MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
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DEPARTMENT OF MATHEMATICS

SUBJECT NAME: STATICS

SUBJECT CODE: SM24B

SEMESTER: IV

PREPARED BY: PROF.S.AROCKIYA PRINCY

STATICS

YEAR: II **SEMESTER: IV**

Learning outcomes:
Students will acquire knowledge about

Particles or body in rest under the given forces.
Forces, equilibrium of a particle and centre of mass of various bodies.

UNIT I
Force- Newtons laws of motion - resultant of two forces on a particle- Equilibrium of a particle

UNIT II
Forces on a rigid body moment of a force general motion of a rigid body- equivalent systems of forces parallel forces forces along the sides of a triangle couples

UNIT III
Resultant of several coplanar forces**-** equation of the line of action of the resultant- Equilibrium
of a rigid body under three coplanar forces- Reduction of coplanar forces into a force and a couple.-problems involving frictional forces

UNIT IV
Centre of mass finding mass centre a hanging body in equilibrium

UNIT V
Hanging strings- equilibrium of a uniform homogeneous string suspension bridge

Introduction

"***Mathematics is the Queen of the Sciences and Number Theory is the Queen of Mathematics***" - Gauss.

Mechanics is a branch of Science which deals with the action of forces on bodies. Mechanics has two branches called Statics and Dynamics.

Statics is the branch of Mechanics which deals with bodies remain at rest under the influence of forces.

Dynamics is the branch of Mechanics which deals with bodies in motion under the action of forces.

Definitions:

Space: The region where various events take place is called a space.

Body: A portion of a matter is called a body.
Rigid body: A body consists of innumerable particles in which the distance between any two particles remains the same in all positions of the body is called a rigid body.

Particle: A particle is a body which is very small whose position at any time coincides with a point.

Motion: If a body changes its position under the action of forces, then it is said to be in motion.

Path of a particle: It is the curve joining the different positions of the particle in space while in motion.

Speed: The rate at which the body describes its path. It is a scalar quantity.

Displacement (vector quantity): It is the change in the positions of a particle in a certain interval.
Velocity (vector quantity): It is the rate of change of displacement.

Acceleration (vector quantity): It is the rate of change of velocity.
Equilibrium: A body at rest under the action of any number of forces on it is said to be in equilibrium.
Equilibrium of two forces

$Q \longleftrightarrow P$

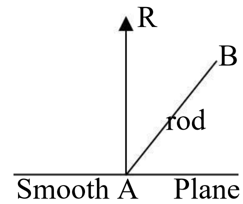
If two forces P, Q act on a body such that they have equal magnitude, opposite directions, same line of action then they are in equilibrium.

Force (vector): Force is any cause which produces or tends to produce a change in the existing state of rest of a body or of its uniform motion in a straight line. Force is represented by a straight line (through the point of application) which has both magnitude and direction.

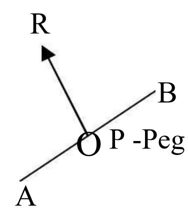
Types of forces: Weight, attraction, repulsion, tension, thrust, friction etc.
By Newton's third law, action and reaction are always equal and opposite.

Directions of Normal Reaction 'R' at the point of contact.

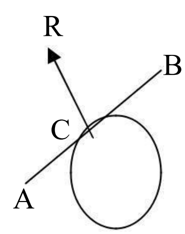
1. When a rod AB is in contact with a smooth plane, R is perpendicular to the plane at the point of contact A.



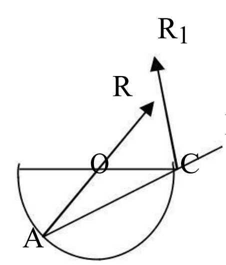
2. When a rod AB is resting on a smooth peg P, R is perpendicular to the rod at the point of contact P.



3. When a rod AB is resting on a smooth sphere, R is normal to the sphere at the point of contact C.



4. When a rod AB is resting on the rim of a hemisphere, with one end A in contact with the inner surface and C in contact with the rim. Then the normal reactions R at A is normal to the spherical surface and passes through the centre O, R_1 at C is perpendicular to the rod.



Regular polygon is a polygon with equal sides. Its vertices lie on a circle.

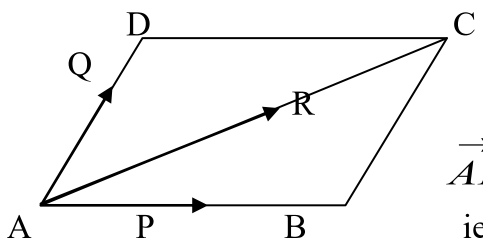
UNIT - I
FORCE

Introduction

Forces are represented by straight lines with magnitude and direction. Forces acting on a rigid body may be represented by straight lines with magnitude and direction passing through the same point and we say the forces are acting at a point. If P_1, P_2, P_3 …….. are the forces acting on a rigid body it is easy to find a single force whose effect is same as the combined effect of P_1, P_2, P_3 …….. Then the single force is called the resultant. P_1, P_2, P_3 ….. are called the components of the resultant. In this section we study some theorems and methods to find the resultant of two or more forces acting at a point.

1.1 Parallelogram law of forces (Fundamental theorem in statics)

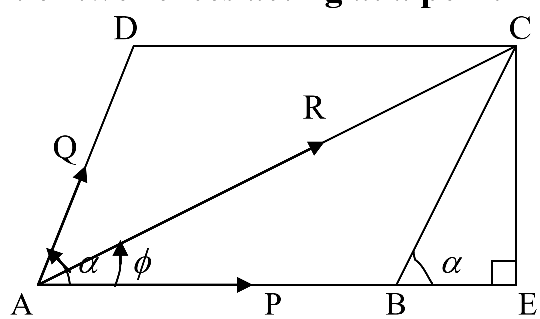
If two forces acting at a point be represented in magnitude and direction by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.



$$\vec{AB} + \vec{AD} = \vec{AC}$$

ie) P + Q = R

The resultant of two forces acting at a point



Let the two forces P and Q acting at A be represented by AB and AD. Let α be the angle between them.

i.e. $\angle BAD = \alpha$

Complete the parallelogram ABCD.

Then the diagonal AC will represent the resultant.

Let $\angle$CAB $= \varphi$

Draw CE $\perp r$ to AB. Now BC = AD = Q.

From the right angled ΔCBE,

$$\sin C\hat{B}E = \frac{CE}{BC} \quad \text{i.e. } \sin\alpha = \frac{CE}{Q}$$

$$\therefore CE = Q\sin\alpha \quad \text{.... ... ... (i)}$$

$$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$$\therefore BE = Q\cos\alpha \quad \text{.... ... ... (ii)}$$

$$R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$$

$$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$

$$= P^2 + 2PQ\cos\alpha + Q^2$$

$$\boxed{\therefore R = \sqrt[2]{\left(P^2 + 2PQ\cos\alpha + Q^2\right)}}$$

$$\boxed{\tan\varphi = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha}}$$

Result 1 If the forces P and Q are at right angles to each other, then $\alpha = 90^{\circ}$;

$$R = \sqrt{P^2 + Q^2} \qquad \tan\varphi = \frac{Q}{P}$$

Result 2 If the forces are equal (i.e.) Q = P, then

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1 + \cos\alpha)}$$

$$= \sqrt{2P^2 . 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

$$\tan\varphi = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2}$$

ie) $\boxed{\varphi = \frac{\alpha}{2}}$

Thus the resultant of two equal forces P, P at an angle α is 2 P cos $\frac{\alpha}{2}$ in a direction bisecting the angle between them.

Result 3 Resultant R is greatest when cos α is greatest.

i.e. when cos $\alpha = 1$ or $\alpha = 0^{o.}$

ie) Greatest value of R is R = P +Q.

R is least when cos α is least.

i.e. when cos $\alpha = -1$ or $\alpha = 180^{o.}$ Least value of R is P~Q.

Problem 1

The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y, Prove that.

$$P = (X^2 + QR)^{1/2} = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

Prove also that, if P + Q + R = 0, Y = X.

Solution:

Let α be the angle between P and Q

Given

$X^2 = P^2 + Q^2 + 2PQ\cos\alpha$ .... (1)

$X^2 = P^2 + R^2 + 2PR\cos\alpha$ .... (2)

$Y^2 = Q^2 + R^2 + 2QR\cos\alpha$ .... (3)

(1) – (2) gives $0 = Q^2 - R^2 + 2P\cos\alpha\,(Q-R)$

i.e. $0 = (Q - R)(Q+R+2P\cos\alpha)$

But $Q \neq R$ and so $Q - R \neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

$$\cos\alpha = -\frac{Q+R}{2P} \quad \text{........ (4)}$$

Substitute (4) in (1),

$$X^2 = P^2 + Q^2 + 2PQ\left[-\left(\frac{Q+R}{2P}\right)\right] = P^2 + Q^2 - Q^2 - QR$$

$$P^2 = X^2 + QR. \text{ i.e. } P = (X^2 + QR)^{1/2}$$

Substitute (4) in (3),

$$Y^2 = Q^2 + R^2 + 2QR\left[-\left(\frac{Q+R}{2P}\right)\right]$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

$$P = \frac{QR(Q+R)}{Q^2+R^2-Y^2}$$

If $P + Q + R = 0$, then $Q + R = -P$,

$$\therefore \text{From (4), } \cos\alpha = -\frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$$

$$\cos\ \alpha = \frac{1}{2} \Rightarrow$$

$$X^2 = P^2 + R^2 + PR \ldots \quad \ldots \quad \ldots (5)$$

$$Y^2 = Q^2 + R^2 + QR \ldots \quad \ldots \quad \ldots (6)$$

(5) – (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$

$$= (P - Q)(P + Q + R)$$

$$= (P - Q).0 = 0$$

$$\therefore X = Y$$

Problem 2

Two forces of given magnitude P and Q act at a point at an angle α. What will be the maximum and minimum value of the resultant?

Solution:

i. Maximum value of the resultant = $P + Q$

ii. Minimum value of the resultant = $P \sim Q$.

Problem 3

The greatest and least magnitudes of the resultant of two forces of constant magnitudes are R and S respectively. Prove that, when the forces act at an angle 2φ, the resultant is of magnitude $\sqrt{R^2\cos^2\varphi + S^2\sin^2\varphi}$

Solution:

Given, R = P + Q, S = P-Q, where P and Q are two forces.

When P and Q are acting at an angle 2φ

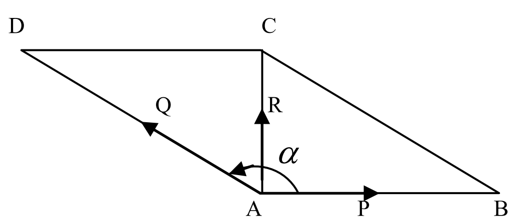
$$\begin{aligned}
\text{Resultant} &= \sqrt{P^2 + Q^2 + 2PQ.\cos 2\varphi} \\
&= \sqrt{\left(P^2 + Q^2\right) + 2PQ\left(\cos^2\varphi - \sin^2\varphi\right)} \\
&= \sqrt{\left(P^2 + Q^2\right)\left(\sin^2\varphi + \cos^2\varphi\right) + 2PQ\left(\cos^2\varphi - \sin^2\varphi\right)} \\
&= \sqrt{\left(P^2 + Q^2 + 2PQ\right)\cos^2\varphi + \left(P^2 + Q^2 - 2PQ\right)\sin^2\varphi} \\
&= \sqrt{R^2\cos^2\varphi + S^2\sin^2\varphi}.
\end{aligned}$$

Problem 4

The resultant of two forces P and Q is at right angles to P. Show that the angle between the forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$

Solution:

Let α be the angle between the two forces P and Q. Given $\varphi = 90^{o}$.



We know, $\tan\varphi = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

i.e. $\tan 90^{o} = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

$$\frac{1}{0} = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$\therefore P+Q\cos\alpha = 0$$

$$\therefore \cos\alpha = -\frac{P}{Q}$$

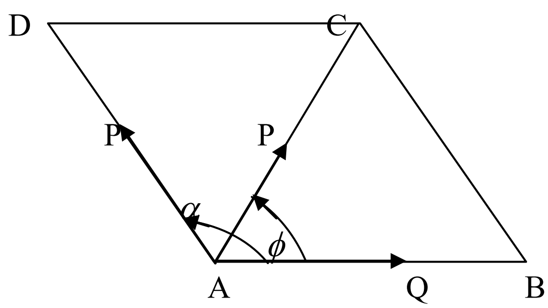
$$\boxed{\therefore \alpha = \cos^{-1}\left(-\frac{P}{Q}\right)}$$

Problem 5

The resultant of two forces P and Q is of magnitude P. Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.

Solution:

Let α be the angle between P and Q



Given, $P^2 = P^2 + Q^2 + 2PQ\cos\alpha$.

$\therefore$ Q (Q+2Pcosα) = 0

$$\boxed{\therefore \cos\alpha = -\frac{Q}{2P}}$$

If **P is doubled,** let R be the new resultant, and φ be the angle between Q and R.

$$\therefore R^2 = (2P)^2 + Q^2 + 2(2P)Q.\cos\alpha$$

$$= 4P^2 + Q^2 + 4PQ\left(-\frac{Q}{2P}\right)$$

$$= 4P^2 + Q^2 - 2Q^2 = 4P^2 - Q^2$$

$$\boxed{\therefore R = \sqrt{4P^2 - Q^2}}$$

$$\tan\varphi = \frac{(2P)\sin\alpha}{Q+(2P)\cos\alpha} = \frac{2P\sin\alpha}{Q+2P\left(-\frac{Q}{2P}\right)}$$

$$\text{i.e. } \tan\varphi = \frac{2P\sin\alpha}{0}$$

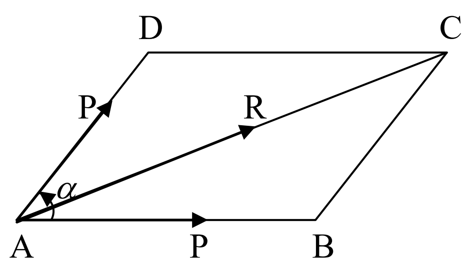
$$\therefore \cos\varphi = 0 \Rightarrow \varphi = 90^0$$

$\therefore$ Q is at right angles to R.

Problem 6

Two equal forces act on a particle, find the angle between them when the square of their resultant is equal to three times their product.

Solution:



Let α be the angle between the two equal forces P, P, and let R be their resultant.

$$\therefore R^2 = P^2 + P^2 + 2P.P.\cos\alpha$$

$$= 2P^2(1+\cos\alpha) = 2P^2 \times 2\cos^2\frac{\alpha}{2}$$

$$\text{i.e. } R^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \boxed{R = 2P\cos\frac{\alpha}{2}}$$

$$\text{Given, } R^2 = 3 \times P \times P = 3P^2$$

$$\therefore 3P^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \cos^2\frac{\alpha}{2} = \frac{3}{4} \Rightarrow \cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{\alpha}{2} = 30^{\circ}$$

$$\Rightarrow \alpha = 60^{0}$$

Problem 7

If the resultant of forces 3P, 5P is equal to 7P find

i. the angle between the forces

ii. the angle which the resultant makes with the first force.

Solution:

Let α be the angle between 3P, 5P

i. Given $(7P)^2 = (3P)^2 + (5P)^2 + 2\,(3P)\,(5P)\,.\cos\alpha$

$$49P^2 = 9P^2 + 25P^2 + 30P^2\cos\alpha$$

$$\therefore 15P^2 = 30P^2\cos\alpha$$

$$\therefore \cos\alpha = \frac{1}{2} \qquad \Rightarrow \alpha = 60^{0}$$

ii. Let φ be the angle between the resultant and 3P.

$$\therefore \tan\varphi = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$= \frac{5P.\sin\alpha}{3P + 5P.\cos\alpha}$$

$$= \frac{5P.\sin 60^{\circ}}{3P + 5P.\cos 60^{\circ}}$$

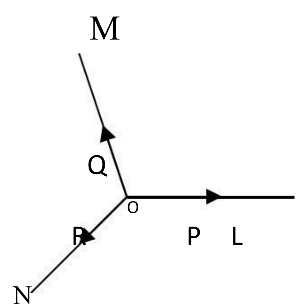
$$= \frac{5 \times \frac{\sqrt{3}}{2}}{3 + \left(5 \times \frac{1}{2}\right)}$$

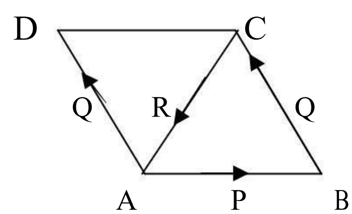
$$\tan\varphi = \frac{5\sqrt{3}}{11}$$

$$\therefore \varphi = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

1.2 Triangle of forces

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.





Let the forces, P,Q,R act at a point O and be represented in magnitude and direction by the sides AB,BC,CA of the triangle ABC.

To prove : They will be in equilibrium.

Complete the parallelogram BADC.

$$P+Q = \overline{AB} + \overline{AD} = \overline{AB} + \overline{BC}$$

$$= \overline{AC}$$

ie) The resultant of the forces P, Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

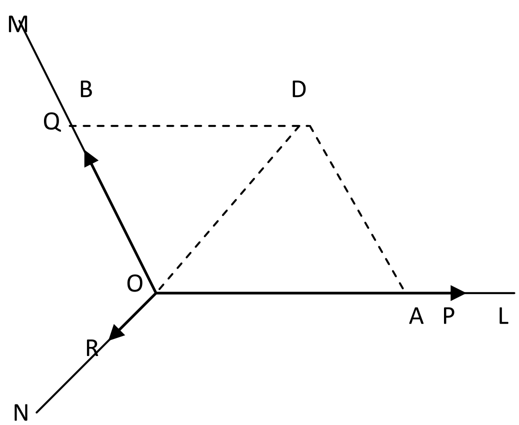
Hence P+Q+R = $\overline{AC} + \overline{CA} = \overline{0}$

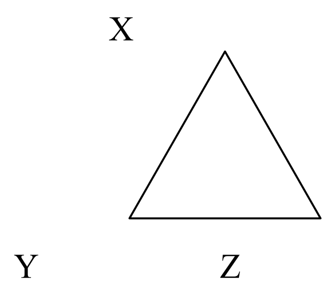
Principle

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

1.3 Lami's Theorem

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.





Proof:

By converse of the triangle of forces, the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.

By sine rule in ΔOAD , we have

$$\frac{OA}{\sin \angle ODA} = \frac{AD}{\sin \angle DOA} = \frac{DO}{\sin \angle OAD} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But $\angle OAD = alt.\angle BOD = 180^0 - \angle MON$

$$\therefore \sin \angle ODA = \sin\left(180^0 - \angle MON\right) = \sin \angle MON \quad \ldots\ldots\ldots (2)$$

Also $\angle DOA = 180^0 - \angle NOL$

$$\therefore \sin \angle DOA = \sin\left(180^0 - \angle NOL\right) = \sin \angle NOL \quad \ldots\ldots\ldots (3)$$

And $\angle OAD = 180^0 - \angle BOA = 180^0 - \angle LOM$

$\therefore \sin \angle OAD = \sin\left(180^0 - \angle LOM\right) = \sin \angle LOM$ ……. (4)

Substitute (2), (3), (4) in (1),

$$\frac{OA}{\sin \angle MON} = \frac{AD}{\sin \angle NOL} = \frac{DO}{\sin \angle LOM}$$

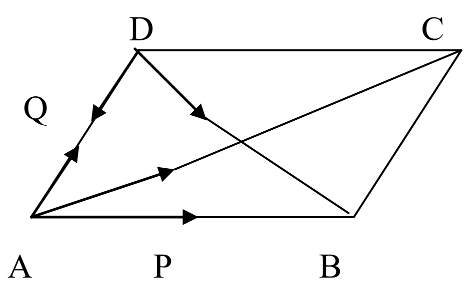
i.e. $\dfrac{P}{\sin \angle MON} = \dfrac{Q}{\sin \angle NOL} = \dfrac{R}{\sin \angle LOM}$

$$\frac{P}{\sin(Q.R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$$

Problem 8

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.

Solution:



Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD. Complete the parallelogram BAD.

Then P+Q= $\overline{AB} + \overline{AD} = \overline{AC}$

P-Q $= \overline{AB} - \overline{AD}$

$= \overline{AB} + \overline{DA}$

$= \overline{DA} + \overline{AB}$

$= \overline{DB}$

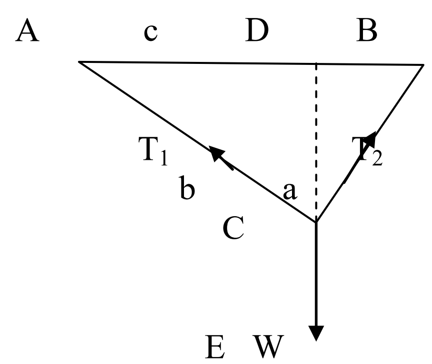
Given $\overline{AC}$ and $\overline{DB}$ are at right angles.
The diagonals AC and BD cut at right angles.

$\therefore$ ABCD must be a rhombus.
$\therefore$ AB = AD.
P = Q.

Problem 9

Let A and B two fixed points on a horizontal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$

Solution



Forces T_1, T_2, W are acting at C.

By Lami's theorem,

$$\frac{T_1}{\sin \angle ECB} = \frac{T_2}{\sin \angle ECA} \text{........(1)}$$

Now $\sin \angle ECB = \sin(180^0 - \angle DCB)$

$= \sin \angle DCB$

$= \sin (90^0 - \angle ABC) = \cos \angle ABC$

$\sin \angle ECA = \sin(180^0 - \angle ACD)$

$= \sin \angle ACD$

$= \sin (90^0 - \angle BAC) = \cos \angle BAC$

$$\frac{T_1}{\cos\angle ABC}=\frac{T_2}{\cos\angle BAC}\quad\therefore\;\frac{T_1}{T_2}=\frac{\cos B}{\cos A}=\frac{\left(\dfrac{c^2+a^2-b^2}{2ca}\right)}{\left(\dfrac{b^2+c^2-a^2}{2bc}\right)}$$

$$\therefore\frac{T_1}{T_2}=\left(\frac{c^2+a^2-b^2}{2ca}\right)\times\left(\frac{2bc}{b^2+c^2-a^2}\right)=\frac{b\left(c^2+a^2-b^2\right)}{a\left(b^2+c^2-a^2\right)}$$

Problem 10

ABC is a given triangle. Forces P,Q,R acting along the lines OA,OB,OC are in equilibrium. Prove that

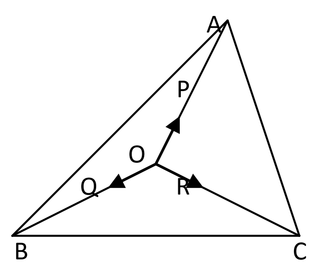
(i)P : Q : R=$a^2\left(b^2+c^2-a^2\right):b^2\left(c^2+a^2-b^2\right):c^2\left(a^2+b^2-c^2\right)$ if O is the cicumcentre of the triangle.

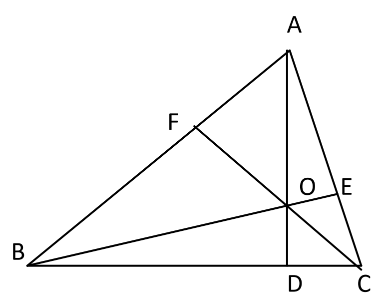
(ii) P : Q : R= $=\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$ if O is the incentre of the triangle.

(iii) P : Q : R= a:b:c if O is the ortho centre of the triangle.

(iv) P : Q : R=OA : OB : OC if O is the centroid of the triangle,

Solution:





By Lami's theorem,

$$\frac{P}{\sin\angle BOC}=\frac{Q}{\sin\angle COA}=\frac{R}{\sin\angle AOB}\qquad\ldots\ldots\ldots\ldots\ldots(1)$$

(i) O is the circumcentre of the Δ ABC

$$\angle BOC=2\angle BAC=2A;\angle COA=2B\;\; and\;\; \angle AOB=2C$$

$$\therefore (1) \Rightarrow \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

i.e. $$\frac{P}{2\sin A\cos A} = \frac{Q}{2\sin B\cos B} = \frac{R}{2\sin C\cos C} \quad \ldots\ldots (2)$$

But $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $\sin A = \dfrac{2\Delta}{bc}$

where Δ is the area of the triangle ABC

$$\therefore 2\sin A\cos A = 2\,\frac{2\Delta\left(b^2+c^2-a^2\right)}{bc \quad 2bc}$$

$$= \frac{2\Delta\left(b^2+c^2-a^2\right)}{b^2c^2}$$

Similarly $2\sin B\cos B = \dfrac{2\Delta\left(c^2+a^2-b^2\right)}{c^2a^2}$

$$2\sin C\cos C = \frac{2\Delta\left(a^2+b^2-c^2\right)}{a^2b^2}$$

Substitute in (2)

$$\frac{P.b^2c^2}{2\Delta\left(b^2+c^2-a^2\right)} = \frac{Q.c^2a^2}{2\Delta\left(c^2+a^2-b^2\right)} = \frac{Ra^2b^2}{2\Delta\left(a^2+b^2-c^2\right)}$$

Divide by $\dfrac{a^2b^2c^2}{2\Delta}$

$$\frac{P}{a^2\left(b^2+c^2-a^2\right)} = \frac{Q}{b^2\left(c^2+a^2-b^2\right)} = \frac{R}{c^2\left(a^2+b^2-c^2\right)}$$

(ii) O is the in-centre of the triangle,

OB and OC are the bisectors of $\angle$B and $\angle$C

$$\therefore \angle BOC = 180^0 - \frac{B}{2} - \frac{C}{2} = 180^0 - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180^0 - \left(90^0 - \frac{A}{2}\right) = 90^0 + \frac{A}{2}$$

Similarly $\angle \text{COA} = 90^0 + \frac{B}{2}, \ \angle AOB = 90^0 + \frac{C}{2}$

$$(1) \Rightarrow \frac{P}{\sin\left(90^0 + \frac{A}{2}\right)} = \frac{Q}{\sin\left(90^0 + \frac{B}{2}\right)} = \frac{R}{\sin\left(90^0 + \frac{C}{2}\right)}$$

i.e. $\dfrac{P}{\cos\frac{A}{2}} = \dfrac{Q}{\cos\frac{B}{2}} = \dfrac{R}{\cos\frac{C}{2}}$

(iii) O is the ortho-centre of the triangle

AD, BE, CF are the altitudes of the triangle

AFOE is a cyclic quadrilateral.

$\therefore \angle FOE + A = 180^0$, $\therefore \angle FOE = 180^0 - A$

$\therefore \angle BOC = 180^0 - A$

Similarly, $\angle COA = 180^0 - B$, $\angle AOB = 180^0 - C$

Hence (1) becomes

$$\frac{P}{\sin\left(180^0 - A\right)} = \frac{Q}{\sin\left(180^0 - B\right)} = \frac{R}{\sin\left(180^0 - C\right)}$$

i.e. $\dfrac{P}{\sin A} = \dfrac{Q}{\sin B} = \dfrac{R}{\sin C}$

i.e. $\dfrac{P}{a} = \dfrac{Q}{b} = \dfrac{R}{c} \quad \left(\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\right)$

(iv) O is the centroid of the triangle

Δ BOC = Δ COA = Δ AOB = $\frac{1}{3}\Delta ABC$

Δ BOC = $\frac{1}{2}OB.OC\sin\angle BOC = \frac{1}{3}\Delta ABC$

$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3OB.OC}$

Similarly, $\sin\angle COA = \frac{2\Delta ABC}{3OC.OA}$, $\sin\angle AOB = \frac{2\Delta ABC}{3OA.OB}$

Hence (1) becomes $\frac{P.3OB.OC}{2\Delta ABC} = \frac{Q.3OC.OA}{2\Delta ABC} = \frac{R.3OA.OB}{2\Delta ABC}$

i.e. P.OB.OC = Q.OC.OA = R.OA.OB

Dividing by OA.OB.OC, we get $\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$.

1.4 Friction

In the previous sections we have studied problems on equilibrium of smooth bodies. Practically no bodies are perfectly smooth. All bodies are rough to a certain extent. Friction is the force that opposes the motion of an object. Only because of this friction we are able to travel along the road by walking or by vehicles. So friction helps motion. It is a tangential force acting at the point on contact of two bodies. To stop a moving object a force must act in the opposite direction to the direction of motion. Such force is called a frictional force. For example if you push your book across your desk, the book will move. The force of the push moves the book. As the books slides across the desk, it slows down and stops moving. When you ride a bicycle the contact between the wheel and the road is an example of dynamic friction.

Definition

If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called ***friction;*** the force exerted is called the ***force of friction***.

Types of Friction

There are three types of friction

1) Statical Friction 2) Limiting Friction 3) Dynamical friction.

1. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium is called ***statical friction***.

2. When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called ***limiting friction***; the equilibrium is said to be limiting equilibrium.

3. When motion ensues by one body sliding over another, the friction exerted is called ***dynamical friction***.

1.5 Laws of Friction

Friction is not a mathematical concept; it is a physical reality.

Law 1 When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

Law 2 When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law 3 The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law 4 The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.

Law 5 (Law of dynamical Friction)

When motion ensues by one body sliding over the other the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

Friction is a passive force: Explain

1) Friction is only a resisting force.
2) It appears only when necessary to prevent or oppose the motion of the point of contact.
3) It can not produce motion of a body by itself, but maintains relative equilibrium.
4) It is a self-adjusting force.
5) It assumes magnitude and direction to balance other forces acting on the body.

Hence, friction is purely a passive force.

Co-efficient of friction

The ratio of the limiting friction to the normal reaction is called the co-efficient of friction. It is denoted by μ

i.e.) $\boxed{\frac{F}{R} = \mu}$ $\Rightarrow$ $\boxed{F = \mu R}$

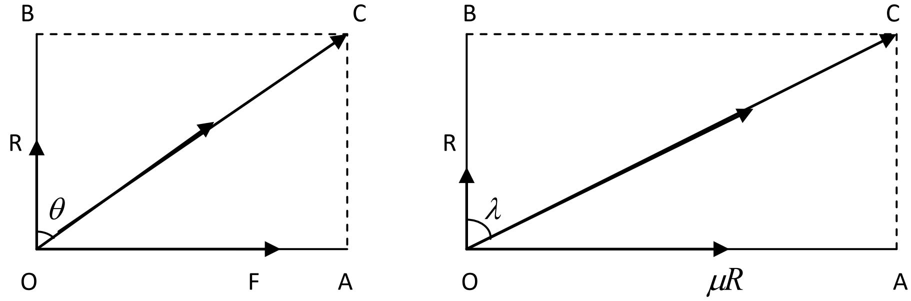
Note: 1) μ depends on the nature of the materials in contact.

2) Friction is maximum when it is limiting. μR is the maximum value of friction.

3) When equilibrium is non-limiting, $F < \mu R$ i.e.) $\frac{F}{R} < \mu$

4) Friction 'F' takes any value from zero upto μR.

Angle of Friction



Let OA = F(Friction), $\overrightarrow{OB} = R$ (Normal reaction) & $\overrightarrow{OC}$ be the resultant of F and R.

If $B\hat{O}C = \theta,\ \tan\theta = \dfrac{BC}{OB} = \dfrac{OA}{OB} = \dfrac{F}{R}$ ……….. (1)

As F increases, θ - increases until F reaches its maximum value μR. In this case, equilibrium is limiting.

Definition

"When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called **the angle of friction** and is denoted by λ"

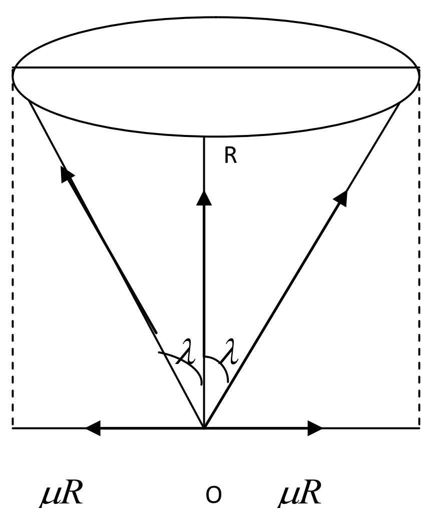
In the limiting equilibrium, $B\hat{O}C = \lambda$ = angle of friction.

$$\therefore \tan\lambda = \frac{BC}{OB} = \frac{OA}{OB} = \frac{\mu R}{R} = \mu$$

$$\boxed{\mu = \tan\lambda}$$

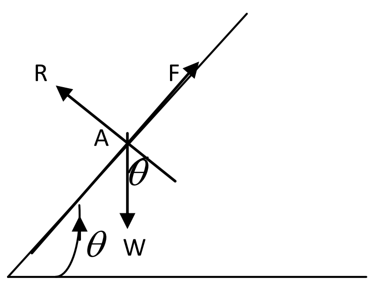
i.e.) The co-efficient of friction is equal to the tangent of the angle of friction.

Cone of Friction



We know, the greatest angle made by the resultant reaction with the normal is λ (angle of friction) where $\lambda = \tan^{-1}(\mu)$. Consider the motion of a body at O (its point of contact) with another. When two bodies are in contact, consider a cone drawn with O as vertex, common normal as the axis of the cone, λ - be the semi-vertical angle of the cone. Now, the resultant reaction of R and μR will have a direction which lies within the surface or on the surface of the cone. It can not fall outside the cone. This cone generated by the resultant reaction is called the ***cone of friction***.

1.6 Equilibrium of a particle on a rough inclined plane.



Let θ - be the inclination of the rough inclined plane, on which a particle of weight W, is placed at A. Forces acting on the particle are,

1) Weight W vertically downwards
2) Normal reaction R, $\perp$ r to the plane.
3) Frictional force F, along the plane upwards (Since the body tries to slip down).

Resolving the forces along and perpendicular to the plane,

$$F = W\sin\theta, \quad R = W\cos\theta$$

$$\therefore \frac{F}{R} = \tan\theta$$

But $\frac{F}{R} < \mu \;\therefore \tan\theta < \mu$

i.e) $\tan\theta < \tan\lambda$

$\therefore \theta < \lambda$

When $\theta = \lambda, \frac{F}{R} = \tan\lambda = \mu$

Hence, it is clear that "***when a body is placed on a rough inclined plane and is on the point of sliding down the plane, the angle of inclination of the plane is equal to the angle of friction***." Now λ is called as the ***angle of repose.***

Thus the angle of repose of a rough inclined plane is equal to the angle friction when there is no external force act on the body.

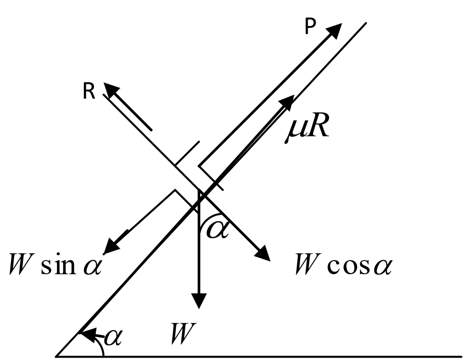
1.7 Equilibrium of a body on a rough inclined plane under a force parallel to the plane.

A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted on by a force parallel to the plane. Find the limits between which the force must lie.

Proof:

Let α be the inclination of the plane, W be the weight of the body& R be the normal reaction.

Case 1: Let the body be on the point of slipping down. Therefore μR acts upwards along the plane.



Let P be the force applied to keep the body at rest.

Resolving the forces along and perpendicular to the plane,

$P + \mu R = W \sin\alpha$ ……………….. (1)

$R = W.\cos\alpha$ ………………… (2)

$$\therefore P = W.\sin\alpha - \mu.W\cos\alpha$$

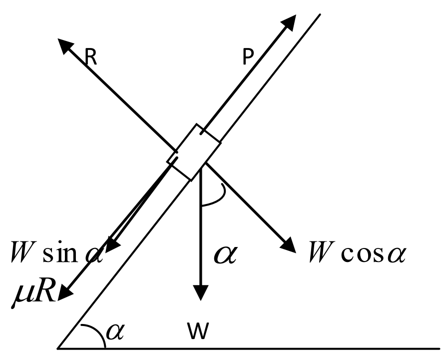
$$= W[\sin\alpha - \tan\lambda.\cos\alpha]$$

$$= \frac{W}{\cos\lambda}[\sin\alpha.\cos\lambda - \cos\alpha\sin\lambda]$$

$$= \frac{W}{\cos\lambda}.\sin(\alpha - \lambda)$$

Let $P_1 = \dfrac{W.\sin(\alpha - \lambda)}{\cos\lambda}$

Case ii Let the body be on the point of moving up. Therefore limiting frictional force μR acts downward along the plane.



Let P be the external force applied to keep the body at rest.

Resolving the force,

$R = W\cos\alpha;\ P = \mu R + W\sin\alpha$

$\therefore P = \mu.W\cos\alpha + W\sin\alpha$

$$= \frac{W}{\cos\lambda}[\sin\lambda\cos\alpha + \cos\lambda.\sin\alpha]$$

$= \dfrac{W}{\cos\lambda}.\sin(\alpha+\lambda)$

Let $P_2 = \dfrac{W}{\cos\lambda}.\sin(\alpha+\lambda)$

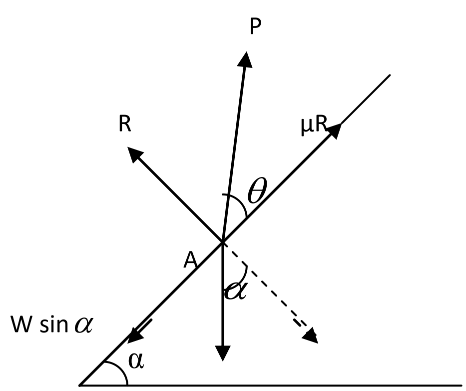
If $P < P_1$, body will move down the plane. If $P > P_2$, body will move up the plane.

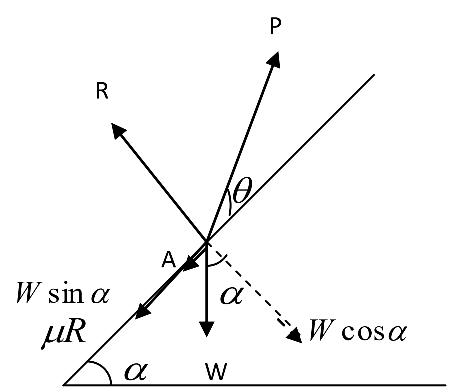
$\therefore$ For equilibrium P must lie between P_1 and P_2.

i.e.) $\boxed{P_1 > P > P_2}$

1.8 Equilibrium of a body on a rough inclined plane under any force.

Theorem: A body is at rest on a rough inclined plane of inclination α to the horizon, being acted on by a force making an angle θ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.





Let α be the inclination of the plane, W be the weight of the body, P – be the force acting at an angle θ with the inclined plane and R – be the normal reaction.

Case i: The body is just on the point of slipping down. Therefore the limiting friction μR acts upwards.

Resolving the forces along and $\perp r$ to the inclined plane,

$P\cos\theta + \mu R = W\sin\alpha$ ……………….. (1)

$P\sin\theta + R = W\cos\alpha$ ………………….. (2)

$\therefore R = W\cos\alpha - P\sin\theta$

$\therefore (1) \Rightarrow P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$

$P(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha)$

$$\therefore P = \frac{W(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

We have $\mu = \tan\lambda$

$$\therefore P = \frac{W(\sin\alpha - \tan\lambda.\cos\alpha)}{\cos\theta - \tan\lambda.\sin\theta}$$

$$= W\frac{(\sin\alpha\cos\lambda - \cos\alpha.\sin\lambda)}{\cos\theta.\cos\lambda - \sin\theta.\sin\lambda}$$

$$= W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$$

Let $P_1 = W.\dfrac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$

Case ii: The body is just on the point of moving up the plane. Therefore μR acts downwards.

Resolving the forces along and $\perp r$ to the plane.

$P\cos\theta - \mu R = W.\sin\alpha$ …………………. (3)

$P\sin\theta + R = W.\cos\alpha$ …………………. (4)

$R = W\cos\alpha - P\sin\theta$

$(3) \Rightarrow P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W.\sin\alpha$

$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$

$$\therefore P = \frac{W(\sin\alpha + \tan\lambda.\cos\alpha)}{(\cos\theta + \tan\lambda.\sin\theta)}$$

$$= \frac{W(\sin\alpha.\cos\lambda + \sin\lambda.\cos\alpha)}{(\cos\theta\cos\lambda + \sin\theta.\sin\lambda)}$$

$$= \frac{W.\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Let $P_2 = \dfrac{W.\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

To keep the body in equilibrium, P_1 and P_2 are the limiting values of P.

Find the least force required to drag the body up the inclined plane

We have, P = $W.\dfrac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

P is least when $\cos(\theta-\lambda)$ is greatest.

i.e.) When $\cos(\theta-\lambda)=1$

i.e.) When $\theta-\lambda=0$

i.e.) When $\theta=\lambda$

$$\boxed{\therefore \textit{Least value of } P = W.\sin(\alpha+\lambda)}$$

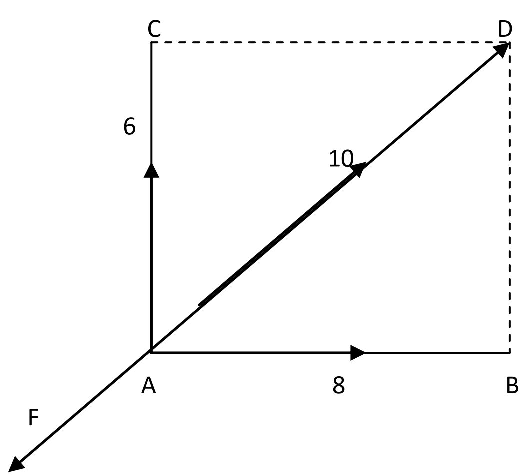
Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

i.e.) *"The best angle of traction up a rough inclined plane is the angle of friction"*

Problem 1

A particle of weight 30 kgs. resting on a rough horizontal plane is just on the point motion when acted on by horizontal forces of 6kg wt. and 8kg. wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.

Solution:



Let AB = 8 and AC = 6 represent the directions of the forces, A being the particle.

The resultant force = $\sqrt{8^2+6^2}$ = 10kg. wt. and this acts along AD, making an angle $\cos^{-1}\left(\frac{4}{5}\right)$ with the 8kg force.

Let F be the frictional force. As motion just begins, magnitude of F is equal to that of the resultant force.

$\therefore F = 10$ ……………… (1)

If R is the normal reaction on the particle,

R = 30 ………………….. (2)

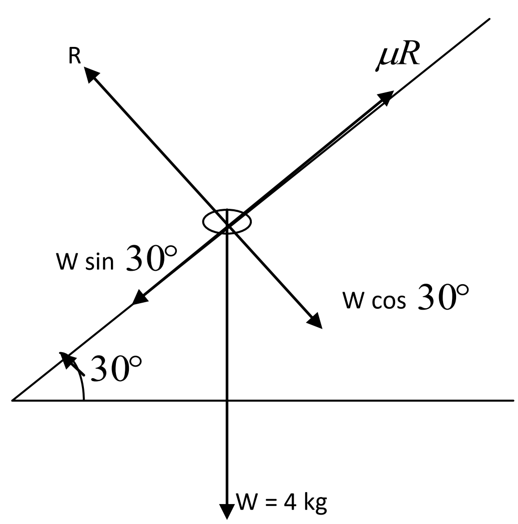
If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$

$10 = \mu.30 \qquad \therefore \mu = \frac{10}{30} = \frac{1}{3}.$

Problem 2

A body of weight 4 kgs. rests in limiting equilibrium on an inclined plane whose inclination is $30°$. Find the coefficient of friction and the normal reaction.

Solution:



Since the body is in limiting equilibrium on the inclined plane, it tries to move in the downward direction along the inclined plane.

$\therefore$ Frictional force μR acts in the upward direction along the inclined plane. Resolving along and $\perp r$ to the plane,

$\mu R = W \sin 30°$ ………………… (1)

$= 4.\frac{\sqrt{3}}{2} = 2\sqrt{3}$

R = $W.\cos 30°$ …………………. (2)

$= 4\frac{1}{2} = 2$

$\frac{(1)}{(2)} \Rightarrow \mu = \frac{1}{\sqrt{3}}$

$\tan\lambda = \frac{1}{\sqrt{3}}, \therefore$ $\boxed{\lambda = 30°}$

UNIT - II
PARALLEL FORCES

2.1 Parallel forces:

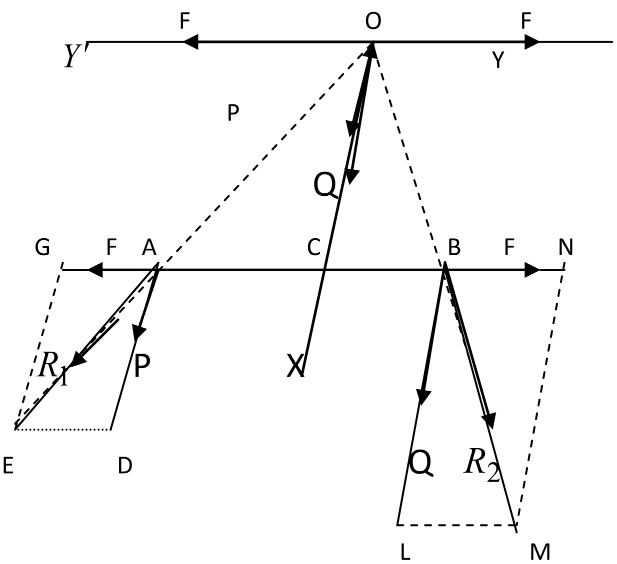
Forces acting along parallel lines are called parallel forces. There are two types of parallel forces known as like and unlike parallel forces. Since the parallel forces do not meet at a point, in this chapter we study methods to find the resultant of two like parallel and unlike parallel forces. Parallel forces acting on a rigid body have a tendency to rotate it about a fixed point. Such tendency is known as moment of the parallel forces. Here we study the theorem on moments of forces about a point.

OC

Definition:

Two parallel forces are said to be **like** if they act in the same direction, they are said to be **unlike** if they act in opposite parallel directions.

The resultant of two like parallel forces acting on a rigid body



Proof:

Let P and Q be two like parallel forces acting at A and B along the lines AD and BL.At A and B, introduce two equal and opposite forces F along AG and BN. These two forces F balance each other and will not affect the system.

Now, R_1 is the resultant of P and F at A and R_2 is the resultant of Q and F at B as in the diagram.

Produce EA and MB to meet at O. At O, draw YOY^1 parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their original components. R_1 at O is equal to F along OY^1 and P along OX. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. The remaining two forces P and Q acting along OX have the resultant **P+Q (sum)** along OX.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles, OAC and AED are similar.

$$\therefore \frac{OC}{AD}=\frac{AC}{ED} \text{ ie) } \frac{OC}{P}=\frac{AC}{F}$$

$$\therefore F.OC = P.AC \text{ .... (1)}$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL}=\frac{CB}{LM} \text{ ie) } \frac{OC}{Q}=\frac{CB}{F}$$

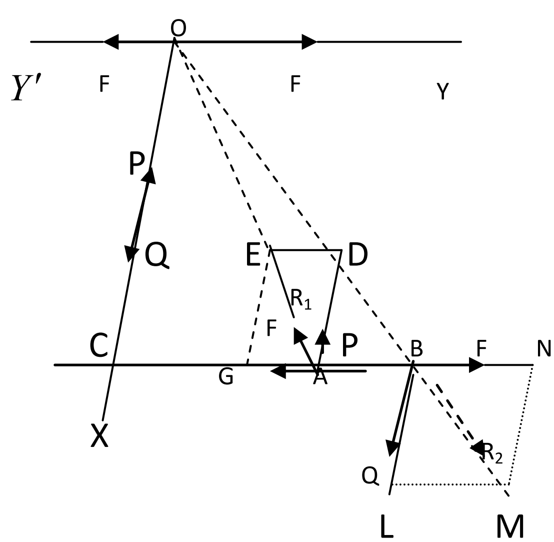
$$\therefore F.OC = Q.CB \text{ .... (2)}$$

(1) & (2) $\Rightarrow$ $\boxed{P.AC = Q.CB}$

ie) $\frac{AC}{CB} = \frac{Q}{P}$

ie) 'C' divides AB internally in the inverse ratio of the forces.

The resultant of two unlike and unequal parallel forces acting on a rigid body:



Proof:

Let P and Q at A and B be two unequal unlike parallel forces acting along AD and BL.

Let P > Q.

At A and B introduce two equal and opposite forces F along AG and BN. These two balances each other and will not affect the system.

Let R_1 be the resultant of F and P at A and R_2 be the resultant of F and Q at B. as in the diagram.

Produce EA and MB to meet at O. At O, draw Y'OY parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their components. R_1 at O is equal to F along OY' and P along XO. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. Now, the remaining forces are P and Q along the same line but opposite directions.

Hence the resultant is **P ~ Q (difference)** along XO.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles OCA and EGA are similar.

$$\therefore \frac{OC}{EG} = \frac{CA}{GA}, \text{ ie) } \frac{OC}{P} = \frac{CA}{F}$$

$$F.OC = P.AC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM}, \text{ ie) } \frac{OC}{Q} = \frac{CB}{F}$$

$$\therefore F.OC = Q.CB \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

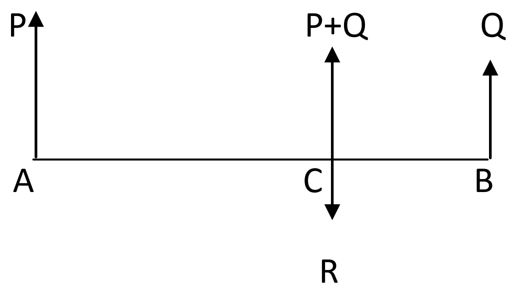
(1) and (2) $\Rightarrow$ $\boxed{P.AC = Q.CB}$

ie) $\frac{CA}{CB} = \frac{Q}{P}$

ie) 'C' divides AB externally.

Note : The effect of two equal and unlike parallel forces can not be replaced by a single force.

The condition of equilibrium of three coplanar parallel forces



Let P, Q, R be the three coplanar parallel forces in equilibrium. Draw a line to meet the forces P, Q, R at the points A, B, C respectively.

Equilibrium is not possible if all the three forces are in the same direction.

Let P + Q be the resultant of P and Q parallel to P. Hence R must be equal and opposite to P + Q.

$\therefore$ R = P + Q (in magnitude, opposite in direction)

$$\therefore P.AC = Q.CB$$

$$\therefore \frac{P}{CB} = \frac{Q}{AC} = \frac{P+Q}{CB+AC} = \frac{R}{AB}$$

Hence, $\boxed{\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}}$

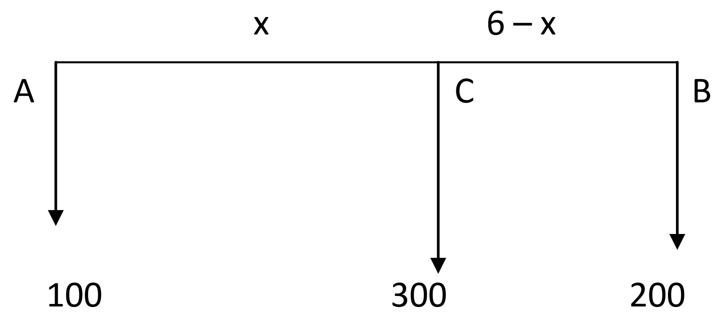
ie) If three parallel forces are in equilibrium then each force is proportional to the distance between the other two.

Note: The centre of two parallel forces is a fixed point through which their resultant always passes.

Problem 1

Two men, one stronger than the other, have to remove a block of stone weighing 300 kgs. with a light pole whose length is 6 metre. The weaker man cannot carry more than 100 kgs. Where the stone be fastened to the pole, so as just to allow him his full share of weight?

Solution:



Let A be the weaker man bearing 100 kgs., B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC = x. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

$\therefore$ 100.AC = 200.BC

i.e. 100x = 200 (6-x) = 1200 – 200x

$\therefore$ 300x = 1200 or x=4

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

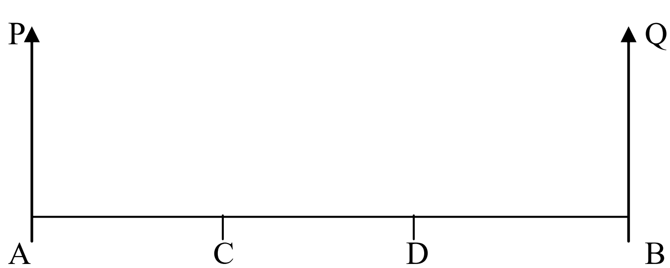
Problem 2

Two like parallel forces P and Q act on a rigid body at A and B respectively.

a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

b) If P and Q be interchanged in position, show that the point of application of the resultant will be displayed along AB through a distance d, where $d = \frac{P-Q}{P+Q}.AB$.

Solution:



Let C – be the centre of the two forces.

Then P. AC = Q.CB ………… (1)

(a) If Q is changed to $\frac{P^2}{Q}$, (P remaining the same), let D be the new centre of parallel forces.

Then P.AD = $\frac{P^2}{Q}$ DB …. …. ….. (2)

Q.AD = P.DB ……………. (3)

Relation (3) shows that D is the centre of two like parallel forces, with Q at A and P at B.

(b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

Let CD = d

From (3), Q. (AC+CD) = P. (CB – CD)

i.e. Q.AC + Q.d = P.CB – P.d

(Q + P).d = P.CB – Q.AC

= P (AB – AC) – Q (AB – CB)

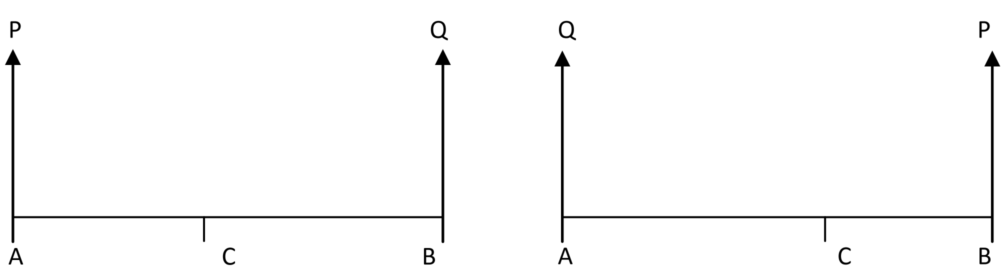
= (P – Q).AB[$\because$P.AC = Q.CB from (1)]

$$\boxed{d = \frac{P-Q}{P+Q}.AB}$$

Problem 3

The position of the resultant of two like parallel forces P and Q is unaltered, when the position of P and Q are interchanged. Show that P and Q are of equal magnitude.

Solution:



Let C be the centre of two like parallel forces P at A and Q at B.

$\therefore$ P.AC = Q.CB ……………… (1)

When P and Q are interchanged, the centre C is not altered (given)

$\therefore$ Q.AC =P.CB ………………. (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{P}{Q} = \frac{Q}{P}$$

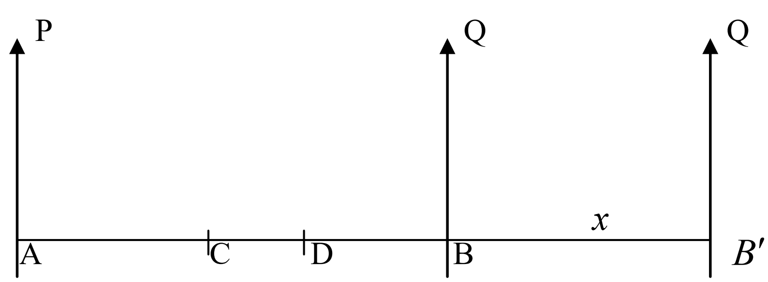
$$\therefore P^2 = Q^2$$

$$\boxed{\therefore P = \pm\ Q}$$

Problem 4

P and Q are like parallel forces. If Q is moved parallel to itself through a distance x, prove that the resultant of P and Q moves through a distance $\dfrac{Qx}{P+Q}$.

Solution:



Let C be the centre of P and Q at A and B.

$\therefore P.AC = Q.CB$ ………… (1)

Let D be the new centre of P at A and Q at B' such that $BB' = x$

$\therefore P.AD = Q.DB'$ …………………… (2)

ie) $P(AC + CD) = Q[DB + BB'] = Q[(CB - CD) + x]$

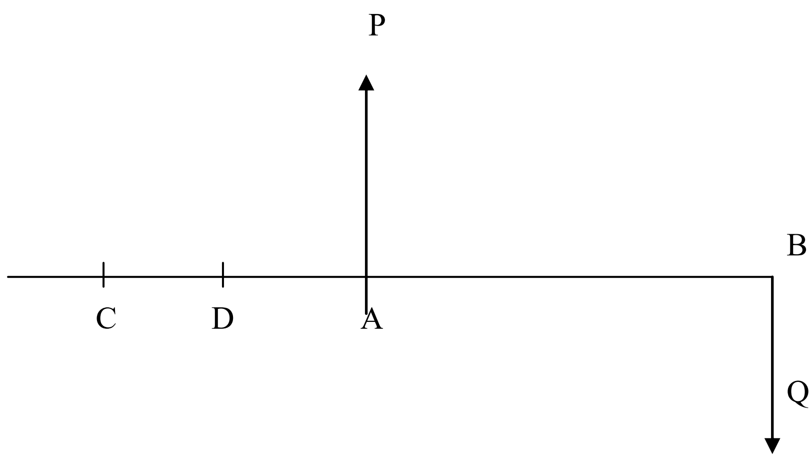
$$\therefore (P+Q)CD = Q.x \text{ using (1)}$$

$$\boxed{\therefore CD = \frac{Qx}{P+Q}}$$

Problem 5

Two unlike parallel forces P and Q (P>Q) acting on a rigid body at A and B respectively be interchanged in position, show that the point application of the resultant in AB will be displayed along AB through a distance $\frac{P+Q}{P-Q} AB$.

Solution:



Let C be the centre of two unlike parallel forces P at A and Q at B.

$$\therefore P.AC = Q.CB \text{ ………………… (1)}$$

Let D be the new centre when P and Q are interchanged in position.

$$\therefore Q.AD = P.DB \text{ ……………….. (2)}$$

i.e.) $Q(AC - CD) = P.(DA + AB)$

i.e.) $Q[(CB - AB) - CD] = P.[(AC - CD) + AB]$

$Q.CB - Q.AB - Q.CD = P.AC - P.CD + P.AB$

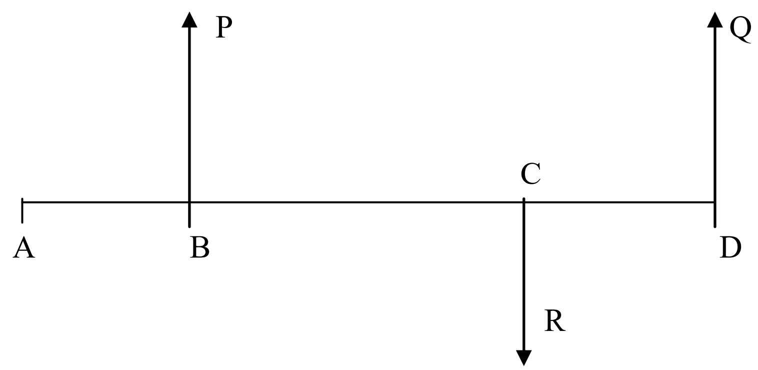
$\therefore (P - Q).CD = (P + Q).AB$ using (1)

$$\boxed{\therefore CD = \frac{P+Q}{P-Q}.AB}$$

Problem 6

A light rod is acted on by three parallel forces P, Q, and R, acting at three points distant 2, 8 and 6 ft. respectively from one end. If the rod is in equilibrium, show that P: Q: R = 1:2:3.

Solution



P, Q, R are parallel forces acting on the rod AD at B, D, C respectively.

Given, AB = 2 ft, AD = 8ft, AC = 6ft.

$\therefore$ BC = 4ft, CD = 2ft, BD = 6ft.

For equilibrium of the rod, each force should be proportional to the distance between the other two.

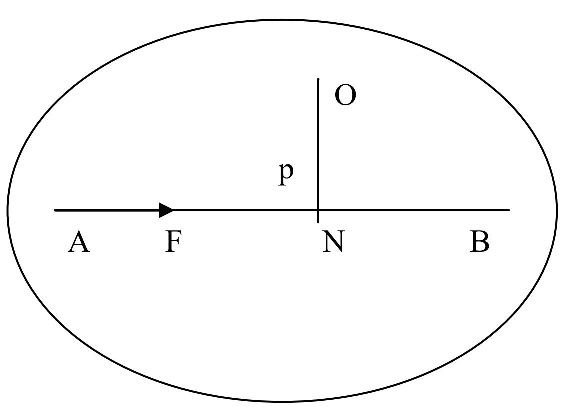
$$\therefore \frac{P}{2} = \frac{Q}{4} = \frac{R}{6} \Rightarrow P:Q:R = 2:4:6$$

$$\boxed{\therefore P:Q:R = 1:2:3}$$

2.2 Moment of a force (or) Turning effect of a force

Definition:

The moment of a force about a point is defined as the product of the force and the perpendicular distance of the point from the line of action of the force.



Moment of F about O = F x ON = F x p.

Note: Moment of F about O is zero if either F = O (or) ON = O.

i.e.) F = 0 (or) AB passes through O.

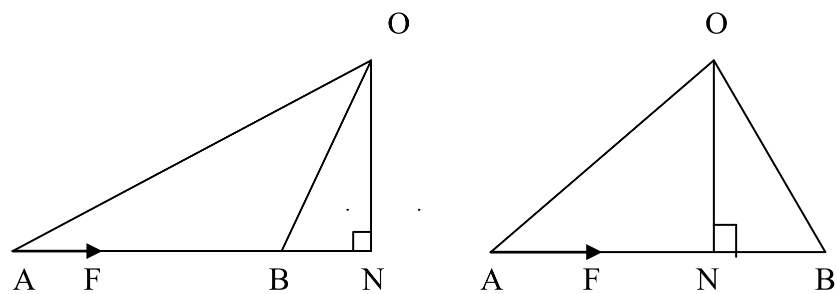
Hence, moment of a force about any point is zero if either

the force itself is zero (or) the force passes through that point.

Physical significance of the moment of a force

It measures the tendency to rotate the body about the fixed point.

Geometrical Representation of a moment



Let AB represent the force F both in magnitude and direction and O be any given point.

$\therefore$ the moment of the force F about O

= F x ON = AB x ON = 2. Δ AOB

= Twice the area of the triangle AOB

Sign of the moment

If the force tends to turn the body in the **anticlockwise** direction, moment is **positive.**

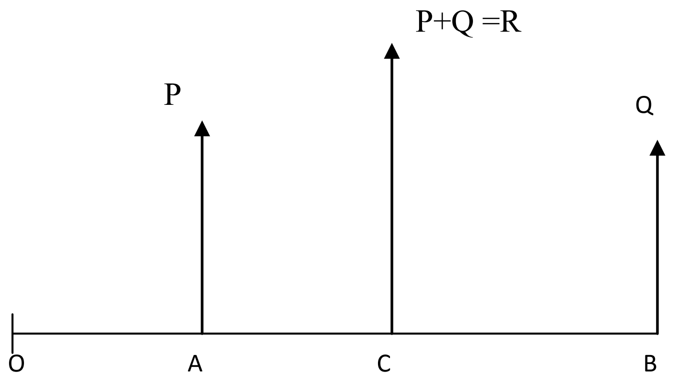
If the force tends to turn the body in the **clockwise** direction, moment is **negative.**

Varignon's Theorem of Moments

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

Proof:

Case 1 Let the forces be parallel and O lies **i) Outside AB**



Let P and Q be the two parallel forces acting at A and B. P + Q be their resultant R acting at C. such that

P.AC = Q.CB ……………... (1)

Algebraic sum of the moments of P and Q about O

= P.OA + Q.OB

= P x (OC – AC) + Q x (OC + CB)

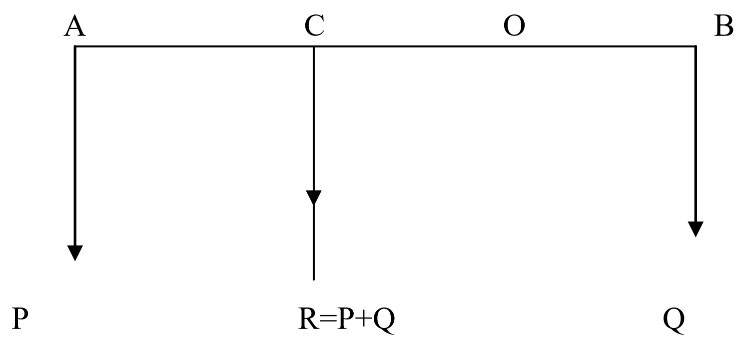
= (P +Q).OC – P.AC +Q.CB

= (P+Q).OC using (1)

= R.OC

= moment of R about O.

ii) P and Q are parallel and O lies within AB



Algebraic sum of the moments of P and Q about O

= P.OA – Q.OB

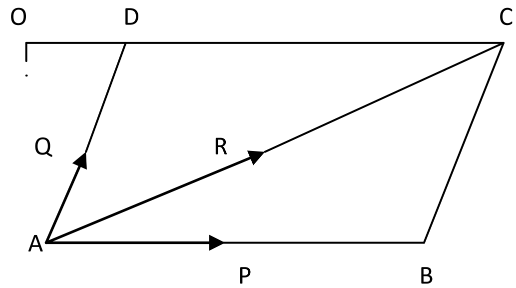
= P. (OC+CA) – Q. (CB – CO)

= (P+Q).OC + P.CA – Q.CB by (1)

= R.OC

= moment of R about O.

Case II iii) P and Q meet at a point and O any point in their plane. O lies outside the angle BAD



Through O, draw a line parallel to the direction of P, to meet the line of action of Q at D. Complete the parallelogram ABCD such that AB, AD represent the magnitude of P and Q and the diagonal AC represents the resultant R of P and Q.

Algebraic sum of the moments of P and Q about O

$= 2.\ \Delta\ AOB + 2.\Delta\ AOD$

$= 2\ \Delta\ ACB + 2.\ \Delta\ AOD$ [$\because \Delta\ AOB = \Delta\ ACB$]

$= 2\ \Delta\ ADC + 2\ \Delta\ AOD$

$= 2\ (\Delta\ ADC + \Delta\ AOD)$

$= 2.\ \Delta\ AOC$

= Moment of R about O.

iv) O lies inside the angle BAD

Algebraic sum of the moments of P and Q about O:

$= 2\ \Delta\ AOB - 2\ \Delta\ AOD$

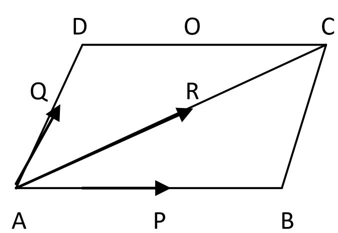
$= 2\ \Delta\ ACB - 2\ \Delta\ AOD$

$= 2\ \Delta\ ADC - 2\ \Delta\ AOD$

$= 2\ (\Delta\ ADC - \Delta\ AOD)$

$= 2.\ \Delta\ AOC$

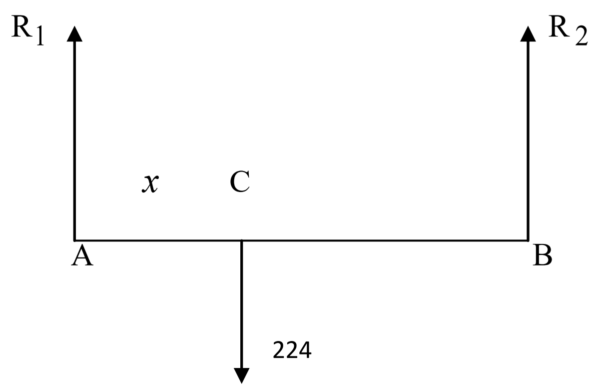
= moment of R about O.



Problem 7

Two men carry a load of 224 kg. wt, which hangs from a light pole of length 8 m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other. What is the pressure on each shoulder?

Solution



AB is the light pole of length 8m. C is the point from which the load of 224 kgs. is hung.

Let AC = x. Then BC = 8 – x. given (8 – x) – x =2

i.e) 8 – 2x = 2 0r 2x = 6.

$\therefore$ x = 3. i.e. AC = 3 and BC = 5.

Let the pressures at A and B be R_1 and R_2 kg. wt. respectively. Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2 and 224 kg. wt. about any point must be equal to zero.

Taking moments about B,

224 CB – R_1.AB = 0

i.e. $224 \times 5 - R_1 \times 8 = 0.$

$$\therefore R_1 = \frac{224 \times 5}{8} = 140.$$

Taking moments about A,

R_2.AB – 224.AC = 0.

i.e. $8R_2 - 224 \times 3 = 0.$

$$\therefore R_2 = \frac{224 \times 3}{8} = 84$$

Problem 8

A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

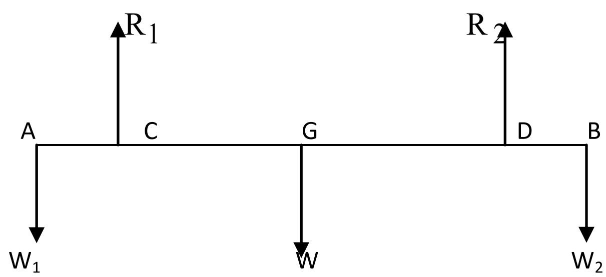
$$\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}.$$

Solution

Let AB be the plank placed upon two vertical props at C and D. CD = b. The weight W of the plank acts at G, the midpoint of AB,

AG = GB = a

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D is zero.



There is upward reaction R_1 at C.

Take moments about C, we have

$W_1 . AC = W.CG$

i.e. $W_1 (AG - CG) = W.CG$

$W_1 .AG = (W + W_1).CG$

i.e. $W_1 . a = (W+W_1)\ CG$

$$CG = \frac{W_1 a}{W + W_1} \ldots\ldots\ldots\ldots\ldots (1)$$

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Take moments about D, we get

$W.GD = W_2\,(GB - GD)$

$GD\,(W+W_2) = W_2.GB = W_2\,.a$

$$GD = \frac{W_2 a}{W + W_2} \ldots\ldots\ldots\ldots (2)$$

$$\boxed{CG + GD = CD = b}$$

$$\therefore \frac{W_1 a}{W + W_1} + \frac{W_2 a}{W + W_2} = b$$

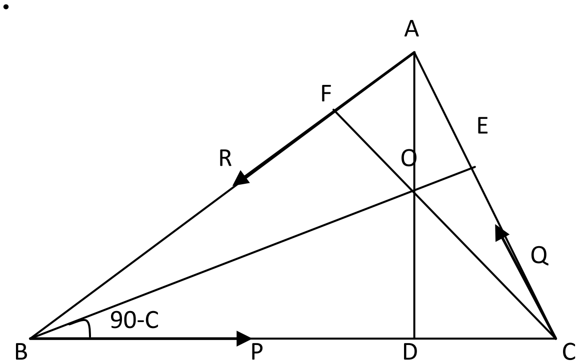
$$\frac{W_1}{W + W_1} + \frac{W_2}{W + W_2} = \frac{b}{a}$$

Problem 9

The resultant of three forces P, Q, R, acting along the sides BC, CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled.

If $\angle A = 120°$, and B = C, show that Q+R = P $\sqrt{3}$.

Solution:



Let AD, BE and CF be the altitudes of the triangle intersecting at O, the orthocentre.

As the resultant passes through O, moment of the resultant about O = O.

$\therefore$ Sum of the moments of P, Q, R about O = O

P.OD+Q.OE+R.OF = 0 ……….. (1)

In rt. $\angle d\Delta BOD, \angle OBD = \angle EBC = 90° - C.$

$$\therefore \tan(90° - C) = \frac{OD}{BD}$$

i.e) cot C = $\frac{OD}{BD}$

OD = BD cot C …………. (2)

From rt. $\angle d\Delta ABD, \cos B = \frac{BD}{AB}$

$$\therefore From(2), OD = c\cos B.\cot C \; = \; c\cos B.\frac{\cos C}{\sin C}$$

$$= \frac{c}{\sin C}.\cos B \cos C$$

$= 2R'\cos B\cos C(\because \frac{c}{\sin C} = 2R'$, R' is the circumradius of the Δ)

Similarly OE = $2R'\cos C\cos A$

and OF = $2R'\cos A\cos B$

Hence (1) becomes

$P.2R'\cos B\cos C + Q.2R'\cos C\cos A + R.2R'\cos A\cos B = 0$

Dividing by $2R'\cos A\cos B\cos C,$

$\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0$ …… (3)

Now, P, Q, R being magnitudes of the forces, are all positive.

(3) may hold good, if at least one of the terms must be negative.

Hence one of the cosines must be negative.

i.e) the triangle must be obtuse angled.

If A = 120° and the other angles equal, then B = C = 30°

Hence (3) becomes

$$\frac{P}{\cos 120°} + \frac{Q}{\cos 30°} + \frac{R}{\cos 30°} = 0$$

i.e. $\dfrac{P}{\left(-\dfrac{1}{2}\right)} + \dfrac{Q+R}{\left(\dfrac{\sqrt{3}}{2}\right)} = 0$

i.e. $P\sqrt{3} = Q + R$

2.3 **Couples**: Definition

Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.

Examples of a couple are the forces used in winding a clock or turning tap. Such forces acting upon a rigid body can have only a rotator effect on the body and they can not produce a motion of translation.

The moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them,

The perpendicular distance (p) between the two equal forces P of a couple is called the **arm of the couple.** A couple each of whose forces is P and whose arm is p is usually denoted by (P, p).

A couple is positive when its moment is positive i.e., if the forces of the couple tend to produce rotation in the anti-clockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

UNIT-III
COUPLES

3.0 INTRODUCTION

In the last unit we have seen that the general method of finding the resultant of two equal and unlike parallel forces fails i.e. the effects of two equal and unlike parallel forces cannot be replaced by a single force. A pair of such forces is called a couple.

3.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by Couple.
- Discuss the theorems on Equilibrium of two couples.
- Describe couples in parallel planes.

3.2 COUPLES

Definition. *Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.*

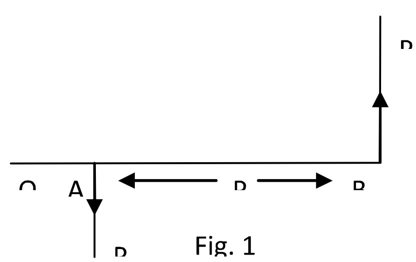


Fig. 1

Examples of a couple are the forces used in winding a clock or turning a tap. Such forces acting upon a rigid body can have only a rotatory effect on the body and they cannot produce a motion of translation.

Let P, p be the magnitudes of the forces forming a couple and O any point in their plane.

Draw OAB perpendicular to the forces to meet their lines of action in A and B.

The algebraic sum of the moments of the forces about O is

$$= P.OB - P.OA$$

$$= P.(OB - OA) = P.AB$$

And this value is independent of the position of O.

Thus the algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and is equal to the product of either of the forces and the perpendicular distance between them. This algebraic sum measures the total turning effect of the couple upon the body and is called *the moment of the couple.*

Thus, the moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.

The perpendicular distance $AB(= p)$ between the two equal forces P of a couple is called *the arm of the couple*. A couple each of whose forces is P and whose arm is p, as in fig. 1 is usually denoted by (P, p).

A couple is positive when its moment is positive i.e. if the forces of the couple tend to produce rotation in the anticlockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

3.3 EQUILIBRIUM OF TWO COUPLES

Theorem.1. *If two couples, whose moments are equal and opposite, act in the same plane upon a rigid body, they balance one another.*

Let (P, p) and (Q, q) be two given couples such that $Pp = Qq$ in magnitude but opposite in sign.

Case 1: Let the forces P and Q be parallel.

Draw a straight line perpendicular to the lines of action of the forces, meeting them at A, B, C, D as in fig. 2.

Since the moments of the couples are equal, we have

$$P.AB = Q.CD \qquad \text{....(1)}$$

The downward like parallel forces P at A and Q at D can be compounded into a single force $P + Q$ acting at L such that

$$P.AL = Q.DL \qquad \text{....(2)}$$

(1)-(2) gives

$$P.(AB - AL) = Q.(CD - DL)$$

i.e. $P.B$L=Q.CL ....(3)

Result (3) shows that the resultant of the upward like parallel forces P at B and Q at C will also pass through L. The magnitude of this resultant is also $(P+Q)$ but it is opposite in direction to the previous resultant. Thus the two resultants balance each other. Hence the four forces forming the couples are in equilibrium.

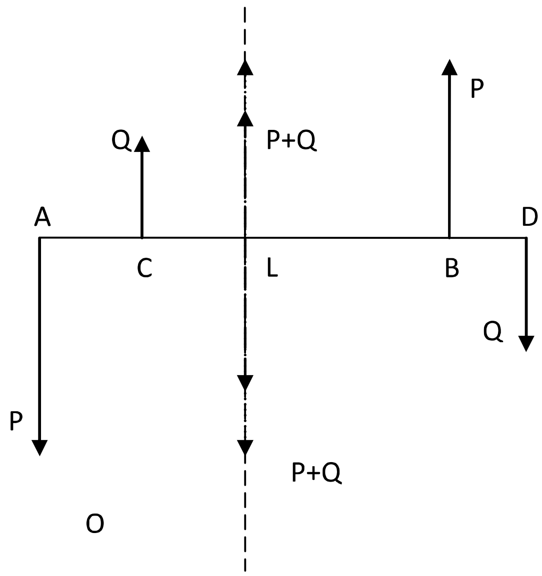


Fig. 2

Case 2: Let the forces P and Q intersect.

Let the two forces P of the couple (P, p) meet the two forces Q of the couples (Q, q) at the points A, B, C, D. Clearly ABCD is a parallelogram.

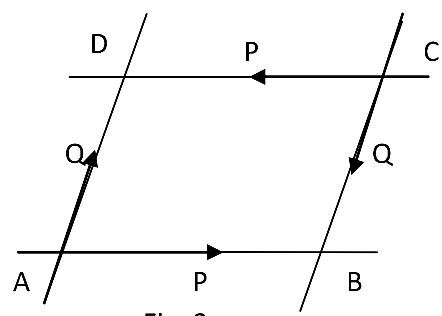


Fig. 3

Let AB represented P on some scale.

As the moments of the two couples are equal, we have

$P.p = Q.q$ …..(1)

Also $AB.p = AD.q$ (each being equal to the area of the || gm. ABCD)……(2)

$(1) \div (2)$ gives

$\frac{P}{AB} = \frac{Q}{AD}$ …..(3)

(3) shows that the side AD will represent Q on the same scale in which the side AB represents P.

The two forces P and Q meeting at A can be compounded by ||gm. Law so that

$(P+Q)$ at $A = \overline{AB} + AD = AC$

Similarly $(P+Q)$ at C $= CD + CB = CA$.

The two resultants AC and CA being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

3.4 EQUIVALENCE OF TWO COUPLES

Theorem 2. *Two couples in the same plane whose moments are equal and of the same sign are equivalent to one another.*

Let (P, p) and (Q, q) be two couples in one plane having the same equal moments in magnitude and direction. Let (R, r) be a third couple, in the same plane, whose moment is equal to the moment of either (P, p) or (Q, q) only in magnitude but opposite in direction. By the previous theorem, the couple (R, r) will balance the couple (P, p). It will also balance the couple (Q, q). Hence the effects of the couples (P, p) and (Q, q) must be the same. In other words, they are equivalent.

This is a fundamental theorem on coplanar couples. Form this, it follows that a couple in a plane can be replaced by any other couple in the same plane, provided that the moment of the latter replacing couple is equal in magnitude and direction to the moment of the first couple. The only important criterion is that the moment of the new couple must be equal to that of the first couple in magnitude and sense.

Thus a couple (P, p) may be replaced by a couple $\left(F, \frac{Pp}{F}\right)$ in the same plane with its constituent forces each equal to F and the arm length begin equal to $\frac{Pp}{F}$. The moment of the couple is $= F\frac{Pp}{F} = P_p$ moment of the first couple. Also one force F may be taken to be acting in any line and direction, the other at the distance $\frac{Pp}{F}$ begin on that side so as to make the sign of the moment same as that of (P, p).

Similarly, the couple (P, p) may be replaced by a couple $\left(\frac{Pp}{x}, x\right)$ with a given arm x anywhere in the plane.

3.5 COUPLES IN PARALLEL PLANES

The effect of a couple upon a rigid body is not altered if it is transferred to a parallel plane provided its moment remains unchanged in magnitude and direction.

Consider a couple of forces P at the ends of arm AB in given plane. Let AL and BM be the line of action of the forces.

In any parallel plane, take a straight line CD equal and parallel toAB.

Then ABCD will be a parallelogram. The diagonals AD and BC will bisect each other, say at O.

At O, introduce two equal and opposite forces of magnitude 2P along EF, parallel to the forces P at A and B. By this, the effect of the given couple is not altered.

Now the unlike parallel forces P along AL and 2P along OE can be compounded into a single force P acting at D, since $\frac{AD}{OD} = \frac{2}{1} = \frac{2P}{P}$. This resultant force P acts along DN in the second plane. Similarly, the unlike parallel forces P along BM and 2P along OF can be compounded into a single force P acting at C along CK. We are therefore left with a couple of forces P at the ends of the arm CD in a plane parallel to that of the original couple.

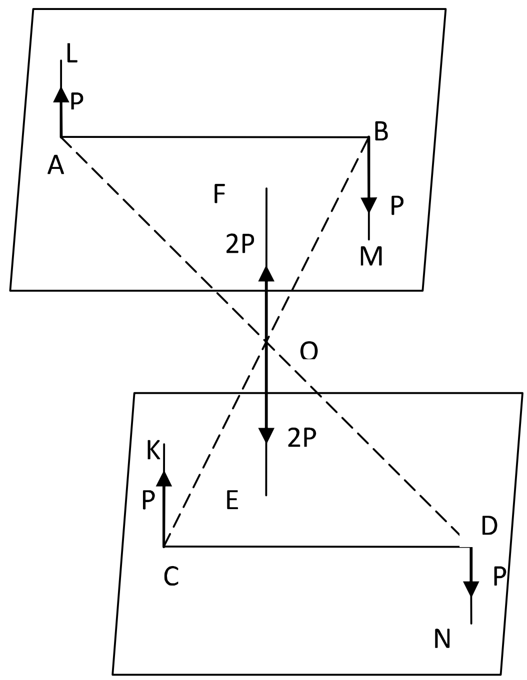


Fig. 4

Thus the given couple with the arm AB is equivalent to another couple of the same moment in a parallel plane, having its arm CD equal and parallel to AB. Now this couple with arm CD can be replaced in its own plane by another couple, provided the moment is unchanged in magnitude and direction as in 6.3. Hence we conclude that a couple in any plane can be replaced by another couple acting in a parallel plane, provided that the moments of the two couples are the same in magnitude and sign.

3.6 REPRESENTATION OF A COUPLE BY A VECTOR

From 6.3 and 6.4, it is clear that a couple is not localized in any particular plane, for it may be replaced by another couple of the same moment in the same plane or in any parallel plane. Thus the effect of a couple remains unaltered so long as its moment remains the same in magnitude and sense, whatever be the magnitude of its constituent forces, the length of its arm and its position in any one of a set of parallel planes in which it may be supposed to act.

A couple is therefore completely specified if we know (i) the direction of the set of parallel plane (ii) the magnitude of its moment (iii) the sense in which it acts. These three aspects of a couple can be conveniently represented by a straight line drawn (i) perpendicular to the set of parallel planes to indicate the direction (ii) of a measured length, to indicate the moment of the couple and (iii) in a definite direction, to indicate the sense of the moment.

3.7 RESULTANT OF A COUPLE AND A PLANE

Theorem 3. *The resultant of any number of couples in the same plane on a rigid body is a single couple whose moment is equal to the algebraic sum of the moment s of the several couples.*

Let $(P_1, p_1), (P_2, p_2), (P_3, p_3)$ etc. be a number of couples acting in the same plane upon a body. Let AB represent the arm p_1 of the first couple (P_1, p_1) whose component forces P_1 act along AC and BD.

The moment of the second couple $(P_2, p_2) = P_2 p_2$. This couple can be replaced by an equivalent couple, having its arm along AB and having its forces AC and BD.

If F is the forces of such a replacing couple,

We have $F.p_1 = P_2.p_2$.

$\therefore F = \frac{P_2 p_2}{p_1}$

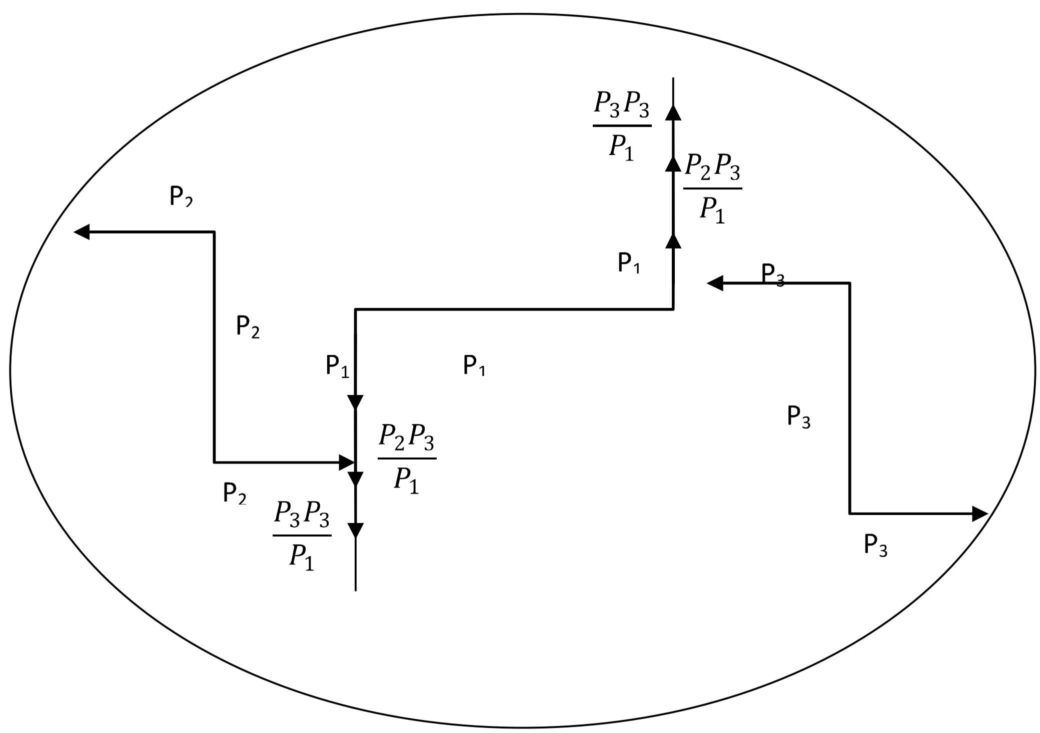


Fig. 5

Thus the couple $P_2 p_2$ is replaced by another couple whose arm coincides with AB and whose component forces along AC and BD are magnitude $\frac{P_2 p_2}{p_1}$.

Similarly the couple (P_3, p_3) is replaced by another couple $\left(\frac{P_3 p_3}{p_1}, \mathrm{p}_1\right)$ with the forces $\frac{P_3 p_3}{p_1}$ along AC and BD. This process is repeated for the other couples.

Finally, we get a single couple with the arm AB, each of whose component forces

$= P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \ldots.$

The moment of this resultant couple

$$= \left(P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \ldots.\right) \times p_1$$

$= P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots.$

= the algebraic sum of the moments of the several couples.

Note. (i) If all the component couples have not the same sign, we have merely to give each its proper sign and the same proof will apply.

(ii) If all the couples do not lie in the same plane but in different parallel planes, they can all be transferred into equivalent couples in one plane parallel to the given planes and then their resultant can be found.

Theorem 4. *A couple and a signal force acting on a body cannot be in equilibrium but they are equivalent to the single force acting at some other point parallel to its original direction.*

Let the given couple be (P, p) and the given force be F lying in the same plane. Let F act along AC.

Replace the couple (P, p) by another couple whose each force is equal to F. If x be the length of the arm of this new couple, its moment $= F.x = Pp.$

$$\therefore x = \frac{Pp}{F}$$

Place this couple such that one of its component forces F acts at A along the line of action of the given force F but in the opposite direction i.e. it acts along AD. The original force F along AC and the force F along AD balance. We are left with a force F acting at B parallel to AC, as the statical equivalent of the system.

Also $AB = x = \frac{Pp}{F}$

Hence the couple (P, p) and the force F are equivalent to an equal force F,

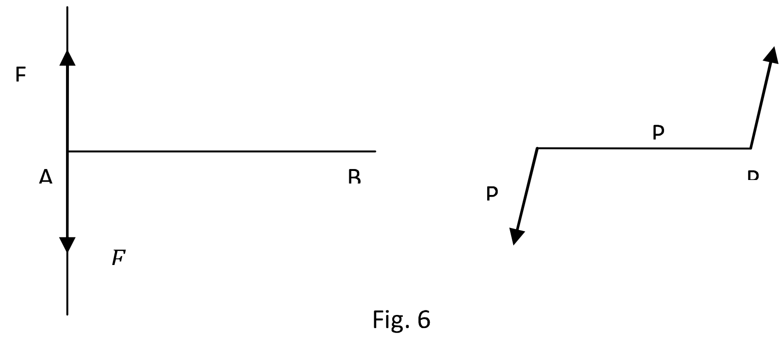


Fig. 6

parallel to its original direction, at a distance $\frac{Pp}{F}$ from its original line of action.

Theorem. 5. *A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body, together with a couple.*

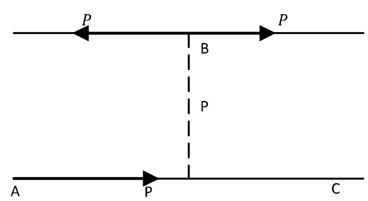


Fig. 7

Let P be a force acting at A along AC and B any arbitrary point. Let p be the distance of B from AC.

At B, apply two equal and opposite forces each equal and parallel to P along BL and BM. These two new forces being equal and opposite, will have no effect on the body. Of the three forces P along BM and P along AC from a couple and the remaining is the force P acting at B, parallel to the original force. Thus the statical equivalent of the original force P at A is an equal and parallel force P at B, together with a couple whose moment is Pp, where p is the perpendicular distance of B from AC.

Note. The moment of the couple is equal to the moment of the original force at A about B.

Theorem 6. *If there forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is twice the area of the triangle.*

Let P, Q, R be three forces acting on a rigid body and represented in magnitude, direction and line of action by the sides BC, CA, AB of the triangle ABC taken in order. Through A draw LAM parallel to BC. At A, along AL and

AM introduce two equal and opposite forces, each equal to P. These two new forces, being equal and opposite, have no effect on the body.

Now the three forces P along AM, Q along CA, and R along AB act at the point A and they are completely represented by the sides of the ΔABC taken in order. Hence, by the triangle of forces, they are in equilibrium. We are left with a force P along AL and a force P along BC. These being two equal and opposite force form a couple whose moment

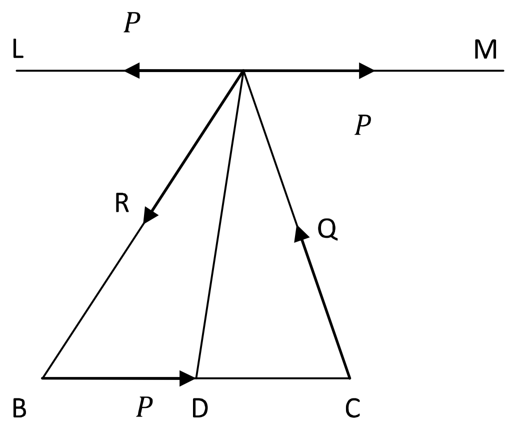


Fig. 8

$$= P.AD = BC.AD = 2\Delta ABC.$$

Theorem 7. If any number of forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a polygon taken in order, they are equivalent to a couple whose moment is twice the area of the polygon.

Let the forces be represented completely by the sides AB, BC, CD, DE, EF and FA of the closed polygon ABCDEF. Join AC, AD and AE.

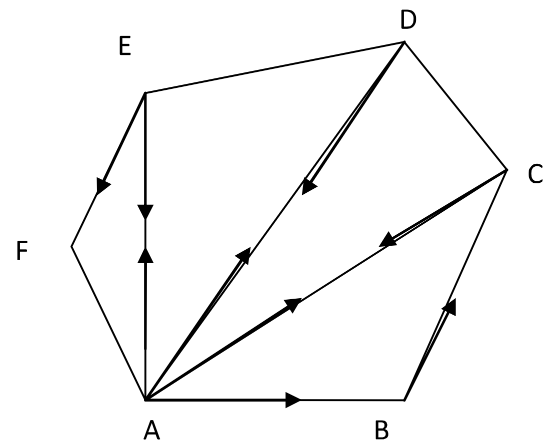


Fig. 9

Introduce along AC, AD and AE,
pairs of equal and opposite forces represented completely by these lines. These new forces do not affect the resultant of the system.

Applying the theorem 6, we have

$\overline{AB} + \overline{BC} + \overline{CA} =$ a couple whose moment is equal to $2\Delta ABC$.

$\overline{AC} + \overline{CD} + \overline{DA} =$ a couple whose moment is equal to $2\Delta ACD$.

$\overline{AD} + \overline{DE} + \overline{EA} =$ a couple whose moment is equal to $2\Delta ADE$.

$\overline{AE} + \overline{EF} + \overline{FA} =$ a couple whose moment is equal to $2\Delta AEF$.

Adding vectorically,

$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FA} =$ resultant of the four couples

= a single couple whose moment is equal to $2(\Delta ABC + \Delta ACD + \Delta ADE + \Delta AEF)$i.e. The resultant is a couple whose moment is equal to twice the area of the polygon ABCDEF.

Example 8. *ABC is an equilateral triangle of side a: D. E. F divide the sides BC, CA, AB respectively in the ratio 2:1. Three forces each equal to P act at D, E, F perpendicularly to the sides and outward from the triangle. Prove that they are equivalent to a couple of moment* $\frac{1}{2}Pa$.

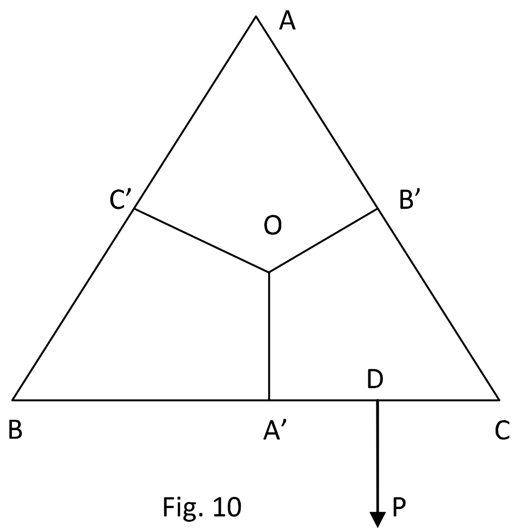


Fig. 10

Let O be the circumcentre (also the orthocentre) of the equilateral Δ and $A^{'}, B^{'}, C^{'}$ the middle points of the sides. $OA^{'}$ is $\perp$ to BC.

Applying theorem 5 , the force P acting at D $\perp$ to BC is equivalent to a parallel force P acting at O along $OA^{'}$ together with a couple whose moment

$$= P.AD = P.(A^{'}C - DC) = P.\left(\frac{a}{2} - \frac{a}{3}\right) = \frac{Pa}{6}$$

Similarly, the force P acting at E $\perp$ to CA is replace by a parallel force P acting at O along $OB^{'}$ together with a couple whose moment $= \frac{Pa}{6}$.

The force P acting at F $\perp$ to AB is replaced by a parallel force P acting at O along $OC^{'}$ together with a couple whose moment $= \frac{Pa}{6}$.

The three equal forces P acting O $\perp$ to the sides of the triangle are in equilibrium by the perpendicular by the perpendicular triangle of forces.

The three couples having the same moment $\frac{Pa}{6}$ each in the same direction are equivalent to a single couple whose moment $= 3 \times \frac{Pa}{6} = \frac{Pa}{2}$.

Example 9. *Five equal forces ac along the sides AB, BC, CD, DE, EF of a regular hexagon. Find the sum of the moments of these forces about a point Q of AF at a distance x from A. Interpret the result and explain why it is so.*

Let a be the length of each side of the regular hexagon. Each interior angle of the regular hexagon = $120°$.

We know that $AB||DE, BD||EF$ and $DC||AF, FB \perp BC, AE$ and DB are $\perp$ to AB.

Let equal force P act along the sides AB, BC, CD, DF and EF. Q is point on AF such that $AQ = x$.

Form Q, draw QL ⊥ to EA and QM ⊥ to BF.

Let AN be ⊥ to BF.

$$FB = FN + NB = a\cos 30° + a\cos 30° + 2a\cos 30°$$

$$AC = AE = BF = 2a\cos 30°$$

Moment of P along AB about Q

$= P.AL = P.x\cos 30°$ (from rt. $\angle d$ ΔAQL)

$= P.x\frac{\sqrt{3}}{2}$ …..(1)

Moment of P along BC about Q

$= P.MB = P.(FB - FM)$

$= P[2a\cos 30° - (a - x)\cos 30°]$

$= P[2a - a + x]\cos 30°$

$= P(a + x)\frac{\sqrt{3}}{2}$ ….(2)

Moment of P along CD about Q

=P. AC (∵ AF∥CD and AC is ⊥ to CD)

$= P.2a\cos 30° = P.2a\frac{\sqrt{3}}{2} = Pa\sqrt{3}$ …..(3)

Moment of P along DE about Q

$= P.EL = P(AE - AL)$

$= P(2a\cos 30° - x\cos 30°)$

$= P(2a - x)\frac{\sqrt{3}}{2}$ …..(4)

Moment of P along EF about Q= $P.MF$

$= P(a - x)\cos 30°$

$= P(a - x)\frac{\sqrt{3}}{2}$ …..(5)

Adding up, the sum of the moments of the five forces about Q

$$= Px\frac{\sqrt{3}}{2} + P(a+x)\frac{\sqrt{3}}{2} + Pa\sqrt{3} + P(2a-x)\frac{\sqrt{3}}{2} + P(a-x)\frac{\sqrt{3}}{2}$$

$$= P\frac{\sqrt{3}}{2}(x+a+x+2a-x+a-x)$$

$= P\frac{\sqrt{3}}{2}$ 6a $= 3Pa\sqrt{3} =$ a constant, independent of x.

The sum of the moments of the five forces about any point on the sixth side AF is constant.

Introduce two equal and opposite forces, each equal to P along the sixth side. These new forces do not affect the resultant of the system. We have now seven forces. The moment of the new force P introduced along AF about Q is =0.

The other six forces act along the sides of the hexagon and are represented in magnitude, direction and line of action by the sides of the hexagon.

Hence by theorem 6.6, they are equivalent to a couple whose moment is $= 2 \times$ area of the hexagon$= 2 \times 6 \times a^2 \frac{\sqrt{3}}{4}$

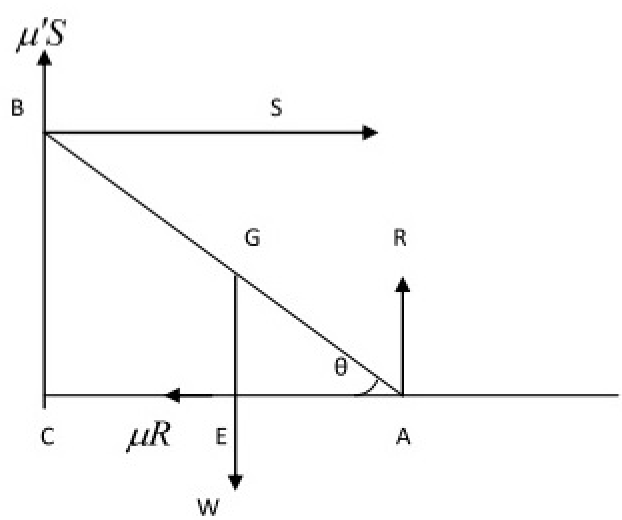
$= 3a^2\sqrt{3} = 3a\sqrt{3}P$ (as P is represented in magnitude by a).

3.8 Problems Involving Frictional Forces

Problem 1

A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$. Find also the reactions at the wall and ground.

Solution:



AB is the uniform ladder, whose weight W is acting at G such that AG = GB. Forces acting are,

1. Weight W
2. Normal reaction R at A
3. Normal reaction S at B
4. μR
5. $\mu' S$

When the ladder is on the point of slipping at both ends, frictional forces $\mu' S, \mu R$ act along CB, AC respectively.

Since the ladder is in equilibrium resultant is zero.

$\therefore$ Resolving horizontally and vertically,

$S = \mu R$ ………. (1)

$R + \mu' S = W$ ………… (2)

$\therefore R + \mu'(\mu R) = W$

$R(1+\mu\mu') = W \Rightarrow$ $\boxed{R = \dfrac{W}{1+\mu\mu'}}$ $\boxed{\therefore S = \dfrac{\mu W}{1+\mu\mu'}}$

By Varigon's theorem on moments, taking moments about A

$$S.BC + \mu' S.AC = W.AE$$

$$S.AB\sin\theta + \mu' S.AB\cos\theta = W.AG.\cos\theta$$

$$S.\sin\theta + \mu' S.\cos\theta = W.\frac{1}{2}.\cos\theta \quad \left[\because AG = \frac{AB}{2}\right]$$

$$\therefore S.\sin\theta = \left[\frac{W}{2} - \mu' S\right].\cos\theta$$

$$\therefore \tan\theta = \frac{W}{2S} - \mu' = \frac{W}{2\left[\dfrac{\mu W}{1+\mu\mu^1}\right]} - \mu^1 = \frac{1+\mu\mu'}{2\mu} - \mu'$$

$$= \frac{1+\mu\mu' - 2\mu\mu'}{2\mu} \qquad \boxed{\tan\theta = \frac{1-\mu\mu'}{2\mu}}$$

Problem 2

In the previous problem, when $\mu = \mu'$ show that $\theta = 90° - 2\lambda,$ where λ is the angle of friction.

Solution:

In the previous problem, we have proved $\tan\theta = \dfrac{1-\mu\mu'}{2\mu}$

Put $\mu = \mu'$, we get

$$\tan\theta = \frac{1-\mu^2}{2\mu} = \frac{1-\tan^2\lambda}{2\tan\lambda};[\because \mu = \tan\lambda]$$

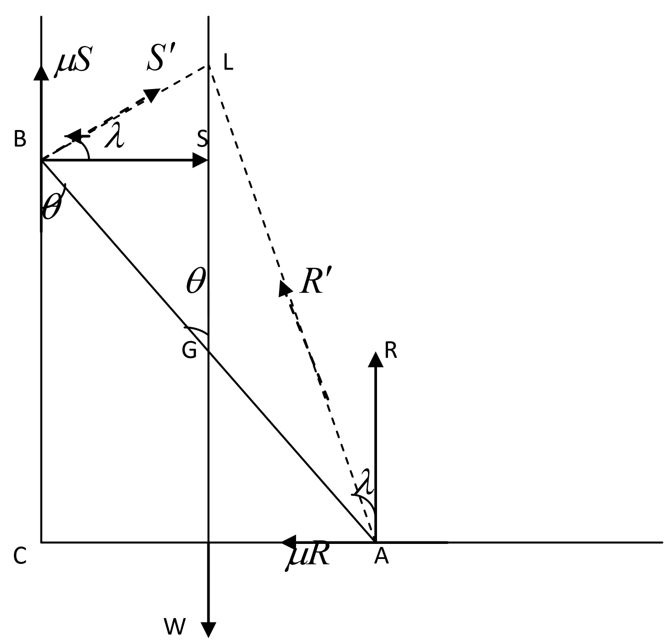
$$= \frac{1}{\tan 2\lambda} = \cot 2\lambda = \tan(90^\circ - 2\lambda)$$

i.e.) $\tan\theta = \tan(90^\circ - 2\lambda)$ $\therefore$ $\boxed{\theta = 90^\circ - 2\lambda}$

Problem 3

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against an equally rough vertical wall. If θ be the inclination of the ladder to the vertical, prove that $\tan\theta = \dfrac{2\mu}{1-\mu^2}$ where μ is the coefficient of friction.

Solution:



When the ladder AB is in limiting equilibrium, five forces are acting as marked in the figure.

1) Weight of the ladder W
2) Normal reaction R at A
3) Normal reaction S at B
4) Frictional force μR
5) frictional force μS

Let R', S' be the resultant reactions of R, μR and S, μS respectively.

$\therefore$ We have 3 forces R', S', W. For equilibrium, they must be concurrent at L.

In $\Delta LAB, L\hat{G}A = 180° - \theta; A\hat{L}G = \lambda$

$$B\hat{L}G = 90 - \lambda, AG : GB = 1:1$$

$\therefore$ By trigonometrical theorem in Δ LBA,

(1+1) $\cot(180° - \theta) = 1.\cot(90° - \lambda) - 1.\cot\lambda$

$$-2.\cot\theta = \tan\lambda - \cot\lambda = \frac{\tan^2\lambda - 1}{\tan\lambda}$$

$$\therefore \cot\theta = \frac{1 - \tan^2\lambda}{2\tan\lambda}$$

i.e.) $\dfrac{1}{\tan\theta} = \dfrac{1-\mu^2}{2\mu}$ $\quad \boxed{\therefore \tan\theta = \dfrac{2\mu}{1-\mu^2}}$

Problem 4

A uniform ladder rests with its lower end on a rough horizontal ground its upper end against a rough vertical wall, the ground and the wall being equally rough and the angle of friction being λ. Show that the greatest inclination of the ladder to the vertical is 2λ.

Solution

In the previous problem, we have proved, $\tan\theta = \dfrac{2\mu}{1-\mu^2}$ But $\mu = \tan\lambda$

$$\therefore \tan\theta = \frac{2\tan\lambda}{1-\tan^2\lambda} = \tan 2\lambda \Rightarrow \boxed{\therefore \theta = 2\lambda}$$

Problem 5

A ladder which stands on a horizontal ground, leaning against a vertical wall, is so loaded that its C. G. is at a distance a and b from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium, its inclination θ to the horizontal is given by $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$ where μ, μ' are the coefficients of friction between the ladder and the ground and the wall respectively.

Solution:

As in problem 5, five forces are acting on the ladder

Here, AG : GB = a: b

$\therefore$ By Trigonometrical theorem in ΔLBA,

$$(b+a).\cot(90+\theta) = b.\cot(90-\lambda') - a.\cot\lambda$$

i.e.) $(a+b)(-\tan\theta) = b.\tan\lambda^1 - a.\cot\lambda$

$$\therefore \tan\theta = \frac{\left(\frac{a}{\mu}\right) - b.\mu'}{a+b} = \frac{a - b.\mu\mu'}{(a+b)\mu}$$

Problem 6

A ladder AB rests with A on a rough horizontal ground and B against an equally rough vertical wall. The centre of gravity of the ladder divides AB in the ratio a: b. If the ladder is on the point of slipping, show that the inclination θ of the ladder to the ground is given by $\tan\theta = \dfrac{a - b\mu^2}{\mu(a+b)}$ where μ is the coefficient of friction.

Solution:

In the previous problem,

Put $\mu = \mu'$ in $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$

$$\boxed{\therefore \tan\theta = \frac{a - b\mu^2}{\mu(a+b)}}$$

Problem 7

A ladder AB rests with A resting on the ground and B against a vertical wall, the coefficients of friction of the ground and the wall being μ and μ' respectively. The centre of gravity G of the ladder divides AB in the ratio 1: n. If the ladder is on the point of slipping at both ends, show that its inclination to the ground is given by $\tan\theta = \dfrac{1-n\mu\mu'}{(n+1)\mu}$.

Solution:

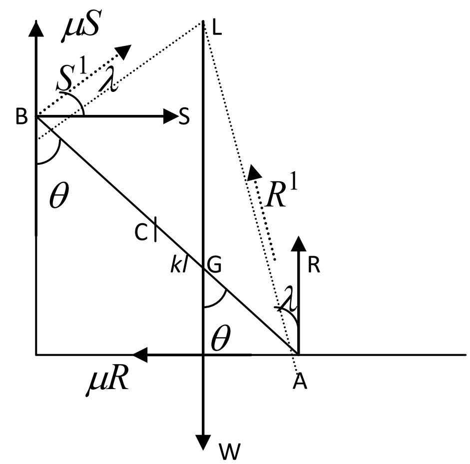
Put a : b = 1 : n in problem7.

$$\therefore \tan\theta = \frac{1-n\mu\mu'}{(1+n)\mu}$$

Problem 8

A ladder of length $2l$ is in contact with a vertical wall and a horizontal floor, the angle of friction being λ at each contact. If the weight of the ladder acts at a point distant kl below the middle point, prove that its limiting inclination θ to the vertical is given by $\cot\theta = \cot 2\lambda - k\,\mathrm{cosec}\,2\lambda.$

Solution:



Forces are acting as marked in the figure. For equilibrium, the three forces R', S', W must be concurrent at L, where W – be the weight of the ladder.

In $\Delta LAB, BC = CA = l; CG = kl.$

$$\therefore BG = BC + CG = l + kl = (1+k)l$$

$B\hat{L}G = 90° - \lambda, L\hat{G}A = 180° - \theta$

$A\hat{L}G = \lambda; GA = CA - CG = l - kl = (1-k)l.$

$BG : GA = (1+k) : (1-k)$

$\therefore$ By Trigonometrical theorem in ΔLBA,

$[(1+k)+(1-k)].\cot(180° - \theta) = (1+k).\cot(90° - \lambda) - (1-k).\cot\lambda.$

$2(-\cot\theta) = (1+k).\tan\lambda - (1-k).\cot\lambda$

$\therefore 2\cot\theta = (1-k)\cot\lambda - (1+k)\tan\lambda$

$$= \frac{(1-k).\cot^2\lambda - (1+k)}{\cot\lambda}$$

$$= \frac{(\cot^2\lambda - 1) - k(\cot^2\lambda + 1)}{\cot\lambda}$$

$$\cot\theta = \frac{(\cot^2\lambda - 1) - k.\operatorname{cosec}^2\lambda}{2.\cot\lambda}$$

$$= \frac{1-\tan^2\lambda}{2\cot\lambda.\tan^2\lambda} - k\left[\frac{1+\cot^2\lambda}{2.\cot\lambda}\right]$$

$$= \frac{1}{\left(\frac{2\tan\lambda}{1-\tan^2\lambda}\right)} - k\left[\frac{1+\tan^2\lambda}{2.\tan^2\lambda.\cot\lambda}\right]$$

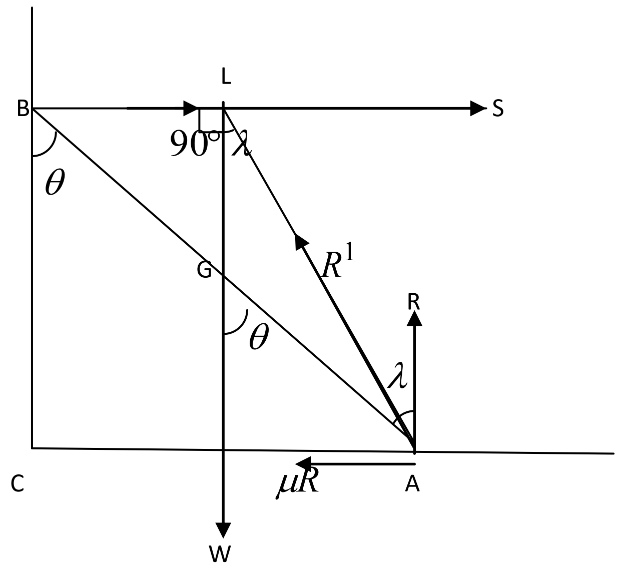
$$= \frac{1}{\tan 2\lambda} - k.\frac{1}{\sin 2\lambda}$$

ie) $\cot\theta = \cot 2\lambda - k.\operatorname{cosec} 2\lambda$

Problem 9

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and with the upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, prove that, $\tan\theta = 2\mu$, where μ is the coefficient of friction.

Solution:



Since the wall is smooth, there is no frictional force. Forces acting on the ladder are i) its weight W, ii) Frictional force μR iii) R at A iv) S at B. For equilibrium, the three forces W, R', S must be concurrent at L. where R^1 is the resultant of R and μR. In triangle LAB,

$$L\hat{G}A = 180° - \theta, A\hat{L}G = \lambda, B\hat{L}G = 90°; BG : GA = 1:1. A\hat{B}C = \theta$$

By Trigonometrical theorem in ΔLAB,

$$(1+1)\cot(180° - \theta) = 1.\cot 90° - 1.\cot\lambda$$

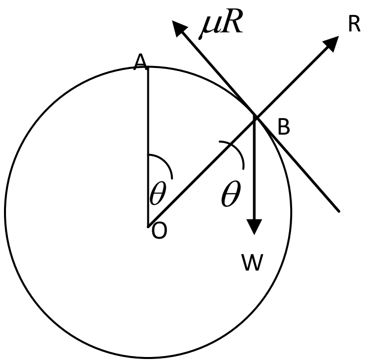
$$-2.\cot\theta = 0 - \cot\lambda$$

$$\therefore \frac{2}{\tan\theta} = \frac{1}{\tan\lambda} \quad \therefore \tan\theta = 2\tan\lambda \quad \text{i.e)} \quad \boxed{\tan\theta = 2\mu}$$

Problem 10

A particle is placed on the outside of a rough sphere whose coefficient of friction is μ. Show that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1}\mu$ with the vertical.

Solution:



Let O be the centre, A the highest point of the sphere and B the position of the particle which is just on the point of motion. Let $\angle AOB = \theta$

The forces acting at B are:

1) the normal reaction R
2) limiting friction μR
3) Its weight W,

Since the particle at B is in limiting equilibrium,

Resolving along the normal OB,

$R = W\cos\theta$ …………………. (1)

Resolving along the tangent at B,

$\mu R = W\sin\theta$ ……………….. (2)

$\frac{(2)}{(1)} \Rightarrow \mu = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\mu}$

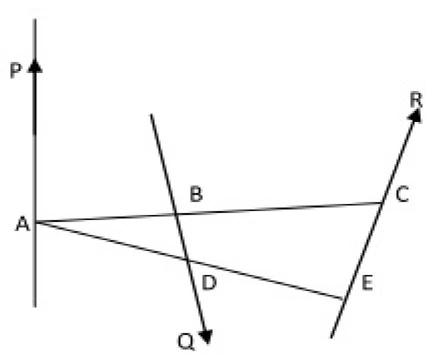
3.9 Equilibrium of three forces acting on a Rigid Body.

In the previous sections we have studied theorems and problems involving parallel forces and forces acting at a point. Here we study three important theorems and solved problems on forces acting on a rigid body and their conditions of equilibrium.

Theorem

If three forces acting on a rigid body are in equilibrium, they must be coplanar.

Proof:



Let the three forces be P,Q,R

Given : They are acting on a rigid body and in equilibrium.

Take 'A' on the force P, and B on the force Q such that AB is not parallel to R.

$\therefore$ Sum of the moments of P, Q, R about AB = 0 [$\therefore$ P,Q, R are in equilibrium]

Now, moment of P and Q about AB = 0 [$\because$ P and Q intersect AB].

$\therefore$ Moment of R about AB = 0, Hence R must intersect AB at a point C

Similarly if D is another point on Q such that AD is not parallel to R, we prove, R must intersect AD at a point E.

Since BC and DE intersect at A, BD, CE, A lie on the same plane. i.e) 'A' lies on the plane formed by Q and R. Since A is an arbitrary point on the force P, every point on the force P lie on the same plane.

ie) P, Q, R lie on the same plane.

Three Coplanar Forces – theorem

If three coplanar forces acting on a rigid body keep it in equilibrium, they must be either concurrent or all parallel.

Proof:

Let P, Q, R be the three forces acting on a rigid body keep it in equilibrium.

$\therefore$ One force must be equal and opposite to the resultant of the other two.

$\therefore$ they must be parallel or intersect.

Case 1: If P and Q are parallel (like or unlike)

Then the resultant of P and Q is also parallel. Hence R must be parallel to P and Q.

Case 2: If P and Q are not parallel: (intersect)

They meet at O. Therefore, by parallelogram law, the third force R must pass through O.

i.e) the three forces are concurrent.

Note: A couple and a single force can not be in equilibrium

Conditions of equilibrium

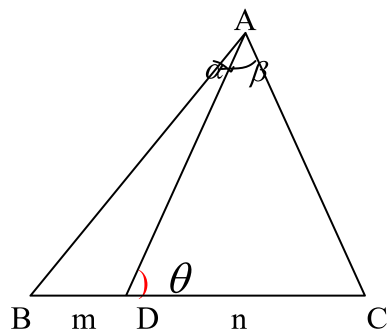
1. If three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two.
2. If three forces in equilibrium are parallel, then each force is proportional to the distance between the other two

Two Trigonometrical theorems

If D is any point on BC of a triangle ABC such that $\frac{BD}{DC}=\frac{m}{n}$ and $\angle ADC=\theta$, $\angle BAD=\alpha, \angle DAC=\beta$ then

1) $(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta$ 2) $(m+n)\cot\theta = n.\cot B - m.\cot C$.

Proof:



1. Given, $\frac{m}{n}=\frac{BD}{DC}=\frac{BD}{DA}.\frac{DA}{DC}$

Using, sine formula in Δ ABD, Δ ADC,

$$\frac{m}{n}=\frac{\sin\angle BAD}{\sin\angle ABD}\times\frac{\sin\angle ACD}{\sin\angle DAC}$$

$$\frac{m}{n}=\frac{\sin\alpha}{\sin(\theta-\alpha)}\times\frac{\sin(\theta+\beta)}{\sin\beta}$$

$$=\frac{\sin\alpha}{\sin\beta}\times\frac{(\sin\theta.\cos\beta+\cos\theta.\sin\beta)}{(\sin\theta\cos\alpha-\cos\theta.\sin\alpha)}$$

Divide by $\sin\alpha.\sin\theta.\sin\beta$

$$\frac{m}{n}=\frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$$

$$\therefore m(\cot\alpha-\cot\theta)=n(\cot\beta+\cot\theta)$$

$$\boxed{(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta}$$

2. $\dfrac{m}{n}=\dfrac{BD}{DA}.\dfrac{DA}{DC}$

$$=\frac{\sin\angle BAD}{\sin\angle ABD}\times\frac{\sin\angle ACD}{\sin\angle DAC}$$

$$=\frac{\sin(\theta-B).\sin C}{\sin B.\sin[180°-(\theta+C)]}=\frac{\sin C.\sin(\theta-B)}{\sin B.\sin(\theta+C)}$$

$$=\frac{\sin C\times(\sin\theta.\cos B-\cos\theta\sin B)}{\sin B(\sin C\cos\theta+\cos C\sin\theta)}$$

Divide by sin B sin C sinθ

$$\frac{m}{n}=\frac{\cot B-\cot\theta}{\cot\theta+\cot C}$$

$$\therefore m(\cot\theta+\cot C)=n(\cot B-\cot\theta)$$

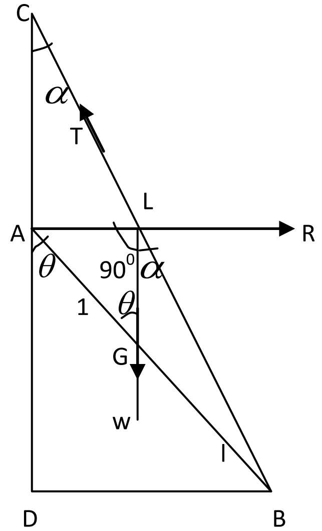
$$\boxed{\therefore (m+n)\cot\theta=n\cot B-m\cot C}$$

Problem 1

A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length *l*, tied to one end of the rod, the other end of the string being attached to a point in the wall: show that the rod can rest inclined to the wall at an angle θ given by $\cos^2\theta=\dfrac{l^2-a^2}{3a^2}$.

What are the limits of the ratio of a: *l* in order that equilibrium may be possible?

Solution:



AB is the rod of length a, with G its centre of gravity and BC is the string of length l.
The forces acting on the rod are:

(i). Its weight W acting vertically downwards through G.

(ii). The reaction R at A which is normal to the wall and therefore horizontal.

iii) The tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L.

Let the string make an angle α with the vertical.

$\therefore \angle ACB = \alpha = \angle GLB.$

$\angle LGB = 180° - \theta \ and \angle ALG = 90°$, AG:GB = 1 :1,

Using the trigonometrical theorem in Δ ALB

$(1+1)\cot(180° - \theta) = 1.\cot 90° - 1.\cot\alpha$

i.e) $-2\cot\theta = -\cot\alpha$

$2\cot\theta = \cot\alpha$ ……………… (1)

Draw BD $\perp$ to CA.

From rt. $\angle d\Delta CDB, BD = BC.\sin\alpha = l.\sin\alpha$

rt. $\angle d\Delta ABD, BD = AB\sin\theta = a\sin\theta$

$\therefore l\sin\alpha = a\sin\theta$ ……….. (2)

Eliminate α between (1) and (2).

We know that $\cos ec^2\alpha = 1 + \cot^2\alpha$ …………………… (3)

(2) $\Rightarrow \sin\alpha = \dfrac{a\sin\theta}{l}$ $\therefore \cos ec\alpha = \dfrac{l}{a\sin\theta}$ …………………… (4)

Substitute (4) and (1) in (3)

$$\frac{l^2}{a^2\sin^2\theta} = 1 + 4\cot^2\theta$$

i.e. $\dfrac{l^2}{a^2} = \sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$

$$\therefore 3\cos^2\theta = \frac{l^2}{a^2} - 1 = \frac{l^2 - a^2}{a^2}$$

$$\therefore \cos^2\theta = \frac{l^2 - a^2}{3a^2} \text{ ………………… (5)}$$

Equilibrium position is possible, if $\cos^2\theta$ positive and less than 1

$\therefore l^2 - a^2 > 0$ i.e. $l^2 > a^2 or a^2 < l^2$ ……………….. (6)

Also $\frac{l^2 - a^2}{3a^2} < 1$ i.e. $l^2 - a^2 < 3a^2 or\ l^2 < 4a^2$

i.e. $a^2 > \frac{l^2}{4}$ …………………… (7)

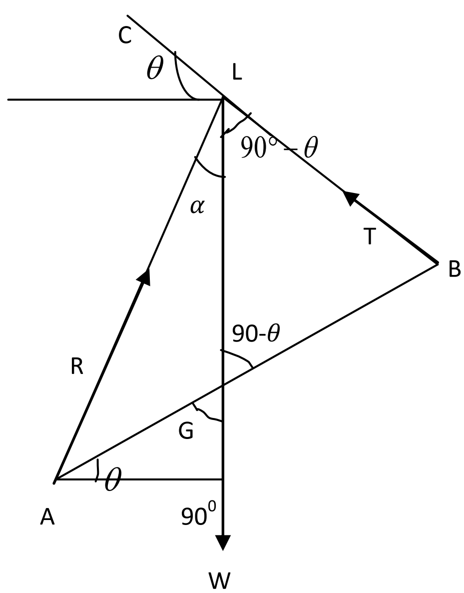
$\frac{l^2}{4} < a^2 < l^2$

[By (6) & (7)] $\frac{1}{4} < \frac{a^2}{l^2} < 1 = \frac{1}{2} < \frac{a}{l} < 1.$

Problem 2

A beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that the reaction at the hinge is $\frac{W}{4}\sqrt{8 + \operatorname{cosec}^2\theta}$

Solution:



Let AB be the beam of weight W and G its centre of gravity.

BC is the string

The force acting on the beam are:

i) Its wt. W acting vertically down wards at G

ii) the tension T along BC

iii) the reaction R at the hinge A.

For equilibrium (i) , (ii) and (iii) must meet at L.

BC and AB make the same angle θ with the horizon.

$\therefore$ They make $90^\circ - \theta$ with the vertical LG,

i.e. $\angle BLG = 90^\circ - \theta = \angle LGB$

Let $\angle ALG = \alpha$

Using trigonometrical theorem in Δ ALB, AG:GB = 1:1

$(1+1)\cot(90^\circ - \theta) = 1.\cot\alpha - 1.\cot(90^\circ - \theta)$

i.e. 2 $\tan\theta = \cot\alpha - \tan\theta$

3 $\tan\theta = \cot\alpha$ ……………… (1)

Applying Lami's theorem at L,

$$\frac{R}{\sin(90^\circ - \theta)} = \frac{W}{\sin(90^\circ - \theta + \alpha)}$$

i.e. $\dfrac{R}{\cos\theta} = \dfrac{W}{\sin\left(90^\circ - \overline{\theta - \alpha}\right)} = \dfrac{W}{\cos(\theta - \alpha)}$

$$\therefore R = \frac{W\cos\theta}{\cos(\theta - \alpha)} = \frac{W\cos\theta}{\cos\theta\cos\alpha + \sin\theta\sin\alpha}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta\cot\alpha + \sin\theta)}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta.3\tan\theta + \sin\theta)} \qquad \text{[By (1)]}$$

$$= \frac{W\cos\theta\operatorname{cosec}\alpha}{3\sin\theta + \sin\theta} = \frac{W\cot\theta}{4}.\operatorname{cosec}\alpha = \frac{W}{4}\cot\theta\sqrt{1+\cot^2\alpha}$$

$$= \frac{W}{4}.\cot\theta\sqrt{1+9\tan^2\theta}$$

$$= \frac{W}{4}\sqrt{\cot^2\theta + 9} = \frac{W}{4}\sqrt{\cot^2\theta + 1 + 8}$$

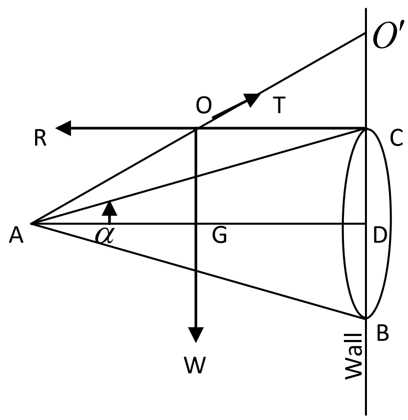
$$= \frac{W}{4}\sqrt{\operatorname{cosec}^2\theta + 8}$$

Problem 3

A solid cone of height h and semi-vertical angle α is placed with its base flatly against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$.

(The centre of gravity of a solid cone lies on its axis and divides it in the ratio 3 : 1 from the vertex.)

Solution:



Let A be the vertex, & height AD = h.

Semi-vertical angle $D\hat{A}C = \alpha$.

G divides AD in the ratio 3: 1

Length AO' is greatest, when the cone is just in the point of turning about C.

At that time, normal reaction R must be perpendicular to the wall.

Since, the cone is in equilibrium, the three forces T, W, R must be concurrent at O.

$\Delta AOG \& \Delta AO'D$ are similar.

$$\therefore \frac{AO'}{AO} = \frac{AD}{AG} = \frac{h}{\left(\frac{3}{4}h\right)} = \frac{4}{3} \qquad \boxed{\therefore AO' = \frac{4}{3}AO} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, OG = CD.

From $\Delta ACD, \tan\alpha = \frac{CD}{AD} = \frac{CD}{h} \quad \therefore CD = h\tan\alpha$

$$\boxed{\therefore OG = h.\tan\alpha}$$

From $\Delta AOG, AO^2 = AG^2 + GO^2$

$$= \left(\frac{3}{4}h\right)^2 + (h.\tan\alpha)^2$$

$$= \frac{9h^2}{16} + h^2.\tan^2\alpha$$

$$= \frac{9h^2 + 16h^2\tan^2\alpha}{16}$$

$$AO^2 = h^2\left(\frac{9}{16} + \tan^2\alpha\right)$$

$$\therefore AO = h.\sqrt{\frac{9}{16} + \tan^2\alpha}$$

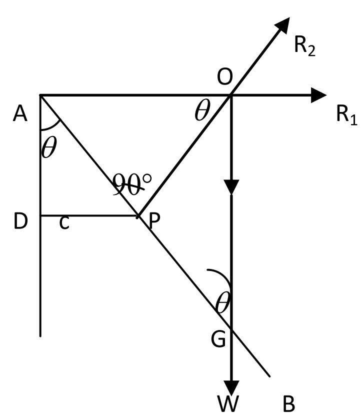
$$(1) \Rightarrow AO' = \frac{4}{3} \times h \times \sqrt{\frac{9}{16} + \tan^2\alpha}$$

$$\boxed{AO' = h.\sqrt{1 + \frac{16}{9}\tan^2\alpha}}$$

Problem 4

A heavy uniform rod of length 2a lies over a smooth peg with one end resting on a smooth vertical wall. If c is the distance of the peg from the wall and θ the inclination of the rod to the wall, show that c = a $\sin^3\theta$

Solution:



Forces acting on the rod AB are

i) Weight W at G $(\downarrow)$

ii) Reaction R_1 at A ($\perp$ to the wall)

iii) Reaction R_2 at the peg P ($\perp$ to the rod)

For equilibrium, W, R_1, R_2 must be concurrent at O.

From rightangled triangle ADP (DP = c)

$$\sin\theta = \frac{c}{AP} \text{ ………………… (1)}$$

From $\Delta AOP, \sin\theta = \dfrac{AP}{AO}$ ………………….. (2)

From $\Delta OGA, \sin\theta = \dfrac{OA}{AG}$ ………………….. (3)

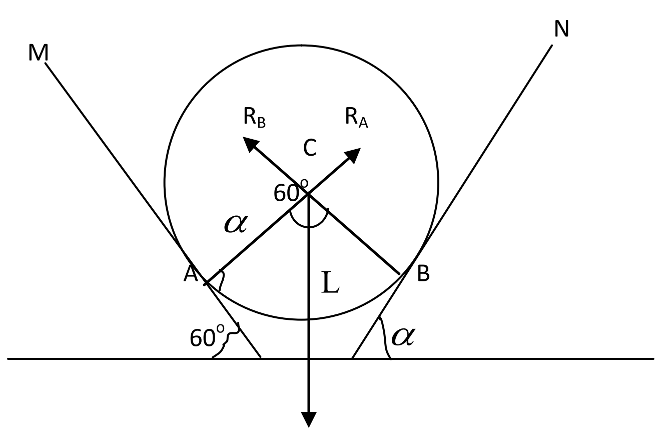
$$(1)\times(2)\times(3) \Rightarrow \sin^3\theta = \frac{c}{AP}\times\frac{AP}{AO}\times\frac{OA}{AG} = \frac{c}{AG} = \frac{c}{a}$$

$\therefore$ $\boxed{c = a\sin^3\theta}$

Problem 5

A heavy uniform sphere rests touching two smooth inclined planes one of which is inclined at $60°$ to the horizontal. If the pressure on this plane is one-half of the weight of the sphere, prove that the inclination of the other plane to the horizontal is $30°$

Solution:



Let the sphere centre C rest on the inclined planes AM and BN. MA makes $60°$ with the horizontal and let NB make an angle α with the horizon.

The forces acting are

i) Reaction R_A at A perpendicular to the inclined plane AM and to the sphere and hence passing through C.

ii) Reaction R_B at B which is normal to the inclined plane BN and to the sphere and hence passing through C.

iii) W, the weight of the sphere acting vertically downwards at C along CL.

Clearly the above three forces meet at C.

Also $\angle ACL = 60° and \ \angle BCL = \alpha$

Applying Lami's theorem,

$$\frac{R_A}{\sin \alpha} = \frac{W}{\sin(60+\alpha)}$$

$$\therefore R_A = \frac{W \sin \alpha}{\sin(60° + \alpha)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

But $R_A = \dfrac{W}{2}$ ……………… (2)

From (1) and (2), we have

$$\frac{W \sin \alpha}{\sin(60° + \alpha)} = \frac{W}{2}$$

i.e. 2 sin $\alpha = \sin(60° + \alpha) = \sin 60° \cos\alpha + \cos 60° \sin \alpha$

i.e. 2 sin $\alpha = \dfrac{\sqrt{3}}{2} \cos \alpha + \dfrac{1}{2} \sin \alpha$ or $4 \sin \alpha = \sqrt{3} \cos \alpha + \sin \alpha$

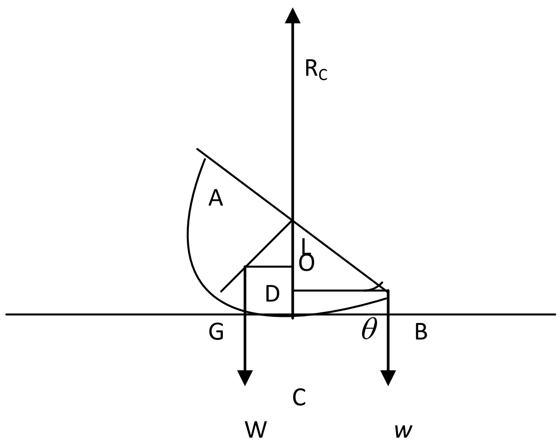
i.e. 3 sin $\alpha = \sqrt{3} \cos\alpha$ or $\dfrac{\sin \alpha}{\cos\alpha} = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$

i.e. tan $\alpha = \dfrac{1}{\sqrt{3}}$ or $\alpha = 30°$

Problem 6

A uniform solid hemisphere of weight W rests with its curved surface on a smooth horizontal plane. A weight w is suspended from a point on the rim of the hemisphere. If the plane base of the rim is inclined to the horizontal at an angle θ, prove that $\tan\theta = \dfrac{8w}{3W}$

Solution:



Draw GL perpendicular to OC and BD perpendicular to OC. Base AB is inclined at an angle θ with the horizontal BD. Forces acting are i) Reaction R_c ii) Weight W at G iii) Weight w at B.

Since these three forces are parallel, and in equilibrium each force is proportional to the distance between the other two.

$$\therefore \frac{W}{BD} = \frac{w}{GL} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $\Delta OBD \Rightarrow BD = OB\cos\theta = r\cos\theta$

Here, OG = $\dfrac{3r}{8}$, r – radius

$$GL = OG.\sin\theta = \frac{3r}{8}\sin\theta$$

$$\therefore (1) \Rightarrow \frac{W}{r\cos\theta} = \frac{w}{\left(\frac{3r}{8}\sin\theta\right)}$$

$$\boxed{\therefore \tan\theta = \frac{8w}{3W}}$$

UNIT - IV
CENTER OF MASS

4.1 Center of Gravity

Earth attracts every body by a constant gravitational force (weight of the body). As a body is composed of many particles so each particle is affected by gravity, hence a large number of forces are acting on the entire body. The point at which the resultant of these forces acts is called center of gravity of the body. It can be defined as an imaginary point in a body of matter where the total weight of the body may be thought to be located.

4.2 Center of Mass

For every system of mass m, there is a unique location in space, where all the mass can be assumed to be located. This place is called the center of mass, and is defined as point with respect to which the linear moment of mass m is zero. It is commonly designated by $c.m$ or C.
Note: Center of mass is independent of gravitational field while center of gravity is affected by gravitational field.
When the gravitational field is uniform, the center of mass is also its center of gravity but if the body is lying in varying gravitational fields, the center of gravity will be shifted from

center of mass towards stronger gravitational field. For example if a stronger gravitational field is found towards right and a weaker gravitational field is found towards left of a body, the center of mass is unmoved but the center of gravity will be shifted towards stronger gravitational field.
Here we will consider only earth's gravitational field that is uniform, hence the center of mass will be the center of gravity of the body.

1. Center of Mass of a System of Two Particles

Consider a regular trihedral system and two particle of mass m_1 and m_2, situated at point P_1 and P_2, whose position vectors relative to origin O are $\vec{r}_1$ and $\vec{r}_2$. At center of mass, the linear moment of mass is zero. Mathematically

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{4.1}$$

2. Center of Mass of a Set of n Particles

Consider a regular trihedral system and a set of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. At center of mass, the sum of linear moments of all masses is zero. Mathematically

$$\sum_{i=1}^{n} m_i\vec{r}_i = 0 \tag{4.2}$$

Theorem 4.1. *Every set of particles has one and only center of mass.*

Proof Consider a regular trihedral system and a system of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. Suppose C is a center of mass of the system and $\vec{r}$ be its position vector relative to O. Then the position vector of P_i relative to C is $\vec{r}_i - \vec{r}$. Then by definition, at C the sum of moments of all masses is zero.

$$\begin{aligned}\sum_{i=1}^{n} m_i\left(\vec{r}_i - \vec{r}\right) &= 0\\ \sum_{i=1}^{n} m_i\vec{r}_i - \vec{r}\sum_{i=1}^{n} m_i &= 0\end{aligned}$$

or we can write

$$\vec{r}\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i\vec{r}_i$$

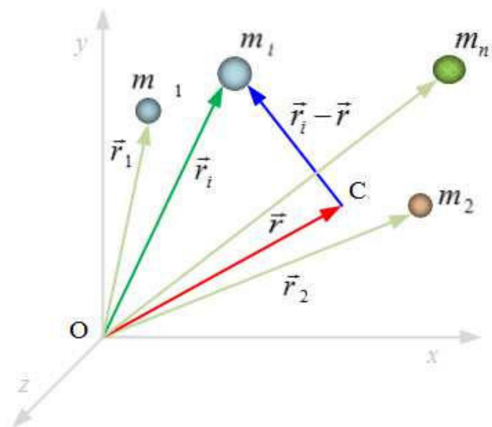


Figure 3: center of mass

therefore

$$\vec{r}_m = \frac{\sum\limits_{i=1}^{n} m_i \vec{r}_i}{\sum\limits_{i=1}^{n} m_i} \qquad (4.3)$$

(4.3) gives the position vector of C relative to O. Let C' be an other center of mass of the system and $\vec{r'}$ be its position vector relative to O. Then the position vector of P_i relative to C is $\vec{r}_i - \vec{r'}$. Then by definition, at C' the sum of moments of all masses is zero.

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{r'}\right) = 0$$

with the same above reasoning, we can write

$$\vec{r'}_m = \frac{\sum\limits_{i=1}^{n} m_i \vec{r}_i}{\sum\limits_{i=1}^{n} m_i} \qquad (4.4)$$

From (4.3) and (4.4),we can write

$$\vec{r'}_m = \vec{r}_m$$

Hence the system has one and only center of mass.

Example 4.1. *Find center of mass of the system consisting of two particles connected by a massless rod given in the following cases:*

(a) Both masses are $1\ kg$ *and length of rod is* $2\ m$*.*

(b) The mass on right from O *is* $2\ kg$ *and the mass on left is* $1\ kg$*. The length of rod is* $2\ m$*.*

(c) Both masses are $1\ kg$*. The mass on right is* $1.5\ m$ *away from* O *and mass on left from* O *is* $1\ m$ *away from* O*.*

Solution: The rod is considered to be 1 *dimensional* object just to understand the concept. And for one dimensional motion $+, -$ signs are enough to represent the direction of a vector.

(a) Both masses are same and length of rod is $2\ m$.

Let the center of the rod be at the origin. Let one mass is at A with position vector $1\ m$ and the other mass is at B with position vector $-1\ m$. The system is shown the Fig. 4. Here $n = 2$. Using (4.3), the center of mass of the system is

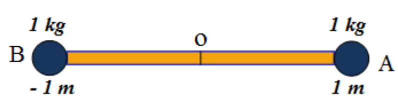


Figure 4: System of 2 particles with same masses and same distances

$$\begin{aligned} \vec{r}_m &= \frac{\sum\limits_{i=1}^{2} m_i \vec{r}_i}{\sum\limits_{i=1}^{2} m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ &= \frac{1(1) + 1(-1)}{1+1} = \frac{0}{2} \\ &= 0\ m \end{aligned}$$

In this case origin is the center of mass.

(b) The mass on right from O is $2\ kg$ and the mass on left from O is $1\ kg$. The length of rod is $2\ m$.

Let the center of the rod be at the origin. Let the mass of 2 kg is at A with position vector 1 and mass of 1 kg is at B with position vector -1. The system is shown the Fig. 5. Here $n = 2$. Using (4.3), the position vector of center of mass of the system is

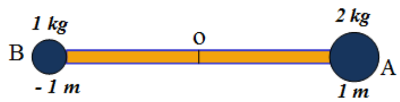


Figure 5: System of 2 particles with different masses and same distances

$$
\begin{aligned}
\vec{r}_m &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\
&= \frac{2(1) + 1(-1)}{1+1} = \frac{1}{2} \\
&= 0.5\ m
\end{aligned}
$$

In this case the center of mass is shifted towards right from origin.

(c) Both masses are 1 kg. The mass on right is 1.5 m away from O and mass on left is 1 m away from O.

Let one mass 1 kg is at A with position vector 1.5 and mass 1 kg is at B with position vector -1. The system is shown the Fig. 10.6. Here $n = 2$. Using (10.4.3), the center of

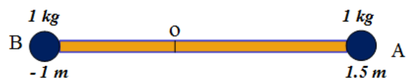


Figure 6: System of 2 particles with same masses and different distances

mass of the system is

$$
\begin{aligned}
\vec{r}_m &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\
&= \frac{1(1.5) + 1(-1)}{1+1} = \frac{0.5}{2} \\
&= 0.25\ m
\end{aligned}
$$

In this case the center of mass is shifted towards right from origin.

3 Cartesian Coordinates of the Center of Mass

Consider a regular trihedral system and a system of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. Then the position vector of P_i is

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k} \qquad (4.5)$$

Suppose C is a center of mass of the system and $\vec{r} = (\bar{x}, \bar{y}, \bar{z})$ be its position vector relative to O. Then

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (4.6)$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \qquad (4.7)$$

$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} \qquad (4.8)$$

In case of plane coordinate system, $z-coordinate$ can be ignored considering xy plane and in case of collinear coordinate system only one coordinate will be sufficient.

Example 2. *A mass of 3 kg is located at $(0,0)$, a mass of 4 kg is located at $(5,4)$ and a mass of 8 kg is located at $(-3,3)$. Find the coordinates of their centre of mass.*

Solution: The given system has three masses and is two dimensional. Here $n = 3$ Let

$$m_1 = 3\ kg$$
$$m_2 = 5\ kg$$
$$m_1 = 8\ kg$$

Then

$$\sum_{i=1}^{3} m_i = m_1 + m_2 + m_3 = 3 + 5 + 8$$
$$= 16\ kg$$

Sum of moments of all masses about $x\ axis$ is

$$\sum_{i=1}^{3} m_i x_i = m_1x_1 + m_2x_2 + m_3x_3 = 3(0) + 4(5) + 8(-3)$$
$$= 0 + 20 - 24 = -4\ kg.m$$

Sum of moments of all masses about $y\ axis$ is

$$\sum_{i=1}^{3} m_i y_i = m_1 y_1 + m_2 y_2 + m_3 y_3 = 3(0) + 4(4) + 8(3)$$
$$= 0 + 16 + 24 = 40\ kg.m$$

using (4.6), the x coordinate of center of mass are

$$\bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i}$$
$$= \frac{-4}{16} = -0.25\ m$$

using (4.7), the y coordinate of center of mass are

$$\bar{y} = \frac{\sum_{i=1}^{3} m_i y_i}{\sum_{i=1}^{3} m_i}$$
$$= \frac{40}{16} = 2.5\ m$$

Hence the centre of mass of the system is located at the point (-0.25, 2.5).

5 Centroid of a Body or System

For a uniform body, the center of mass is called centroid. In this case, if body or system has n particles of equal masses $m_1 = m_2 = ... = m_n$, situated at point $P_1, P_2, ..., P_n$, then its center of mass or centroid is

$$\vec{r} = \frac{\sum_{i=1}^{n} \vec{r}_i}{n}$$
$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\sum_{i=1}^{n} x_i}{n}, \frac{\sum_{i=1}^{n} y_i}{n}, \frac{\sum_{i=1}^{n} z_i}{n} \right) \qquad (\ 5.1)$$

5.1 Center of Mass of a System of n Particles in Plane or Space

If n mass points are not necessarily on a line but are in a plane or in space with position vectors $\vec{r}_1, \vec{r}_2, ...\vec{r}_n$, then its center of mass is

$$\vec{r}_m = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...m_n\vec{r}_n}{m_1 + m_2 + ...m_n} \qquad (\ 5.2)$$

The center of mass of a system of two particles of masses m_1, m_2 is

$$\vec{r}_m = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (5.3)$$

But (5.3) gives the position vector of the point dividing the directed line segment from $\vec{r}_1$ to $\vec{r}_2$ in the ratio $m_1 : m_2$. Hence the center of mass of two particles of masses m_1, m_2 divides the directed line segment from $\vec{r}_1$ to $\vec{r}_2$ in the ratio $m_1 : m_2$.
In case of two equal masses $m_1 = m_2 = m$, the centroid is

$$\vec{r}_m = \frac{\vec{r}_1 + \vec{r}_2}{2} \qquad (5.4)$$

Example 5.1. *Consider three masses $2, 3, 4$ kg are situated at P_1, P_2, P_3 having position vectors $\hat{i}$, $2\hat{i} - \hat{j}$ and $3\hat{i} + \hat{j} - 4\hat{k}$. What will be their centroid and center of mass?*

Solution The position vector of the centroid is

$$\begin{aligned}\vec{r}_m &= \frac{\hat{i} + 2\hat{i} - \hat{j} + 3\hat{i} + \hat{j} - 4\hat{k}}{3} \\ &= \frac{6\hat{i} - 4\hat{k}}{3}\end{aligned}$$

And the position vector of the center of mass is

$$\begin{aligned}\vec{r}_m &= \frac{2\left(\hat{i}\right) + 3\left(2\hat{i} - \hat{j}\right) + 4\left(3\hat{i} + \hat{j} - 4\hat{k}\right)}{2 + 3 + 4} \\ &= \frac{19\hat{i} - \hat{j} - 12\hat{k}}{9}\end{aligned}$$

Hence the coordinates of the center of mass is $\left(\frac{19}{9}, -\frac{1}{9}, -\frac{4}{3}\right)$.

6 Center of Mass of a Continuous Distribution of Matter

The formulae obtained in the preceding article are applicable in the case of discrete systems only. If we are to find the center of mass of a continuous distribution of matter forming a body, integration methods explained below are to be employed. First of all consider one dimensional object.

6.1 Center of Mass of One Dimensional Object

Consider a body (a line or curve) of mass m and length l in one dimension. We subdivide the object into n parts. Take a small element of length ds with r be its position vector (see Fig. 7), then mass of small element is

$$dm = \rho ds$$

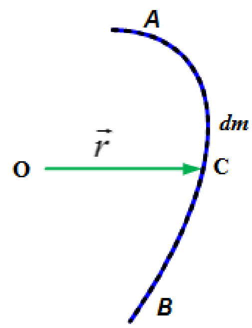


Figure 7: Center of mass of 1 dimensional system

Where ρ is the density of the body. Then the center of mass of the body is

$$\begin{aligned}\bar{r} &= \frac{\int\limits_s \vec{r}dm}{\int\limits_s dm} \\ &= \frac{\int\limits_s \vec{r}dm}{m} \qquad (6.1)\end{aligned}$$

Where the integration has to be performed over the entire body.
More clearly if one dimensional system is x axis and the mass m is from x_1 to x_2, then its total length is $l = x_2 - x_1$. Consider small element dm having length dx, having position vector $\vec{x}$ from the orion. Let ρ be the its density at $\vec{r}$, then

$$\rho = \frac{dm}{dx}$$

or small element is

$$dm = \rho dx$$

then the center of mass of is

$$\begin{aligned}\bar{x} &= \frac{\int\limits_{x_1}^{x_2} \vec{x}\rho dx}{\int\limits_{x_1}^{x_2} \rho dx} \\ &= \frac{1}{m}\int\limits_{x_1}^{x_2} \vec{x}\rho dx \qquad (6.2)\end{aligned}$$

Where $m = \int\limits_{x_1}^{x_2} \rho dx$ is the total mass of the body.
If the body is homogenous (has uniform distribution of mass), then the center of mass of is

$$
\begin{aligned}
\bar{x} &= \frac{\int\limits_{x_1}^{x_2} \vec{x} dx}{\int\limits_{x_1}^{x_2} dx} \\
&= \frac{1}{l} \int\limits_{x_1}^{x_2} \vec{x} dx \qquad (6.3)
\end{aligned}
$$

Example 6.1. *Find center of mass of a uniform rod of mass m kg of length a m.*

Solution: A uniform rod of mass m kg of length a m is shown in Fig. 8 Consider a small element mass dm of width dx at a distance x from origin O. Here

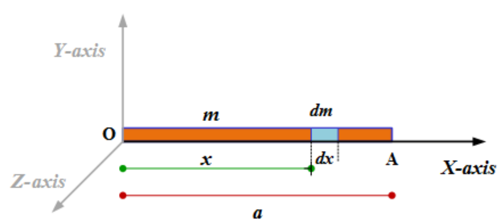


Figure 8: rod of length a

$$
\begin{aligned}
\vec{x} &= x \\
m &= m\ kg \\
l &= a \\
x_1 &= 0 \\
x_2 &= a
\end{aligned}
$$

Since the rod is uniform, using (6.3) the center of mass of the rod is

$$\begin{aligned}
\bar{x} &= \frac{1}{a}\int_0^a x dx \\
&= \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a \\
&= \frac{1}{a}\frac{(a)^2}{2} \\
&= \frac{1}{2}a\ m
\end{aligned}$$

Hence the center of mass of the rod is its mid point.

6.2 Center of Mass of Two Dimensional Object

Consider a lamina of mass m and area A in cartesian coordinate system. We subdivide the lamina into n rectangles by drawing lines parallel to coordinate axes. Take a small rectangle of area ds (see Fig. 9), then mass of small element is

$$dm = \rho ds$$

Let $r_i = (x_i, y_i)$ be any point in it. Then the center of mass of the lamina is

$$\begin{aligned}
\vec{r}_m &= \frac{\int_s \vec{r} dm}{\int_s dm} \\
&= \frac{\int_s \vec{r} dm}{m} \qquad (6.4)
\end{aligned}$$

Where the integration has to be performed over the entire body.

More clearly if two dimensional system is in xy plane and the mass m has dimensions $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$, then its total area is $A = (x_2 - x_1)(y_2 - y_1)$. Consider small element dm having area $dA = dxdy$. Let $r_i = (x_i, y_i)$ be any point with density ρ in it. Then

$$\rho = \frac{dm}{dA}$$

or small element is

$$dm = \rho dA$$

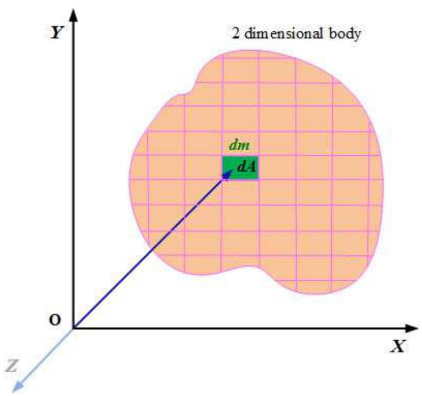


Figure 9: Center of mass of 2 dimensional system

Then the center of mass of the lamina is

$$\bar{r} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r}\rho dA}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \rho dA} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r}\rho dA}{m} \quad (6.5)$$

Where $m = \int_s \rho dA$ is the total mass of the lamina.
If the body homogenous, then the center of mass of the lamina is

$$\vec{r}_m = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r} dA}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} dA} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r} dA}{A} \quad (6.6)$$

Where $A = \int_s dA$ is the total area of the lamina.

Example 6.2. *Find the center of mass of a uniform rectangular lamina.*

Solution
Let $OABC$ be a rectangular lamina of mass m and OA (along x $axis$) and OB along y $axis$

Let $OA = 2a$ and $OC = 2b$ Area of lamina is

$$A = 4ab$$

Consider a small element of surface area $dA = dxdy$ at a distance y from $x\ axis$ Here

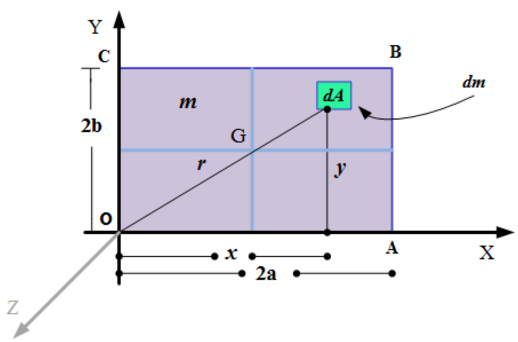


Figure 10: rectangular lamina

$$\begin{aligned}
\vec{r} &= \langle x, y \rangle \\
m &= m\ kg \\
A &= 4ab \\
x_1 &= 0 \\
x_2 &= 2a \\
y_1 &= 0 \\
y_2 &= 2b
\end{aligned}$$

Since lamina is uniform, using (6.6) the center of mass of the rod is

$$
\begin{aligned}
\vec{r}_m &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2} \vec{r} dA}{A} \\
\langle \bar{x}, \bar{y} \rangle &= \frac{\int\limits_{0}^{2a}\int\limits_{0}^{2b} \langle x, y \rangle dx dy}{4ab} \\
&= \frac{1}{4ab} \left\langle \left[\frac{x^2}{2}\right]_0^{2a} \int\limits_0^{2b} dy, \left[x\right]_0^{2a} \int\limits_0^{2b} y dy \right\rangle \\
&= \frac{1}{4ab} \left\langle \left[2a^2\right] \left[y\right]_0^{2b}, (2a) \left[\frac{y^2}{2}\right]_0^{2b} \right\rangle \\
&= \frac{1}{4ab} \left\langle \left[2a^2\right](2b), (2a)\left[2b^2\right] \right\rangle \\
&= \langle a, b \rangle
\end{aligned}
$$

Hence the center of mass or centroid of rectangular lamina is (a, b)

6.3 Center of Mass of Three Dimensional Object

Consider a three dimensional rigid body of mass m and volume V in a regular trihedral system. We subdivide the lamina into n rectangular parallelepipeds by drawing planes parallel to coordinate axes. One such parallelepiped of volume ds is shown in Fig. 10.11. The mass of small element is

$$dm = \rho ds$$

Let $r_i = (x_i, y_i, z_i)$ be any point within it. Then the center of mass of the body is

$$
\begin{aligned}
\bar{r} &= \frac{\int\limits_s \vec{r} dm}{\int\limits_s dm} \\
&= \frac{\int\limits_s \vec{r} dm}{m} \qquad (6.7)
\end{aligned}
$$

Where the integration has to be performed over the entire body, and $m = \int\limits_s dm$ is the total mass of the body.

More clearly if three dimensional system is in xyz space and the mass m has dimensions $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$ and $z_1 \leq z \leq z_2$, then its total volume is $V =$

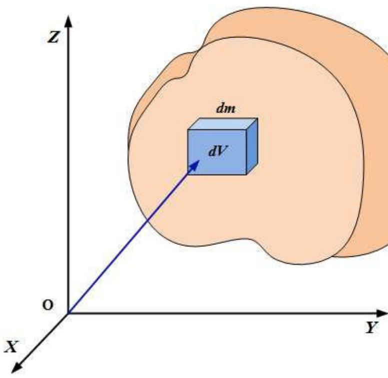


Figure 11: Center of mass of 3 dimensional system

$(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)$. Consider small element dm having volume $dV = dxdydz$, Let $r_i = (x_i, y_i, z_i)$ be any point with density ρ in it. Then

$$\rho = \frac{dm}{dV}$$

or small element is

$$dm = \rho dV$$

Then the center of mass of the body is

$$\begin{aligned}\vec{r}_m &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \vec{r}\rho dV}{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \rho dV} \\ &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \vec{r}\rho dV}{m} \qquad (6.8)\end{aligned}$$

Where $m = \int\limits_{s} \rho dV$ is the total mass of the body.

If the body homogenous, then the center of mass of the body is

$$\begin{aligned}\vec{r}_m &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \vec{r} dV}{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} dV} \\ &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \vec{r} dV}{V} \qquad (6.9)\end{aligned}$$

Where $V = \int_s dm$ is the total volume of the body.

Example 6.3. *Find the center of mass of a uniform cube.*

Solution

Consider a uniform cube of mass m with OA along $x\ axis$, OB along $y\ axis$ and OC along $z\ axis$. Let $OA = a$, $OB = a$ and $OC = a$ Volume of the cube is

$$V = a^3$$

Consider a small volume $dV = dxdydz$ at a distance r from origin O. Here

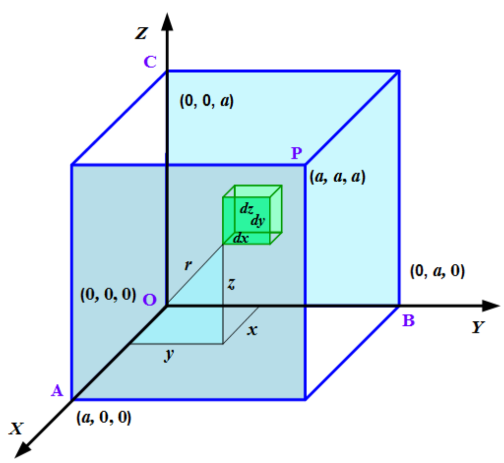


Figure 12: A cube with edges along coordinate axis.

$$\begin{aligned}
\vec{r} &= \langle x, y, z \rangle \\
m &= m\ kg \\
V &= a^3 \\
x_1 &= 0 \\
x_2 &= a \\
y_1 &= 0 \\
y_2 &= a \\
z_1 &= 0 \\
z_2 &= a
\end{aligned}$$

Since cube is uniform, using (10.6.9) the center of mass of the rod is

$$\begin{aligned}
\vec{r}_m &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} \vec{r}dV}{V} \\
\langle \bar{x}, \bar{y} \rangle &= \frac{\int\limits_0^a\int\limits_0^a\int\limits_0^a \langle x, y, z\rangle dxdydz}{a^3} \\
&= \frac{1}{a^3}\left\langle \left[\frac{x^2}{2}\right]_0^a \int\limits_0^a\int\limits_0^a dydz, \left[x\right]_0^a \int\limits_0^a\int\limits_0^a ydydz, \left[x\right]_0^a \int\limits_0^a\int\limits_0^a zdydz \right\rangle \\
&= \frac{1}{a^3}\left\langle \left[\frac{a^2}{2}\right] \left[y\right]_0^a \int\limits_0^a dz, (a)\left[\frac{y^2}{2}\right]_0^a \int\limits_0^a dz, a\left[y\right]_0^a \int\limits_0^a zdz \right\rangle \\
&= \frac{1}{a^3}\left\langle \frac{1}{2}a^3\left[z\right]_0^a, \frac{1}{2}a^3\left[z\right]_0^a, a^2\left[\frac{z^2}{2}\right]_0^a \right\rangle \\
&= \frac{1}{a^3}\left\langle \frac{1}{2}a^4, \frac{1}{2}a^4, \frac{1}{2}a^4 \right\rangle \\
&= \langle \frac{a}{2}, \frac{a}{2}, \frac{a}{2} \rangle
\end{aligned}$$

Hence the center of mass or centroid of a uniform cube is $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$

7 Symmetry and Center of Mass

If a body possesses some sort of symmetry, then it is too much easy to compute the position of its center of mass. We first explain the concept of symmetry and shall therefore show how to use this concept in determining the center of mass of a body.

7.1 Symmetry with respect to a Point

A body is said to be symmetric with respect to a point O if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that O is the middle point of the line segment $PP^{'}$ and $\rho(P) = \rho(P^{'})$, $i.e.$, the density of the body at the points P and $P^{'}$ is the same. Such symmetry is called central symmetry and the point O is called the center of symmetry.
It follows that a uniform body is symmetric with respect to origin O if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(-x, -y, -z)$ in the body such that O is the middle point of the line segment $PP^{'}$ and $\rho(P) = \rho(P^{'})$.
Examples
1. A uniform rod is symmetric with respect to its mid point, hence its mid point is center

of mass as shown in example 10.6.1.
2. A uniform circular lamina is symmetric with respect to its geometric center.
3. A uniform solid sphere or spherical shell is symmetric with respect to its geometric center.

7.2 Symmetry with respect to a Line

A body is said to be symmetric with respect to a line l if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that l bisects the line segment $PP^{'}$ perpendicularly and $\rho(P) = \rho(P^{'})$. Such symmetry is called axial symmetry and the line l is called the axis of symmetry.
In particular, a uniform body is symmetric with respect to the z axis if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(-x, -y, z)$ in the body such that z axis is the right bisector of the line segment $PP^{'}$.
Examples
1. A uniform circular cylinder is symmetric with respect to its axis.
2. A uniform solid sphere or spherical shell is symmetric with respect to its axis.
A uniform lamina is symmetric with respect to x axis if and only if for every point $P(x, y)$ of the body there exist a point $P^{'}(x, -y)$ of the lamina. In this case its center is the center of mass as shown in example 6.3.

7.3 Symmetry with respect to a Plane

A body is said to be symmetric with respect to a plane p if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that p bisects the line segment $PP^{'}$ perpendicularly and $\rho(P) = \rho(P^{'})$. Such symmetry is called axial symmetry and the line l is called the axis of symmetry.
In particular, a uniform body is symmetric with respect to the xy plane if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(x, y, -z)$ in the body such that xy plane bisects of the line segment $PP^{'}$.
Examples
1. A uniform solid or hollow ellipsoid is symmetric with respect to each of its principal planes.
2. A uniform solid sphere or spherical shell is symmetric with respect to each of its diametral plane (planes passing through the center).

8 Centroid of a Plane Region

The centroid C is a point which defines the geometric center of an object. The centroid coincides with the center of mass or the center of gravity only if the material of the body is

homogenous (density or specific weight is constant throughout the body). If an object has an axis of symmetry, then the centroid of object lies on that axis.
Consider the region bounded by the curve $y = f(x)$, the $x - axis$, the line $x = a$ and the line $x = b$ as shown in Fig. 13. Let the density of the region is 1. Then by (10.1.4) the

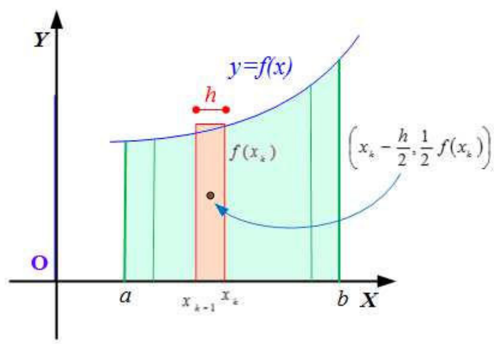


Figure 13: Plane region.

total mass of the system is the total area of the region

$$m = A$$

For uniform distribution of mass the area under the curve is

$$A = \int_a^b f(x)dx$$

Hence the mass of the region is

$$m = \int_a^b f(x)dx$$

As area under the curve is obtained by using approximation method. In this method the interval $[a, b]$ is divided into n subintervals of length h. Then h is

$$h = \frac{b-a}{n} \quad (8.1)$$

Let $x_k, k = 1, 2, ...n$ be the endpoints of each subinterval. Next construct a rectangle on each subinterval and find its area to approximate the area under the curve. On interval $x_{k-1} \leq x \leq x_k$, the rectangle has height $f(x_k)$ and width $x_k - x_{k-1} = h$. Its area is its mass

$$dA = hf(x_k) = dm$$

And its center of mass is its geometric center, given by

$$C_I \quad = \quad \left(x_k - \frac{h}{2}, \frac{1}{2}f(x_k)\right)$$

The mass moment about $y - axix$ of this rectangle is

$$dM_y \quad = \quad hf(x_k)\left(x_k - \frac{h}{2}\right)$$

We can imagine that center of mass of each rectangle is its geometric center. The mass moment about $y - axix$ of all n rectangles is

$$M_x \quad = \quad \sum_{k=1}^{n}\left[hf(x_k)\left(x_k - \frac{h}{2}\right)\right]$$

Similarly the mass moment about $x - axix$ of all n rectangles is

$$M_y \quad = \quad \sum_{k=1}^{n}\left[hf(x_k)\frac{1}{2}f(x_k)\right]$$

Taking limit $n \to \infty$, the sum of areas of all rectangles approaches to true area under the curve, and in the same way the moments about $y - axix$ and $x - axix$ of the rectangles approaches to true moments of area under the curve. As $n \to \infty$, by (10.8.1) $h \to 0$. Hence $x_k - \frac{h}{2} \to x_k$. Thus for a plane region bounded by $y = f(x)$, the $x - axis$, the line $x = a$ and the line $x = b$, the moments about $x - axis$ is

$$\begin{aligned} M_x &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[hf(x_k)\frac{1}{2}f(x_k)\right] \\ &= \int_a^b xf(x)dx \qquad (\quad 8.2) \end{aligned}$$

and $y - axis$ is

$$\begin{aligned} M_y &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[hf(x_k)\left(x_k - \frac{h}{2}\right)\right] \\ &= \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx \qquad (\quad 8.3) \end{aligned}$$

Hence the x coordinate of center of mass is

$$\bar{x} \quad = \quad \frac{M_x}{m} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx} \qquad (\quad 8.4)$$

the y coordinate of center of mass is

$$\bar{y} \quad = \quad \frac{M_y}{m} = \frac{\int\limits_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int\limits_a^b f(x)dx} \qquad (8.5)$$

Example 8.1. *Find the center of mass of a plane region bounded by the curve $y = \sqrt{x}$, the $x - axis$, the line $x = 1$ and the line $x = 3$.*

Solution
The given data is

$$\begin{aligned} f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\ a &= 1 \\ b &= 3 \end{aligned}$$

The plane region bounded by the curve $y = \sqrt{x}$, the $x - axis$, the line $x = 1$ and the line $x = 3$ is shown in Fig. 14. The area of the region is

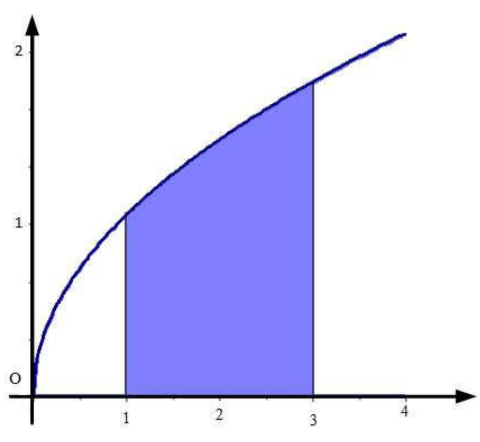


Figure 14: Plane region.

$$
\begin{aligned}
A &= \int_a^b f(x)dx \\
&= \int_1^3 (x)^{\frac{1}{2}} dx \\
&= \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^3 \\
&= \frac{2}{3}\left[(3)^{\frac{3}{2}} - 1\right] \\
&= 2.8\ units^2
\end{aligned}
$$

Using (8.2) the mass moments about $x - axis$ is

$$
\begin{aligned}
M_x &= \int_a^b xf(x)dx \\
&= \int_1^3 x(x)^{\frac{1}{2}} dx = \int_1^3 (x)^{\frac{3}{2}} dx \\
&= \left[\frac{(x)^{\frac{5}{2}}}{\frac{5}{2}}\right]_1^3 \\
&= \frac{2}{5}\left[(3)^{\frac{5}{2}} - 1\right] \\
&= 5.8\ units^3
\end{aligned}
$$

Using (8.3) the mass moments about $y - axis$ is

$$
\begin{aligned}
M_y &= \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx \\
&= \int_1^3 \frac{1}{2}\left[\sqrt{x}\right]^2 dx \\
&= \int_1^3 \frac{1}{2} x dx \\
&= \frac{1}{2}\left[\frac{x^2}{2}\right]_1^3 \\
&= 2\ units^3
\end{aligned}
$$

Using (8.4) the x coordinate of center of mass is

$$\begin{aligned}\bar{x} &= \frac{M_x}{m} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx} \\ &= \frac{5.8}{2.8} \\ &= 2.1 \; units\end{aligned}$$

Using (8.5) the y coordinate of center of mass is

$$\begin{aligned}\bar{y} &= \frac{M_y}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)dx} \\ &= \frac{2}{2.8} \\ &= 0.7 \; units\end{aligned}$$

Hence the center of mass is $(2.1, 0.7)$ unit.

Exercises

1. The density of glass of mass $10kg$ is 3140 kg/m^3. Determine its volume.
2. A mass of 5 kg is located at $(1, 0, -1)$, a mass of 4 kg is located at $(2, 5, 4)$ and a mass of 2 kg is located at $(4, -3, 1)$. Find the coordinates of their centre of mass.
3. A square of side a has particles of masses 1 kg, 2 kg, 3 kg, 4 kg at its vertices. Find the center of mass of the system.
4. Find the center of mass of a uniform rectangular lamina whose center is the origin of the coordinate system.
5. Find the center of mass of a uniform triangular lamina.
6. Find the center of mass of a uniform circular disc whose center
   (a) is the origin of the coordinate system.
   (b) lies on x axis of the coordinate system and passing through the origin.
   (c) lies on y axis of the coordinate system and passing through the origin.
7. Find the center of mass of a uniform elliptic disc whose center lies on the origin.
8. Find the center of mass of a plane region bounded by
   (a) lines $y = 2x,\ y = -2x$ and $x = 2$.
   (b) the curve $y = \sqrt{x}$, the $x - axis$ and the line $x = 4$.
   (c) the curve $y = x^2$, the $x - axis$ and the lines $x = 1, x = 2$.

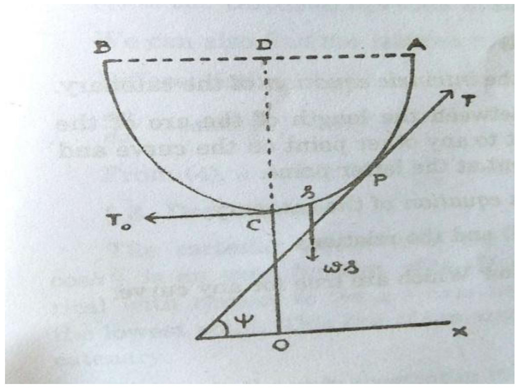
UNIT - V
HANGING STRINGS

1. Equilibrium of Strings

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a *catenary*. If the weight per unit length of the chain or string is constant, the catenary is called the *uniform or common catenary*.

2. Equation of the common catenary:

A uniform heavy inextensible string hangs freely under the action of gravity; to find the equation of the curve which it forms.



Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest, of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the are CP = s

Let ω be the weight per unit length of the chain.

Consider the equilibrium of the portion CP of the chain.

The forces acting on it are:

(i) Tension T_0 acting along the tangent at C and which is therefore horizontal.

(ii) Tension T acting at P along the tangent at P making an angle Ψ with OX.

(iii) Its weight **ws** acting vertically downwards through the C.G. of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight **ws** must pass through the point of the intersection of T and T_o.

Resolving horizontally and vertically, we have

$$T\cos\Psi = T_o \ldots \ldots \quad (1)$$

$$\text{and } T\sin\Psi = \mathbf{ws} \ldots \ldots (2)$$

Dividing (2) by (1), $\tan\Psi = \frac{\mathbf{ws}}{T_0}$

Now it will be convenient to write the value of T_o the tension at the lowest point, as $T_o = wc$ … … (3) where *c* is a constant. This means that we assume T_o, to be equal to the weight of an unknown length *c* of the cable.

Then $\tan\Psi = \frac{\boldsymbol{ws}}{\boldsymbol{wc}} = \frac{\boldsymbol{s}}{\boldsymbol{c}}$

$\therefore$ S = ctanΨ … … … (4)

Equation (4) is called the *intrinsic* equation of the catenary.

It gives the relation between the length of the area of the curve from the lowest point to any other point on the curve and the inclination of the tangent at the latter point.

To obtain the *certesian equation* of the catenary,

We use the equation (4) and the relations

$\frac{dy}{ds} = \sin\Psi$ and $\frac{dy}{dx} = \tan\Psi$ which are true for any curve.

Now $\frac{dy}{d\Psi} = \frac{dy}{ds}.\frac{ds}{d\Psi}$

$= \sin\Psi \frac{d}{d\Psi} c\tan\Psi$

$= \sin c\sec^2\Psi = c\sec\Psi\tan\Psi$

$\therefore y = \int c\sec\Psi\tan\Psi\, d\Psi + A$

$= c\sec\Psi + S$

If $y = c$ when $\Psi = 0$, then $c = c\sec 0 + A$

$\therefore A = 0$

Hence y = csec Ψ … … … (5)

$\therefore y^2 = c^2\sec\Psi = c^2(1 + \tan^2\Psi)$

$= c^2 + s^2$ … … (6)

$\frac{dy}{dx} = \tan\Psi = \frac{s}{c} = \frac{\sqrt{y^2-c^2}}{c}$

$\therefore \frac{dy}{\sqrt{y^2-c^2}} = \frac{dx}{c}$

Integrating, $\cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + B$

When x = 0, y = c

i.e. $\cosh^{-1} 1 = 0 + B$ or $B = 0$

$\therefore \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c}$

i.e. $y = c\cosh\left(\frac{x}{c}\right)$ … … (7)

(7) is the Cartesian equation to the catenary.

We can also find the relation connecting s and x.

Differentiating (7).

$$\frac{dy}{dx} = c\sinh\frac{x}{c}.\ \frac{1}{c} = \sinh\frac{x}{c}$$

From (4), $s = c\tan\Psi = c.\ \frac{dy}{dx} = c\sinh\frac{x}{c} \ldots (8)$

Definitions:

The Cartesian equation to the catenary is $y = c\cosh\frac{x}{c}$. $\cosh\frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the axis of the catenary.

Since c is the only constant, in the equation, it is called the *parameter* of the catenary and it determines the size of the curve.

The lowest point *C* is called the vertex of the catenary. The horizontal line at the depth *c* below the vertex (which is taken by us the x – axis) is called the directrix of the catenary.

If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the span and the distance CD (i.e. the depth of the lowest point C below AB) is called the sag.

3. Tension at any point:

We have derived the equations

$T\cos\Psi = T_0 \ldots \ldots \ldots \ldots (1)$

And $T\sin\Psi = ws \ldots \ldots \ldots \ldots (2)$

We have also put $T_0 = wc \ldots \ldots \ldots (3)$

Equation (3) shows that the tension at the lowest point is a constant and is equal to the weight of a portion of the string whose length is equal to the parameter of the catenary. From the equation (1), we find that the horizontal component of the tension at any point on the curve is equal to the tension at the lowest point and hence is a constant.

From equation (2), we deduce that the vertical component of the tension at any point is equal to ws i.e. equal to the weight of the portion of the string lying between the vertex and the point. ($\therefore$ s = are CP)

Squaring (1) and (2) and then adding,

$T^2 = T^2{}_0 + w^2s^2$

$=w^2c^2+w^2a^2$

$=w^2(c^2+s^2)$

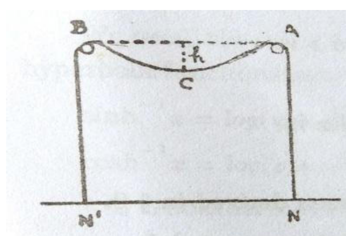
$=w^2y^2$ using equation (6) of page 377

$\therefore T = wy$ ... … … (4)

Thus the tension at any point is proportional to the height of the point above the origin. It is equal to the weight of a portion of the string whose length is equal to the height of the point above the directrix.

Important Corollary:

Suppose a long chains is thrown over two smooth pegs A and B and is in equilibrium with the portions AN and B*N*' hanging vertically. The potion BCA of the chain will from a catenary.



The tension of the chain is unaltered by passing overt the smooth peg A. The tension at A can be calculated by two methods.

On one side (i.e. from the catenary portion), Tension at A = $w.y$ where y is the height of A above the directrix.

On the other side, tension at A = weight of the free part AN hanging down

$= w.$ AN

$\therefore y$=AN

In other words, N is on the directrix of the catenary.

Similarly N' is on the directrix.

Hence if a long chain is thrown over two smooth pegs and is in equilibrium, the free ends must reach the directrix of the catenary formed by it.

Important Formulae:

The Cartesian coordinates of a point P on the catenary are (x, y) and its intrinsic coordinates are (s, Ψ). Hence there are four variable quantities we can have a relation connecting any two of them. There will be ${}_4C_2 = 6$ such relations, most of them having been already derived. We shall derive the remaining. It is worthwhile to collect these results for ready reference.

(i) The relation connecting x and y is

$y = c\cosh\frac{x}{c}$ … … (1)

and this is the Cartesian equation to the catenary.

(ii) The relation connecting s and Ψ is

$s = c\tan\Psi$ … … (2)

(iii) The relation connecting y and Ψ is

$y = c\sec\Psi$ … … … (3)

(iv) The relation connecting y and s is

$y^2 = c^2 + s^2$ …. … … (4)

(v) The relation connecting s and x is

$s = c\sinh\frac{x}{c}$

(vi) We have $y = c\cosh\frac{x}{c}$ and $y = c\sec\Psi$,

$\therefore \sec\Psi = \cosh\frac{x}{c}$

$\therefore \frac{x}{c} = \cosh -1(\sec\Psi)$

$= \log(sec\Psi + \sqrt{sec^2\Psi - 1}$

$= \log(sec\Psi + \tan\Psi)$

$\therefore x = clog\ (sec\Psi + tan\Psi)$ … … (6)

This relation can also be obtained thus:

$\frac{dx}{d\Psi} = \frac{dx}{ds}.\frac{ds}{d\Psi}$

$= \cos\Psi.\frac{d}{d\Psi}(c\tan\Psi)$ since $\frac{dx}{ds} = \cos\Psi$ for any curve

$= \cos\Psi.\ C\sec2\Psi - c\sec\Psi$

Integrating, x = $\int c\sec\Psi\, d\Psi + D$

$= c\log(\sec\Psi + \text{ran}\Psi) + D$

At the lowest point, $\Psi = 0$ and $x = 0$

$\therefore 0 = c\log(\sec 0 + \tan 0 + D$

i.e. $0 = D$

$\therefore x = c\log(\sec\Psi + \tan\Psi)$

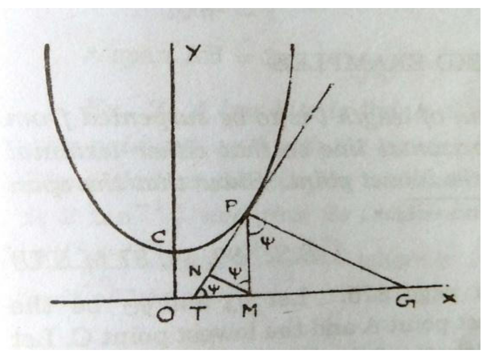
(vii) The tension at any point = wy … … (7), where y is the distance of the point from the directrix.

(viii) The tension at the lowest point = wc … … (8)

$\sinh^{-1} x = \log(x+\sqrt{x^2+1})$

$\cosh^{-1} x = \log(x+\sqrt{x^2-1})$

4. Geometrical Properties of the Common catenary:



Let P be any point on the catenary $y = c\cosh\frac{x}{c}$.

PT is the tangent meeting the directrix (i.e. the x axis) at T.

angle PTX = Ψ

PM (=y) is the ordinate of P and PG is the normal at P.

Draw MN ⊥ to PT.

From ΔPMN. MN = PMcosΨ

$= y\cos\Psi$

$=c\sec\Psi\cos\Psi$

$=c$=constant

i.e. *The length of the perpendicular from the foot of the ordinate on the tangent at any point of the catenary is constant.*

Again $\tan\Psi = \frac{PN}{MN} = \frac{PN}{C}$

$\therefore$ PN = $c\tan\Psi = S$ arc CP

$PM^2 = NM^2 + PN^2$

$\therefore y^2 = c^2+s^2$, a relation already obtained.

If is the radius of curvature of the catenary at P,

$P=\frac{ds}{d\Psi} = \frac{d}{d\Psi}(c\tan\Psi) = c\sec^2\Psi$

Let the normal at P cut the x axis at G.

Then PG. $\cos\Psi$ = PM = y

$\therefore PG = \frac{y}{\cos\Psi} = c\sec\Psi.\sec\Psi = c\sec^2\Psi$

$\therefore \rho = PG$

Hence the radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.

Problem 1

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1}$

Solution:

Tension at A = wy_A

And tension at C $= w.y_C$ since T = wy at any point

Now $w.y_A = n.w.y_C$

$\therefore y_A = ny_C = nc$

But $y_A = c\cosh\frac{x_A}{c} = nc$

$\therefore \cosh\frac{x_A}{c} = n$

or $\frac{x_A}{c} = \cosh^{-1} n = \log(n + \sqrt{n^2 - 1})$

$\therefore x_A = c\log(n + \sqrt{n^2 - 1}) \ldots \ldots \ldots (1)$

We have to find c.

$y^2_A = c^2 + s^2_A$, s_A denoting the length of CA.

$= c^2 + \frac{l^2}{4}$ (as total length = l)

i.e. $n^2c^2 = c^2 + \frac{l^2}{4}$

or $c^2 = \frac{l^2}{4(n^2-1)}$

$\therefore c = \frac{l^2}{2\sqrt{n^2-1}} \ldots \ldots (2)$

Substituting (2) in (1),

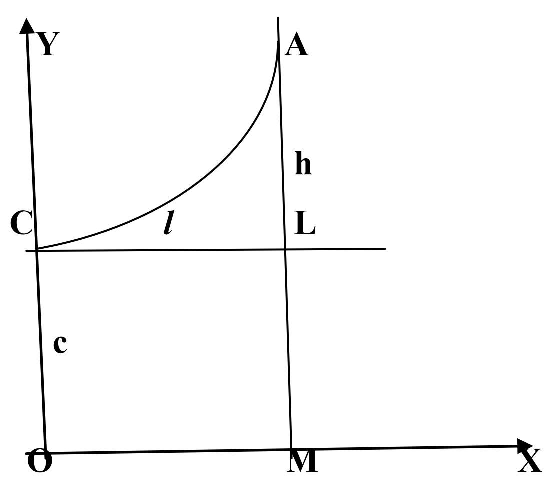
$$x_A = \frac{l^2}{2\sqrt{n^2-1}} \log(n + \sqrt{n^2 - 1})$$

$\therefore$ span AB $= 2x_A = \frac{l}{\sqrt{n^2-1)}} \log(n + \sqrt{n^2 - 1})$

Problem 2

A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite, the inclination of the wire to the ground is $2\tan^{-1}\frac{h}{l}$ and that its tensions there and at the ground are $\frac{w(l^2+h^2)}{2h}$ and $\frac{w(l^2-h^2)}{2h}$ where w is the weight of the wire per unit of length.

Solution:



C is the vertex of the catenary CA, A being the kite. The origin O is taken at a depth c below C.

Then $y_A = c + h$ and $s_A = \text{arc } CA = l$

Since $y^2 = c^2 + s^2$, we have $(c+h)^2 = c^2 + l^2$

i.e. $h^2+2ch = l^2$

or $c = \frac{l^2-h^2}{2h}$ … … … (1)

We know that $s = c \tan \Psi$ … … …. …. …. (2)

Applying (2) at the point A, we have

$l = c. \tan \Psi_A$

$\therefore \operatorname{Tan} \Psi_A = \frac{l}{c} = \frac{2hl}{l^2-h^2}$ substituting for c from (1)

$$= \frac{2(\frac{h}{l})}{1-(\frac{h}{l})2} \text{ … … (3)}$$

But $\tan\Psi = \frac{2\tan\frac{\Psi}{2}}{1-\tan^2\frac{\Psi}{2}}$ … … (4)

Comparing (3) and (4), we find that

$\tan\frac{\Psi}{2}$ at A = $\frac{h}{l}$

$\therefore \frac{\Psi}{2} = \tan^{-1}\frac{h}{l}$

or Ψ at A = $2\tan^{-1}\frac{h}{l}$

The tension at A = $w.y_A$

$= w.(c + h)$

$$= w\left(\frac{l^2-h^2}{2h} + h\right) = \frac{w(l^2+h^2)}{2h}$$

Problem 3

A uniform chain of length l is to have its extremities fixed at two points in the same horizontal line. Show that the span must be $\frac{l}{\sqrt{8}} \log(3+\sqrt{8})$ in order that the tension at each support shall be three times that at the lowest point.

Solution:

Put n = 3 in problem number 13.

Problem 4

A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension A is twice that at the lowest point, show that the span AB is $\frac{l}{\sqrt{3}}\log(2+\sqrt{3})$

Solution:

Put n = 2 in problem number 13.

Problem 5

A uniform chain of length $2l$ hangs between two points A and B on the same level. The tension both at A and B is five times that at the lowest point. Show that the horizontal distance between A and B is $\frac{l}{\sqrt{6}}\log(5+2\sqrt{3})$

Solution:

Put n = 5 and length = $2l$ in problem number 13.

Problem 6

If T is the tension at any point P and T_0 is the tension at the lowest point C then prove that $T^2 - T_0^2 = W^2$ where W is the weight of the arc CP of the string.

Solution:

Given T is the tension at P. Let w be the weight per unit length and y is the ordinate of P.

Then $T = wy$.

Also $T_0 = wc$

$$\therefore T^2 - T_0^2 = w^2y^2 - w^2c^2$$
$$= w^2(y^2 - c^2)$$
$$= w^2s^2$$
$$= W^2$$

5. Suspension Bridges:

In the case of a suspension bridge the main load is the weight of the roadway. We have two chains hung up so as to be parallel, their ends being firmly fixed to supports. From different points of these chains, hang supporting chains or rods which carry the roadway of the bridge. These supporting rods are spaced at equal horizontal distances from one another and so carry equal loads. The weight of the chain itself and the weights of the supporting rods may be neglected in comparison with that of the horizontal roadway. The weight supported by each of the rods may therefore be taken to be the weight of equal portions of the roadway. Hence the figure of each chain of a suspension bridge approximates very closely to that of a parabola.