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1 - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{0 + (1 - 2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\} - \left\{0 + (1 + 2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\}\right]$$

$$= \frac{(-1)^n l^2}{l\,n^2\pi^2}\left[1 - 2l - 1 - 2l\right]$$

$$= \frac{(-1)^n l}{n^2\pi^2}\left[-4l\right] = \frac{4\,l^2\,(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l}(x-x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{-(l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}-\left\{-(-l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{-(-1)^n l}{l\, n\pi}\left[l-l^2+l+l^2\right]$$

$$= \frac{(-1)^{n+1}}{n\pi}[2l] = \frac{2l\,(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}\left(\frac{-2l^2}{3}\right)+\sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{l}+\frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$(i.e.)\ \ f(x) = \frac{-l^2}{3}+\frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}-\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}-\frac{1}{4^2}\cos\frac{4\pi x}{l}+\ldots\ldots\ldots\right]$$

$$+\frac{2l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l}-\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}-\frac{1}{4}\sin\frac{4\pi x}{l}+\ldots\ldots\ldots\right]$$

8 **Obtain the Fourier series of f(x) = 1-x^2 over the interval (-1,1).**

Sol The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_{0}^{1}(1-x^2)dx = 2\left[x-\frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1}(1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)+(-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx \quad =0, \text{ since } f(x)\sin(n\pi x) \text{ is odd.}$$

The Fourier series of f(x) is

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

9 **Find the Fourier series for the function** $f(x) = \begin{cases} 1+x, & -2 \le x \le 0 \\ 1-x, & 0 \le x \le 2 \end{cases}$

Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Sol . $f(-x) = 1 - x$ in (–2, 0)
$= f(x)$ in (0, 2)
and $f(-x) = 1 + x$ in (0, 2)
$= f(x)$ in (–2, 0)
Hence $f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (1-x)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^2$$

$$= [(2-2)-(0)]$$

$$= 0$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 (1-x)\cos\frac{n\pi x}{2}dx$$

$$= \left[(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{4(-1)^n}{n^2\pi^2}\right\} - \left\{0 - \frac{4}{n^2\pi^2}\right\}\right]$$

$$= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n^2\pi^2} \cos \frac{n\pi x}{2}$$

$$= \frac{4}{\pi^2}\left[\frac{2}{1^2}\cos\frac{\pi x}{2} + 0 + \frac{2}{3^2}\cos\frac{3\pi x}{2} + 0 + \frac{2}{5^2}\cos\frac{5\pi x}{2} + ...\right]$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + ...\right]$$

Put $x = 0$ in the above series we get

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}...........\right] \quad \text{--------------- (1)}$$

But $x = 0$ is the point of discontinuity. So we have

$$f(0) = \frac{f(0-) + f(0+)}{2} = \frac{(1)+(1)}{2} = \frac{2}{2} = 1$$

Hence equation (1) becomes

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}...........\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}................$$

$$(i.e.)\ \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

10 **Obtain the sine series for** $f(x) = \begin{cases} x & in \quad 0 \le x \le \frac{l}{2} \\ l - x & in \quad \frac{l}{2} \le x \le l \end{cases}$

Sol Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l} (l-x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \frac{2}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}$$

$$=\frac{2}{l}\left[\left\{-\left(\frac{l}{2}\right)\left(\frac{l.\cos\frac{n\pi}{2}}{n\pi}\right)+\left(\frac{l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right\}-\{0+0\}\right]+\frac{2}{l}\left[\{0-0\}-\left\{-\left(\frac{l}{2}\right)\left(\frac{l.\cos\frac{n\pi}{2}}{n\pi}\right)-\left(\frac{l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right\}\right]$$

$$=\frac{2}{l}\left[\frac{2l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$b_n=\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$=\sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

$$=\frac{4l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin\frac{\pi x}{l}+0+\frac{1}{3^2}\sin\frac{3\pi}{2}\sin\frac{3\pi x}{l}+0+\frac{1}{5^2}\sin\frac{5\pi}{2}\sin\frac{5\pi x}{l}+0+.......................\right]$$

$$=\frac{4l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l}-\frac{1}{3^2}\sin\frac{3\pi x}{l}+\frac{1}{5^2}\sin\frac{5\pi x}{l}+.......................\right]$$

11 **Find the Fourier series of** $f(x)=\begin{cases}1, & 0<x<\pi\\ 2, & \pi<x<2\pi\end{cases}$

Sol Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}(1)\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}(2)\,dx$$

$$=\frac{1}{\pi}[x]_0^{\pi}+\frac{2}{\pi}[x]_{\pi}^{2\pi}$$

$$=\frac{1}{\pi}[(\pi-0)]+\frac{2}{\pi}[(2\pi-\pi)]$$

$$=1+2=3$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}(2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}+\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$=\frac{1}{\pi}(0-0)+\frac{2}{\pi}(0-0)$$

$$=0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \;= \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{-1}{n\pi}[(-1)^n - 1] - \frac{2}{n\pi}[1-(-1)^n]$$

$$= \frac{1}{n\pi}\left[-(-1)^n + 1 - 2 + 2(-1)^n\right]$$

$$= \frac{(-1)^n - 1}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty}\left[0.\cos nx + \frac{(-1)^n - 1}{n\pi}\sin nx\right]$$

$$= \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\dots\dots\dots\right]$$

12 **Find the Fourier series expansion of** $f(x) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l - x, & \frac{l}{2} < x < l \end{cases}$

Sol . Let $2L = l \Rightarrow L = \frac{l}{2}$, then the given function becomes

$$f(x) = \begin{cases} x, & 0 < x < L \\ 2L - x, & L < x < 2L \end{cases}$$

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx \;= \frac{1}{L}\int_0^{L}(x)\,dx + \frac{1}{L}\int_L^{2L}(2L - x)\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^{L} + \frac{1}{L}\left[\frac{(2L-x)^2}{-2}\right]_L^{2L}$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0\right] + \frac{1}{L}\left[0 - \frac{L^2}{-2}\right]$$

$$= \frac{L}{2} + \frac{L}{2} = L$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^{L} x\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_L^{2L}(2L-x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[(x)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L + \frac{1}{L}\left[(2L-x)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_L^{2L}$$

$$= \frac{1}{L}\left[\left\{0 + \frac{(-1)^n L^2}{n^2\pi^2}\right\} - \left\{0 + \frac{L^2}{n^2\pi^2}\right\}\right] + \frac{1}{L}\left[\left\{0 - \frac{L^2}{n^2\pi^2}\right\} - \left\{0 - \frac{(-1)^n L^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{L}\frac{L^2}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^{L} x\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_L^{2L}(2L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[(x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L + \frac{1}{L}\left[(2L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_L^{2L}$$

$$= \frac{1}{L}\left[\left\{-\frac{(-1)^n L^2}{n\pi} + 0\right\} - \{0+0\}\right] + \frac{1}{L}\left[\{0-0\} - \left\{-\frac{(-1)^n L^2}{n\pi} - 0\right\}\right]$$

$$= 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty}\left(\frac{2L[(-1)^n - 1]}{n^2\pi^2}\cos\frac{n\pi x}{L} + 0\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{L} + 0 - \frac{2}{3^2}\cos\frac{3\pi x}{L} + 0 - \frac{2}{5^2}\cos\frac{5\pi x}{L} + 0 - \dots\dots\dots\right]$$

$$= \frac{L}{2} - \frac{4L}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \dots\dots\dots\right]$$

$$(i.e.)\ f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{6\pi x}{l} + \frac{1}{5^2}\cos\frac{10\pi x}{l} + \dots\dots\dots\right]$$

13 **Find the Fourier series expansion of** $f(x) = \begin{cases} l - x, & 0 < x < l \\ 0, & l < x < 2l \end{cases}$

Hence deduce the value of the series (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots$

Sol

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{l}(l-x)\,dx + \frac{1}{l}\int_l^{2l}(0)\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^l$$

$$= \frac{1}{-2l}\left[0 - l^2\right]$$

$$= \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\cos\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\left\{0 - \frac{(-1)^n l^2}{n^2\pi^2}\right\} - \left\{0 - \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$= \frac{l}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\sin\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\{0 - 0\} - \left\{-\frac{l^2}{n\pi} - 0\right\}\right]$$

$$= \frac{l}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{l}{4} + \sum_{n=1}^{\infty}\left(\frac{l[(-1)^{n+1}+1]}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$= \frac{l}{4} + \frac{l}{\pi^2}\left[\frac{2}{1^2}\cos\frac{\pi x}{l} + 0 + \frac{2}{3^2}\cos\frac{3\pi x}{l} + 0 + \frac{2}{5^2}\cos\frac{5\pi x}{l} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\ldots\ldots\ldots\right]$$

$$(i.e.)\ f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\ldots\ldots\right]$$

$$+ \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\ldots\ldots\ldots\right] \text{--------(1)}$$

Put $x = \frac{l}{2}$ (which is point of continuity) in equation (1), we get

$$l - \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}(0) + \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 4\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{l}{2} = \frac{l}{4} + \frac{l}{\pi}\left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{l}{2} - \frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$$

Put $x = l$ in equation (1) we get

$$f(l) = \frac{l}{4} + \frac{2l}{\pi^2}\left[-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \ldots\ldots\ldots\ldots\right] \text{ --------------- (2)}$$

But $x = l$ is the point of discontinuity. So we have

$$f(l) = \frac{f(l-) + f(l+)}{2} = \frac{(0)+(0)}{2} = 0$$

Hence equation (2) becomes

$$0 = \frac{l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right]$$

$$-\frac{l}{4} = -\frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\ldots$$

HALF RANGE FOURIER SERIES

- **Half Range Fourier Sine Series defined in $[0,\pi]$:**

  The Fourier half range sine series in $[\mathbf{0},\boldsymbol{\pi}]$is given by $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

  Where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

This is Similar to the Fourier series defined for odd function in $[-\pi,\pi]$

- **Half Range Fourier Sine Series defined in $[0,l]$:**

  The Fourier half range sine series in $[\mathbf{0},\boldsymbol{\pi}]$is given by $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

  Where $b_n=\frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx$

This is Similar to the Fourier series defined for odd function in $[-l,l]$

- **Half Range Fourier cosine Series defined in $[0,\pi]$:**

  The Fourier half range cosine series in $[\mathbf{0},\boldsymbol{\pi}]$is given by

  $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$

  Where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$

  $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

This is Similar to the Fourier series defined for even function in $[-\pi,\pi]$

- **Half Range Fourier cosine Series defined in $[0,l]$:**

  The Fourier half range cosine series in $[\mathbf{0},\boldsymbol{l}]$is given by

  $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

  Where $a_0=\frac{2}{l}\int_0^{l} f(x)dx$

  $a_n=\frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}dx$

This is Similar to the Fourier series defined for even function in $[-l,l]$

Problems

1 **Find the half range sine series for $f(x) = 2$ in $0 < x < \pi$.**

Sol

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} 2\sin nx\,dx$$

$$=\frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}=\frac{-4}{n\pi}\left[(-1)^n-1\right]=\frac{4}{n\pi}\left[1-(-1)^n\right]$$

Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n\pi} \sin nx$$

$$= \frac{4}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \dots\dots\dots\dots\dots\right]$$

$$= \frac{8}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\dots\dots\dots\dots\right]$$

2 **Expand *f(x) = cos x, 0 < x < π* in a Fourier sine series.**

Sol Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin nx \cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]\,dx\,, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(n+1)x}{n+1}\right) + \left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= -\frac{1}{\pi}\left[(-1)^n\left\{\frac{-1}{n+1} + \frac{-1}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{2n}{n^2-1}\right\} + \left\{\frac{2n}{n^2-1}\right\}\right]$$

$$b_n = \frac{2n}{\pi(n^2-1)}\left[(-1)^n + 1\right], \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}(1-1) = 0$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0+\sum_{n=2}^{\infty}\frac{2n[(-1)^n+1]}{\pi(n^2-1)}\sin nx$$

$$=\frac{2}{\pi}\left[\frac{4\sin 2x}{3}+0+\frac{8\sin 4x}{15}+0+\frac{12\sin 6x}{35}+0+..................\right]$$

$$=\frac{8}{\pi}\left[\frac{\sin 2x}{3}+\frac{2\sin 4x}{15}+\frac{3\sin 6x}{35}+..................\right]$$

3 **Find the half range cosine series for the function f*(x)* =*x* (π – *x*) in 0<*x*< π.**

Sol Half range Fourier cosine series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\,dx$$

$$=\frac{2}{\pi}\left[\frac{\pi x^2}{2}-\frac{x^3}{3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)-(0-0)\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$=\frac{\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right)-(\pi-2x)\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left\{0+\frac{(-\pi)(-1)^n}{n^2}+0\right\}-\left\{0+\frac{(\pi)(1)}{n^2}+0\right\}\right]$$

$$=\frac{2\pi}{\pi n^2}\left[-(-1)^n-1\right]$$

$$=-\frac{2}{n^2}\left[(-1)^n+1\right]$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$=\frac{1}{2}\left(\frac{\pi^2}{3}\right)+\sum_{n=1}^{\infty}-\frac{2}{n^2}\left[(-1)^n+1\right]\cos nx$$

$$=\frac{\pi^2}{6}-2\left[0+\frac{2\cos 2x}{2^2}+0+\frac{2\cos 4x}{4^2}+0+\frac{2\cos 6x}{6^2}+0+\ldots\ldots\ldots\ldots\right]$$

$$=\frac{\pi^2}{6}-4\left[\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\ldots\ldots\ldots\ldots\right]$$

4 **Find the half range cosine series for the function *f(x)* = *x* in 0 < x < *l*.**

Sol

Half range Fourier cosine series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{l}\int_0^l x\,dx=\frac{2}{l}\left[\frac{x^2}{2}\right]_0^l=\frac{2}{l}\left[\frac{l^2}{2}-0\right]=l$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[(x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[\left\{0+\frac{(-1)^n l^2}{n^2\pi^2}\right\}-\left\{0+\frac{l^2}{n^2\pi^2}\right\}\right]$$

$$=\frac{2l}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l[(-1)^n-1]}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l}{2}+\frac{2l}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{l}+0-\frac{2}{3^2}\cos\frac{3\pi x}{l}+0-\frac{2}{5^2}\cos\frac{5\pi x}{l}+0-\ldots\ldots\ldots\ldots\right]$$

$$(i.e.)\ f(x)=\frac{l}{2}-\frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+\ldots\ldots\ldots\ldots\right]$$

5 **Find the half range sine series of *f(x)* = *x cos x* in *(0, π)*.**

Sol

Fourier sine series is $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,(2\sin nx\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx\ , \quad n \neq 1$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx\ ,\ n \neq 1$$

$$b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[x\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n+1}}{n+1} + 0\right\} - \{0+0\}\right] + \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n-1}}{n-1} + 0\right\} - \{0+0\}\right]$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^n}{n-1}$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{2n}{(n+1)(n-1)}\right]$$

$$(i.e.) b_n = \frac{2n(-1)^n}{n^2-1}\ ,\ n \neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi\left(\frac{-1}{2}\right) + 0\right\} - \{0+0\}\right] = -\frac{1}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

$$= -\frac{1}{2}\sin x + 2\left[\frac{2\sin 2x}{3} - \frac{3\sin 3x}{8} + \frac{4\sin 4x}{15} + \dots\dots\dots\dots\right]$$

6 **Obtain the half range cosine series for *f(x)* = (*x* – 2)^2 in the interval 0 <*x* < 2. Deduce that** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Sol Half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (x-2)^2\,dx$$

$$= \left[\frac{(x-2)^3}{3}\right]_0^2$$

$$= \left[0 - \left\{\frac{-8}{3}\right\}\right] = \frac{8}{3}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 (x-2)^2 \cos\frac{n\pi x}{2}\,dx$$

$$= \left[(x-2)^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - [2(x-2)]\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + (2)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \left[\{0+0-0\} - \left\{0 - \frac{16}{n^2\pi^2} - 0\right\}\right]$$

$$= \frac{16}{\pi^2 n^2}$$

$$f(x) = \frac{8}{6} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$(i.e.)\ f(x) = \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \dots\dots\dots\right]$$

Put $x = 0$ in equation (1) we get

$$f(0) = \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\dots\dots\right]$$

But $x = 0$ is the point of discontinuity. So we have

$$f(x) = \frac{(x+2)^2 + (x-2)^2}{2}$$

$$f(0) = \frac{(0+2)^2 + (0-2)^2}{2} = \frac{(4)+(4)}{2} = 4$$

Hence equation becomes

$$4=\frac{4}{3}+\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$4-\frac{4}{3}=\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\frac{8}{3}=\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots$$

Put $x = 2$ in equation we get

$$f(2)=\frac{4}{3}+\frac{16}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

But $x = 2$ is the point of discontinuity. So we have

$$f(x)=\frac{(x-2)^2+(2-x)^2}{2}$$

$$f(2)=\frac{(2-2)^2+(2-2)^2}{2}=0$$

Hence equation becomes

$$0=\frac{4}{3}+\frac{16}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$-\frac{4}{3}=-\frac{16}{\pi^2}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\ldots\ldots\ldots\ldots\ldots$$

we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\right]$$

$$\frac{3\pi^2}{12}=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots$$

$$(i.e.)\ \frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

UNIT IV
FOURIER TRANSFORMS

- Fourier integral theorem,
- Fourier sine and cosine integrals
- Fourier transforms
- Fourier sine and cosine transform
- Inverse transforms
- Finite Fourier transforms

Introduction

The Fourier transform named after Joseph Fourier, is a mathematical transformation employed to transform signals between time (or spatial) domain and frequency domain, which has many applications in physics and engineering. It is reversible, being able to transform from either domain to the other. The term itself refers to both the transform operation and to the function it produces.

In the case of a periodic function over time (for example, a continuous but not necessarily sinusoidal musical sound), the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients. They represent the frequency spectrum of the original time-domain signal. Also, when a time-domain function is sampled to facilitate storage or computer-processing, it is still possible to recreate a version of the original Fourier transform according to the Poisson summation formula, also known as discrete-time Fourier transform. See also Fourier analysis and List of Fourier-related transforms.

Integral Transform

The integral transform of a function f(x) is given by

$$I[f(x)] \text{ or } F(s) = \int_a^b f(x)k(s,x)dx$$

Where k(s, x) is a known function called **kernel of the transform**

s is called the **parameter of the transform**

f(x) is called the **inverse transform of F(s)**

Fourier transform

$k(s,x) = e^{isx}$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Laplace transform

$k(s,x) = e^{-sx}$

$$L[f(x)] = F(s) = \int_0^{\infty} f(x)e^{-sx}dx$$

Henkel transform

$k(s,x) = xJ_n(sx)$

$$H[f(x)] = H(s) = \int_0^{\infty} f(x)xJ_n(sx)dx$$

Mellin transform

$k(s,x) = x^{s-1}$

$$M[f(x)] = M(s) = \int_0^{\infty} f(x)x^{s-1}dx$$

DIRICHLET'S CONDITION

A function f(x) is said to satisfy Dirichlet's conditions in the interval (a,b) if

1. f(x) defined and is single valued function except possibly at a finite number of points in the interval (a,b)
2. f(x) and $f^1(x)$ are piecewise continuous in (a,b)

Fourier integral theorem

If f(x) is a given function defined in (-*l*,*l*) and satisfies the Dirichlet conditions then

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda$$

Proof:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})+\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})$$

where

$$a_0=\frac{1}{L}\int_{-L}^{L}f(t)\,dt$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos(\frac{n\pi x}{L})\,dt$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin(\frac{n\pi x}{L})\,dt$$

Substituting the values in f(x)

$$f(x)=\frac{1}{L}\int_{-L}^{L}f(t)[1+2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{L})(t-x)]\,dt---------(1)$$

But cosine functions are even functions

$$\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)=1+2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{L})(t-x)-------(2)$$

Substituting equation (2) in (1)

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L}f(t)\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)\,dt$$

$$\frac{n\pi}{L}=\lambda$$

$$\underset{n\to\infty}{Lt}\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)=\int_{-\infty}^{\infty}\cos\lambda(t-x)\,d\lambda=2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)[2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda]\,dt$$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)\,d\lambda dt$$

Fourier Sine Integral

If f (t) is an odd function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{0}^{\infty}f(t)\sin\lambda t dt d\lambda$$

Fourier Cosine Integral

If f (t) is an even function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{0}^{\infty}f(t)\cos\lambda t dt d\lambda$$

Problems

1 **Express** $f(x)=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$ **as a Fourier integral. Hence evaluate** $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}$ **and also find the value of** $\int_0^\infty \frac{\sin\lambda}{\lambda}$

Sol

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-1}^{1}\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\frac{2}{\lambda}\sin\lambda\cos\lambda x d\lambda$$

$$\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}f(x)=\{\begin{matrix}\frac{\pi}{2},|x|<1\\0,|x|>1\end{matrix}$$

$$|x|=1$$

$$\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}\left[\frac{1+0}{2}\right]=\frac{\pi}{4}$$

$$x=0$$

$$\int_0^\infty\frac{\sin\lambda}{\lambda}d\lambda=\frac{\pi}{2}$$

2 **Using Fourier Integral show that** $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty\frac{\lambda^2+2}{\lambda^4+2}\cos\lambda x d\lambda$

Sol

$$f(x)=e^{-x}\cos x$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty f(t)\cos\lambda t\,dt]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty e^{-t}\cos t\cos\lambda t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\cos\lambda x[\int_0^\infty e^{-t}(\cos(\lambda+1)t+\cos(\lambda-1)t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\cos\lambda x[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\lambda^2+2}{\lambda^4+2}\cos\lambda x d\lambda$$

FOURIER TRANSFORMS

The complex form of Fourier integral of any function f(x) is in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)e^{i\lambda t}dtd\lambda$$

Replacing λ by s

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}ds\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

Let

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

Here F(s) is called Fourier transform of f(x) and f(x) is called inverse Fourier transform of F(s)

Alternative Definitions

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ist}dt, f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx, f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}ds$$

Fourier Cosine Transform
Infinite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos stdt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C[f(t)]\cos sxds$$

Finite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{l}}\int_0^{l} f(t)\cos(\frac{n\pi t}{l})dt$$

$$f(x) = \frac{1}{l}F_C(0) + \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_C(s)\cos(\frac{n\pi x}{l})$$

Fourier Sine Transform
Infinite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin stdt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S[f(t)]\sin sxds$$

Finite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{l}}\int_0^l f(t)\sin(\frac{n\pi t}{l})dt$$

$$f(x) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_S(s)\sin(\frac{n\pi x}{l})$$

Alternative Definitions:

$$1. F_C(s) = \int_0^{\infty} f(x)\cos sx dx, f(x) = \frac{2}{\pi}\int_0^{\infty} F_C(s)\cos sx ds$$

$$2. F_S(s) = \int_0^{\infty} f(x)\sin sx dx, f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(s)\sin sx ds$$

Properties of Fourier Transforms

1 **Linear Property:** $F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$

Proof

$$F[af_1(x) + bf_2(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [af_1(x) + bf_2(x)]e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)e^{ist}dt + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(x)e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$$

2 **Shifting Theorem:** (a) $F[f(x-a)] = e^{ias}F(s)$

(b) $F[e^{iax}f(x)] = F(s+a)$

Proof
(a)

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-a)e^{ist}dt$$

$$t - a = z$$
$$dt = dz$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{isz}e^{ias}dz$$

$$F[f(x-a)] = e^{ias}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{isz}dz$$

$$F[f(x-a)] = e^{ias}F(s)$$

(b)

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}e^{iat}dt$$

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(a+s)t}dt$$

$$F[e^{iax}f(x)] = F(s+a)$$

3 **Change of scale property:** $F[f(ax)] = \frac{1}{a} F(\frac{s}{a}) (a > 0)$

Proof $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at) e^{ist} dt$

$at = z$

$dt = \frac{1}{a} dz$

$F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{i\left(\frac{s}{a}\right)z} dz$

$F[f(ax)] = \frac{1}{a} F(\frac{s}{a})$

4 **Multiplication Property:** $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

Proof $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$

$\frac{dF}{ds} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t.f(t) e^{ist} dt$

$\frac{d^2F}{ds^2} = \frac{i^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2.f(t) e^{ist} dt$

continuing

$\frac{d^nF}{ds^n} = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n.f(t) e^{ist} dt$

$F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

5 **Modulation Theorem:** $F[f(x) \cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right], F[s] = F[f(x)]$

Proof $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos at \, e^{ist} dt$

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{e^{iat} + e^{-iat}}{2}\right] e^{ist} dt$

$F[f(x)] = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(s+a)t} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(s-a)t} dt\right]$

$F[f(x) \cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

Problems

1 **Find the Fourier transform of** $f(x)=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$

Hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$

Sol:

$$F[f(x)]=\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)]=\int_{-1}^{1} 1.e^{isx}dx$$

$$F[f(x)]=\left|\frac{e^{isx}}{is}\right|_{-1}^{1}$$

$$F[f(x)]=\frac{e^{is}-e^{-is}}{is}=2\frac{\sin s}{s}$$

$$f(x)=\frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s]e^{-isx}ds$$

$$f(x)=\frac{1}{2\Pi}\int_{-\infty}^{\infty} 2\frac{\sin s}{s}e^{-isx}ds$$

$$f(x)=\frac{1}{\Pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}e^{-isx}ds$$

$$\frac{1}{\Pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}e^{-isx}ds=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$$

$$x=0$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}ds=\Pi$$

$$\int_0^{\infty} \frac{\sin s}{s}ds=\frac{\Pi}{2}$$

2 **Find the Fourier transform of** $f(x)=\{\begin{matrix}1-x^2,|x|\le 1\\0,|x|>1\end{matrix}$

Hence evaluate $\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \int_{-1}^{1} (1-x^2)e^{isx}dx$$

$$F[f(x)] = \left|(1-x^2)\frac{e^{isx}}{is} - 2x\frac{e^{isx}}{(is)^2} + 2\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$F[f(x)] = 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right)$$

$$F[f(x)] = \frac{-4}{s^3}(s\cos s - \sin s)$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s]e^{-isx}ds$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \begin{cases} 1-x^2, |x| \le 1 \\ 0, |x| > 1 \end{cases}$$

$$x = 1/2$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}[\cos\frac{s}{2} - i\sin\frac{s}{2}]ds = -\frac{3\Pi}{8}$$

$$\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}ds = -\frac{3\Pi}{16}$$

3 **Find the Fourier transform of $e^{-a^2x^2}$ .Hence deduce that $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform**

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x^2-isx/a^2)}dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x-isx/2a^2)^2}e^{-s^2/4a^2}dx$$

$$t = a(x - isx/2a^2)$$

$$dx = dt/a$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-t^2}e^{-s^2/4a^2}\frac{dt}{a}$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a}\sqrt{\Pi}$$

$$F[f(x)] = \frac{\sqrt{\Pi}}{a}e^{-s^2/4a^2}$$

$$a^2 = 1/2$$

$$F[e^{-x^2/2}] = \sqrt{2\Pi}e^{-s^2/2}$$

Hence $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform

4 **Find the Fourier cosine transform** e^{-x^2} **.**

Sol:

$$F_c(e^{-x^2}) = \int_0^\infty e^{-x^2}\cos sxdx = I$$

$$\frac{dI}{ds} = -\int_0^\infty xe^{-x^2}\sin s\,xdx = \frac{1}{2}\int_0^\infty(-2xe^{-x^2})\sin s\,xdx$$

$$\frac{dI}{ds} = \frac{-s}{2}\int_0^\infty e^{-x^2}\cos sxdx = \frac{-s}{2}I$$

$$\frac{dI}{I} = \frac{-s}{2}ds$$

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$$\log I = \int\frac{-s}{2}ds + \log c = \frac{-s^2}{4} + \log c = \log(ce^{-s^2/4})$$

$$I = ce^{-s^2/4}$$

$$\int_0^\infty e^{-x^2}\cos sxdx = ce^{-s^2/4}$$

$$s = 0$$

$$c = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\Pi}}{2}$$

$$\int_0^\infty e^{-x^2}\cos sxdx = \frac{\sqrt{\Pi}}{2}e^{-s^2/4}$$

5 **Find the Fourier sine transform** $e^{-|x|}$ **.Hence show that**

$$\int_0^\infty\frac{x\sin mx}{1+x^2}dx = \frac{\Pi e^{-m}}{2}\text{ ,m>0}$$

Sol: x being positive in the interval $(0,\infty)$

$$e^{-|x|} = e^{-x}$$

$$F_s(e^{-x}) = \int_0^\infty e^{-x}\sin sxdx = \frac{s}{1+s^2}$$

$$f(x) = \frac{2}{\Pi}\int_0^\infty F_s(e^{-x})\sin sxds$$

$$f(x) = \int_0^\infty\frac{s}{1+s^2}\sin sxds$$

$$e^{-x} = \int_0^\infty\frac{s}{1+s^2}\sin sxds$$

Replace x by m

$$e^{-m} = \frac{2}{\Pi}\int_0^\infty\frac{s}{1+s^2}\sin smds$$

$$\int_0^\infty\frac{s}{1+s^2}\sin smds = \frac{\Pi}{2}e^{-m}$$

$$\int_0^\infty\frac{x}{1+x^2}\sin mxds = \frac{\Pi}{2}e^{-m}$$

6 **Find the Fourier cosine transform** $f(x) = \begin{cases} x, 0 < x < 1 \\ 2-x, 1 < x < 2 \\ 0, x > 2 \end{cases}$.

Sol:

$$F_c(f(x)) = \int_0^\infty f(x)\cos sx dx$$

$$F_c(f(x)) = \int_0^1 x\cos sx dx + \int_1^2 (2-x)\cos sx dx + \int_2^\infty 0.\cos sx dx$$

$$F_c(f(x)) = \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}\right)$$

$$F_c(f(x)) = \frac{2\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2}$$

7 **If the Fourier sine transform of** $f(x) = \frac{1-\cos n\Pi}{(n\Pi)^2}$ **then find f(x).**

Sol:

$$f(x) = \frac{2}{\Pi}\sum_{n=1}^{\infty} F_s(n)\sin nx$$

$$F_s(n) = \frac{1-\cos n\Pi}{(n\Pi)^2}$$

$$f(x) = \frac{2}{\Pi}\sum_{n=1}^{\infty} \frac{1-\cos n\Pi}{(n\Pi)^2}\sin nx$$

$$f(x) = \frac{2}{\Pi^3}\sum_{n=1}^{\infty} \frac{1-\cos n\Pi}{n^2}\sin nx$$

Convolution Theorem

Definition

The convolution of two functions f(x) and g(x) is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Perseval's Identity

If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

PROBLEMS

Problem 1 If the Fourier transform of $f(x)$ is $F(s)$ then, what is Fourier transform of $f(ax)$?

Solution:

Fourier transform of $f(x)$ is

$$F(s)=F(f(x))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F(f(ax))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

Put $t=ax$

$dt=adx$

$$F(f(ax))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is(t/a)}\frac{dt}{a}$$

$$=\frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist/a}dt$$

$$=F(f(ax))=\frac{1}{a}.F\left(\frac{s}{a}\right).$$

Problem 2 Find the Fourier sine transform of e^{-3x}.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\ dx$$

$$F_s(e^{-3x})=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-3x}\sin sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{e^{-3x}}{s^2+9}(-3\sin sx-s\cos x)\right\}_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+9}\right)\left[\therefore\int e^{ax}\sin\ bx\ dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]\right].$$

Problem 3 Find the Fourier sine transform of $f(x)=e^{-ax}$, $a>0$. Hence deduce that $\int_0^{\infty}\frac{x\sin\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}$.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

$$F_s(e^{-ax})=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

By inverse Sine transform, we get

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx\ ds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sx\ ds$$

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}f(x)=\int_0^{\infty}\frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}e^{-ax}=\int_0^{\infty}\frac{s\sin sx}{s^2+a^2}ds$$

Put $a=1,\quad x=\alpha$

$$\frac{\pi}{2}e^{-\alpha}=\int_0^{\infty}\frac{s\sin sx}{s^2+1}ds$$

Replace 's' by 'x'

$$\int_0^{\infty}\frac{s\sin sx}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}.$$

Problem 4 Prove that $F_C\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_C(s+a)+F_C(s-a)\right]$.

Solution:

$$F_c(s)=F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos sx\ dx$$

$$F_c\left[f(x)\cos ax\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos ax\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\left[\frac{\cos(a+s)x+\cos(a-s)x}{2}\right]dx$$

$$=\frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos(s+a)x\,dx\right\}+\frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos(s-a)x\,dx\right\}$$

$$=\frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right].$$

Problem 5 Find the Fourier cosine transform of $f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x\geq a\end{cases}$.

Solution:

$$F_c(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^{a}\cos x\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{a}\left[\frac{\cos(s+1)x+\cos(s-1)x}{2}\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right],\text{ provided } S\neq 1,\ S\neq -1.$$

Problem 6 Find $F_C\left(xe^{-ax}\right)$ and $F_S\left(xe^{-ax}\right)$.

Solution:

$$F_c\left(xe^{-ax}\right)=\frac{d}{ds}F_s\left[f(x)\right]$$

$$F_c\left(xe^{-ax}\right)=\frac{d}{ds}F_s\left[e^{-ax}\right]$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-ax}\sin sx\,dx\right]$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right].$$

$$F_s\left[xe^{-ax}\right]=-\frac{d}{ds}\left[F_c e^{-ax}\right]\left(\therefore F_s\left(xf\left(x\right)\right)=-\frac{d}{ds}\left(F_c\left(f\left(x\right)\right)\right)\right)$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx\right]$$

$$=-\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(s^2+a^2\right)^2}\right].$$

Problem 7 If $F(s)$ is the Fourier transform of $f(x)$, then prove that the Fourier transform of $e^{ax}f(x)$ is $F(s+a)$.

Solution:

$$F(s)=F\left(f(x)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left(e^{iax}f(x)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(a+s)x}f(x)dx$$

$$=F(s+a).$$

Problem 8 Find the Fourier cosine transform of $e^{-2x}+3e^{-x}$.

Solution:

Let $f(x)=e^{-2x}+3e^{-x}$

$$F_c\left(f(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\;dx$$

$$F_c\left[e^{-2x}+3e^{-x}\right]=\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-2x}\cos sx\,dx+\int_0^\infty 3e^{-x}\cos sx\,dx\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+\frac{3}{s^2+1}\right].$$

Problem 9 State convolution theorem.

Solution:

If F(s) and G(s) are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F\left[f(x)*g(x)\right]=F\left[f(x)\right]F\left[g(x)\right]$

Problem 10 Derive the relation between Fourier transform and Laplace transform.

Solution:

Consider $f(t)\begin{cases} e^{-xt}g(t), & t>0 \\ 0\quad , & t<0 \end{cases}$ $\qquad -(1)$

The Fourier transform of $f(x)$ is given by

$$F\left[f(t)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-xt}g(t)e^{ist}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(\text{is}-x)t} g(t)\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-pt} g(t)\,dt \text{ where } p = x - \text{is}$$

$$=\frac{1}{\sqrt{2\pi}} L(g(t)) \left[\therefore L\left(f(t)=\int_{0}^{\infty} e^{-st} f(t)\,dt\right)\right]$$

$\therefore$ Fourier transform of $f(t)=\frac{1}{\sqrt{2\pi}}\times$ Laplace transform of g(t) where g(t) is defined by (1).

Problem 11 Find the Fourier sine transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\ dx$$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{x}\sin sx\ dx$$

Let $sx=\theta$

$sdx = d\theta;\ \theta: 0 \to \infty$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\text{s}}{\theta}\sin\theta\frac{d\theta}{s}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin\theta}{\theta}\ d\theta\left[\therefore \int_0^{\infty}\frac{\sin\theta}{\theta}d\theta=\frac{\pi}{2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}}.$$

Problem 12 Find $f(x)$ if its sine transform is $e^{-as}, a>0$.

Solution:

$F_s(f(x))=F(s)$

Given that $F_s(f(x))=e^{-as}$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_s(s)\sin x\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-as}\sin sx\,ds$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+s^2}(-a\sin sx - x\cos\ sx)\right]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{x}{a^2+x^2}\right).$$

Problem 13 Using Parseval's Theorem find the value of $\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)^2}dx, a>0$. Find the Fourier transform of $e^{-a|x|}, a>0$.

Solution:

Parseval's identify is $\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2ds$

Result : $F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$

$$\int_{-\infty}^{\infty} \left|\left(e^{-ax}\right)\right|^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right)^2 ds$$

$$2\int_{0}^{\infty} \left(e^{-ax}\right)^2 dx = 2\int_{0}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right)^2 ds$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^{\infty} = \frac{2}{\pi}\int_0^{\infty} \frac{s^2}{\left(s^2+a^2\right)^2} ds$$

$$i.e., \int_0^{\infty} \frac{s^2}{\left(s^2+a^2\right)^2} ds = \frac{\pi}{2}\left(\frac{0+1}{2a}\right) = \frac{\pi}{4a}.$$

$$\therefore \int_0^{\infty} \frac{x^2}{\left(x^2+a^2\right)^2}\, dx = \frac{\pi}{4a}.$$

Problem 14 Find the Fourier sine transform of $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$.

Solution:

The Fourier sine transform of $f(x)$ is given by $F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin\, sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 \sin\, sx\, dx + \int_1^{\infty} 0 \sin sx\, dx\right\} = \sqrt{\frac{2}{\pi}}\left[\frac{-\cos sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{-\cos s}{s} + \frac{1}{s}\right\} = \sqrt{\frac{2}{\pi}}\left[\frac{1}{s} - \frac{\cos s}{s}\right].$$

Problem 15 Find the Fourier transform of $e^{-a|x|}$, a > 0

Solution:

$$F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, \left(\cos sx + i \sin sx\right) dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, \cos sx\, dx \quad \left[\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx = 0,\ odd\ function\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, \cos sx\, dx$$

$$F\left(e^{-a|x|}\right) = \frac{2}{\sqrt{2\pi}}\left(\frac{a}{a^2+s^2}\right).$$

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\mathrm{i}sx}dx$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,dx+\int_{-a}^{-a}\left(a-|x|\right)e^{\mathrm{i}sx}dx+\int_{a}^{\infty}0\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{-a}\left(a-|x|\right)\left(\cos sx+\mathrm{i}\sin sx\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{-a}\left(a-|x|\right)\cos sx\,dx+0 \qquad \left[\therefore \int a\sin sx\ \&\ |x|\sin sx \text{ are } \mathit{odd\ \ functions}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{a}\left(a-x\right)\cos sx\ dx\right]$$

$$f(x)=\frac{2}{\sqrt{2\pi}}\left[\left(a-x\right)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[0-\frac{\cos sx}{s^2}+\frac{1}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right] \qquad -(1)$$

By inverse Fourier transform $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)\ e^{-isx}\ ds$.

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right]e^{-\mathrm{i}sx}ds \text{ Put } x=0$$

$$f(0)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\frac{as}{2}}{s^2}ds$$

$$\frac{\pi a}{4}=\int_{0}^{\infty}\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}ds$$

Put $a=2$

$$\frac{\pi}{2}=\int_{0}^{\infty}\frac{\sin^2 s}{s^2}ds \qquad [\ \therefore \text{s is a dummy variable, we can replace it by 't'}]$$

i.e $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$.

Problem 17 **(i)** Prove that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier transform.

(ii) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$. Hence evaluate $\int_{0}^{\infty}\frac{\sin s}{s}ds$.

Solution:

(i) $f(x)=e^{-x^2/2}$

$$F(s)=F\left(f(x)\right)$$

$$F(s)=F\left(f(x)\right)=\frac{1}{\sqrt{2\pi}}\ f(x)e^{\mathrm{is}x}\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{\mathrm{is}x}\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2+\mathrm{is}x+\frac{i^2s^2}{2}-\frac{i^2s^2}{2}}\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mathrm{is})^2}{2}}\ e^{\frac{-s^2}{2}}\ dx$$

Let $\quad y=\dfrac{x-\mathrm{is}}{\sqrt{2}} \qquad x=\infty \Rightarrow y=\infty$

$dx=\sqrt{2}dy \qquad x=-\infty \Rightarrow y=-\infty$

$$F(s)\ =\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-y^2}e^{-s^2/2}\sqrt{2}\ dy$$

$$=\frac{2e^{-s^2/2}}{\sqrt{\pi}}\int_{0}^{\infty} e^{-y^2}\ dy$$

$$=\frac{2e^{-s^2/2}}{\sqrt{\pi}}\int_{0}^{\infty} e^{-y^2}\ dy=\frac{2e^{-s^2/2}}{\sqrt{\pi}}\times\frac{\sqrt{\pi}}{2} \qquad \left[\int_{0}^{\infty} e^{-x^2}\ dx=\frac{\sqrt{\pi}}{2}\right]$$

$F(s)=e^{-s^2/2}$ *i.e.* $e^{-x^2/2}$ is self reciprocal hence proved.

(ii). Fourier transform of $f(x)$ is

$$F\left(f(x)\right)=\frac{1}{\sqrt{2\pi}}=\int_{-\infty}^{\infty} f(x)e^{\mathrm{is}x}dx$$

$$=\frac{1}{\sqrt{2\pi}}=\int_{-a}^{a} e^{\mathrm{is}x}dx$$

$$=\frac{1}{\sqrt{2\pi}}=\int_{-a}^{a}\left(\cos sx-i\sin sx\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}=\int_{0}^{a}\cos sx\,dx \qquad (\therefore \sin sx \text{ is } an\,odd\ fn.)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\cos sx\ dx$$

$$=\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin\ sx}{s}\right]_0^a$$

$$F(s)=\frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin\ as}{s}\right]$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} (\cos sx - i \sin sx) dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx \, dx - (0) \left[\therefore \frac{\sin as}{s} \sin sx \text{ is } odd \right]$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin as}{s} \right) \cos sx \, ds$$

$put \ a = 1, \ x = 0$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin s}{s} ds$$

$$\frac{\pi}{2} \times 1 = \int_{0}^{\infty} \frac{\sin s}{s} ds \quad (\therefore f(x) = 1, -a \le x \le a)$$

$$\therefore \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}.$$

Problem 18. Find the Fourier cosine transform of $f(x)$ defined as

$$f(x) = \begin{cases} x \ \ for & 0 < x < 1 \\ 2 - x \ \ for & 1 < x < 2 \\ 0 \ \ for & x > 2 \end{cases}$$

Solution. By definition of Fourier Transform

$$\begin{aligned} F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\left(x \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right)_0^1 + \left((2-x) \frac{\sin sx}{s} + \frac{\cos sx}{s^2} \right)_1^2 \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin s}{s} - 0 - \frac{\cos s - \cos 0}{s^2} \right) + \left(0 - (1) \frac{\sin s}{s} + \frac{\cos 2s - \cos s}{s^2} \right) \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{\cos 2s - 2\cos s + 1}{s^2} \right] \end{aligned}$$

Problem 19

Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b; \\ 0, & x < a. \text{ and } x > b \end{cases}$

Solution.

$$\begin{aligned} F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ikx} e^{isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{i(k+s)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b \\ &= \frac{i}{(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right] \end{aligned}$$

Problem 20. State and Prove convolution theorem on Fourier transforms

Statement: The Fourier transforms of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$F(f(x)*g(x)) = F[f(x)]F[g(x)]$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt \;\; e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t) e^{isx}\, dx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} g(x-t) e^{isx}\, dx \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F(g(x-t))\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} F(g(t))\, dt \qquad \left[\therefore f(g(x-t)) = e^{ist} F(g(t)) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist}\, dt\, G(s) \qquad \left[\because F(g(t)) = G(s) \right]$$

$$= F(f*g) = F(s).G(s). \qquad \left[\because F(f(t)) = F(s) \right].$$

Problem 21

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| \ge a \end{cases}$ and hence evaluate

(i) $\displaystyle\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right) dt$ **(ii)** $\displaystyle\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right)^2 dt$

Solutions

Fourier transform of $f(x)$ is

$$F\left(f(x)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[0+\int_{-a}^{a}\left(a^2-x^2\right)e^{isx}dx+0\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a^2-x^2\right)\left(\cos sx+i\sin x\right)dx\right]$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}\left(a^2-x^2\right)\cos sx\,dx\;\left[\therefore\left(a^2-x^2\right)\sin sx \text{ is } an\ odd\ fn.\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(a^2-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[0-\frac{2a\cos as}{s^2}+\frac{2a\sin as}{s^3}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{2a\cos as+2\sin as}{s^3}\right]$$

$$F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as-as\cos as}{s^3}\right]\qquad -(1)$$

By inverse Fourier transforms,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}dx$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as-as\cos as}{s^3}\right)\left(\cos sx-i\sin sx\right)ds$$

$$f(x)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin as-as\cos as}{s^3}\cos sx\,dx$$ ($\therefore$ the second terms is on odd function)

Put $a=1$

$$f(x)=\frac{2}{\pi}\times 2\int_{-\infty}^{\infty}\frac{\sin s-s\cos s}{s^3}\cos sx\,ds \qquad \left[f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|\ge 1\end{cases}\right]$$

Put $x=0$

$$f(0)=\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s-s\cos s}{s^3}ds \qquad \left[\begin{aligned}f(0)&=1-0\\&=1\end{aligned}\right]$$

$$1=\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s-s\cos s}{s^3}ds$$

$$\frac{\pi}{4}\square\int_{0}^{\infty}\frac{\text{sint - t cost}}{t^3}\,\text{dt}.$$

Hence (i) is proved. Using Parseval's identify

$$\int_{-\infty}^{\infty}\left|F(s)\right|^2 ds=\int_{-\infty}^{\infty}\left|f(x)\right|^2\,dx$$

$$\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-a}^{a}\left|a^2 - x^2\right| dx$$

$$\int_{-\infty}^{\infty}\frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1}\left(1-x^2\right)^2 dx$$

$$2\times\frac{8}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1}\left(1-x\right)^2 dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x+\frac{x^5}{4}-\frac{2x^3}{3}\right]_0^1$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{16}\times 2\left(\frac{8}{15}\right) = \frac{\pi}{15}$$

Put $a = 1$

Put s = t

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$ Hence (ii) is proved.

Problem: 22

Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ and hence find the value of

(i) $\int_{0}^{\infty}\frac{\sin^2 t}{t^2} dt$. **(ii)** $\int_{0}^{\infty}\frac{\sin^4 t}{t^4} dt$.

Solution:

The Fourier transform of $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{is}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx \left[\therefore (1-|x|) \sin sx \text{ is an odd } fn.\right]$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ (1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ -\frac{\cos s}{s^2} + \frac{1}{s^2} \right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] - (1)$$

(i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] (\cos ssx - \mathrm{i} \sin sx) (by\ (1))$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx\, ds \ \ (\therefore \text{Second term is odd})$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx\, ds$$

$$Put x = 0$$

$$1-|0| = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds$$

$$\int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{2\sin^2(s/2)}{s^2} ds = \frac{\pi}{2}$$

$$put \quad t = s/2 \qquad ds = 2dt$$

$$\int_0^{\infty} \frac{2\sin^2 t}{(2t)^2} 2dt = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right) \right]^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = 2\int_0^1 (1-x)^2 dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2} \right)^2 ds = \left[2\left(\frac{1-x}{-3} \right)^3 \right]_0^1$$

$$\frac{16}{\pi}\int_0^\infty\left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds=\frac{2}{3}; Let\, t=s/2, dt=\frac{ds}{2}$$

$$\frac{16}{\pi}\int_0^\infty\left(\frac{\sin t}{2t}\right)^4 2\,dt=\frac{2}{3}$$

$$\frac{16}{16\pi}\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt=\frac{1}{3}$$

$$\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}.$$

Problem 23 **(i)** Find the Fourier sine transform of $e^{-|x|}$. Hence prove that $\int_0^\infty\left(\frac{x\sin\alpha x}{1+x^2}\right)dx=\frac{\pi}{2}e^{-\alpha},\ \alpha>0$.

(ii) Find the Fourier sine transform of $e^{-ax}\left(a>0\right)$. Hence find (a) $F_C\left(xe^{-ax}\right)$ and (b) $F_S\left(\frac{e^{-ax}}{x}\right)$.

Solution:

(i) $F_s\left(f\left(x\right)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty f\left(x\right)\sin sx\,dx$

$$F_s\left(e^{-|x|}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)\sin sx\,dx$$

Result: $\int_0^\infty e^{ax}\sin bx dx=\frac{b}{a^2+b^2}$

By Fourier sine inversion formula, we have

$$f\left(x\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s\left(s\right)\sin\ sx\,ds$$

$$e^{-x}=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)\sin\ sx\,ds$$

$$=\frac{2}{\pi}\int_0^\infty\frac{s\sin sx}{s^2+1}ds$$

$$\int_0^\infty\frac{s\sin sx}{s^2+1}ds=\frac{\pi}{2}e^{-x}\qquad put\ \ x=a$$

$$\int_0^\infty\frac{s\sin sa}{1+s^2}ds=\frac{\pi}{2}e^{-a}$$

Replace S by x

$$\int_0^\infty\frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}.$$

(ii) . $F_s\left(f\left(x\right)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty f\left(x\right)\sin sx\,dx$

$$F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-ax}}{a^2+s^2}\left(-a\sin sx - s\cos sx\right)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right) \ldots\ldots.(1)$$

By Property

$$F_s\left[x\ f(x)\right] = -\frac{d}{ds}\left[F_c\left(f(x)\right)\right]$$

$$F_c\left[x\ f(x)\right] = \frac{d}{ds}F_s\left(f(x)\right)$$

(a) To Find $F_c\left[x e^{-ax}\right]$

$$F_c\left[x e^{-ax}\right] = \frac{d}{ds}F_s\left(e^{-ax}\right)$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a^2+s^2-2s^2}{\left(a^2+s^2\right)^2}\right]$$

$$F_c\left[x.f(x)\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(a^2+s^2\right)^2}\right]$$

(b) To find $F_s\left[\frac{e^{-ax}}{x}\right]$

$$F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-at}}{t}\sin st\ dt \qquad -(1)$$

$$F(s) \qquad = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-at}}{t}\sin st\ dt$$

Diff. on both sides w.r to ‘s’ we get

$$\frac{d}{ds}\left(F(s)\right) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-at}}{t}\sin st\ dt \quad \left[\therefore \int_0^\infty e^{-ax}\cos bx\ dx = \frac{a}{b^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left\{\frac{e^{-at}}{t}\sin st\right\}dt$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{te^{-at}\cos st}{t}dt$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at}\cos st\ \ dt$$

$$\frac{d}{ds}F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at}\cos st\ \ dt \quad = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating w.r. to ‘s’ we get

$$F(s) = \sqrt{\frac{2}{\pi}}\int\frac{a}{s^2+a^2}ds + c$$

$$= \sqrt{\frac{2}{\pi}}\,a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$$

But $F(s) = 0$ When $s = 0 \therefore c = 0$ from (1)

$$\therefore F(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right).$$

Problem 24 **(i)** Find the Fourier transform of $e^{-a^2x^2}$ Hence prove that $e^{\frac{-x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.

(ii) Find the Fourier cosine transform of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under cosine transform. Also find $F\left(\frac{1}{\sqrt{|x|}}\right)$.

Solution:

(i) $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\text{is}x}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{\text{is}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2\right)+\text{is}x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-\text{is}x\right)}\,dx \qquad -(1)$$

Consider $a^2x^2 - isx$

$$= (ax)^2 - 2(ax)\frac{(\text{is})}{2a} + \left(\frac{\text{is}}{2a}\right)^2 - \left(\frac{\text{is}}{2a}\right)^2$$

$$= \left(ax - \frac{\text{is}}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad -(2)$$

Sub: (2) in (1) ,We get

$$F[f(x)] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{\text{is}}{2a}\right)+\frac{s^2}{4a^2}\right]}dx$$

$$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{\text{is}}{2a}\right)^2}dx$$

$$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-t^2}\frac{dt}{a} \qquad Let\ t = ax - \frac{\text{is}}{2a},\ dt = adx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}\right]$$

$$= \frac{1}{a\sqrt{2}}e^{-\frac{s^2}{4a^2}} \qquad -(3)$$

$$Put\ a = \frac{1}{\sqrt{2}}\ in\ (3)$$

$$F\left[e^{-x^2/2}\right] = e^{-S^2/2}$$

$\therefore e^{-S^2/2}$ is self reciprocal with respect to Fourier Transforms.

(ii). $F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$

$$F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} x^{n-1}\cos sx\ dx \qquad -(1)$$

We know that $\Gamma(n) = \int_0^{\infty} e^{-y}y^{n-1}dy$

Put $y = ax$, we get $\int_0^{\infty} e^{-ax}(ax)^{n-1}adx = \Gamma(n)$

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$$

Put a = is

$$\therefore \int_0^\infty e^{-\text{is}\,x}x^{n-1}dx = \frac{\Gamma(n)}{(\text{is})^n}$$

$$\int_0^\infty (\cos sx - i\sin sx)\,x^{n-1}dx = \frac{\Gamma(n)}{s^n}i^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - \text{i}\sin\frac{n\pi}{2}\right]$$

Equating real parts, we get

$$\int_0^\infty x^{n-1}\cos sx dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad -(2)$$

Using this in (1) we get

$$F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\,\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine transform

To find $F\left\{\frac{1}{\sqrt{1x1}}\right\}$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e\frac{\text{is}\,x}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{x}}(\cos sx + \text{is}\,insx)\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{x}}\cos sx\,dx \quad [\therefore The\,\text{sec}\,ond\;term\;odd]$$

Put n =1/2 in (2), we get

$$\int_0^\infty \frac{\cos sx}{\sqrt{x}}\,dx = \frac{\Gamma 1/2}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2s}}$$

$$\therefore F\left\{\frac{1}{\sqrt{|x|}}\right\} = \sqrt{\frac{2}{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2s}} = \frac{1}{\sqrt{s}}.$$

Problem 25 **(i)** Find $f(x)$ if its Fourier sine Transform is $\dfrac{e^{-as}}{s}$.

(ii) Using Parseval's Identify for Fourier cosine and sine transforms of e^{-ax}, evaluate

(a). $\displaystyle\int_0^\infty \frac{1}{\left(a^2+x^2\right)^2}dx$ **(b).** $\displaystyle\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)^2}dx$

Solution:

(i) Let $F_s\left(f(x)\right)=\dfrac{e^{-as}}{s}$

Then $f(x)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \frac{e^{-as}}{s}\sin sx\ dx \qquad -(1)$

$\therefore \dfrac{df}{dx}=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty e^{-as}\cos sx\,ds=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}$

$\therefore F(x)=\sqrt{\dfrac{2}{\pi}}a\displaystyle\int\frac{dx}{a^2+x^2}$

$=\sqrt{\dfrac{2}{\pi}}\tan^{-1}\left(\dfrac{x}{a}\right)+c \qquad -(2)$

At $x=0$, $f(0)=0$ using (1)

(2) => $f(0)=\sqrt{\dfrac{2}{\pi}}\tan^{-1}(0)+c \quad \therefore c=0$

Hence $f(x)=\sqrt{\dfrac{2}{\pi}}\tan^{-1}\left(\dfrac{x}{a}\right)$.

(ii) (a) To find $\displaystyle\int_0^\infty \frac{dx}{\left(a^2+x^2\right)^2}$

$F_c\left(e^{-ax}\right)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty e^{-ax}\cos sx\,dx$

$=\sqrt{\dfrac{2}{\pi}}\left[\dfrac{e^{-ax}}{a^2+s^2}\left(-a\cos sx+s\sin sx\right)\right]_0^\infty$

$F_c\left(e^{-ax}\right)=\sqrt{\dfrac{2}{\pi}}\left(\dfrac{a}{a^2+s^2}\right) \qquad -(1)$

By Parseval's identify.

$\displaystyle\int_0^\infty \left|f(x)\right|^2dx=\int_0^\infty \left|F_c(s)\right|^2ds$

$\displaystyle\int_0^\infty e^{-2ax}dx=\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2 ds,\ from\ (1)$

$\left[\dfrac{e^{-2ax}}{-2a}\right]_0^\infty=\dfrac{2}{\pi}a^2\displaystyle\int_0^\infty\frac{ds}{\left(a^2+s^2\right)^2}$

$\dfrac{1}{2a}=\dfrac{2a^2}{\pi}\displaystyle\int_0^\infty\frac{ds}{a^2+s^2}$

$i.e\ \displaystyle\int_0^\infty\frac{dx}{\left(a^2+x^2\right)^2}=\frac{\pi}{4a^3}$ [Replace s by x]

(b) To find $\displaystyle\int_0^\infty\frac{x^2}{\left(a^2+x^2\right)^2}\ dx$

$$F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx-s\cos sx)\right]_0^{\infty}$$

$$F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$$

By parseval's identify

$$\int_0^{\infty}\left|f(x)^2\right|dx=\int_0^{\infty}\left|F_s(f(x))\right|^2 ds$$

$$\int_0^{\infty}\left(e^{-ax}\right)^2 dx=\frac{2}{\pi}\int_0^{\infty}\left(\frac{s}{a^2+s^2}\right)^2 ds$$

$$i.e\int_0^{\infty}\frac{s}{\left(a^2+s^2\right)}ds=\frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}=\frac{\pi}{2}\times\frac{1}{2a}$$

$$\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)^2}dx=\frac{\pi}{4a}\qquad (\text{Re}\,place\,'s'\,by\,'x').$$

Problem 26 **(i).** Find the Fourier cosine transform of $e^{-ax}\cos ax$

(ii). Evaluate **(a).** $\int_0^{\infty}\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}dx$ **(b).** $\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$, using Fourier cosine and sine transform.

Solution:

(i) $F_c\left(f(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\ dx$

$$F_c\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\ dx$$

$$F_c(s)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

By Modulation Theorem,

$$F_c\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_c(a+s)+F_c(a-s)\right]$$

$$F_c\left[e^{-ax}\cos ax\right]=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left\{\frac{a}{a^2+(a+s)^2}+\frac{a}{a^2+(a-s)^2}\right\}\right]$$

$$=\frac{1}{2}\times\sqrt{\frac{2}{\pi}}\times a\left\{\frac{a^2+(a-s)^2+a^2+(a+s)^2}{\left[a^2+(a+s)^2\right]\left[a^2+(a-s)^2\right]}\right\}.$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{4a^2+2s^2}{s^4+4a^4}\right]$$

$$F_c\left[e^{-ax}\cos ax\right]=\frac{2a}{\sqrt{2\pi}}\left[\frac{2a^2+s^2}{s^4+4a^4}\right].$$

(ii) (a) Let $f(x)=e^{-x}$ and $g(x)=e^{-2x}$

$$F_c\left(e^{-x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{s^2+1}(-\cos x+s\sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1} \right] \qquad -(1)$$

$$F_c\left(e^{-2x}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2}{s^2+4} \right) \qquad -(2)$$

$$\therefore \int_0^\infty f(x) g(x) dx = \int_0^\infty F_c\left(f(x)\right) F_c\left(g(x)\right) ds$$

$$\int_0^\infty e^{-x} e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{1}{s^2+1} . \frac{1}{s^2+4} \right) ds \left(from (1) \& (2) \right)$$

$$\int_0^\infty e^{-3x} dx = \frac{4}{\pi} \int_0^\infty \frac{ds}{\left(s^2+1\right)\left(s^2+4\right)} ds$$

$$\int_0^\infty \frac{ds}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{\pi}{4} \left[\frac{e^{-3x}}{-3} \right]_0^\infty = \frac{\pi}{4} \left(\frac{1}{3} \right)$$

$$\int_0^\infty \frac{ds}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{\pi}{12}.$$

(b) To find $\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)} dx.$

Let

$$\left(f(x)\right) = e^{-ax}, g(x) = e^{-bx}$$

$$F_s\left(g(x)\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right) \qquad -(1)$$

$$F_s\left(g(x)\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+b^2} \right) \qquad -(2)$$

$$\int_0^\infty f(x) g(x) dx = \int_0^\infty F_s\left[f(x)\right] . F_s\left[g(x)\right] ds \ \ From (1) \text{ and} (2)$$

$$\int_0^\infty e^{-ax} e^{-bx} dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)} ds$$

$$\int_0^\infty \frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)} ds = \frac{\pi}{2} \int_0^\infty e^{-(a+b)x} dx$$

$$i.e \int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)} dx = \frac{\pi}{2} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = \frac{\pi}{2(a+b)}.$$

Problem 27 **(i).** Find Fourier transform of $e^{-a|x|}$ and hence deduce that

(a). $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt = \frac{\pi}{2a}e^{-a|x|}$ **(b).** $F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$.

(ii) . Find Fourier cosine transform of $e^{-ax}\sin ax$.

Solution:

(i) Fourier transform of $f(x)$ is

$$F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left(\cos sx + i\sin sx\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx \qquad \left[\therefore e^{-a|x|}\sin sx \text{ is } odd\ fn.\right]$$

$$F\left[e^{-a|x|}\right] = \frac{2}{\sqrt{2\pi}}\left[\frac{a}{a^2+s^2}\right] = F(s) \qquad -(1)$$

(a) Using Fourie rinverse transform,

$$e^{-a|x|} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\left(\cos sx + i\sin sx\right)ds$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx}{a^2+s^2}ds + 0 \qquad \left[\therefore \frac{\sin sx}{s^2+a^2} \text{ is } an\ odd\ fn.\right]$$

$$= \frac{2a}{\pi}\int_{-\infty}^{\infty}\frac{\cos xt}{a^2+t^2}dt \qquad \left(\text{Re}\,place\,'s'\ by\ 't'\right)$$

$$\frac{\pi}{2a}e^{-a|x|} = \int_{-\infty}^{\infty}\frac{\cos xt}{a^2+t^2}dt.$$

(b) . Find Fourier cosine transform of $e^{-ax}\sin ax$.

To prove $F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$

Property:

$$F\left[x^n f(x)\right] = (-i)^n\frac{d^nF}{ds^n}$$

$$F\left[x\,f(x)\right] = -i\frac{dF(s)}{ds}$$

$$F\left[xe^{-a|x|}\right] = \frac{-i\,dF\left(e^{-a|x|}\right)}{ds}$$

$$= -i\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right)$$

$$= -ia\sqrt{\frac{2}{\pi}}\left(\frac{-2s}{\left(a^2+s^2\right)^2}\right) = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(a^2+s^2\right)^2}.$$

(ii) Find the Fourier cosine transform of $e^{-ax}\sin ax$.

$$F_c\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-ax}\sin ax\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\,\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty e^{-ax}\left[\sin(s+a)x - \sin(s-a)x\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s+a)^2}\right\} \qquad \left[\therefore \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{\left(a^2+(s+a)^2\right)(s+a) - (s-a)\left(a^2+(s+a)^2\right)}{\left(a^2+(s+a)^2\right)\left(a^2+(s-a)^2\right)}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{2a^2s+s^3-2as^2+2a^3+as^2-2a^2s-2s^2-s^3-2as^2+2a^3+s^2a+2sa^2}{4a^4+2a^2s^3-4a^3s+2a^2s^2+s^4-2as^3+4a^3s+2as^2-4a^2s^2}\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\left\{\frac{2a^3-as^2}{4a^4+s^4}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a\left(2a^2-s^2\right)}{4a^4+s^4}\right).$$

Problem 28 (i). State and Prove Parseval's Identity in Fourier Transform.

(ii). Find Fourier cosine transform of e^{-x^2}

Solution:

(i) Parseval's identity:

Statement: If $F(s)$ is the Fourier transform of $f(x)$, then $\int_{-\infty}^{\infty}\left|f(x)\right|^2dx = \int_{\infty}^{\infty}\left|F(s)\right|^2$

$$\text{Proof by convolution theorem } F\left[f*g\right] = F(s)G(s)$$

$$f*g = F^{-1}\left[F(s)G(s)\right]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)e^{isx}\,ds \quad -(1)$$

Put $x = 0$ and $g(-t) = \overline{f(t)}$, then it follows that $G(s) = \overline{F(s)}$

$\therefore (1)$ *becomes*

$$\int_{-\infty}^{\infty}\left[f(t)\overline{f(t)}\right]dt = \int_{-\infty}^{\infty}\left[F(s)\overline{F(s)}\right]ds$$

$$\text{i.e.} \int_{-\infty}^{\infty}\left|f(t)\right|^2 dt = \int_{-\infty}^{\infty}\left|F(s)\right|^2 ds$$

$$\text{i.e.} \int_{-\infty}^{\infty}\left|f(t)\right|^2 dx = \int_{-\infty}^{\infty}\left|F(s)\right|^2 ds$$

(ii)

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\times\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\quad \text{RP of}\quad e^{-isx}dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}e^{isx}\,dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2+isx}dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2+isx}\frac{e^{\frac{s^2}{4}}}{e^{\frac{s^2}{4}}}dx$$

$$=\text{R.P of }e^{-s^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2+isx+s^2/4}dx$$

$$=\text{R.P of }e^{-s^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-is/2)^2}dx$$

Put $\frac{x-is}{2}=t\quad dx=dt$

When $t=-\infty \qquad y=-\infty$

$t=\infty \qquad y=\infty$

$$F_c\left[f(x)\right]=R.P\ \text{ of }\frac{e^{-s^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$\text{R.P.of }\frac{e^{-s^2/4}}{\sqrt{2\pi}}\times\sqrt{\pi}\qquad\qquad \left[\therefore\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}\right]$$

$$=\frac{e^{-s^2/4}}{\sqrt{2}}$$

$$\Rightarrow F_c\left[e^{-x^2}\right]=\frac{e^{-s^2/4}}{\sqrt{2}}.$$

Problem 29 (i). Find the Fourier transform of $\frac{\sin ax}{x}$ and hence prove that $\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx=4a\pi$.

(ii). Find $f(x)$, if the Fourier transform of $F(s)$ is $\frac{2\sin 3(s-2\pi)}{(s-2\pi)}$.

Solution:

(i) $F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$F\left[\frac{\sin ax}{x}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin ax}{x}\right]e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin ax}{x}\right](\cos sx+\sin x)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\times\pi$$

$$F\left[\frac{\sin ax}{x}\right]=\sqrt{2\pi} \qquad -(1)$$

$$\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{\sin(a+s)x}{x}+\frac{\sin(a-s)x}{x}\right]dx=\left\{\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]\right\}\right] if\ a+s>0\ \&\ a-s>0$$

$$if\ a+s>0\ \&\ a-s<0\ or\ a+s<0\ \&\ a-s>0$$

By Parseval's identity

$$\int_{-\infty}^{\infty}\left|f(x)\right|^2 dx=\int_{-\infty}^{\infty}\left|F(s)\right|^2 ds$$

$$\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx=\int_{-a}^{a}\left|\sqrt{2\pi}\right|^2 ds=2\pi\left[s\right]_{-a}^{a}=2\pi(a+a)=4\pi a$$

$$i.e.\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx=4\pi a.$$

(ii) Let us find $F^{-1}\left\{\frac{2\sin 3s}{s}\right\}$

$$F^{-1}\left\{\frac{2\sin 3s}{s}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}e^{-isx}ds$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}(\cos sx+i\sin sx)ds=\frac{1}{\pi}\int_0^{\infty}\frac{2\sin 3s\cos sx}{s}ds$$

(By the property of odd and even function)

$$=\frac{1}{\pi}\int_0^{\infty}\left\{\frac{\sin(3+x)s}{s}+\frac{\sin(3-x)}{s}\right\}ds$$

$$=\frac{1}{\pi}\left[\int_0^{\infty}\frac{\sin(3+x)s}{s}ds+\int_0^{\infty}\frac{\sin(3-x)s}{s}ds\right]$$

$$=\begin{cases}\frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] & if\ \ 3+x>0\ \&\ 3-x>0\\ & if\ \ 3+x>0\ \&\ 3-x<0\ \ or\\ 0 & \ \ 3+x<0\ \&\ 3-x>0\end{cases}$$

$$\left[\therefore\int_0^{\infty}\frac{\sin mx}{x}dx=\frac{\pi}{2}\ or-\frac{\pi}{2}\ \text{according as } m>0 \text{ or } m<0.\right]$$

$$=\begin{cases}1 & if\ \ -3<x<3\\ 0 & if\ \ x<-3\ or\ x>3\end{cases}$$

$$=\begin{cases}1 & if\ \ |x|<3\\ 0 & if\ \ |x|>3\end{cases} \qquad -(1)$$

By the shitting property, $F\left\{e^{iax}f(x)\right\}=F(s-a)$

$$e^{iax}f(x)=F^{-1}\left\{F(s-a)\right\}$$

Thus $F^{-1}\left\{\frac{2\sin\left[3(s-2\pi)\right]}{s-2\pi}\right\}=e^{i2\pi}F^{-1}\left(\frac{2\sin 3s}{s}\right)$

$$=e^{i2\pi}\times\begin{cases}1\ if\ |x|<3\\ 0\ if\ |x|>3\end{cases}$$

$$=e^{i2\pi}\times\begin{cases}e^{i2\pi x} & if\ |x|<3\\ 0 & if\ |x|>3\end{cases}\ .$$

UNIT V
Z-TRANSFORM

- Definition of Z-transforms
- Elementary properties
- Inverse Z-transform
- Convolution theorem
- Formation and solution of difference equations.

Introduction

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science and economics. These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

Definition

If the function u_n is defined for discrete value and $u_n = 0$ for n<0 then the Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=1}^{\infty} u_n z^{-n}$$

The inverse Z-transform is written as

$$u_n = Z^{-1}[U(z)]$$

Properties of the z transform

For the following

$$Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n] z^{-n} = F(z) \; Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

Linearity:

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$
which follows from definition of z-transform.

Time Shifting

If we have $f[n] \Leftrightarrow F(z)$ then $f[n - n_0] \Leftrightarrow z^{-n_0} F(z)$

The ROC of Y(z) is the same as F(z) except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

Proof: Let $y[n] = f[n - n_0]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} f[n - n_0] z^{-n}$$

Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k] z^{-k-n_0} = z^{-n_0} F[z]$$

Multiplication by an Exponential Sequence

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of FX(z) is $r_R < |z| < r_L$, then the ROC of Y(z) is $r_R < |z/z_0| < r_L$, i.e., $|z_0| r_R < |z| < |z_0| rL$

Proof: $Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of X(z) is

rR<|z|<rL, then the ROC of Y(z) is
rR< |z/z$_0$| <rL, i.e., |z$_0$|rR < |z| < |z$_0$|rL

Differentiation of X(z)

If we have $f[n] \Leftrightarrow F(z)$ then $nf[n] \xleftrightarrow{z} -z\dfrac{dF(z)}{z}$ and ROC = R_f

Proof:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} = -z\sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n-1} = \sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} \xleftrightarrow{z} nf[n]$$

Some Standard Z-Transform

	Sequence	z - transform
1	$\delta[n]$	1
2	$u[n]$	$\dfrac{z}{z-1}$
3	b^n	$\dfrac{z}{z-b}$
4	$b^{n-1}u[n-1]$	$\dfrac{1}{z-b}$
5	e^{an}	$\dfrac{z}{z-e^{a}}$
6	n	$\dfrac{z}{(z-1)^2}$
7	n^2	$\dfrac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\dfrac{bz}{(z-b)^2}$
9	$e^{an} n$	$\dfrac{z e^{a}}{(z-e^{a})^2}$
10	$\sin(an)$	$\dfrac{\sin(a)\, z}{z^2 - 2\cos(a)\, z + 1}$
11	$b^n \sin(an)$	$\dfrac{\sin(a)\, b z}{z^2 - 2\cos(a)\, b z + b^2}$
12	$\cos(an)$	$\dfrac{z(z-\cos(a))}{z^2 - 2\cos(a)\, z + 1}$
13	$b^n \cos(an)$	$\dfrac{z(z-b\cos(a))}{z^2 - 2\cos(a)\, b z + b^2}$

Problems

1 **Find the z transform of $3n + 2 \times 3^n$**

Sol From the linearity property

$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$

and from the Table 1

$$Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad Z\{3^n\} = \frac{z}{(z-3)}$$

($r^n$with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

2 **Find the z-transform of each of the following sequences:**

(a) $\boldsymbol{x(n) = 2^n u(n) + 3(\tfrac{1}{2})^n u(n)}$

(b) $\boldsymbol{x(n) = \cos(n\omega_0) u(n)}$.

Sol (a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{(1-2z)\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write

$x(n) = \cos(n\omega_0)\, u(n) = \tfrac{1}{2}(e^{jn\omega 0} + e^{-jn\omega 0})\, u(n)$

Therefore, the z-transform is

$$X(z) = \frac{1}{2}\frac{1}{1-e^{jn\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jn\omega_0}z^{-1}}$$

with a region of convergence |z| >1. Combining the two terms together, we have

$$X(z) = \frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)\,z^{-1}+z^{-2}}$$

3 **Determine $f_n$by Infinite Series and Partial Fraction Expansion**

$$F(z) = \frac{2z}{(z-2)(z-1)^2}$$

Sol

$$F(z) = \frac{2z}{z^3-4z^2+5z-2}$$

Now divide (long division) with the polynomials written in descending powers of z

$2z^{-2}+8z^{-3}+22z^{-4}+52z^{-5}+114z^{-6}+\ldots$

$Z^3-4z^2+5z-2 \,|\, 2z$

$2z-8+10z^{-1}-4z^{-2}$

$8-10z^{-1}+04z^{-2}$

$8-32z^{-1}+40z^{-2}-16z^{-3}$

$22z^{-1}-36z^{-2}+016z^{-3}$

$22z^{-1}-88z^{-2}+110z^{-3}-44z^{-4}$

$52z^{-2}-094z^{-3}+044z^{-4}$

$52z^{-2}-208z^{-3}+260z^{-4}-104z^{-5}$

$114z^{-3}-216z^{-4}+104z^{-5}$

$$\therefore F(z)=\sum_{n=0}^{\infty} f_n z^{-n} = 2z^{-2} + 8z^{-3} + 22z^{-4} + 52z^{-5} + 114z^{-6} + \ldots$$

And the time sequence for f_n is

n	0	1	2	3	4	5	6	…
f_n	0	0	2	8	22	52	114	…

NOTE This method does NOT give a closed form for the answer, but it is a good method for finding the first few sample values or to check out that the closed form given by another method at least starts out correctly.

$$F(z)=\frac{2z}{(z-2)(z-1)^2}=\frac{k_1 z}{z-2}+\frac{k_2 z}{z-1}+\frac{k_3 z}{(z-1)^2}$$

To find k1 multiply both sides of the equation by (z-2), divide by z, and let z→2

$$\frac{2z}{(z-1)^2}=k_1 z+\frac{k_2 z(z-2)}{z-1}+\frac{k_3 z(z-2)}{(z-1)^2}$$

$$\frac{2}{(z-1)^2}=k_1+\frac{k_2(z-2)}{z-1}+\frac{k_3(z-2)}{(z-1)^2}$$

$$\left.\frac{2}{(z-1)^2}\right|_{z=2}=k_1+\left.\frac{k_2(z-2)}{z-1}\right|_{z=2}+\left.\frac{k_3(z-2)}{(z-1)^2}\right|_{z=2}$$

k_1 = 2

Similarly to find k_3 multiply both sides by $(z-1)^2$, divide by z, and let z→1

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$

k_3 = -2

Finding k2 requires going back to Equation A above and taking the derivative of both sides

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$

$$-\frac{2}{(z-2)^2}=k_1\left[\frac{2(z-1)}{z-2}-\frac{2(z-1)^2}{(z-2)^2}\right]+k_2$$

Now again let z→1

k_2 = -2

$$\therefore F(z)=\frac{2z}{z-2}-\frac{2z}{z-1}-\frac{2z}{(z-1)^2}$$

Convolution theorem

If $u_n = Z^{-1}[U(z)]$ and $v_n = Z^{-1}[V(z)]$ then $Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_n v_{n-m} = u_n * v_n$

Where the symbol * denotes the convolution operation

Proof We have

$u_n = Z^{-1}[U(z)]$ and $v_n = Z^{-1}[V(z)]$

$$U(z).V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + ... + u_n z^{-n} + ...) x (v_0 + v_1 z^{-1} + v_2 z^{-2} + ... + v_n z^{-n} + ...)$$

$$U(z).V(z) = \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + ... + u_n v_0) z^{-n}$$

$$U(z).V(z) = Z(u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + ... + u_n v_0)$$

$$Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_n v_{n-m} = u_n * v_n$$

EX **Use convolution theorem to evaluate** $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$

Sol

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n, Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{1}{(z-a)} - \frac{1}{(z-b)}\right\} = a^n * b^n$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \sum_{m=0}^{n} a^n . b^{n-m}$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = b^n \frac{(a/b)^{n+1} - 1}{(a/b) - 1}$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{a^{n+1} - b^{n+1}}{a - b}$$

Formation and solution of difference equations.

Take the Z-transform of both sides of the difference equations and the given conditions

Transpose all terms without $U(z)$ to the right

Divide by the coefficient of $U(z)$ getting $U(z)$ as a function of z

Express this function in terms of Z-transforms of known functions and take inverse Z-transform of both sides

This gives u_n as a function of n which is desired solution

Ex **Using Z-transform solve** $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ **with** $u_0 = 0, u_1 = 1$

Sol

$$Z(u_n) = U(z), Z(u_{n+1}) = z[U(z) - u_0]$$

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

$$Z(3^n) = z/(z-3)$$

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z[U(z) - u_0] + 3U(z) = z/(z-3)$$

$$U(z)(z^2 + 4z + 3) = z + z/(z-3)$$

$$\frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$U(z) = \frac{3z}{8(z+1)} + \frac{z}{24(z-3)} - \frac{5z}{12(z+3)}$$

$$u_n = \frac{3}{8} Z^{-1}\left[\frac{z}{(z+1)}\right] + \frac{1}{24} Z^{-1}\left[\frac{z}{(z-3)}\right] - \frac{5}{12} Z^{-1}\left[\frac{z}{(z+3)}\right]$$

$$u_n = \frac{3}{8}(-1)^n + \frac{1}{24}(3)^n - \frac{5}{12}(-3)^n$$