# MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

**Block No.8, College Road, Mogappair West, Chennai – 37**

**Affiliated to the University of Madras**
**Approved by the Government of Tamil Nadu**
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# DEPARTMENT OF MATHEMATICS

**SUBJECT NAME: ALGEBRA**

**SUBJECT CODE: SM21A**

**SEMESTER: I**

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## ALGEBRA

**YEAR: I** **SEMESTER: I**

**Learning Outcomes:**
**Students will acquire**

Basic ideas on Theory of Equations, Matrices and Theory of Numbers.
Knowledge to solve theoretical and applied problems.

**UNIT I**

Theory of Equations: Polynomial equations with Imaginary and irrational roots- Relation between roots and coefficients- Symmetric functions of roots in terms of coefficients.

**UNIT II**

Reciprocal equations - Standard form-Increase or Decrease the roots of the given equation -

**UNIT III**

Summation of Series : Binomial- Exponential -Logarithmic series (Theorems without proof):

**UNIT IV**

Symmetric- Skew Symmetric- Hermitian- Skew Hermitian- Orthogonal Matrices- Eigen values
& Eigen Vectors- Similar matrices- Cayley - Hamilton Theorem.

**UNIT V**

Prime number and Composite number - Divisors of a given number N- Euler's function (without proof) - Integral part of a real number - Congruences.

# UNIT - I THEORY OF EQUATION

## Theory of Equations:

**Every equation $f(x) = 0$ of the $n^{th}$ degree has** $n$ roots

Let $f(x)$ be the polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_n$.

n n roots

We assume that every equation $f(x) = 0$ has at least one root real or imaginary

Let $\alpha_1$ be a root of $f(x) = 0$.

Then $f(x)$ is exactly divisible by $x - \alpha_1$, so that

$$f(x) = (x - \alpha_1)\,\phi_1(x)$$

where $\phi_1(x)$ is a rational integral function of degree $n - 1$.

Again $\phi_1(x) = 0$ has a root real or imaginary and let that root be $a_2$.

Then $\phi_1(x)$ is exactly divisible by $x - \alpha_2$, so that

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rational integral function of degree $n - 2$.

$$\therefore f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x).$$

By continuing in this way, we obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)\,\phi_n(x)$$

where $\phi_n(x)$ is of degree $n - n$, i.e., zero

$\therefore \phi_n(x)$ is a constant.

Equating the coefficients of $x^n$ on both sides we get

$$\phi_n(x) = \text{coefficients of } x^n$$

$$= a_0$$

$\therefore f(x) = a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$.

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanished when x has any one of the values $\alpha_1, \alpha_2, \ldots \alpha_n$. If x is given any value different from any one of these $n$ roots, then no factor of $f(x)$ can vanish and the equation is not satisfied. Hence $f(x) = 0$ cannot have more than $n$ roots.

**Example. 1**. If α be a real root of the cubic equation $x^3 + px^2 + qx + r = 0$, of which the coefficients are real, show that the other two roots of the equation are real, if

$$p^2 \geq 4q + 2p\alpha + 3\alpha^2.$$

Solution.

Since α is a root of the equation, $x^3 + px^2 + qx + r$ is exactly divisible by $x - \alpha$.

∴ Let $x^3 + px^2 + qx + r \equiv (x - \alpha)(x^2 + ax + b)$.

Equating the coefficients of powers of x on both sides, we get

$$p = -\alpha + a$$

$$q = -a\alpha + b$$

$$r = -b\alpha$$

∴ $a = p + \alpha$ and $b = q + a\alpha = q + \alpha(p + \alpha)$

$$= q + p\alpha + \alpha^2.$$

The other two roots of the equation are the roots of

$x^2 + (p + \alpha)x + q + p\alpha + \alpha^2 = 0$

Which are real if $(p + \alpha)^2 - 4(q + p\alpha + \alpha^2) \geq 0$

i.e., $p^2 - 2p\alpha - 4q - 3\alpha^2 \geq 0$

i.e., $p^2 \geq 4q + 2p\alpha + 3\alpha^2$.

**Example 2**. If $x_1, x_2, x_3 \ldots x_n$ are the roots of the equation $(a_1 - x)(a_2 - x)\ldots(a_n - x) + k = 0$, then show that $a_1, a_2, \ldots, a_n$ are the roots of the equation

$$(x_1 - x)(x_2 - x)\ldots(x_n - x) - k = 0.$$

Solution.

Since $x_1, x_2, x_3 \ldots x_n$ are the roots of the equation

$$(a_1 - x)(a_2 - x)\ldots(a_n - x) + k = 0$$

We have

$(a_1 - x)(a_2 - x)\ldots(a_n - x) + k \equiv (x_1 - x)(x_2 - x)\ldots(x_n - x)$

∴ $(x_1 - x)(x_2 - x)\ldots(x_n - x) - k \equiv (a_1 - x)(a_2 - x)\ldots(a_n - x)$.

∴ $a_1, a_2, a_3 \ldots a_n$ are the roots of

$(x_1 - x)(x_2 - x)\ldots(x_n - x) - k = 0$.

**Example. 3**. Show that if $a$, $b$, $c$ are real, the roots of

$\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3}{x}$ are real.

Solution.

Simplifying we get

$$x(x+b)(x+c)+x(x+c)(x+a)+x(x+a)(x+b)$$
$$-3(x+a)(x+b)(x+c)=0$$

Let f($x$) be the expression on the left-hand side. It can easily be seen that $f$(x) is a quadratic function of .

$$\therefore f(-a)=-a(b-a)(c-a)$$
$$f(-b)=-b(c-b)(a-b)$$
$$f(-c)=-c(a-c)(b-c).$$

Without loss of generally let us assume that a> b> c and a, b, c are all positive. Then $a-b$, $b-c$, $a-c$ are positive.

$\therefore$ $f(-a) = -$ ve.

$f(-b) = +$ ve.

$f(-c) = -$ ve.

$\therefore$ The equation has at least one real root between $-a$ and $-b$, and another between $-b$ and $-c$.

The equation can have only two roots since $f(x) = 0$ is a quadratic equation.

$\therefore$ The roots of the equations are real.

**Exercises**

1. If $x^3 + 3px + q$ has a factor of the form $x^2 - 2ax + a^2$, show that $q^2 + 4p^3 = 0$.
2. If $px^3 + qx + r$ has a factor of the form $x^2 + ax + 1$, prove that $p^2 = pq + r^2$.
3. If $px^5 + qx^2 + r$ has a factor of the form $x^2 + ax + 1$, prove that
   $(p^2 - r^2)(p^2 - r^2 + qr) = p^2 q^2$.
4. If $a$, $b$, $c$ are all positive, show that all the roots of

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{x}$$ are real.

5. If $a > b > c > d$ and E, A, B, C, D are positive, show that the equation

$$E + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} = 0$$

has no root between $a$ and $b$, one root between $b$ and $c$ and one between $c$ and $d$ and if

$E > 0$, there is $a$ root $> d$ and if $E < 0$, there is a root $< a$.

6. If $a < b < c < d$, show that the roots of $(x - a)(x - c) = k(x - b)(x - d)$ are real for all values of k.

**In an equation with rational coefficients, imaginary roots occur in pairs.**

Let the equation be f(x) = 0 and let $\alpha + i\beta$ be an imaginary root of the equation. We shall show that $\alpha - i\beta$ is also a root.

We have $(x - \alpha - i\beta)(x - \alpha + i\beta) = (x - \alpha)^2 + \beta^2$ ………(1)

If f(x) is divided by $(x - \alpha)^2 + \beta^2$, let the quotient be Q(x) and the remainder be Rx + R'

Here Q(x) is of degree (n – 2).

$\therefore f(x) = \{(x - \alpha)^2 + \beta^2\} Q(x) + Rx + R'$ ………(2)

Substituting $(\alpha + i\beta)$ for x in the equation (2), we get

$f(\alpha + i\beta) = \{(\alpha + i\beta - \alpha)^2 + \beta^2\} Q(\alpha + i\beta) + R(\alpha + i\beta) + R'$

$= R(\alpha + i\beta) + R'$

But $f(\alpha + i\beta) = 0$ since $\alpha + i\beta$ is a root of f(x) = 0.

Therefore

$R(\alpha + i\beta) + R' = 0.$

Equating to zero the real and imaginary parts

$R\alpha + R' = 0$ and $R\beta = 0$.

Since $\beta \neq 0$, R = 0 and so R' = 0

$\therefore f(x) = \{(x - \alpha)^2 + \beta^2\}Q(x)$.

$\therefore \alpha - i\beta$ is also a root of f(x) = 0.

**Solved Problems**

**1.** Form a rational cubic equation which shall have for roots 1, $3 - \sqrt{-2}$.

Solution.

Since $3 - \sqrt{-2}$ is a root of the equation, $3 + \sqrt{-2}$ is also a root. So we

have to form an equation whose roots are 1, $3 - \sqrt{-2}$, $3 + \sqrt{-2}$.

Hence the required equation is $(x - 1)(x - 3 - \sqrt{-2})(x - 3 + \sqrt{-2}) = 0$

$$(x - 1)\{(x - 3)^2 + 2\} = 0$$

$$(x - 1)(x^2 - 6x + 11) = 0$$

$$x^3 - 7x^2 + 17x - 11 = 0.$$

**2.** Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ of which one root is $-1 + \sqrt{-1}$.

Solution.

Imaginary roots occur in pairs. Hence $-1 - \sqrt{-1}$ is also a root of the equation.

Therefore the expression on the left side of equation has the factors

$(x + 1 - \sqrt{-1})(x + 1 + \sqrt{-1})$.

The expression on the left side is exactly divisible by $(x + 1)^2 + 1$, i.e., $x^2 + 2x + 2$.

Dividing $x^4 + 4x^3 + 5x^2 + 2x - 2$ by $x^2 + 2x + 2$, we get the quotient $x^2 + 2x - 1$.

Therefore $x^4 + 4x^3 + 5x^2 + 2x - 2 = (x^2 + 2x + 2)(x^2 + 2x - 1)$.

Hence the other roots are obtained from $x^2 + 2x - 1 = 0$.

Thus the other roots are $-1 \pm \sqrt{2}$.

**3.** Show that $\frac{a^2}{x-\alpha}+\frac{b^2}{x-\beta}+\frac{c^2}{x-\gamma}-$ x + δ = 0 has only real roots if a, b, c, α, β, γ, δ are real.

Solution.

If possible let p + iq be a root. Then p – iq is also root.

Substituting these values for x, we have

$$\frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}+\frac{c^2}{p+iq-\gamma}-p-iq+\delta=0 \quad ……(1)$$

$$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}+\frac{c^2}{p-iq-\gamma}-p+iq+\delta=0 \quad ……(2)$$

Substituting (2) from (1), we get

$$-\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0$$

$$-2iq\left\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right\}=0$$

This is only possible when q = 0 since the other factor cannot be zero. In that case the roots are real.

**In an equation with rational coefficients irrational roots occur in pairs.**

Let f(x) = 0 denotes the equation and suppose that a + $\sqrt{b}$ is a root of the equation where a and b are rational and $\sqrt{b}$ is irrational. We now show that a – $\sqrt{b}$ is also a root of the equation

$$(x - a - \sqrt{b})(x - a + \sqrt{b}) = (x - a)^2 - b \quad …….. (1)$$

If f(x) is divided by $(x - a)^2 - b$, let the quotient be Q(x) and the remainder be Rx + R'.

Here Q(x) is a polynomial of degree (n – 2).

$$\therefore f(x) = \{(x - a)^2 - b\}\, Q(x) + Rx + R' \quad ……….(2)$$

Substituting a + $\sqrt{b}$ for x in (2), we get

$$f(a+\sqrt{b}) = \{(a+\sqrt{b}-a)^2 - b\}\, Q(a+\sqrt{b}) + R(a+\sqrt{b}) + R'$$

$$= R(a+\sqrt{b}) + R'$$

but $f(a+\sqrt{b}) = 0$, since $a+\sqrt{b}$ is a root of $f(x) = 0$.

$\therefore \; Ra + R' + R\sqrt{b} = 0.$

Equating the rational and irrational parts, we have

$Ra + R' = 0$ and $R = 0$.

$\therefore R' = 0.$

Hence $f(x) = \{(x-a)^2 - b\}Q(x)$.

$= (x - a - \sqrt{b})(x - a + \sqrt{b})Q(x)$.

$\therefore a - \sqrt{b}$ is a root of $f(x) = 0$.

**Solved Problems**

**Example 1.** Frame an equation with rational coefficients, one of whose root is $\sqrt{5}+\sqrt{2}$

Solution.

Then the other roots are $\sqrt{5}-\sqrt{2}, -\sqrt{5}+\sqrt{2}, -\sqrt{5}-\sqrt{2}$

Hence the required equation is $(x-\sqrt{5}-\sqrt{2})(x-\sqrt{5}+\sqrt{2})(x+\sqrt{5}+\sqrt{2})(x+\sqrt{5}-\sqrt{2}) = 0$

i.e. $\{(x-\sqrt{5})^2 - 2\}\{(x+\sqrt{5})^2 - 2\} = 0$

i.e. $(x^2 - 2x\sqrt{5} + 3)(x^2 + 2x\sqrt{5} + 3) = 0$

i.e. $(x^2 + 3)^2 - 4x^2.5 = 0$

i.e. $x^4 - 14x^2 + 9 = 0.$

**Example 2.** Solve the equation $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$ given that one of the roots is $1 -\sqrt{5}$ .

Solution.

Since the irrational roots occur in pairs, $1 +\sqrt{5}$ is also a root. The factors corresponding to these roots are

$(x - 1 +\sqrt{5})(x - 1 -\sqrt{5})$, i.e. $(x - 1)^2 - 5$

i.e. $x^2 - 2x - 4$.

Dividing $x^4 - 5x^3 + 4x^2 + 8x - 8$ by $x^2 - 2x - 4$, we get the quotient $x^2 - 3x + 2$.

Therefore $x^4 - 5x^3 + 4x^2 + 8x - 8 = (x^2 - 2x - 4)(x^2 - 3x + 2)$

$= (x^2 - 2x - 4)(x - 1)(x - 2)$

The roots of the equation are $1 \pm \sqrt{5}$, 1, 2.

**Example 3**. Form the equation with rational coefficients whose roots are

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

(ii) $-\sqrt{3} + \sqrt{-2}$.

Solution :

(i) $1 + 5\sqrt{-1}, 5 - \sqrt{-1}$

Then the other roots are $1 + 5\sqrt{-1}, 5 - \sqrt{-1}, 1 - 5\sqrt{-1}, 5 + \sqrt{-1}$

Hence the equation is

$(x -1 + 5\sqrt{-1})(x -1 - 5\sqrt{-1})(x - 5 - \sqrt{-1})(x - 5 + \sqrt{-1}) = 0$

$\{(x - 1)^2 - (5\sqrt{-1})^2\}\left\{(x - 5)^2 - \left(\sqrt{-1}\right)^2\right\} = 0$

$(x^2 - 2x +26)(x^2 - 10x + 26) = 0$

$x^4 - 12x^3 + 72x^2 - 312x + 676 = 0.$

(ii) $-\sqrt{3} + \sqrt{-2}$

Then the other roots are $-\sqrt{3} + \sqrt{-2}, -\sqrt{3} - \sqrt{-2}, \sqrt{3} + \sqrt{-2}, \sqrt{3} - \sqrt{-2}$

$$\left\{\left(x + \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\} \left\{\left(x - \sqrt{3}\right)^2 - \left(\sqrt{-2}\right)^2\right\} = 0$$

$$(x^2 + 2\sqrt{3}x + 5)(x^2 - 2\sqrt{3}x + 5) = 0$$

$$x^4 - 2x^2 + 25 = 0.$$

**Example 4.** Solve : $x^4 - 4x^3 + 8x + 35 = 0$ given that $2 + i\sqrt{3}$ is a root of it.

Solution.

Since the irrational roots occur in pair, $2 - i\sqrt{3}$ is also a root.

The factors corresponding to these roots are $(x - 2)^2 - (i\sqrt{3})^2$

$$x^2 - 4x + 7.$$

Dividing $x^4 - 4x^3 + 8x + 35$ by $x^2 - 4x + 7$, we get the equation $x^2 + 4x + 5$

$$x^4 - 4x^3 + 8x + 35 = (x^2 - 4x + 7)(x^2 + 4x + 5)$$

The roots of the equation are $2 \pm i\sqrt{3}, -2 \pm i$

**Example** 5.Solve the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$ given that one root is $\sqrt{2} - \sqrt{-1}$.

Solution.

Then the other roots are $\sqrt{2} - \sqrt{-1}, \sqrt{2} + \sqrt{-1}, -\sqrt{2} - \sqrt{-1}, -\sqrt{2} + \sqrt{-1}$

$$\left\{\left(x - \sqrt{2}\right)^2 - \left(\sqrt{-1}\right)^2\right\}\left\{\left(x + \sqrt{2}\right)^2 - \left(\sqrt{-1}\right)^2\right\} = 0$$

$$(x^2 - 2\sqrt{2}x + 3)(x^2 + 2\sqrt{2}x + 3) = 0$$

$$x^4 - 2x^2 + 9 = 0$$

Dividing $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81$ by $x^4 - 2x^2 + 9$ we get the equation $2x^2 - 3x + 9$

$$2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = (x^4 - 2x^2 + 9)(2x^2 - 3x + 9)$$

The roots of the equation are $\sqrt{2} \pm \sqrt{-1}, , -\sqrt{2} \pm \sqrt{-1}, 3\left(\frac{1 \pm i\sqrt{7}}{4}\right)$

Exercises

1. Find the equation with rational coefficients whose roots are
   (i) $4\sqrt{3}, 5 + 2\sqrt{-1}$ .
   (ii) $\sqrt{-1} - \sqrt{5}$.

2. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$ given that $1 + \sqrt{-1}$ is a root of it
3. Solve the equation $x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0$ given that one root is$\sqrt{2} - \sqrt{3}$.

   Answer : 1. (i)$x^4 - 10x^3 - 19x^2 + 480x - 1392= 0$,(ii) $x^4 - 8x^2 + 36 = 0$, 2. $-2 \pm \sqrt{3}$, 3. $2 \pm\sqrt{3}$, $2 \pm\sqrt{5}$.

**Relation between the roots and coefficient of equations.**

Let the equation be $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$.If this equation has the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then we have

$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

$= (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$

$= x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1\alpha_2 x^{n-2} - \ldots + (-1)^n \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

$= x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^nS_n$

Where $S_r$ is the sum of the products of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ taken r at a time.

Equating the coefficients of like powers on both sides, we have

$-p_1 = S_1$ = sum of the roots.

$(-1)^2p_2 = S_2$ = sum of the products of the roots taken two at a time.

$(-1)^3p_3 = S_3$ = sum of the products of the roots taken three at a time.

$(-1)^np_n = S_n$ = product of the roots.

If the equation is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$.

Divide each term of the equation by $a_0$.

The equation becomes $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$

and so we have

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

$$\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

These n equations are of no help in the general solution of an equation but they are often helpful in the solution of numerical equations when some special relation is known to exist among the roots. The method is illustrated in the examples given below.

**Example 1.** Show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Arithmetical progression if $2p^3 - 9pq + 27r = 0$ show that the above condition is satisfied by the equation $x^3 - 6x^2 + 13x - 10 = 0$. Hence or otherwise solve the equation.

Solution.

Let the roots of the equation $x^3 + px^2 + qx + r = 0$ be $\alpha - \delta, \alpha, \alpha + \delta$.

We have from the relation of the roots and coefficients

$$\alpha - \delta + \alpha + \alpha + \delta = -p$$

$$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$$

$$(\alpha - \delta)\alpha(\alpha + \delta) = -r.$$

Simplifying these equation, we get

$$3\alpha = -p \qquad \ldots(1)$$

$$3\alpha^2 - \delta^2 = q \qquad \ldots(2)$$

$$\alpha^3 - \alpha\delta^2 = -r. \qquad \ldots(3)$$

From (1), $\alpha = -\frac{p}{3}$.

From (2), $\delta^2 = 3\left(-\frac{p}{3}\right)^2 - q = \frac{p^2}{3} - q$ .

Substituting these value in (3), we get

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

i.e.,$2p^3 - 9pq + 27r = 0$.

In the equation $x^3 - 6x^2 + 13x - 10 = 0$.

$p = -6, q = 13, r = -10$.

Therefore $2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)13 + 27(-10) = 0$

The condition is satisfied and so the roots of the equation are in arithmetical progression. In this case the equations (1), (2), (3) become

$3\alpha = 6$

$3\alpha^2 - \delta^2 = 13$

$\alpha^3 - \alpha\delta^2 = 10.$

$\alpha = 2$ , $12 - \delta^2 = 13$

Therefore $\delta^2 = -1$

i.e., $\delta = \pm i$.

The roots are $2 - i, 2, 2 + i$.

**Example 2.** Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression.

Solution.

Let the roots of the equation be $\frac{k}{r}$, k, kr.

Therefore $\frac{k}{r} + k + kr = -\frac{3b}{a}$ …..(1)

$\frac{k^2}{r} + k^2 + k^2r = \frac{3c}{a}$ ….(2)

$k^3 = -\frac{d}{a}$ …..(3)

From (1), $k\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}$.

From (2), $k^2\left(\frac{1}{r} + 1 + r\right) = \frac{3c}{a}$.

Divided one by the other, we get $k = -\frac{c}{b}$

Substituting this value of k in (3), we get $\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$.

Therefore $ac^3 = b^3d$.

In the equation $27x^3 + 42x^2 - 28x - 8 = 0$

$\frac{k}{r} + k + kr = -\frac{42}{27}$

$\frac{k^2}{r} + k^2 + k^2r = -\frac{28}{27}$

$k^3 = \frac{8}{27}$

$\therefore \quad k = \frac{2}{3}$.

Substituting the value of k in(4), we get

$$\frac{2}{3}\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$$

$$3r^2 + 10r + 3 = 0$$

$$(3r + 1)(r + 3) = 0$$

Therefore $r = -\frac{1}{3}$ or $r = -3$.

For both the value of r, the roots are $-2, \frac{2}{3}, -\frac{2}{9}$.

**Example 3.** Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression.

Solution.

Let the roots be α, β, γ.

Then $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

i.e., $2\gamma\alpha = \beta\gamma + \alpha\beta$ ……(1)

From the relation between the coefficients and the roots we have

$$\alpha + \beta + \gamma = \frac{18}{81} \quad \text{…….(2)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{36}{81} \quad \text{……..(3)}$$

$$\alpha\ \beta\ \gamma = -\frac{8}{81} \quad \text{….(4)}$$

From (1) and (3), we get

$$2\gamma\alpha + \gamma\alpha = -\frac{36}{81}$$

$$3\gamma\alpha = -\frac{36}{81}$$

Therefore $\gamma\alpha = -\frac{4}{27}$ …….(5)

Substituting this value of γα in (4), we get

$$\beta\left(-\frac{4}{27}\right) = -\frac{8}{81}$$

Therefore $\beta = \frac{2}{3}$.

From (2), we have

$$\alpha + \gamma = \frac{18}{81} - \frac{2}{3} = -\frac{4}{9} \quad \text{……(6)}$$

From (5)and (6), we get

$\alpha = \frac{2}{9}$ and $\gamma = -\frac{2}{3}$

The roots are $\frac{2}{9}, \frac{2}{3}$ and $-\frac{2}{3}$.

**Example 4.** If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Solution.

Let the roots of the equation be α, β, γ and δ

Then $\alpha + \beta = \gamma + \delta$ …..(1)

From the relation of the coefficients and the roots, we have

$\alpha + \beta + \gamma + \delta = -p$ ………(2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ ………..(3)

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$ ………..(4)

$\alpha\beta\gamma\delta = s$ ………..(5)

From (1) and (2), we get

$2(\alpha + \beta) = -p$ ………(6)

(3) can be written as

$\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$

i.e., $(\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q$ ……(7)

(4) can be written as

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$

$(\alpha\beta + \gamma\delta)(\alpha + \beta) = -r$ ……(8)

From (6) and (7), we get

$\alpha\beta + \gamma\delta + \frac{p^2}{4} = q$

$\therefore \quad \alpha\beta + \gamma\delta = q - \frac{p^2}{4}$ …..(9)

From (8), we get

$$-\frac{p}{2}(\alpha\beta + \gamma\delta) = -r$$

$$\alpha\beta + \gamma\delta = \frac{2r}{p} \qquad \ldots\ldots\ (10)$$

Equating (9) and (10), we get

$$q - \frac{p^2}{4} = \frac{2r}{p}$$

$$4pq - p^3 = 8r$$

$$p^3 + 8r = 4pq.$$

**Example 5.** Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.

Solution.

Let the roots of the equation be α, β, γ and δ

Here $\gamma = -\delta$

i.e., $\gamma + \delta = 0$ ……(1)

From the relation of the roots and coefficients

$$\alpha + \beta + \gamma + \delta = 2 \qquad \ldots\ldots\ldots(2)$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 4 \qquad \ldots\ldots\ldots..(3)$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -6 \qquad \ldots\ldots\ldots..(4)$$

$$\alpha\beta\gamma\delta = -21 \qquad \ldots\ldots\ldots..(5)$$

from (1) and (2), we get $\alpha + \beta = 2$ …….(6)

(3) can be written as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 4$

$$\alpha\beta + \gamma\delta = 4 \qquad \ldots\ldots(7)$$

(4) can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -6$

$\gamma\delta(\alpha + \beta) = -6$ ..........(8)

from (6) and (8), we get $\gamma\delta = -3$ …..(9)

but $\gamma + \delta = 0$ $\therefore$ $\gamma = \sqrt{3}$ , $\delta = -\sqrt{3}$.

From (7) and (9), we get $\alpha\beta = 7$

$\therefore$ $\alpha$ and $\beta$ are the roots of x2 – 2x + 7 = 0.

$\therefore$ $\alpha = 1 + \sqrt{-6}$ , $\beta = 1 - \sqrt{-6}$

Therefore the roots of the equation are $\pm\sqrt{3}$, $1 \pm\sqrt{-6}$.

**Example 6**. Find the condition that the general bi quadratic equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have two pairs of equal roots.

Solution.

Let the roots be $\alpha, \alpha, \beta, \beta$.

From the relations of coefficients and roots

$2\alpha + 2\beta = -\frac{4b}{a}$ …………….(1)

$\alpha^2 + \beta^2 + 4\alpha\beta = \frac{6c}{a}$ …………(2)

$2\alpha\beta^2 + 2\alpha^2\beta = -\frac{4d}{a}$ …………….(3)

$\alpha^2\beta^2 = \frac{e}{a}$ …………(4)

From (1), we get $\alpha + \beta = -\frac{2b}{a}$ ……..(5)

From (3), we get $2\alpha\beta(\alpha + \beta) = -\frac{4d}{a}$

$\therefore \alpha\beta = \frac{d}{b}$ ………(6)

From (5) and (6), we get that $\alpha, \beta$ are the roots of the equation $x^2 + \frac{2b}{a}x + \frac{d}{b} = 0$

$$\therefore ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2$$

Comparing coefficients

$$6c = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}$$

$\therefore$ $3abc = a^2d + 2b^3$ and $eb^2 = ad^2$.

Exercises

1. Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$ whose roots are in harmonic progression.
2. Find the values of a and b for which the roots of the equation $4x^4 - 16x^3 + ax^2 + bx - 7 = 0$ are in arithmetical progression.
3. The roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in geometrical progression. Find them.
4. Solve $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, it being given that the sum of two of the roots is equal to the sum of the other two.
5. If two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ are equal in value but differ in sign, show that $r^2 + p^2s = pqr$.
6. Show that the four roots, α, β, γ and δ of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ will be connected by the relation $\alpha\beta + \gamma\delta = 0$ if $p^2s + r^2 = 4qs$.
7. Solve the equation $x^4 - 2x^3 - 3x^2 + 4x - 1 = 0$given that the product of two of the roots is unity.

   Answer : 1.1, $\frac{1}{2}$, $\frac{1}{3}$, 2.a = 4 or $-\frac{4}{9}$, b = 24 or $\frac{296}{9}$, 3. $\frac{1}{4}$, $\frac{1}{2}$, 1, 4. – 1, 5, 1, 3, 7. $\frac{3\pm\sqrt{5}}{2}$, $\frac{-1\pm\sqrt{5}}{2}$

**Symmetric function of the roots**

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ be the roots of the equation.

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots. + p_n = 0.$$

We have learned that

$$S_1 = \Sigma\, \alpha_1 = -\, p_1$$

$$S_2 = \Sigma\, \alpha_1\alpha_2 = p_2$$

$S_3 = \Sigma\, \alpha_1 \alpha_2 \alpha_3 = -\, p_3$

……………………..

……………………..

Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

**Example 1.** If α, β, γ are the roots of the equations $x^3 + px^2 + qx + r = 0$, Express the value of $\Sigma\, \alpha^2 \beta$ in terms of the coefficients.

Solution.

We have $\alpha + \beta + \gamma = -\, p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -\, r.$

$$\Sigma\, \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)\, (\alpha + \beta + \gamma) - 3\alpha\beta\gamma$$

$$= q\, (-\, p) - 3\, (-\, r)$$

$$= 3r - pq.$$

**Example 2.** If α, β, γ, δ be the roots of the bi quadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, Find (1) $\Sigma\, \alpha^2$ , (2) $\Sigma\, \alpha^2\, \beta\gamma$, (3) $\Sigma\, \alpha^2\, \beta^2$ , (4) $\Sigma\, \alpha^3\, \beta$ and (5) $\Sigma\, \alpha^4$.

Solution.

The relations between the roots and the coefficients are

$\alpha + \beta + \gamma + \delta = -\, p.$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\, r$

$$\alpha\beta\gamma\delta = s.$$

$$\Sigma\, \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (\Sigma\, \alpha)^2 - 2\, \Sigma\, \alpha\beta$$

$$= p^2 - 2q.$$

$$\Sigma\, \alpha^2\, \beta\gamma = (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)\,(\alpha + \beta + \gamma + \delta) - 4\, \alpha\beta\gamma\delta$$

$$= (\Sigma\, \alpha\beta\gamma)\,(\Sigma\, \alpha) - 4\, \alpha\beta\gamma\delta$$

$$= pr - 4s.$$

$$\Sigma\, \alpha^2\, \beta^2 = \alpha^2\, \beta^2 + \alpha^2\, \gamma^2 + \alpha^2\, \delta^2 + \beta^2\, \gamma^2 + \beta^2\, \delta^2 + \gamma^2\, \delta^2$$

$$= (\Sigma\, \alpha\beta)^2 - 2\, \Sigma\, \alpha^2\, \beta\gamma - 6\, \alpha\beta\gamma\delta$$

$$= q^2 - 2\,(pr - 4s) - 6s$$

$$= q^2 - 2pr + 2s.$$

$$\Sigma\, \alpha^3\, \beta = (\Sigma\, \alpha^2)\,(\Sigma\, \alpha\beta) - \Sigma\, \alpha^2\, \beta\gamma$$

$$= (p^2 - 2q)\, q - (pr - 4s)$$

$$= p^2q - 2q^2 - pr + 4s.$$

$$\Sigma\, \alpha^4 = (\Sigma\, \alpha^2)^2 - 2\, \Sigma\, \alpha^2\, \beta^2$$

$$= (p^2 - 2q)^2 - 2\,(q^2 - 2pr + 2s)$$

$$= p^4 - 4p^2q + 2q^2 + 4pr - 4s.$$

**Example 3.** If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, from the equation whose roots are αβ, βγ, and γα.

Solution.

The relations between the roots and coefficients are

$$\alpha + \beta + \gamma = -a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\alpha\beta\gamma = -c.$$

The required equation is

$$(x - \alpha\beta)(x - \beta\gamma)(x - \gamma\alpha) = 0$$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)x - \alpha^2\beta^2\gamma^2 = 0$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha) + x\,\alpha\beta\gamma(\alpha + \beta + \gamma) - (\alpha\beta\gamma)^2 = 0$

i.e., $x^3 - bx^2 + acx - c^2 = 0$

**Example 4.** If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, from the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$.

Solution.

We have $\alpha + \beta + \gamma = -p$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r.$$

In the required equation

$S_1$ = Sum of the roots $= \beta + \gamma - 2\alpha + \gamma + \alpha - 2\beta + \alpha + \beta - 2\gamma$

$= 0.$

$S_2$ = Sum of the products of the roots taken two at a time

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta) + (\beta + \gamma - 2\alpha)(\alpha + \beta - 2\gamma) + (\alpha + \beta - 2\gamma)(\gamma + \alpha - 2\beta)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta) + 2$ similar terms

$= (-p - 3\alpha)(-p - 3\beta) + (-p - 3\alpha)(-p - 3\gamma) + (-p - 3\gamma)(-p - 3\beta)$

$= (p + 3\alpha)(p + 3\beta) + (p + 3\alpha)(p + 3\gamma) + (p + 3\gamma)(p + 3\beta)$

$= 3p^2 + 6p(\alpha + \beta + \gamma) + 9(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 3p^2 + 6p(-p) + 9q$

$= 9q - 3p^2.$

$S_3$ = Products of the roots

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)(\alpha + \beta + \gamma - 3\gamma)$

$= (-p - 3\alpha)(-p - 3\beta)(-p - 3\gamma)$

$= -\{p^3 + 3p^2(\alpha + \beta + \gamma) + 9p(\alpha\beta + \beta\gamma + \gamma\alpha) + 27\alpha\beta\gamma\}$

$= -\{p^3 + 3p^2(-p) + 9pq - 27r\}$

$= 2p^2 - 9pq + 27r$

Hence the required equation is

$x^3 - S_1x^2 + S_2x - S_3 = 0$

i.e., $x^3 + (9q - 3p^2)x - (2p^3 - 9pq + 27r) = 0.$

**Example 5.** If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$ prove that

(1) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = r - pq$
(2) $\alpha^3 + \beta^3 + \gamma^3 = -p^3 + 3pq - 3r.$

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

(1). $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = [-(p + \alpha)(p + \beta)(p + \gamma)]$

Since $\alpha + \beta + \gamma = -p$ $\therefore \alpha + \beta = -p - \gamma$

$$= -[p^3 + p^2(\alpha + \beta + \gamma) + p(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma]$$

$$= -[p^3 + p^2 \times -p + pq - r] = -[p^3 - p^3 + pq - r] = r - pq.$$

(2). $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)]$

$\sum \alpha^3 = \sum \alpha\,[\sum \alpha^2 - \sum \alpha\beta] + 3\alpha\beta\gamma;$

But $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

Therefore $\sum \alpha^3 = \sum \alpha\,[(\sum \alpha)^2 - 3\sum \alpha\beta] + 3\alpha\beta\gamma;$ $= -p[p2 - 3q] - 3r = -p3 + 3pq - 3r.$

**Example 6.** If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ find the values of

(1) $\sum \frac{1}{\beta+\gamma}$.

(2) $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma}$

Solution.

Since α, β, γ are the roots of the equation $x^3 + qx + r = 0$.

We have $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

Therefore $\beta + \gamma = -\alpha$

(1). $\sum \frac{1}{\beta+\gamma} = \sum \frac{1}{-\alpha} = -\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] = -\frac{\sum \alpha\beta}{\alpha\beta\gamma} = \frac{-q}{-r} = \frac{q}{r}$

(2). $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \sum \frac{(\beta+\gamma)^2 - 2\beta\gamma}{\beta+\gamma} = \frac{\sum\left[\alpha^2 + 2\frac{r}{\alpha}\right]}{-\alpha} = \frac{\sum \alpha^3 + 2r}{-\alpha^2} = -\sum \alpha - 2\sum \frac{r}{\alpha^2}$

$= -2r\sum \frac{1}{\alpha^2}$; since $\sum \alpha = 0$

But $\sum \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{(\sum \alpha\beta)^2}{(\alpha\beta\gamma)^2}$ since $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \sum \alpha^2\beta^2 + 2\alpha\beta\gamma\sum \alpha = \sum \alpha^2\beta^2$; since $\sum \alpha = 0$

$\sum \alpha^2\beta^2 = q^2$ ;$\sum \frac{1}{\alpha^2} = \frac{q^2}{r^2} = \frac{q^2}{r^2}$

$\therefore \sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{-2q^2r}{r^2} = \frac{-2q^2}{r}$.

**Example 7.** If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$ find the value of

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$

(2). $\sum(\beta + \gamma - \alpha)^2$.

Solution.

Since α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$

We have $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$.

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} = \frac{\alpha(\beta^2+\gamma^2)+\gamma(\alpha^2+\beta^2)+\beta(\alpha^2+\gamma^2)}{\alpha\beta\gamma}$

$= \frac{\sum \alpha^2\beta}{\alpha\beta\gamma}$

But $\sum \alpha^2\beta = (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$\frac{\sum \alpha^2\beta}{\alpha\beta\gamma} = \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma}{\alpha\beta\gamma} = \frac{qp-3r}{r}$.

(2). $\sum(\beta + \gamma - \alpha)^2 = \sum(\alpha + \beta + \gamma - 2\alpha)^2 = \sum(p - 2\alpha)^2 = \sum(p^2 + 4\alpha^2 - 4\alpha\beta)$

$= 3p^2 + 4\sum \alpha^2 - 4p\sum \alpha\beta$

$= 3p^2 + 4\left[\left(\sum \alpha\right)^2 - 2\sum \alpha\beta\right] - 4p^2$

$= 3p^2 + 4p^2 - 8q - 4p^2$

$= 3p^2 - 8q$.

**Example 8.** If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$find the value of

$$\Sigma \frac{1}{\alpha^2\beta^2}$$

Solution.

Since α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$

We have $\alpha + \beta + \gamma = \frac{-b}{a}$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\Sigma \frac{1}{\alpha^2\beta^2} = \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{(\alpha\beta\gamma)^2} = \frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\left(\frac{d}{a}\right)^2}$$

$$= \frac{b^2-2ac}{d^2}$$

**Exercises**

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ find the value of

   (1) $(\beta + \gamma - \alpha)^3 + (\gamma + \alpha - \beta)^3 + (\alpha + \beta - \gamma)^3$.

   (2) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$.

2. If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$,

   Evaluate (1) $\Sigma\, \alpha^2\beta\gamma$, (2) $\Sigma\, (\beta + \alpha + \delta)^2$ and (3) $\Sigma\, \frac{1}{\alpha^2}$.

   Answer : 1. (1).$24r - p^3$, (2).$\frac{2rp-q^2}{r}$ , 2. (1).$pr - 4s$ , (2).$3p^2 - 2q$,(3).$\frac{r^2-2qr}{s}$

# UNIT - II RECIPROCAL EQUATION

**Reciprocal roots.**

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n = 0.$$

We have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots p_n \equiv (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n).$$

Put $x = \frac{1}{y}$, we have

$$\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \ldots\ldots + p_n$$

$$= \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\ldots\ldots\left(\frac{1}{y} - \alpha_n\right)$$

Multiplying throughout by $y^n$, we have

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0$$

$$= (\alpha_1\ \alpha_2 \ldots\ldots \alpha_n)\left(\frac{1}{\alpha_1} - y\right)\left(\frac{1}{\alpha_2} - y\right)\ldots\ldots\left(\frac{1}{\alpha_n} - y\right)$$

Hence the equation

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \ldots\ldots + p_1y + 1 = 0 \text{ has roots } \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$$

**Reciprocal equation.**

If an equation remains unaltered when x is changed into its reciprocal, it is called reciprocal equation.

Let $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0.$ ......(1)

be a reciprocal equation. When x is changed into its reciprocal $\frac{1}{x}$, we get the transformed equation

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \ldots\ldots + p_1x + 1 = 0$$

$$x^n + \frac{p_{n-1}}{p_n}x^{n-1} + \frac{p_{n-2}}{p_n}x^{n-2} + \ldots. + \frac{p_1}{p_n}x + \frac{1}{p_n} = 0 \quad \ldots.(2)$$

Since (1) is a reciprocal equation , it must be the same as (2),

$$\therefore \ \frac{p_{n-1}}{p_n} = p_1, \frac{p_{n-2}}{p_n} = p_2 \ldots \frac{p_1}{p_n} = p_{n-1} \text{ and } \frac{1}{p_n} = p_n.$$

$$\therefore \ p_n^2 = 1.$$

$$\therefore \ p_n = \pm 1.$$

**Case i.** $p_n = 1.$

Then $p_{n-1} = p_1, p_{n-2} = p_2, p_{n-3} = p_3, \ldots\ldots$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

**Case ii.** $p_n = -1$, we have

$$p_{n-1} = -p_1, p_{n-2} = -p_2, \ldots. p_1 = -p_{n-1}.$$

In this case the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

**Standard form of reciprocal equations.**

If α be a root of a reciprocal equation, $\frac{1}{\alpha}$ must also be a root, for it is a root of the transformed equation and the transformed equation is identical with the first equation, Hence the roots of a reciprocal equation occur in pairs

$$\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \ldots$$

When the degree is odd one of its roots must be its own reciprocal.

$$\gamma = \frac{1}{\gamma}$$

i.e., $\gamma^2 = 1$.

i.e., $\gamma = \pm 1$.

If the coefficients have all like signs, then $-1$ is a root ; if the coefficients of the terms equidistant from the first and last have opposite signs, then + 1 is a root. In either case the degree of an equation can be depressed by unity if we divide the equation by x +1 or by x – 1. The depressed equation is always a reciprocal equation of even degree with like signs for its coefficients.

If the degree of a given reciprocal equation is even , say n = 2m and if terms equidistant from the first and last have opposite signs, then

$$p_m = - p_m.$$

i.e., $p_m = 0$ , so that in this type of reciprocal equations, the middle term is absent. Such an equation may be written as

$$x^{2m} - 1 + p_1 x(x^{2m-2} - 1) + \ldots\ldots 0 .$$

Dividing by $x^2 - 1$ , this reduces to a reciprocal equation of like signs of even degree. Hence all reciprocal equations may be reduced to an even degree reciprocal equation with like sign, and so an even degree reciprocal equation with like signs is considered as the standard form of reciprocal equations.

**A reciprocal equation of the standard form can always be depressed to another of half the dimensions.**

It has been shown in the previous article that all reciprocal equations can be reduced to a standard form, in which the degree is even and the coefficients of terms equidistant from the beginning and the end are equal and have the same sign.

Let the standard reciprocal equation be

$$a_0 x^{2m} + a_1 x^{2m-1} + a_2 x^{2m-2} + \ldots\ldots a_m x^m + \ldots + a_1 x + a_0 = 0.$$

Dividing by $x^m$ and grouping the terms equally distant from the ends, we have

$$a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots + a_{m-1}\left(x + \frac{1}{x}\right) + a_m = 0$$

Let $x + \frac{1}{x} = z$ and $x^r + \frac{1}{x^r} = X_r$

We have the relation $X_{r+1} = z \cdot X_r - X_{r-1}$.

Giving r in succession the values 1, 2, 3, …

We have $X_2 = z X_1 - X_0 = z^2 - 2$

$X_3 = z X_2 - X_1 = z^3 - 3z$

$X_4 = z X_3 - X_2 = z^4 - 4z^2 + 2$

$X_5 = z X_4 - X_3 = z^5 - 5z^3 + 5z$

and so on. Substituting these values in the above equation. We get an equation of the $m^{th}$ degree in *z*. To every root of the reduced equation in *z*, correspond two roots of the reciprocal equation. Thus if *k* be a root of the reduced equation, the quadratic $x + \frac{1}{x} = k$, i.e., $x^2 - kx + 1 = 0$ gives the two corresponding roots $\frac{k \pm \sqrt{k^2-4}}{2}$ of the given reciprocal equation.

**Example 1.** Find the roots of the equation $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$.

Solution.

This is a reciprocal equation of odd degree with like signs.

$\therefore$ (x+1) is a factor of $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1$

The equation can be written as

$$x^5 + x^4 + 3x^4 + 3x^3 + 3x^2 + 3x + x + 1 = 0$$

i.e., $x^4(x+1) + 3x^3(x+1) + 3x(x+1) + 1(x+1) = 0$

i.e., $(x+1)(x^4 + 3x^3 + 3x + 1) = 0$.

$\therefore$ x +1 = 0 or $x^4 + 3x^3 + 3x + 1 = 0$.

Dividing by $x^2$, we get $x^2 + 3x + \frac{3}{x} + \frac{1}{x^2} = 0$

$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$ .

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$

$\therefore$ $z^2 - 2 + 3z = 0$

$\therefore$ $z = \frac{-3 \pm \sqrt{17}}{2}$ .

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

i.e., $2x^2 + \left(-3 + \sqrt{17}\right) x + 2 = 0$

or $2x^2 + \left(-3 - \sqrt{17}\right) x + 2 = 0$.

From these equations x can be found.

**Example 2.** Solve the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.

Solution.

This is a reciprocal equation of odd degree with unlike signs.

Hence x – 1 is a factor of the left- hand side.

The equation can be written as follows:

$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^2 + 5x^2 - 5x + 6x - 6 = 0$

i.e., $6x^4 (x - 1) + 5x^3(x - 1) - 38x^2 (x - 1) + 5x(x - 1) + 6 (x - 1) = 0$

i.e., $(x - 1) (6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore$ x – 1 = 0 or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ .

We have to solve the equation $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

Dividing by $x^2$, $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

i.e., $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$

Put $x + \frac{1}{x} = z$. $\therefore\ x^2 + \frac{1}{x^2} = z^2 - 2.$

The equation becomes

$$6(z^2 - 2) + 5z - 38 = 0$$

i.e., $6z^2 + 5z - 50 = 0$

i.e., $(2z-5)(3z+10) = 0.$

$\therefore\quad x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = -\frac{10}{3}$

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 + 10x + 3 = 0$

i.e., $(2x - 1)(x - 2) = 0$ or $(3x + 1)(x + 3) = 0$

i.e., $x = \frac{1}{2}$ or 2 or $-\frac{1}{3}$ or $-3$.

$\therefore$ The roots of the equation are $1, \frac{1}{2}, 2, -\frac{1}{3}$ and $-3$.

**Example 3.** Solve the equation $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0.$

Solution.

There is no mid-term and this is a reciprocal equation of even degree with unlike signs. We can easily see that $x^2 - 1$ is a factor of the expression on left-hand side of the equation.

The equation can be written as

$$6(x^6 - 1) - 35x(x^4 - 1) + 56x^2(x^2 - 1) = 0$$

i.e., $6(x^2 - 1)(x^4 + x^2 + 1) - 35x(x^2 - 1) + (x^2 + 1)) + 56x^2(x^2 - 1) = 0$

i.e., $(x^2 - 1)(6x^4 - 35x^3 + 62x^2 - 35x + 6) = 0$

i.e., $x = 1$ or $-1$ or $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

Dividing by $x^2$, we get $6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$.

$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0$.

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$.

$\therefore$ $6(z^2 - 2) - 35z + 62 = 0$

i.e., $6z^2 - 35z - 50 = 0$

i.e., $(3z - 10)(2z - 5) = 0$

$z = \frac{10}{3}$ or $\frac{5}{2}$.

$\therefore$ $x + \frac{1}{x} = \frac{10}{3}$ or $x + \frac{1}{x} = \frac{5}{2}$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(x - 3)(3x - 1) = 0$ or $(x - 2)(2x - 1) = 0$

i.e., $x = 3$ or $\frac{1}{3}$ or $2$ or $\frac{1}{2}$

$\therefore$ The roots of the equation are $1, -1, 3, \frac{1}{3}, 2$ and $\frac{1}{2}$.

**Exercises**

Solve the following equations:-

1. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.
2. $x^4 + 3x^3 - 3x - 1 = 0$.
3. $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$.
4. $2x^5 + x^4 + x + 1 = 12x^2(x + 1)$.
5. $x^5 - 5x^3 + 5x^2 - 1 = 0$.

6. $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0$.

Answer : 1.3 ± √8, 2 ± √3, 2. ± 1, $\frac{-3\pm\sqrt{5}}{2}$, 3. 2, $\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}, \frac{-1\pm\sqrt{-3}}{2}$, 4. $-1, -2, -\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}$,

5.1, 1, 1, $\frac{-3\pm\sqrt{5}}{2}$, 6. ± 1, $\frac{-1\pm i\sqrt{3}}{2}, \frac{-1\pm i\sqrt{3}}{2}$.

**Transformation in general.**

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equations f (x) = 0, it is required to find an equation whose roots are

$$\phi(\alpha_1), \phi(\alpha_2), \ldots, \phi(\alpha_n).$$

The relation between a root x of f (x) = 0 and a root y of the required equation is y = $\phi$ (x).

Now if x be eliminated between f (x) = 0 and y = $\phi$ (x), an equation in y is obtained which is the required equation.

By means of the relations between the roots and coefficients of an equation we can establish a relation between the corresponding roots given and the required equations. A few examples will illustrate the methods of procedure.

**Example 1.** If α , β , γ are the roots of the equation $x^3 + px^2 + qx + r \equiv 0$, from the equation whose roots are $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$.

Solution.

We have $\alpha - \frac{1}{\beta\gamma}$

$= \alpha - \frac{\alpha}{\alpha\beta\gamma}$

$= \alpha - \frac{\alpha}{-r}$ since $\alpha\beta\gamma = -r$

$= \alpha + \frac{\alpha}{r}$.

$\therefore$ y $= x + \frac{x}{r}$.

∴ The required equation is obtained by eliminating x between the equations

**Horner's method**

This method can be used to determine both the commensurable and the incommensurable roots of a numerical equation. First we shall explain the method for obtained the positive root. The procedure is to determine the root figure by figure, first the integral part and then the first decimal place, then the second decimal place and so on until the root terminates or the root has been obtained to the required degree of approximation. The main principle involved in this method is diminishing the roots by certain known quantities by successive transformations. In this method the successive transformations can be exhibited in a compact form and the roots can be obtained to any number of places of decimals required.

First we have to find by trial two consecutive integers between which a real positive root of the equation lies. This will give the integral part of the root. Let it be a first diminish all the roots of the equation by a. Then the transformed equation will have a root between 0 and 1. In order to avoid decimal in the working, all the roots of this transformed equation are multiplied by 10. Then the new transformed equation has a root between 0 and 10. By trial find the integers between which the root lies and thus find the integral part of the root. Let it be b. Then diminish the roots be b and again multiply the roots by 10 and continue the process till we get the root to the number of decimal we required.

**Example 1.** The equation $x^3 - 3x + 1 = 0$ has a root between 1 and 2. Calculate it to three places of decimals.

Solution.

Since the roots lies between 1 and 2, the integral part of the root is 1. Diminish the root of the equation by 1.

| | | | | |
|---|---|---|---|---|
| | 1 | 0 | – 3 | 1 |
| (1 | | | | |
| | | 1 | 1 | – 2 |
| | | 1 | – 2 | - 1 |
| | | 1 | 2 | |
| | | 2 | 0 | |

$$\frac{1}{3}$$

The transformed equation is $x^3 + 3x^2 - 1 = 0$

This equation has therefore a root between 0 and 1.

Multiply the roots of this equation by 10.

Then the equation transforms into $x^3 + 30x^2 - 1000 = 0$

We can easily see that a root of this equation lies between 5 and 6. Diminish the roots of the equation by 5.

| | | | | |
|---|---|---|---|---|
| (5 | 1 | 30 | 0 | – 1000 |
| | | 5 | 175 | 875 |
| | | 35 | 175 | – 125 |
| | | 5 | 200 | |
| | | 40 | 375 | |
| | | 5 | | |
| | | 45 | | |

The transformed equation is $x^3 + 45x^2 + 375x - 125 = 0$.

This equation has therefore a root between 0 and 1.

Multiply the roots of the equation by 10.

Then the equation transforms into $x^3 + 450x^2 + 37500x - 125000 = 0$.

We can easily see that a root of this equation lies between 3 and 4.

Diminish the roots of this equation by 3.

| | | | | |
|---|---|---|---|---|
| 1 | 450 | 37500 | – 125000 | (3 |

| | | |
|---|---|---|
| 3 | 1359 | 116577 |
| 453 | 38859 | – 8423 |
| 3 | 1368 | |
| 456 | 40227 | |
| 3 | | |
| 459 | | |

The transformed equation is $x^3 + 459x^2 + 40227x – 8423 = 0$.

Multiply the roots be 10.

Then the equation transforms into $x^3 + 4590x^2 + 4022700x – 8423000 = 0$.

We can easily see that a root of this equation lies between 2 and 3.diminish the root be 2

| | | | | |
|---|---|---|---|---|
| 1 | 4590 | 4022700 | – 8423000 | (2 |
| | 2 | 9184 | 80637668 | |
| | 4592 | 4031884 | – 359232 | |
| | 2 | 9188 | | |
| | 4594 | 4041072 | | |
| | 2 | | | |
| | 4596 | | | |

The transformed equation is $x^3 + 4596x^2 + 4041072x – 359232 = 0$

Multiply the roots by 10. Then the equation transforms into

$x^3 + 45960x^2 + 404107200x – 359232000 = 0$

We can easily see that a root of this equation lies between 0 and 1. We can stop with this since we require the root correct to three decimal places. Thus the root correct to three decimal places is 1.532. In the actual presentation we need write only the coefficients of the various transformed equations omitting completely the powers of x. The series of arithmetical operations is represented as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 0 | – 3 | 1 | (1.5320 |
| | 1 | 1 | – 2 | |
| | 1 | – 2 | – 1000 | |
| | 1 | 2 | 875 | |
| | 2 | 0 | – 125000 | |
| | 1 | 175 | 116577 | |
| | 30 | 175 | – 8423000 | |
| | 5 | 200 | 8063768 | |
| | 35 | 37500 | – 359232000 | |
| | 5 | 1359 | | |
| | 40 | 38859 | | |
| | 5 | 1368 | | |
| | 450 | 4022700 | | |
| | 3 | 9184 | | |
| | 453 | 4031884 | | |
| | 3 | 9188 | | |
| | 456 | 401407200 | | |
| | 3 | | | |
| | 4590 | | | |
| | 2 | | | |
| | 4592 | | | |

2

4594

2

45960

**Example 2**. Find the positive root of the equation $x^3 – 2x^3 – 3x – 4 = 0$ correct to three places of decimals.

Solution.

by Descartes' rule of signs, there can be at most only one positive root and we can easily see that it lies between 3 and 4. The process is exhibited as follows:

| | | | |
|---|---|---|---|
| -2 | – 3 | – 4 | (3.2842 |
| 3 | 3 | 0 | |
| 1 | 0 | – 4000 | |
| 3 | 12 | 2688 | |
| 4 | 1200 | -1312000 | |
| 3 | 144 | 1242752 | |
| 70 | 1344 | - 69248000 | |
| 2 | 148 | 64746224 | |
| 72 | 149200 | – 4501776000 | |
| 2 | 6144 | 3243903688 | |
| 74 | 155344 | – 1257872312 | |
| 2 | 6208 | | |
| 760 | 16155200 | | |
| 8 | 31356 | | |

| | |
|---|---|
| 768 | 16186556 |
| 8 | 31392 |
| 776 | 1621794800 |
| 8 | 157044 |
| 7840 | 1621951844 |
| 4 | |
| 7844 | |
| 4 | |
| 7848 | |
| 4 | |
| 78520 | |
| 2 | |
| 78522 | |

∴ The roots correct to three decimal places is 3.284

Exercises

1. Find the positive root, correct to two decimal places of the equation $x^3 + 3x^2 + 2x - 5 = 0$.
2. Find the real root of $x^3 + 6x = 2$ to three places of decimals
3. Find the root between 0 and 1 correct to three places of decimal of the equation $x^3 + 18x - 6 = 0$.
4. Find the root of the equation $x^3 - 5x - 11 = 0$ which lies between 2 and 3 correct to two places of decimals.

   Answers : 1.0.90, 2.0.327, 3. 0.33, 4. 2.99.

# UNIT- III SUMMATION OF SERIES

*Summation of series using Binomial, Exponential and Logarithmic series.*

**BINOMIAL SERIES**

When n is a positive integer $(x + a)^n$ can be expanded as $(x+a)^n = x^n + {}_nC_1 . x^{n-1}a + {}_nC_2 . x^{n-2}a^2 + \ldots + {}_nC_r . x^{n-r} . a^r + \ldots + a_n$.. This is known as the binomial theorem for the positive integer n. When n is a rational number $(1 + x)^n$ can be expanded as an infinite series when $-1 < x < 1$ (i.e) $|x| < 1$ and it is given by $(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots$ (1)

This is known as binomial series for $(1+ x)^n$ where n is a rational number.

**General term**

The $(r + 1)^{th}$ term in the expansion is often denoted by

$U_{r+1}$ or $T_{r+1}$ . $U_{r+1} = {}_nC_r \, x^{n-r} \, a^r$

We may obtain any particular term by giving r particular values. Thus the first term is obtained by writing $r = 0$, the second by writing r = 1 and so on . So the $(r +1)^{th}$ term is called the general term.

Thus we get $(x+a)^n = \sum_{r=0}^{n} {}_nC_r x^{n-r} a^r$

**Note:-**

(1) The expansion contains (n + 1) terms.
(2) The numbers ${}_nC_0, {}_nC_1 \ldots . {}_nC_r \ldots .. {}_nC_n$ are called the Binomial Coefficients. They are sometimes written as $C_0$ , $C_1$ , $C_n$.These binomial coefficients are all integers since ${}_nC_r$ is the number of combinations of *n* things taken *r* at a time.
(3) Since $C_0 = C_n$ , $C_1 = C_{n-1}$ , …… $C_r = C_{n-r}$ , the coefficients of terms equidistant from the beginning and the end of the expansion are equal.

**Summation of various series involving Binomial Coefficients**

It is convenient to write the Binomial theorem in the form

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_r x^r + \ldots C_n x^n.$$

We can seen in the expansion that the coefficients of terms which are equidistant from the beginning and the end are equal.

$\therefore\ C_0 = C_n = 1$ , $C_1 = C_{n-1} = n$ …. and in general.

$$C_r = C_{n-r} = \frac{n!}{r!(n-r)!}.$$

**Some important particular cases of the Binomial expansion.**

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots..$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots..$$

$$(1-x)^{-3} = \frac{1}{2}\{1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots.\}$$

$$(1-x)^{-4} = \frac{1}{6}\{1.2.3 + 2.3.4x + 3.4.5x^2 + 4.5.6x^3 + \ldots\}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots$$

$$(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\ldots$$

**Application of the Binomial theorem to the summation of series.**

We have proved when $|x| < 1$ , for all values of n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

**Solved problems**

**Example 1.** Find the sum to infinity of the series $1+\frac{3}{4}+\frac{3}{4}\cdot\frac{5}{8}+\frac{3}{4}\cdot\frac{5}{8}+\frac{7}{12}+\ldots$

Solution.

The factors in the numerators form an A.P with common difference 2: we therefore divide each of these by 2.

Each of the factors in the denominator has 4 for a factor; removing 4 from each will leave a factorial . Hence we have

$$1+\frac{\frac{3}{2}}{1}\cdot\frac{2}{4}+\frac{\frac{3}{2}\cdot\frac{5}{2}}{1.\ 2}\cdot\left(\frac{2}{4}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1.\ 2.\ 3}\cdot\left(\frac{2}{4}\right)^3+\ldots\ldots$$

i.e., $1+\frac{\frac{3}{2}}{1!}\cdot\frac{1}{2}+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\left(\frac{2}{4}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\left(\frac{2}{4}\right)^3+\ldots\ldots$

Put $n=\frac{3}{2}$ and $x=\frac{1}{2}$.

Then the series becomes

$$1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\ldots$$

$$=(1-x)^{-n}$$

$$=(1-\tfrac{1}{2})^{-3/2}$$

$$=2\sqrt{2}.$$

**Example 2.** Sum the series to infinity $\frac{1\ .\ 4}{5\ .\ 10}-\frac{1\ .\ 4\ .\ 7}{5\ .\ 10\ .\ 15}+\frac{1\ .\ 4\ .\ 7\ .\ 10}{5\ .\ 10\ .\ 15\ .\ 20}\ldots..$

Solution.

The numerators form an A.P . with 3 as common difference and the denominators are factorials, each of whose factors has been multiplied by 5.

∴ The series can be written as

$$S = \frac{\frac{1}{3}.\frac{4}{3}}{1\;.\;2}\;.\left(-\frac{3}{5}\right)^2 + \frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}}{1\;.\;2\;.\;3}\;.\left(-\frac{3}{5}\right)^3 + \frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}.\frac{10}{3}}{1\;.\;2\;.\;3\;.\;4}\;.\left(-\frac{3}{5}\right)^4 + \ldots.$$

Put $n = \frac{1}{3}$ and $x = -\frac{3}{5}$.

$$\therefore\; S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \frac{n(n+1)(n+2)(n+3)}{4!}x^4 \ldots..$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3$$

$$= (1-x)^{-n} - 1 - nx$$

$$= (1+\tfrac{3}{5})^{-1/3} - 1 + \frac{1}{3}\,.\,\frac{3}{5} = \frac{1}{2}\,(5)^{1/3} - \frac{4}{5}.$$

**Example 3.** Sum the series to infinity. $\frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + \ldots\ldots$

Solution.

The factors in the numerator form an A.P. with common difference 6 and those of the denominator an A.P with common difference 8.

Let S be the sum of the series.

$$\text{Then } S = \frac{\frac{15}{6}}{2}.\left(\frac{6}{8}\right) + \frac{\frac{15}{6}.\frac{21}{6}}{2\;.3}.\left(\frac{6}{8}\right)^2 + \frac{\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{2.3.4}.\left(\frac{6}{8}\right)^3 + \ldots..$$

The factors of the denominators do not begin with 1. Hence one additional factor , namely unity, has to be introduced into the denominator of each coefficient. The number of factors in the numerator is to be the same as that of the factors in the denominator. So we have to introduce an additional factor in the numerator also, which factor is clearly $\frac{9}{6}$.

$$\therefore\; \frac{9}{6}S = \frac{\frac{9}{6}.\frac{15}{6}}{1.2}\left(\frac{6}{8}\right) + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^3 + \ldots$$

Since the index of x in every term must be the same as the number of factors in the numerator or denominator of the coefficient, we have

$$S.\,\frac{9}{6}.\frac{6}{8} = \frac{\frac{9}{6}.\frac{15}{6}}{2!}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{3!}\left(\frac{6}{8}\right)^3 + \ldots$$

Put $\frac{9}{6} = n$ and $x = \frac{6}{8}$.

$$\therefore \frac{9}{6} S = \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots\ldots$$

$$= 1 + \frac{n}{1!} + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots\ldots - (1+nx)$$

$$= (1-x)^{-n} - (1+nx)$$

$$= (1 - \frac{6}{8})^{-9/6} - (1 + \frac{9}{6} . \frac{6}{8})$$

$$= (\frac{1}{4})^{-3/2} - (1 + \frac{9}{8})$$

$$= \frac{47}{8}.$$

$$\therefore \quad S = \frac{47}{9}.$$

**Example 4.** Find the sum of to infinity of the series $\frac{1}{24} - \frac{1.3}{24.32} + \frac{1.3.5}{24.32.40} - \ldots.$

Solution.

Proceeding as in the previous example, we get

$$S = \frac{\frac{1}{2}}{3} . \left(\frac{2}{8}\right) + \frac{\frac{1}{2}.\frac{3}{2}}{3.4} . \left(\frac{2}{8}\right)^2 + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3.4.5} . \left(\frac{2}{8}\right)^3 + \ldots..$$

In order to express this in the standard binomial form, the factor 1 . 2 must be inserted in each denominator, and two additional factors must be then inserted in each numerator to secure that the number of factors in the numerator is the same as that in the denominator. In order that the factors of the numerator may remain in A.P. the additional factors(which should be the same in each term) must be $-\frac{3}{2}, \frac{1}{2}$.

$$\therefore \quad -\frac{3}{2} . -\frac{1}{2} . S . \frac{1}{1.2} = \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}}{1.23} . \left(\frac{2}{8}\right) - \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.2.3.4} . \left(\frac{2}{8}\right)^2 + \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3.4.5} . \left(\frac{2}{8}\right)^3$$

The index of x should be the same as the number of factors in the numerator.

$\therefore$ The series is to be multiplied by $\left(\frac{2}{8}\right)^2$.

$\therefore\ -\frac{3}{2}.-\frac{1}{2}.S.\frac{1}{2}.\left(\frac{2}{8}\right)^2$

$= \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}}{3!}\left(\frac{2}{8}\right)^3 - \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{4!}.\left(\frac{2}{8}\right)^4 + \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{5!}.\left(\frac{2}{8}\right)^5 + \ldots\ldots$

i.e., $\frac{3S}{128} = \frac{n(n+1)(n+2)}{3!}x^3 - \frac{n(n+1)(n+2)(n+3)}{4!}x^4 + \ldots\ldots$

If $n = -\frac{3}{2}$, $x = \frac{2}{8}$.

$\therefore\ \frac{3S}{128} = -(1+x)^{-n} + \{1 - nx + \frac{n(n+1)}{2!}x^2\}$

$= -(1+\frac{2}{8})^{3/2} + \{1 + \frac{3}{2}.\frac{2}{8} + \frac{-\frac{3}{2}.-\frac{1}{2}}{2!}\left(\frac{2}{8}\right)^2\}$

$= \frac{-5\sqrt{5}}{8} + 1 + \frac{3}{8} + \frac{3}{128}$

$= \frac{179}{128} - \frac{-5\sqrt{5}}{8}$.

$\therefore\ S = \frac{1}{3}(179 - 80\sqrt{5})$.

**Exercises**

Find the sum to infinity of the following series:

(1) $\frac{3}{1} + \frac{3.5}{1.2}.\frac{1}{3} + \frac{3.5.7}{4.8.12} + \ldots\ldots$

(2) $\frac{3}{50} + \frac{3.18}{50.100} + \frac{3.18.33}{50.100.150} + \ldots..$

(3) $\frac{5}{3.6} + \frac{5.7}{3.6.9} + \frac{5.7.9}{3.6.9.12} + \ldots\ldots$

(4) $\frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + \ldots..$

(5) $\frac{5}{3.6}.\frac{1}{4^2} + \frac{5.8}{3.6.9}.\frac{1}{4^3} + \frac{5.8.11}{3.6.9.12}.\frac{1}{4^4} + \ldots.$

(6) $\frac{1}{2^3(3!)} - \frac{1.3}{2^4(4!)} + \frac{1.3.5}{2^5(5!)} + \ldots\ldots$

Answers: 1. $3^{5/3} - 3$, 2. $\left(\frac{10}{7}\right)^{1/5} - 1$, 3. $\sqrt{3} - \frac{2}{3}$, 4.$\frac{1}{5}\left\{8(27)^{1/4} - 17\right\}$,
5.$\frac{1}{2}\left\{\left(\frac{4}{3}\right)^{2/3} - \frac{7}{6}\right\}$, 6.$\frac{23}{24} - \frac{2}{3}\sqrt{2}$.

**Sum of coefficients.**

If f (x) can be expanded as an ascending series in x, we can find the sum of the list (n+1) coefficients.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$\therefore \frac{f(x)}{1-x} = (a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots) . (1 + x + x^2 + \ldots)$

$\therefore$ Coefficient of $x^n$ in $\frac{f(x)}{1-x} = a_0 + a_1 + a_2 + \ldots + a_n$.

Thus, to find the sum of the first (n+1) coefficients in the expansion of f (x), we have only to find the coefficient of $x^n$ of the expansion of $\frac{f(x)}{1-x}$.

**Example 1.** Find the sum of the coefficients of the first (r +1) term in the expansion of $(1-x)^{-3}$.

Solution.

The required result is the coefficient of $x^r$ in the expansion of $\frac{(1-x)^{-3}}{1-x}$.

i.e., in the expansion of $(1-x)^{-4}$

i.e., in $1 + 4x + \frac{4.5}{2!}x^2 + \frac{4.5.6}{3!}x^3 + \ldots + \frac{(r+1)(r+2)(r+3)}{3!}x^r$

$\therefore$ Sum of the (r +1) coefficients in the expansion of

$(1-x)^{-3}$ is $\frac{(r+1)(r+2)(r+3)}{6}$.

Therefore coefficient of $x^r$ = coeff.of $x^{2n}$

$= -3 + 4(-1)^n$

$= - 3 + 4(-1)^{r/2}$.

5. If x is so small that its square and higher powers may be neglected prove that

$\frac{\sqrt{1+x}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4 - \frac{10x}{3}$ .(nearly)

Solution.

$\frac{\sqrt{1+x}(4-3x)^{3/2}}{(8+5x)^{1/3}} = (1+x)^{1/2} 4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2} 8^{1/3}\left(1+\frac{5x}{8}\right)^{-1/3}$

$= 4\left(1+\frac{1}{2}x+\cdots\right)\left(1-\frac{9}{8}x+\cdots\right)\left(1-\frac{5x}{24}+\cdots\right)$

$= 4\left[1+x\left(\frac{1}{2}-\frac{9}{8}-\frac{5}{24}\right)\right]$ (neglecting $x^2$ and higher power of x)

$= 4-\frac{10x}{3}$.

6. Show that $1 + n\left(\frac{2a}{1+a}\right)+\frac{n(n+1)}{1.2}\left(\frac{2a}{1+a}\right)^2 + \ldots\ldots = \left(\frac{1+a}{1-a}\right)^n$.

Solution.

Put $\frac{2a}{1+a} = y$.

Then L.H.S = $1+\frac{ny}{1!}+\frac{n(n+1)}{2!}y^2+\cdots$

$= (1-x)^{-p/q}$ where p = n; a = 1 and $\frac{x}{a} = y$. Hence x = y.

Hence L.H.S = $(1-y)^{-n} = \left(1-\frac{2a}{1+a}\right)^{-n} = \left(\frac{1-a}{1+a}\right)^{-n} = \left(\frac{1+a}{1-a}\right)^n$ = R.H.S

7. Prove that $1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\cdots = 2\left[1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\right.$

$\ldots$

Solution.

L.H.S = $1+\frac{n}{1!}\left(\frac{2}{3}\right)+\frac{n(n+1)}{2!}\left(\frac{2}{3}\right)^2+\ \ldots$

$= \left(1-\frac{2}{3}\right)^{-n} = \left(\frac{1}{3}\right)^{-n} = 3^n$

R.H.S = $2^n\left[1+\frac{n}{1!}\left(\frac{1}{3}\right)+\frac{n(n+1)}{2!}\left(\frac{1}{3}\right)^2+\ \ldots\right]$

$= 2^n\left(1-\frac{1}{3}\right)^{-n} = 2^n\left(\frac{2}{3}\right)^{-n} = 3^n$

L.H.S = R.H.S.

9. Sum to infinity the series $1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + .......$

Solution.

Let $S = 1 + \frac{1}{5} + \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} + .......$

Therefore $S = 1 + \frac{1}{1!}\left(\frac{1}{5}\right) + \frac{1.4}{2!}\left(\frac{1}{5}\right)^2 + \frac{1.4.7}{3!}\left(\frac{1}{5}\right)^3 + \cdots$

$= (1 - x)^{-p/q}$ where $p = 1$; $q = 3$ and $\frac{x}{q} = \frac{1}{5}$. Hence $x = \frac{3}{5}$

Therefore $S = \left(1 - \frac{3}{5}\right)^{-1/3} = \left(\frac{2}{5}\right)^{-1/3} = \left(\frac{5}{2}\right)^{1/3}$.

10. Sum to $\infty$ the series $\left(\frac{1}{2}\right)^2 + \frac{1}{2!}\left(\frac{1}{2}\right)^4 + \frac{1.3}{3!}\left(\frac{1}{2}\right)^6 + \cdots$

Solution.

Let $S = \left(\frac{1}{2}\right)^2 + \frac{1}{2!}\left(\frac{1}{2}\right)^4 + \frac{1.3}{3!}\left(\frac{1}{2}\right)^6 + \cdots$

Therefore $S = \frac{1}{1!}\left(\frac{1}{4}\right) + \frac{1}{2!}\left(\frac{1}{4}\right)^2 + \frac{1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

$-S = \frac{-1}{1!}\left(\frac{1}{4}\right) + \frac{-1.1}{2!}\left(\frac{1}{4}\right)^2 + \frac{-1.1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

$-S + 1 = 1 + \frac{-1}{1!}\left(\frac{1}{4}\right) + \frac{-1.1}{2!}\left(\frac{1}{4}\right)^2 + \frac{-1.1.3}{3!}\left(\frac{1}{4}\right)^3 + \cdots$

$= (1 - x)^{-p/q}$ where $p = 1$; $q = 2$ and $\frac{x}{q} = \frac{1}{4}$

Hence $x = \frac{1}{2}$. Hence $-S+1 = \left(1 - \frac{1}{2}\right)^{1/2}$

$= \left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}$. Hence $S = 1 - \frac{1}{\sqrt{2}}$.

11. Sum to $\infty$ the series $\frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + .......$

Solution.

Let $S = \frac{3}{18} + \frac{3.7}{18.24} + \frac{3.7.11}{18.24.30} + .......$

$= \frac{3}{3}\left(\frac{1}{6}\right) + \frac{3.7}{3.4}\left(\frac{1}{6}\right)^2 + \frac{3.7.11}{3.4.5}\left(\frac{1}{6}\right)^3 + \cdots$

Therefore $\frac{S(-5)(-1)}{1.2}\left(\frac{1}{6}\right)^2 = \frac{(-5)(-1)3}{3!}\left(\frac{1}{6}\right)^2 + \frac{(-5)(-1)3.7}{4!}\left(\frac{1}{6}\right)^4 + \cdots$

$$\frac{5S}{72} + 1\frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2$$

$$= 1 + \frac{(-5)}{1!}\left(\frac{1}{6}\right) + \frac{(-5)(-1)}{2!}\left(\frac{1}{6}\right)^2 + \frac{(-5)(-1)3}{3!}\left(\frac{1}{6}\right)^3 + \cdots$$

$\frac{5S}{72} + \left(1 - \frac{5}{6} + \frac{5}{72}\right) = (1 - x)^{-p/q}$ where p = 5;q = 4 and $\frac{x}{q} = \frac{1}{6}$ and hence x = $\frac{2}{3}$

Therefore $\frac{5S}{72} + \frac{17}{72} = \left(1 - \frac{2}{3}\right)^{5/4}$

$$\therefore \frac{5S}{72} = \left(\frac{1}{3}\right)^{5/4} - \frac{17}{72}$$

$$\therefore S = \frac{72}{5}\left[\frac{3^{-5/4}(72) - 17}{72}\right] = \frac{72}{5}\left[\frac{3^{-5/4}(3)^2 8 - 17}{72}\right]$$

$$= \frac{72}{5}\left[\frac{3^{3/4}(8) - 17}{72}\right] = \frac{72}{5}\left[\frac{8(27)^{1/4} - 17}{72}\right]$$

$$S = \frac{1}{5}\left(8(27)^{1/4} - 17\right).$$

**Exponential Series**

We will learn some series which can be summed up by exponential series. We have proved that for all real values of x.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + .. \text{ to } \infty \quad \ldots\ldots\ldots\ldots..(1)$$

In particular when x = 1 , we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + .. \text{ to } \infty \quad \ldots\ldots\ldots\ldots..(2)$$

and when x = −1 , we have

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n . \frac{1}{n!} + .. \text{ to } \infty \quad \ldots\ldots\ldots\ldots..(3)$$

Changing x into $-x$ in series (1) , we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + (-1)^n . \frac{x^n}{n!} + \cdots \quad \ldots\ldots\ldots\ldots..(4)$$

Adding (1) and (4) , we get

$$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \text{ to } \infty \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Subtracting (4) from (1) , we get

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ to } \infty \quad \ldots\ldots\ldots\ldots(6)$$

When x = 1 , series (5) and (6) become

$$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(7)$$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(8)$$

**Note**. It can be verified that e is an irrational number whose value lies between 2 and 3. Further the value of e correct to four places of decimals is given by e = 2.7183. We shall use these series to find the sums of certain series. The different methods are illustrated by the following worked examples..

**Example.** Sum the series $1 + \frac{1+3}{2!} + \frac{1+3+3^3}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots$ to $\infty$.

Solution.

Let $u_n$ be the $n^{th}$ term of the series and S be the sum to infinity of the series.

$$\therefore \qquad u_n = \frac{1+3+3^2+ \ldots\ldots + 3^{n-1}}{n!}$$

$$= \frac{3^n-1}{3-1} \cdot \frac{1}{n!}$$

$$= \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

$$\therefore \qquad u_1 = \frac{1}{2}\left(\frac{3^1}{1!} - \frac{1}{1!}\right)$$

$$u_2 = \frac{1}{2}\left(\frac{3^2}{2!} - \frac{1}{2!}\right)$$

$$u_3 = \frac{1}{2}\left(\frac{3^3}{3!} - \frac{1}{3!}\right)$$

…………………

…………………

$$u_n = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

……………….

……………….

$$S = \frac{1}{2}\left(\frac{3^1}{1!}+\frac{3^2}{2!}+\cdots+\frac{3^n}{n!}+\cdots\right)-\frac{1}{2}\left(\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots\right)$$

$$=\frac{1}{2}(\,e^3-1)-\,\frac{1}{2}(e-1)$$

$$=\,\frac{1}{2}e(\,e^2-1).$$

**Exercises**

1. Show that $(1+\frac{1}{2!}+\frac{1}{4!}+\cdots)^2 = \,(1+\frac{1}{3!}+\frac{1}{5!}+\cdots)^2$
2. Show that $\frac{e+1}{\mathrm{e}-1}=\frac{\frac{1}{1!}+\frac{1}{3!}+\cdots}{\frac{1}{2!}+\frac{1}{4!}+\cdots}$.
3. Show that 2 { 1 + $\frac{(\log_e n)^2}{2!}$ + $\frac{(\log_e n)^4}{4!}$ $+$…..} = $\left(n+\frac{1}{\mathrm{n}!}\right)$.
4. Show that $\sum_1^\infty \frac{n-1}{\mathrm{n}!}=1$.

If the given series is $\sum_{n=0}^{\infty} f(n).\frac{x^n}{\mathrm{n}!}$ where f (n) is a polynomial in n of degree r , we can find constants $\mathrm{a}_0, \mathrm{a}_1, \ldots \ldots \mathrm{a}_r$ so that

$$f(n) = \mathrm{a}_0+\mathrm{a}_1 n+\mathrm{a}_2 n(n-1)\ \ldots.+\mathrm{a}_r n(n-1)\ \ldots.(n-r+1) \text{ and then}$$

$$\sum_{n=0}^{\infty} f(n).\frac{x^n}{\mathrm{n}!} = \mathrm{a}_0\sum_{n=0}^{\infty}\frac{x^n}{\mathrm{n}!}+\mathrm{a}_1\sum_{n=0}^{\infty}\frac{x^n}{(\mathrm{n}-1)!}+\ \ldots.+\mathrm{a}_r\sum_{n=0}^{\infty}\frac{x^n}{(\mathrm{n}-\mathrm{r})!}$$

$$= \mathrm{a}_0.e^x+\mathrm{a}_1 x.e^x+\ldots..+\mathrm{a}_r.x^r.e^x$$

$$= (\mathrm{a}_0+\mathrm{a}_1 x+\mathrm{a}_2 x^2+\ \ldots.\mathrm{a}_r x^r)\,e^x$$

**Example 1.** Sum the series $\sum_{n=0}^{\infty}\frac{(n+1)^3}{\mathrm{n}!}.\ x^n$.

Solution.

Put $(n+1)^3 = \mathrm{A}+\mathrm{B}\,n+\mathrm{C}\,n\,(n-1)+\mathrm{D}\,n\,(n-1)(n-2)$.

Putting $n = 0$ , 1 , 2 and equating the coefficients of $n^3$, we get

A = 1, B = 7, C = 6, D = 1.

Let the sum of the series be S.

$$S = \sum_0^\infty \frac{1+7n+6n(n-1)+n(n-1)(n-2)}{n!} x^n$$

$$= \sum_0^\infty \frac{x^n}{n!} + 7\sum_0^\infty \frac{x^n}{(n-1)!} + 6\sum_0^\infty \frac{x^n}{(n-2)!} + \sum_0^\infty \frac{x^n}{(n-3)!}$$

Now $\sum_0^\infty \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$

$$\sum_0^\infty \frac{x^n}{(n-1)!} = x + \frac{x^2}{1!} + \frac{x^3}{2!} \ldots = x.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-2)!} = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} \ldots = x^2.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-3)!} = x^3 + \frac{x^4}{1!} + \frac{x^5}{2!} \ldots = x^3.e^x$$

∴ $S = (1 + 7x + 6x^2 + x^3)\, e^x$.

**Example 2.** Sum the series $\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!} \ldots\ldots\ldots + \frac{1^2+2^2+\cdots+n^2}{n!} + \ldots\ldots$

Solution.

Let the $n^{th}$ term of the series be $u_n$ and the sum to infinity be S.

Then $u_n = \frac{1^2+2^2+\cdots+n^2}{n!} = \frac{n(n+1)(2n+1)}{6} \frac{1}{n!}$

Let $n(n+1)(2n+1) = A + Bn + Cn(n-1) + Dn(n-1)(n-2)$.

∴ A = 0, B = 6, C = 9, D = –2.

∴ $S = \sum_{n=1}^\infty \frac{6n+9n(n-1)+2n(n-1)(n-2)}{6} \frac{1}{n!}$

$$= \sum_{n=1}^\infty \frac{1}{(n-1)!} + \frac{3}{2}\sum_{n=1}^\infty \frac{1}{(n-2)!} + \frac{1}{3}\sum_{n=1}^\infty \frac{1}{(n-3)!}$$

$= e + \frac{3}{2}e + \frac{1}{3}e$

$= \frac{17}{6}e.$

**Exercises**

1. Show that the sum to infinity of the series

   $2^2 + \frac{3^2}{1!}x + \frac{4^2}{2!}x^2 + \frac{5^2}{3!}x^3 + \ldots = e^x(x^2 + 5x + 4).$

2. Find the sum to infinity of the series

   (1) $\frac{3.5}{1!}x + \frac{4.6}{2!}x^2 + \frac{5.7}{3!}x^3 + \ldots\infty$

   (2) $1.2 + 2.3x + 3.4.\frac{x^2}{2!} + 4.5.\frac{x^3}{3!}\ldots$

3. Sum to infinity the following series:-

   (1) $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \ldots$

   (2) $\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \ldots$

   (3) $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots$

   (4) $\frac{1.2}{1!} + \frac{2.3}{2!} + \frac{3.4}{3!} + \frac{4.5}{4!} + \ldots$

4. Show that

   (1) $5 + \frac{2.6}{1!} + \frac{3.7}{2!} + \frac{4.8}{3!} + \ldots$ to $\infty = 13e$.

   (2) $\frac{1^2.\ 2^2}{1!} + \frac{2^2.\ 3^2}{2!} + \frac{3^2.\ 4^2}{3!} + \ldots$ to $\infty = 27e$.

   (3) $\sum_{n=1}^{\infty} \frac{n^3-n+1}{n!} = 5e - 1.$

   Answers : 2.(1).$(x^2 + 7x + 8)\ e^x$, (2). $(x^2 + 4x + 2)e^x$, 3.(1).$\frac{3e}{2}$,(2).15e, (3).e + 1,(4). 3e.

**Example 1.** Sum the series $\sum_{n=1}^{\infty} \frac{n^2+3}{n+2} \cdot \frac{x^n}{n!}$.

Solution.

Let the sum of the series be S.

Then S = $\sum_{n=1}^{\infty} \frac{(n^2+3)(n+1)}{(n+2)!} \cdot x^n.$

Let $(n^2+3)(n+1) = \mathrm{A} + \mathrm{B}(n+2) + \mathrm{C}(n+2)(n+1) + \mathrm{D}(n+2)(n+1)n.$

We can easily find that A = $-7$, B = 7, C = $-2$ and D = 1.

Then S = $\sum_{n=1}^{\infty} \frac{-7+7(n+2)-2(n+2)(n+1)+(n+2)(n+1)n}{(n+2)!} \cdot x^n.$

$$= -7\sum_{n=1}^{\infty}\frac{x^n}{(n+2)!} + 7.\sum_{n=1}^{\infty}\frac{x^n}{(\mathrm{n}+1)!} \; -2\sum_{n=1}^{\infty}\frac{x^n}{\mathrm{n}!} + \sum_{n=1}^{\infty}\frac{x^n}{(\mathrm{n}-1)!}$$

Now $\sum_{n=1}^{\infty}\frac{x^n}{(n+2)!} = \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{(n+2)!} + \ldots$

$$= \frac{1}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right).$$

$$\sum_{n=1}^{\infty}\frac{x^n}{(\mathrm{n}+1)!} = \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots + \frac{x^n}{(n+1)!} + \ldots$$

$$= \frac{1}{x}(e^x - 1 - x).$$

$$\sum_{n=1}^{\infty}\frac{x^n}{\mathrm{n}!} = \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = e^x - 1.$$

$$\sum_{n=1}^{\infty}\frac{x^n}{(\mathrm{n}-1)!} = x + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{(n-1)!} + \ldots = xe^x$$

$$\therefore \; \mathrm{S} = \frac{-7}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right) + \frac{7}{x}(e^x - 1 - x) - 2\,(e^x - 1) + xe^x$$

$$= \frac{e^x}{x^2}(x^3 - 2x^2 + 7x - 7) + \frac{7}{2x^2}(3x^2 + 2).$$

**Example 2.** Sum the series $\frac{5}{1!} + \frac{7}{3!} + \frac{9}{5!} + \ldots.$

Solution.

The $n^{th}$ term $\mathrm{u}_n = \frac{(2n+3)}{(2n-1)!}$

Put $2n + 3 = \mathrm{A}(2n-1) + \mathrm{B}$.

Then A = 1 and B = 4.

$$\therefore\ u_n = \frac{2n-1+4}{(2n-1)!}$$

$$= \frac{2n-1}{(2n-1)!} + \frac{4}{(2n-1)!}$$

$$= \frac{1}{(2n-2)!} + \frac{4}{(2n-1)!}$$

$$\therefore\ u_1 = 1 + \frac{4}{1!}$$

$$u_2 = \frac{1}{2!} + \frac{4}{3!}$$

$$u_3 = \frac{1}{4!} + \frac{4}{5!}$$

…………..

…………..

Sum to infinity = $\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right) + 4\left(\frac{1}{1!} + \frac{1}{3!} + \ldots\right)$

$$= \frac{1}{2}\left(e + \frac{1}{e}\right) + 4 \cdot \frac{1}{2} \cdot \left(e - \frac{1}{e}\right)$$

$$= \frac{5}{2}e - \frac{3}{2e}.$$

**Example 3.** Prove that the infinite series $\frac{2\frac{1}{2}}{1!} - \frac{3\frac{1}{3}}{2!} + \frac{4\frac{1}{4}}{3!} - \frac{5\frac{1}{5}}{4!} + \ldots.. = \frac{1+e}{e}$.

Solution.

Let $u_n$ be the $n^{th}$ term of the series and S be the sum of the series to infinity.

Then $u_n = (-1)^{n+1} \frac{(n+1)\frac{1}{n+1}}{n!}$

$$= (-1)^{n+1} \frac{(n+1)^2+1}{(n+1)!}.$$

Put $n^2 + 2n + 2 = A + B\,(n+1) + C\,(n+1)\,n$.

$\therefore$ A = 1 , B = 1 , C = 1.

$\therefore\ u_n = (-1)^{n+1}\frac{1+(n+1)+(n+1)n}{(n+1)!}$

$= (-1)^{n+1} . \left\{\frac{1}{(n+1)!} + \frac{1}{n!} + \frac{1}{(n-1)!}\right\}.$

$\therefore\ S = \sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{(n+1)!} + \sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{n!} + \sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{(n-1)!}$

Now $\sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{(n+1)!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots. = e^{-1}$

$\sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \ldots = -e^{-1} + 1$

$\sum_{n=1}^{\infty}(-1)^{n+1}.\ \frac{1}{(n-1)!} = 1 - \frac{1}{1!} + \frac{1}{2!} \ldots = e^{-1}.$

$\therefore\ S = 1 + e^{-1}$

$= \frac{e+1}{e}$ .

**Exercises**

1. Show that

   (1) $\sum_{n=1}^{\infty} \frac{n-1}{n+2} . \frac{x^n}{n!} = \frac{1}{x^2}\{(x^2 - 3x - 3)\ e^x + \frac{1}{2}x^2 - 3\}.$

   (2) $\sum_{n=1}^{\infty} \frac{(2n-1)}{(n+3)n!} = \frac{1}{2}(43 - 15e)$

2. Sum to infinity the series

   (1) $\frac{3}{1!} + \frac{4}{3!} + \frac{5}{5!} + \frac{6}{7!} + \ldots..$

   (2) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots..$

   (3) $\frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \frac{9}{8!} + \ldots..$

3. Show that $\sum_{0}^{\infty} \frac{5n+1}{(2n+1)!} = \frac{e}{2} + \frac{2}{e}$

4. Prove that $\frac{2^2}{1!} + \frac{2^4}{3!} + \frac{2^6}{5!} = \frac{e^4-1}{e^2}.$

5. Show that $\log_e 2 - \frac{1}{2!}(\log_e 2)^2 + \frac{1}{3!}(\log_e 2)^3 \ldots\ldots = \frac{1}{2}$.

Answer : 2(1).$\frac{1}{e}$,(2).$\frac{1}{2}(3e - 2e^{-1})$,(3).$\frac{1}{2e}$.

By equating the coefficients of like powers of x in the expansions of function of x in two different ways, we can derive some identities. The following examples will illustrated the method:

**Example 1.** By expanding $(e^x - 1)^n$ in two ways or otherwise prove that

$n^r - {}_{n}C_1(n-1)^r + {}_{n}C_2(n-2)^r - \ldots\ldots = 0$ where r < n.

What is the sum of the above series when r = n?

Solution.

$$(e^x - 1)^n = e^{nx} - {}_{n}C_1 e^{(n-1)x} + \ldots.$$

$$= 1 + nx + \frac{(nx)^2}{1!} + \ldots \frac{(nr)^r}{r!} + \ldots - {}_{n}C_1\left[1 + (n-1)x + \frac{\{(n-1)x\}^2}{2!} + \cdots \frac{\{(n-1)x\}^r}{r!} + \ldots\right.$$

$$\left. + {}_{n}C_2\left[1 + (n-2)x + \frac{\{(n-2)x\}^2}{2!} + \cdots \frac{\{(n-2)x\}^r}{r!} + \cdots\right]\ldots\right.$$

Coefficient of $x^r$ in the expansion of $(e^x - 1)^n$

$$= \frac{n^r}{r!} - {}_{n}C_1.\frac{(n-1)^r}{r!} + {}_{n}C_2.\frac{(n-2)^r}{r!} - \ldots\ldots$$

$$= \frac{1}{r!}\{n^r - {}_{n}C_1(n-1)^r + {}_{n}C_2(n-2)^r \ldots.\}$$

Again $(e^x - 1)^n = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots - 1)^n$

$$= \left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots\right)^n$$

$$= x^n\left(\frac{1}{1!} + \frac{x}{2!} + \cdots \frac{x^{n-1}}{n!} + \cdots\right)^n.$$

All terms in the expansion contain $x^n$ and the higher power of x.

$\therefore$ If r < n , there will be no term containing $x^r$ in the expansion.

$$\therefore \frac{1}{r!}\{n^r - {}_{\mathrm{n}}C_1(n-1)^r + {}_{\mathrm{n}}C_2(n-2)^r \ldots\} = 0$$

i.e., $n^r - {}_{\mathrm{n}}C_1(n-1)^r + {}_{\mathrm{n}}C_2(n-2)^r \ldots = 0$

If r = n , then

$$\frac{1}{n!}\{n^n - {}_{\mathrm{n}}C_1(n-1)^n + {}_{\mathrm{n}}C_2(n-2)^n \ldots\}$$

= Coefficient of $x^n$ in the expansion of $x^n\left(\frac{1}{1!}+\frac{x}{2!}+\cdots\right)^n$

= 1 .

$$\therefore n^n - {}_{\mathrm{n}}C_1(n-1)^n + {}_{\mathrm{n}}C_2(n-2)^n \ldots = n!$$

**Example 2.** Show that if $a^r$ be the coefficient of $x^n$ in the expansion of $e^{e\,x}$ , then

$$a_r = \frac{1}{r!}\left\{\frac{1^r}{1!}+\frac{2^r}{2!}+\frac{3^r}{3!}\right\}.$$

Hence show that

$$(i)\ \frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\ldots = 5e$$

$$(ii)\ \frac{1^4}{1!}+\frac{2^4}{2!}+\frac{3^4}{3!}+\ldots = 15e.$$

Solution.

$$e^{e\,x} = 1 + e^x + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + \frac{(e^x)^4}{4!} + \ldots$$

$$= 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \frac{e^{4x}}{4!} + \ldots$$

$$= 1 + \left(1 + x + \frac{x^2}{2!} + \cdots \frac{x^r}{r!} + \ldots\right) + \frac{1}{2!}\left(1 + 2x + \frac{2^2x^2}{2!} + \cdots \frac{2^r x^r}{r!} + \ldots\right)$$

$$+ \frac{1}{3!}\left(1 + 3x + \frac{3^2x^2}{2!} + \cdots \frac{3^r x^r}{r!} + \ldots\right) + \ldots$$

Hence the coefficient of $x^r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!}\right\}$.

Again

$$e^{e^x} = e^{1+x+\frac{x^2}{2!}+\cdots} = e.e^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$= e.\{1 + (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) + \frac{1}{2!}(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^2$$

$$+ \frac{1}{3!}(x + \frac{x^2}{2!} + \cdots)^3 + \ldots\}$$

Coefficient of $x^3 = e\left(\frac{1}{3!} + \frac{1}{2!}.2.\frac{1}{2!} + \frac{1}{3!}\right)$

$$= \frac{e}{3!}(1 + 3 + 1) = \frac{5e}{3!}.$$

We have shown that the coefficient of $x^3$

$$= \frac{1}{3!}\left(\frac{1^3}{1!} + \frac{2^3}{2!} + \cdots\right)$$

$$\therefore \quad \frac{1}{3!}\left(\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \cdots\right) = \frac{5e}{3!}$$

$$\therefore \quad \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \ldots.. = \frac{5e}{3!}.$$

Similarly equating the coefficient of $x^4$ , we get the second result.

**Example 3.** Prove that if n is a positive integer

$$1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2.2^2}x^2 - \frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3 + \cdots$$

$$= e^x\{1 - \frac{n+1}{1^2}x + \frac{(n+1)(n+2)}{1^2.2^2}x^2 - \frac{(n+1)(n+2)(n+3)}{1^2.2^2.3^2}x^3 + \ldots.\}.$$

Solution.

$$e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots..$$

$$(1-\frac{x}{y})^n = 1 - n.\frac{x}{y} + \frac{n(n-1)}{2!}.(\frac{x}{y})^2 - \frac{n(n-1)(n-2)}{3!}(\frac{x}{y})^3 + \ldots..$$

$$\therefore\ 1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2.2^2}x^2 - \frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3 + \cdots$$

= the term independent of y in the product of $e^y(1-\frac{x}{y})^n$.

$$e^y(1-\frac{x}{y})^n = e^x.e^{y-x}.\frac{(y-x)^n}{y^n}$$

$$= e^x.\{1 + \frac{(y-x)}{1!} + \frac{(y-x)^2}{2!} + \ldots.\}\frac{(y-x)^n}{y^n}$$

$$= e^x\left\{\frac{(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots}{y^n}\right\}$$

The term containing $y^n$ in the expression

$$(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots$$

is $y^n - \frac{{}_{n+1}C_1}{1!}y^n.x + \frac{{}_{n+2}C_2}{2!}y^n\ x^2\ldots.$

$\therefore$ Term independent of y in $e^y(1-\frac{x}{y})^n$ is

$$e^x\{1 - \frac{{}_{n+1}C_1}{1!}x + \frac{{}_{n+2}C_2 . x^2}{2!} - \ \ldots.\}$$

$$= e^x\{1 - \frac{(n+1)}{(1!)^2}x + \frac{(n+2)(n+1)}{(2!)^2}x^2 - \ \ldots.\}.$$

Hence the required result.

**Exercises**

1. Show that , if n is a positive integer

   $n\ .\ 1^{n+1} - \frac{n(n-1)}{2!}.2^{n+1} + \frac{n(n-1)(n-2)}{3!}.3^{n+1} - \ldots. = (-1)^n.n.\frac{(n+1)!}{2}.$

2. Find the coefficient of $x^r$ in the expansion of $\frac{e^{nx}-1}{1-e^{-x}}$, n being a positive integer and find the values of

(1) $1^2 + 2^2 + 3^2 + \ldots .. + n^2$

(2) $1^3 + 2^3 + 3^3 + \ldots .. + n^3$

(3) $1^4 + 2^4 + \ldots .. + n^4$

3. By means of the identity $e^{x^2+\frac{1}{x^2}+2} = e^{(x+\frac{1}{x})^2}$ show that

$$e^2\{ 1 + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} \ldots .\} = 1 + \frac{2!}{(1!)^3} + \frac{4!}{(2!)^3} + \frac{6!}{(3!)^3} + \ldots .$$

[ Left side = term independent of x in $e^2 . e^{x^2} . e^{x^{\frac{1}{2}}}$

$$e^{(x+\frac{1}{x})^2} = 1 + \frac{(x+\frac{1}{x})^2}{1!} + \frac{(x+\frac{1}{x})^4}{2!} + \frac{(x+\frac{1}{x})^6}{4!} + \ldots \ldots$$

Term independent of x in the above expansion

$$= 1 + \frac{{}_2C_1}{1!} + \frac{{}_4C_2}{2!} + \frac{{}_6C_3}{3!} + \ldots ..]$$

Answer : 2(1).$\frac{n(n+1)(2n+1)}{6}$,(2).$\frac{n^2(n+1)^2}{4}$,(3).$\frac{n(n+1)(6n^3+9n^2+n-1)}{60}$.

## Extra problems

1. Find the coefficient of $x^n$ in $\frac{a+be^x+ce^{2x}}{e^{3x}}$.

Solution.

$$\frac{a+be^x+ce^{2x}}{e^{3x}} = (a + be^x + ce^{2x})e^{-3x}$$

$$= ae^{-3x} + be^{-2x} + ce^{-x}$$

$$= a\left(1 - \frac{(3x)}{1!} + \frac{(3x)^2}{2!} - \cdots + \frac{(-1)^n(3x)^n}{n!} + \cdots\right) + b\left(1 - \frac{(2x)}{1!} + \frac{(2x)^2}{2!} - \cdots + \frac{(-1)^n(2x)^n}{n!} + \cdots\right) + c\left(1 - \frac{(x)}{1!} + \frac{(x)^2}{2!} - \cdots + \frac{(-1)^n(x)^n}{n!} + \cdots\right)$$

∴ coefficient of $x^n$ in $\frac{a+be^x+ce^{2x}}{e^{3x}}$ is $= \frac{(-1)^n}{n!}[a3^n + b2^n + c]$.

2. What is the coefficient of $x^n$ in the expansion of $(1 + x)e^{1+x}$ in ascending powers of x.

Solution.

$(1 + x)e^{1+x} = (1 + x)\, e\, e^x$

$$= e(1 + x)\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} \ldots\right]$$

Therefore coefficient of $x^n$ in $(1 + x)e^{1+x}$ is $= e\left[\frac{1}{n!} + \frac{1}{(n-1)!}\right]$

$= e\left[\frac{1}{n!}+\frac{n}{n!}\right]=\frac{e}{n!}(1+n)$ .

3. Prove that $\log 2-\frac{(\log 2)^2}{2!}+\frac{(\log 2)^3}{3!}-\cdots=\frac{1}{2}$.

Solution.

Put log 2 = y.

Therefore L.H.S = $y-\frac{y^2}{2!}+\frac{y^3}{3!}-\cdots$

$= -\left[-\frac{y}{1}+\frac{y^2}{2!}-\frac{y^3}{3!}+\cdots\right]$

$= -(e^{-y}-1) = 1-e^{-\log 2} = 1-\frac{1}{2}=\frac{1}{2}$.

4. Prove that $\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{1+\frac{1}{3!}+\frac{1}{5!}+\cdots}=\frac{e-1}{e+1}$

Solution.

L.H.S = $\frac{\frac{1}{2}(e+e^{-1})}{\frac{1}{2}(e-e^{-1})}=\frac{e^2+1-2e}{e^2-1}$

$=\frac{(e-1)^2}{(e+1)(e-1)}=\frac{e-1}{e+1}$.

5. Show that if a > 1. $S=1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\cdots=\frac{e^a-e}{a-1}$.

Solution.

$n^{th}$ term $T_n=\frac{1+a+a^2+\cdots+a^{n-1}}{n!}=\frac{a^n}{n!(a-1)}$.

Therefore $T_n=\left(\frac{1}{a-1}\right)\left[\frac{a^n}{n!}-\frac{1}{n!}\right]$ ............. (1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1=\left(\frac{1}{a-1}\right)\left[\frac{a}{1!}-\frac{1}{1!}\right]$

$T_2=\left(\frac{1}{a-1}\right)\left[\frac{a^2}{2!}-\frac{1}{2!}\right]$

$T_3=\left(\frac{1}{a-1}\right)\left[\frac{a^3}{3!}-\frac{1}{3!}\right]$

... ... ... ... ...

... ... ... ... ...

Adding we get

$S=\left(\frac{1}{a-1}\right)\left[\left(\frac{a}{1!}+\frac{a^2}{2!}+..\right)-\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\right]$

$=\left(\frac{1}{a-1}\right)[(e^a-1)-(e-1)]=\frac{e^a-e}{a-1}$.

6. Prove that $S=1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\cdots=\frac{3e}{2}$.

Solution.

$n^{th}$ term $T_n=\frac{1+2+\cdots+n}{n!}$

$=\frac{n(n+1)}{2n!}=\frac{n+1}{2(n-1)!}$.

Let n+ 1 =A +B(n – 1).

Putting n = 1 and n = 0 we get A = 2; B = 1.

Therefore $T_n = \frac{2+(n-1)}{2(n-1)!}$

Therefore $T_n = \frac{1}{(n-1)!} + \frac{1}{2(n-2)!}$ ................(1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1 = 1$

$T_2 = \frac{1}{1!} + \frac{1}{2!}$

$T_3 = \frac{1}{2!} + \left(\frac{1}{2}\right)\frac{1}{1!}$

$T_4 = \frac{1}{3!} + \left(\frac{1}{2}\right)\frac{1}{2!}$

... ... ... ... ...

... ... ... ... ...

Adding we get $S = \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \frac{1}{2}\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$

$= e + \frac{1}{2}e = \frac{3e}{2}$.

7. Find $S = \sum_{n=1}^{\infty} \frac{n-1}{(n+2)n!} x^n$.

Solution.

Here the n[th] term $T_n = \frac{n-1}{(n+2)n!} x^n$

$= \frac{n^2-1}{(n+2)!} x^n$.

Now, let n2 – 1 = A + B(n + 2) + C(n +2)(n +1).

We get A = 3, B = - 3, C = 1

Therefore $T_n = \frac{3}{(n+2)!} x^n - \frac{3}{(n+1)!} x^n + \frac{1}{n!} x^n$ ...............................(1)

Putting n = 1, 2, 3, ... in (1)we get

$T_1 = \frac{3}{3!} x - \frac{3}{2!} x + \frac{1}{1!} x$

$T_2 = \frac{3}{4!} x^2 - \frac{3}{3!} x^2 + \frac{1}{2!} x^2$

$T_3 = \frac{3}{5!} x^3 - \frac{3}{4!} x^3 + \frac{1}{3!} x^3$

... ... ... ... ...

... ... ... ... ...

Adding we get

$S = 3\left[\frac{x}{3!} + \frac{x^2}{4!} + \cdots\right] - 3\left[\frac{x}{2!} + \frac{x^2}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$

$= \frac{3}{x^2}\left[\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] - \frac{3}{x}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + \left[\frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$

$= \frac{3}{x^2}\left(e^x - \frac{x^2}{2!} - \frac{x}{1!} - 1\right) - \frac{3}{x}\left(-\frac{x}{1!} - 1\right) + (e^x - 1)$

$= \frac{3}{x^2}e^x - \frac{3}{2} - \frac{3}{x} - \frac{3}{x^2} - \frac{3}{x}e^x + 3 + \frac{3}{x} + e^x - 1$

$= e^x\left(\frac{3}{x^2} - \frac{3}{x} + 1\right) - \frac{3}{x^2} + \frac{1}{2} = e^x\left(\frac{3-3x+x^2}{x^2}\right) + \left(\frac{x^2-6}{2x^2}\right).$

$= \frac{2e^x(x^2-3x+3)+(x^2-6)}{2x^2}.$

8. Show that $\frac{1^2 2^2}{1!} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \cdots = 27e.$

Solution.

$n^{th}$ term $T_n = \frac{n^2(n+1)^2}{n!} = \frac{n(n+1)^2}{(n-1)!}$

let $n(n+1)^2 = A + B(n-1) + C(n-1)(n-2) + D(n-1)(n-2)(n-3)$

we get A = 4;B = 14; C = 8; D = 1.

Therefore $T_n = \frac{4}{(n-1)!} + \frac{14}{(n-2)!} + \frac{8}{(n-3)!} + \frac{1}{(n-4)!}.$

$T_1 = \frac{4}{1}$

$T_2 = \frac{4}{1!} + 14$

$T_3 = \frac{4}{2!} + \frac{14}{1!} + 8$

$T_4 = \frac{4}{3!} + \frac{14}{2!} + \frac{8}{1!} + 1$

$T_5 = \frac{4}{4!} + \frac{14}{3!} + \frac{8}{2!} + \frac{1}{1!}$

… … … … …

… … … … …

Adding we get

Therefore $S = 4\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + 14\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + 8\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) +$

$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right)$

= 4e +14e +8e + e = 27e.

**Logarithmic series**

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{1.2}{3!}x^3 - \frac{1.2.3}{4!}x^4 \ldots..$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$$

**Modification of the logarithmic series.**

If $-1 < x < 1$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots.. + (-1)^{n-1}\frac{x^n}{n} + \ldots \qquad \ldots\ldots\ldots(1)$$

It is convenient to remember the form of the series in the case in which x is negative.

Thus

$$\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \ldots\ldots$$

$$= -(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots)$$

i.e., $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$ ..........(2)

Adding the series (1) and (2),

$$\log(1+x) - \log(1-x) = 2x + 2.\frac{1}{3}x^3 + 2.\frac{1}{5}x^5 + \ldots\ldots$$

i.e., $\log\frac{1+x}{1-x} = 2\,(x + \frac{x^3}{2} + \frac{x^5}{3} + \cdots)$

$$\log(1+x) + \log(1-x) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots\right)$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Using the different forms of the logarithmic series we can find the sums of the certain series. The following examples will illustrate the methods of such summation.

**Example 1.** Show that if $x > 0$ . $\log x = \frac{x-1}{x+1} + \frac{1}{2}\cdot\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\cdot\frac{x^3-1}{(x+1)^3} + \ldots..$

Solution.

$$\text{R.H.S.} = \frac{x}{x+1} + \frac{1}{2}\cdot\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}\cdot\left(\frac{x}{x+1}\right)^3 \ldots.. - \left\{\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} + \frac{1}{3}\cdot\frac{1}{(x+1)^3}\cdots\right\}$$

$$= -\log\,(1 - \frac{x}{x+1}) + \log\,(1 - \frac{x}{x+1})$$

$$= -\log\frac{1}{x+1} + \log\frac{x}{x+1}$$

$$= \log\left\{\left(\frac{x}{x+1}\right) + \frac{1}{x+1}\right\}$$

$$= \log x\ .$$

The expansion is valid when

$\left|\frac{x}{x+1}\right| < 1$ and $\left|\frac{1}{x+1}\right| < 1$ , $\left|\frac{x}{x+1}\right|$ is always less than 1.

When $\left|\frac{1}{x+1}\right| < 1$ , $|x+1| > 1$ , i.e., $|x| > 0$

∴ When $x > 0$, the expansion is valid .

**Example 2.** Show that $\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \ldots.$

Solution.

Right side expression can be written as

$$\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \ldots. + 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \ldots..$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{6}\cdot\left(\frac{1}{2}\right)^6 + \ldots.. + 1 + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{7}\cdot\left(\frac{1}{2}\right)^6 + \ldots.$$

$$= \frac{1}{2}\cdot x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \ldots.. + 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \ldots. \text{ When } x = \frac{1}{2}$$

$$= \frac{1}{2}\left\{ x^2 + \frac{1}{2}\cdot x^4 + \frac{1}{3}x^6 + \ldots..\right\} + \frac{1}{x}\left\{ x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots\right\}$$

$$= -\frac{1}{2}\log(1-x^2) + \frac{1}{2x}\log\frac{1+x}{1-x}.$$

∴ The series $= -\frac{1}{2}\log\left(1-\frac{1}{4}\right) + \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$ , since $x = \frac{1}{2}$.

$$= -\frac{1}{2}\log\frac{3}{4} + \log 3$$

$$= \frac{1}{2}\log 9 - \frac{1}{2}\log\frac{3}{4}$$

$$= \frac{1}{2}\log\left(\frac{9.4}{3}\right)$$

$$= \frac{1}{2}\log 12$$

$$= \log\sqrt{12}.$$

**Example 3.** If $a$ , $b$ , $c$ denote three consecutive integers, show that

$$\log_e b = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3} \cdot \frac{1}{(2ac+1)^3} + \ldots.$$

Solution.

$$\text{Right side} = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{2ac+1}{2ac}$$

$$= \frac{1}{2}\log(ac) + \frac{1}{2}\log\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log ac \,.\, \frac{ac+1}{ac}$$

$$= \frac{1}{2}\log(ac+1).$$

If $a$ , $b$ , $c$ denote three consecutive integers then $b = a + 1$ and $b = c - 1$

$\therefore$ $a = b - 1$ ; $c = b + 1$.

$\therefore$ $ac = b^2 - 1$ , i.e., $ac + 1 = b^2$.

$\therefore$ $\frac{1}{2}\log(ac+1) = \frac{1}{2}\log(b^2) = \log b.$

**Exercises**

1. Show that

$$\log\frac{a+x}{a-x} = \frac{2ax}{a^2+x^2} + \frac{1}{3} \cdot \left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5} \cdot \left(\frac{2ax}{a^2+x^2}\right)^5 + \ldots..$$

2. Sum the series $\frac{1}{2x-1} + \frac{1}{3} \cdot \frac{1}{(2x-1)^3} + \frac{1}{5} \cdot \frac{1}{5(2x-1)^5} + \ldots.$

3. Show that when $-1 < x < \frac{1}{3}$

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) = \frac{2x}{1-x} - \frac{1}{2} \cdot \left(\frac{2x}{1-x}\right)^2 + \frac{1}{3} \cdot \left(\frac{2x}{1-x}\right)^3 \ldots.$$

4. Show that

$$\log(x+2h) = 2\log(x+h) - \log(x) - \left\{\frac{h^2}{(x+h)^3} + \frac{h^4}{2(x+h)^3} + \frac{h^6}{3(x+h)^3} + \cdots\right\}.$$

5. Show that

$$\log_e\left(1+\frac{1}{n}\right)^2 = 1 - \frac{1}{2(n+1)} - \frac{1}{2.3(n+1)^2} - \frac{1}{3.4(n+1)^3} \ldots\ldots \infty.$$

6. Show that $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots\ldots$

7. Sum the series $(1+\frac{1}{2}) + \left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9} + \left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2} + \ldots\ldots$ to infinity.

8. Sum to infinity the series $\sum\left(\frac{1}{2n+1}+\frac{1}{(2n)!}\right)x^{2n+1}$ , $(x^2 < 1)$.

9. Prove that $\sum_1^\infty \frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right) = \frac{1}{2}\log_e 10$.

Answer : 2. $\frac{1}{2}log\left(\frac{x}{x-1}\right)$, 7. 9log 3 – 12log 2, 8. $\frac{1}{2}\left[log\frac{1+x}{1-x} + x(e^x + e^{-x})\right]$.

**Series which can be summed up by the logarithmic series.**

We can split the general term into partial fractions and using the result

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$ We can sum certain series. The following examples will illustrate the method.

**Example 1.** Sum the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)2n(2n+1)}$ .

Solution.

Let S be the sum of the series and $u_n$ be the $n^{\text{th}}$ term.

Then $u_n = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2}\cdot\frac{1}{2n+1}$

$\therefore\ u_1 = \frac{1}{2}\cdot\frac{1}{1} - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}$

$u_2 = \frac{1}{2}\cdot\frac{1}{3} - \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{5}$

$u_3 = \frac{1}{2}\cdot\frac{1}{5} - \frac{1}{6} + \frac{1}{2}\cdot\frac{1}{7}$

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Adding the last fraction of a term with the first fraction of the next term, we get

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots..$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots.$$

$$= -\frac{1}{2} + \log 2.$$

**Example 2.** Show that $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \ldots. \infty = 3 \log 2 - 1.$

Solution.

Let S be the sum of the series and $u_n$ be the $n^{\text{th}}$ term of the series.

Then $u_n = \frac{2n+3}{(2n-1)(2n+1)}$.

Splitting $u_n$ into partial fractions, we get

$$u_n = 2 \cdot \frac{1}{2n-1} - 3 \cdot \frac{1}{2n} + 1 \cdot \frac{1}{2n+1}$$

Giving values 1 , 2 , 3 , …. in $u_n$ , we have

$$u_1 = 2 \cdot \frac{1}{1} - 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3}$$

$$u_2 = 2 \cdot \frac{1}{3} - 3 \cdot \frac{1}{4} + 1 \cdot \frac{1}{5}$$

$$u_3 = 2 \cdot \frac{1}{5} - 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{7}$$

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$$\therefore \; S = 2 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{4} + 3 \cdot \frac{1}{5} \ldots.$$

$$= 2 + 3(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5})$$

$= 2 + 3(1 - \frac{1}{2} + \frac{1}{3} \ldots \ldots - 1)$

$= 2 + 3\ (\log 2 - 1)$

$= -1 + 3 \log 2.$

**Exercises**

Show that the sum of the series to infinity

1. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{3.6} + \ldots. = \log 2$
2. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots. = 2 - \log 2.$
3. $\frac{1}{1.2.3} + \frac{5}{3.4.5} + \frac{9}{5.6.7} + \frac{13}{7.8.9} + \ldots. = \frac{5}{2} - 3 \log 2.$
4. $\frac{1}{2.3.4} + \frac{5}{4.5.6} + \frac{9}{6.7.8} + \ldots. = \frac{3}{4} - \log 2$

If $k$ is a positive integer and $|x| < 1$ , then

$$\sum_{n=1}^{\infty} \frac{x^2}{n+k} = \frac{x}{1+k} + \frac{x^2}{2+k} + \frac{x^3}{3+k} + \frac{x^4}{4+k} + \ldots\ldots$$

$$= \frac{1}{x^k}\left(\frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \cdots \infty\right)$$

$$= \frac{1}{x^k}\ \{\ x + \frac{x^2}{2} + \ldots. + \frac{x^k}{k} + \frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \ldots\ \infty$$

$$-(\ x + \frac{x^2}{2} + \ldots. + \frac{x^k}{k})\ \}$$

$$= \frac{1}{x^k}\ \{ -\log(1 - x) - (\ x + \frac{x^2}{2} + \ldots. + \frac{x^k}{k})\ \}$$

$$= -\frac{1}{x^k}\ \{\ \log(1 - x) + x + \frac{x^2}{2} + \ldots. + \frac{x^k}{k}\ \}$$

Similarly $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = -\frac{1}{x}\ \{\ \log(1 - x) + x\}$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2}\ \{\ \log(1 - x) + x + \frac{x^2}{2}\ \}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+3} = -\frac{1}{x^3}\ \{\ \log(1 - x) + x + \frac{x^2}{2} + \frac{x^3}{3}\ \}$$

Using these result we can sum certain series. The following examples will illustrate the method.

**Example 1.** Sum the series $\sum_{n=1}^{\infty} \frac{n^3+n^2+1}{n(n+2)} x^n$ when $|x| < 1$ .

Solution.

Split $\frac{n^3+n^2+1}{n(n+2)}$ into partial fractions.

We have $S = \sum_{n=1}^{\infty} \{ (n-1) + \frac{1}{2} . \frac{1}{n} + \frac{3}{2} . \frac{1}{n+2} \} x^n$

$= \sum_{n=1}^{\infty} \{ (n-1)x^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{x^n}{n+2}$ .

$\sum_{n=1}^{\infty} \{ (n-1)x^n = x^2 + 2x^3 + 3x^4 + ..... \infty$

$= x^2 ( 1 + 2x + 3x^2 + .... \infty )$

$= x^2 (1-x)^2 = \frac{x^2}{(1-x)^2}$.

$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$.

$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2} \{ \log(1-x) + x + \frac{x^2}{2} \}$

$\therefore S = \frac{x^2}{(1-x)^2} - \frac{1}{2} \log(1-x) - \frac{3}{2x^2} \{ \log(1-x) + x + \frac{x^2}{2} \}$.

**Example 2.** Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)(n+2)}$ .

Solution.

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2} . \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} . \frac{1}{n+2}$$

Let S be the sum of the series

$$S = \sum_{1}^{\infty} \left( \frac{1}{2} . \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} . \frac{1}{n+2} \right) (-1)^{n+1} x^n$$

$$= \frac{1}{2} \cdot \sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n} - \sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+1} + \frac{1}{2} \sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+2}.$$

We have

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \ldots.. = \log\,(1+x)$$

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+1} = \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \ldots\ldots = \frac{1}{x}\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \ldots.\right)$$

$$= \frac{1}{x}\,\{-\log\,(1+x) + x\}$$

$$\sum_1^\infty \frac{(-1)^{n+1}\, x^n}{n+2} = \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \ldots\ldots = \frac{1}{x^2}\left\{\frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \ldots.\right\}$$

$$= \frac{1}{x^2}\,\{\log\,(1+x) - x + \frac{x^2}{2}\}.$$

$$\therefore\ S = \frac{1}{2}\log\,(1+x) - \frac{1}{x}\{-\log\,(1+x) + x\} + \frac{1}{2x^2}\,\{\log\,(1+x) - x + \frac{x^2}{2}\}$$

$$= \frac{1}{2}\log\,(1+x)\,(1 + \frac{2}{x} + \frac{1}{x^2}) - (\frac{3}{4} + \frac{1}{2x}).$$

**Exercises**

1. Prove that the sum of the infinite series whose $n^{th}$ term is $\frac{1}{n(n+1)} \cdot \frac{1}{2^n}$ is $1 - \log 2$.
2. Sum the series

   (1) $\sum_1^\infty \frac{n^2+1}{n(n+2)} x^n$.

   (2) $\sum_1^\infty \frac{(n+1)^3}{n(n+3)} x^n$.

   (3) $\sum_1^\infty \frac{n^2}{(n+1)(n+2)} x^n$.
3. Show that

   (1) $\frac{3}{1.2.2} - \frac{4}{2.3.2^2} + \frac{5}{3.4.2^3} - \ldots. = 4\log\frac{3}{2} - 1$.
4. Show that

   (1) $\sum_{r=1}^\infty \frac{4r-1}{2r(2r-1)} \cdot \frac{1}{3^{2r}} = \log 3 - \frac{4}{3}\log 2$.

$$+\frac{1}{3}\left(\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ \ldots.\right)^3$$

$=\frac{x}{2}+\frac{x^2}{24}+$ terms in $x^4$ and higher powers of $x$.

Hence $y=\frac{1}{2}+\frac{x}{24}$.

**Exercises**

1. Evaluate $\lim_{x\to 0}\frac{e^x-\log(e+ex)}{x^2}$.
2. Evaluate $\lim_{x\to 0}\frac{e^x-\log_e(1+x)(1+2x)}{5x^3}$.
3. Find $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$.
4. Find the limit as $x\to 1$ of $\frac{\log x}{x^2-3x+2}$.
5. Evaluate $\lim_{x\to 0}\frac{(2+x)\log(1+x)+(2-x)\log(1-x)}{x^4}$.
6. Evaluate $\lim_{n\to\infty}(1+\frac{3}{n^2}+\frac{1}{n^3})^{n^2}$ .
7. Find the value, when $x$ tends to the limit 1 of the expression $\log(x^{5/2}-1)-\log(x^{3/2}-1)$.
8. Show that when $x$ is small , log $\{(1+x)^{1/3}+(1-x)^{1/3}$ is approximately equal to $\log 2-\frac{x^2}{9}$.
9. By using the fact that $(1+\frac{x}{n})^n=e^{x\log(1+\frac{x}{n})}$ prove that $(1+\frac{x}{n})^n+(1-\frac{x}{n})^{-n}=2e^x\ \{1+\frac{1}{n^2}\left(\frac{x^2}{3}+\frac{x^4}{8}\right)\}$.

   Answer : 1.2, 2. $\frac{1}{10}$, 3. $\frac{3}{2}$, 4 – 1, 5. .$-\frac{1}{3}$ ,6. $e^3$, 7 $log\left(\frac{5}{3}\right)$.

**Extra problems**.

1. Show that $\left[\frac{a-b}{a}\right]+\frac{1}{2}\left[\frac{a-b}{a}\right]^2+\frac{1}{3}\left[\frac{a-b}{a}\right]^3+\ldots=\log_e a-\log_e b$.

   Solution.

   Put $\frac{a-b}{a}=x$.

Therefore L.H.S = x + $\frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

= - log(1 – x)

= - log$\left(1 - \frac{a-b}{a}\right)$ = - log$\left(\frac{b}{a}\right)$ = log$\left(\frac{a}{b}\right)$

= log a – log b.

= R.H.S.

2. Prove that log $\sqrt{\frac{n+1}{n}} = \left(\frac{1}{2n+1}\right) + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots$

Solution.

Let $\frac{1}{2n+1}$ = x.

Therefore R.H.S = $\frac{1}{2}log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n-1}}\right)$

= $\frac{1}{2}log\left(\frac{2n+2}{2n}\right) = \frac{1}{2}log\left(\frac{n+1}{n}\right)$

= log $\sqrt{\frac{n+1}{n}}$.

= L.H.S.

3. Show that $\frac{3}{10}\left[log10 + \frac{1}{2^7} + \frac{1}{2}.\frac{1}{2^{14}} + \frac{1}{3}.\frac{3^2}{2^{21}} + \cdots\right]$ = log 2.

Solution.

L.H.S = $\frac{1}{10}\left[3log10 + \left(\frac{3}{2^7}\right) + \frac{1}{2}.\left(\frac{3}{2^7}\right)^2 + \frac{1}{2}.\left(\frac{3}{2^7}\right)^3 + \cdots\right]$

= $\frac{1}{10}\left[log1000 - log\left(1 - \frac{3}{2^7}\right)\right]$

= $\frac{1}{10}\left[log1000 - log\left(\frac{125}{2^7}\right)\right]$

= $\frac{1}{10}\,log\left(\frac{1000 \times 2^7}{125}\right)$

= $\frac{1}{10}\,log2^{10}$ = log 2 = R.H.S.

4. Sum to infinity the series $\left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5} + \frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$

Solution.

$$\left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{5} + \frac{1}{6}\right)\left(\frac{1}{9^2}\right) + \cdots$$

= $\left[1 + \left(\frac{1}{3} - \frac{1}{9}\right) + \frac{1}{5}\left(\frac{1}{9^2}\right) + \cdots\right] + \left[\frac{1}{2} + \frac{1}{4}\left(\frac{1}{9}\right) + \frac{1}{6}\left(\frac{1}{9^2}\right) + \cdots\right]$

= $3\left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots\right] + \frac{9}{2}\left[\frac{1}{9} + \frac{1}{2}\left(\frac{1}{9}\right)^2 + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \cdots\right]$

= $3\left[\frac{1}{2}log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\right] - \frac{9}{2}log\left(1 - \frac{1}{9}\right)$

$= \frac{3}{2}\log - \frac{9}{2}\ log\left(\frac{8}{9}\right) = \frac{3}{2}\left[log2 - 3log\left(\frac{8}{9}\right)\right]$

$= \frac{3}{2}[log2 - 3log8 + 3log9]$

$= \frac{3}{2}[log2 - 9log2 + 6log3]$

$= \frac{3}{2}[6log3 - 8log2]$

= 9log3 – 12log2.

5. Prove that $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

Solution.

Put $x = \frac{1}{n+1}$

Then L.H.S $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

$= -\log(1 - x) = -\log\left(1 - \frac{1}{n+1}\right) = -\log\left(\frac{n}{n+1}\right)$

$= \log\left(\frac{n+1}{n}\right) = \log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

= R.H.S.

6. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ prove that $x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

Solution.

$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (i.e) $y = \log(1 + x)$

$e^y = 1 + x$

Therefore $x = e^y - 1 = \left[\frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right] - 1.$

Therefore $x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

7. If $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \cdots$ and $|x| < 1$ show that $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Solution.

$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \cdots$

$= -\left[-\frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots\right]$

$= -[e^{-y} - 1]$

Thus $x = 1 - e^{-y}$

$e^{-y} = 1 - x$

$-y = \log_e(1 - x)$

$y = -\log_e(1 - x)$

Therefore y = x + $\frac{x^2}{2} + \frac{x^3}{3} + \cdots$ .

8. If log $(1 - x + x^2)$ be expanded in ascending powers of x in the form $a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$ prove that $a_3 + a_6 + a_9 + \ldots\ldots = \frac{2}{3}\log 2$.

Solution.

$\log(1 - x + x^2) = \log\left[\frac{1+x^3}{1+x}\right]$

$= \log(1+x^3) - \log(1+x)$

$= \left[x^3 - \frac{(x^3)^2}{2} + \cdots + \frac{(-1)^{n-1}(x^3)^n}{n} + \cdots\right] - \left[x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots\right]$

Coefficient of $x^{3n}$ is $a_{3n} = \frac{(-1)^{n-1}}{n} - \frac{(-1)^{3n-1}}{3n}$

$= \frac{(-1)^{n-1}}{n}\left[1 - \frac{1}{3}\right]$

$= (-1)^{n-1}[\frac{2}{3n}]$ .....................(1)

Putting n = 1, 2, 3, ... in (1) and adding we get

$a_3 + a_6 + a_9 + \ldots.. = \frac{2}{3}\left[1 - \frac{1}{2} + \frac{1}{3} - \cdots\right]$

$= \frac{2}{3}\ log_e 2.$

9. Show that if x > 0 log x = $\frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots$

Solution.

R.H.S = $\left(\frac{x}{x+1}\right) + \frac{1}{2}\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}\left(\frac{x}{x+1}\right)^3 + \cdots$

$+ \left[-\left(\frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)^2 - \frac{1}{3}\left(\frac{1}{x+1}\right)^3 - \cdots\right]$

$= -\log\left[1 - \frac{x}{x+1}\right] + \log\left[1 - \frac{1}{x+1}\right]$

$= -\log\left[\frac{1}{x+1}\right] + \log\left[\frac{x}{x+1}\right]$

= log x

= L.H.S.

10. If f(x) = x + $\frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$ where -1 < x < 1.

(i) Represent f(x) as a logarithmic function

(ii) Hence prove $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$

Solution. (i) For – 1 < x < 1 we have

$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

$\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$

$\log(1 + x) - \log(1 - x) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right]$

$$\frac{1}{2}\log\left[\frac{1+x}{1-x}\right] = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$f(x) = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]$

(ii) Now, $f\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}\log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$

$= \frac{1}{2}\log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$

$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)^2$

$= 2\ f(x)$.

11. Sum the series to infinity $\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots\ldots\ldots\ldots$

Solution.

$\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots\ldots\ldots\ldots$

$= \frac{1}{\log_e 3} - \frac{1}{\log_e 9} + \frac{1}{\log_e 27} - \frac{1}{\log_e 81} + \cdots$

$= \frac{1}{\log_e 3} - \frac{1}{2\log_e 3} + \frac{1}{3\log_e 3} - \frac{1}{4\log_e 3} + \cdots$

$= \frac{1}{\log_e 3}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= \frac{\log_e 2}{\log_e 3} = \log_e 2 \times \log_3 e = \log_3 2.$

12. Show that $(1 + x)^{1+x} = 1 + x + x^2 + \frac{1}{2}x^3$ neglecting and higher powers of x. Also find an approximate value of $(1.01)^{1.01}$.

Solution.

$(1 + x)^{1+x} = e^{\log(1+x)^{1+x}}$

$= e^{(1+x)\log(1+x)}$

$\approx e^{(1+x)\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3\right)}$

$\approx e^{x+\frac{1}{2}x^2-\frac{1}{6}x^3}$

$\approx 1 + \left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right) + \frac{1}{2!}\left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)^2 + \frac{1}{3!}\left(x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)^3$

$\approx 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{2!}(x^2 + x^3) + \frac{1}{3!}x^3$

$\approx 1 + x + x^2 + \frac{1}{2}x^3$

Put x = .01 in the result.

$(1.01)^{1.01}$ = 1 + .01 + .0001+ $\frac{1}{2}$(.000001) = 1.0101005.

13. Prove S = $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \cdots = 2 - log2.$

Solution.

Here $T_n = \frac{1}{n(2n+1)}$

$T_n = \frac{A}{n} + \frac{B}{2n+1}$

We can find A = 1; B = - 2

Therefore $T_n = \frac{1}{n} - \frac{2}{2n+1}$ ..............................(1)

Putting n = 1, 2, 3, ... in (1) we get

$T_1 = \frac{1}{1} - \frac{2}{3}$

$T_2 = \frac{1}{2} - \frac{2}{5}$

$T_3 = \frac{1}{3} - \frac{2}{7}$

... ... ... ...

Therefore S = $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

$= 1 - \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

= 1 – [log 2 – 1]

= 2 – log 2.

14. Prove S = $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \cdots$ = log 4 – 1

Solution.

$T_n = (-1)^{n-1}\left[\frac{1}{n(n+1)}\right]$

We have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$T_n = (-1)^{n-1}\left[\frac{1}{n} - \frac{1}{n+1}\right]$ ..........................(1)

Putting n = 1, 2, 3, ....in (1) we get

$T_1 = \frac{1}{1} - \frac{1}{2}$

$T_2 = -\frac{1}{2} + \frac{1}{3}$

$T_3 = \frac{1}{3} - \frac{1}{4}$

… … … …

… … … …

Therefore $S = 1 + 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)$

$= 1 + 2(\log 2 - 1)$

$= \log 4 - 1.$

15. Prove that $\log\left(1 + \frac{1}{n}\right)^n = 1 - \frac{1}{2(n+1)} - \frac{1}{2.3(n+1)^2} - \frac{1}{3.4(n+1)^3} - \cdots$

Solution.

Put $\frac{1}{n+1} = x$

Therefore R.H.S $= 1 - \frac{1}{2}x - \frac{1}{2.3}x^2 - \frac{1}{3.4}x^3 - \cdots$

$= 1 - \left(1 - \frac{1}{2}\right)x - \left(\frac{1}{2} - \frac{1}{3}\right)x^2 - \left(\frac{1}{3} - \frac{1}{4}\right)x^3 - \cdots$

$= \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right) + \left(1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots\right)$

$= -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right) + \frac{1}{x}\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$

$= \log(1 - x) - \frac{1}{x}\log(1 - x)$

$= \left(1 - \frac{1}{x}\right)\log(1 - x)$

$= (1 - n - 1)\log\left(1 - \frac{1}{n+1}\right)$

$= -n\log\left(\frac{n}{n+1}\right) = \log\left(\frac{n+1}{n}\right)^n$

= L.H.S.

# UNIT - IV MATRICES

**Matrix :** A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{12}.........a_{2n} \\ ......................... & \\ ........................ & \\ a_{m1} & a_{m2}.......a_{mn} \end{bmatrix}_{m \times n}$ $= [a_{ij}]_{mxn}$ where $1 \le i \le m$, $1 \le j \le n$.

**Order of the Matrix:** The number of rows and columns represents the order of the matrix. It is denoted by mxn, where m is number of rows and n is number of columns.

**Types of Matrices:**

**Row Matrix:** A Matrix having only one row is called a "Row Matrix".

Eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

**Column Matrix:** A Matrix having only one column is called a "Column Matrix" .

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

**Null Matrix:** A= $\left[a_{ij}\right]_{mxn}$ such that $a_{ij}=0$ $\forall$ i and j. Then A is called a "Zero Matrix". It is denoted by Omxn.

Eg: $O_{2x3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Rectangular Matrix:** If A=$\left[a_{ij}\right]_{mxn}$, and $m \neq n$ then the matrix A is called a "Rectangular Matrix"

Eg : $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

**Square Matrix:** If A=$\left[a_{ij}\right]_{mxn}$ and m = n then A is called a "Square Matrix".

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is a 2x2 matrix

**Lower Triangular Matrix:** A square Matrix $A_{nXn} = \left[a_{ij}\right]_{nxn}$ is said to be lower Triangular of $a_{ij} = 0$ if i <j i.e. if all the elements above the principle diagonal are zeros.

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is a Lower triangular matrix

**Upper Triangular Matrix:** A square Matrix $A = \left[a_{ij}\right]_{nxn}$ is said to be upper triangular of $a_{ij} = 0$ if i.>j. i.e. all the elements below the principle diagonal are zeros.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an Upper triangular matrix

**Triangle Matrix:** A square matrix which is either lower triangular or upper triangular is called a triangle matrix.

**Principal Diagonal of a Matrix:** In a square matrix, the set of all $a_{ij}$, for which i = j are called principal diagonal elements. The line joining the principal diagonal elements is called principal diagonal.

**Note:** Principal diagonal exists only in a square matrix.

**Diagonal elements in a matrix:** A= $[a_{ij}]_{nxn}$, the elements $a_{ij}$ of A for which i = j.

i.e. $a_{11}$, $a_{22}$….$a_{nn}$ are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

**Diagonal Matrix:** A Square Matrix is said to be diagonal matrix, if $a_{ij} = 0$ for $i \neq j$ i.e. all the elements except the principal diagonal elements are zeros.

**Note:** 1. Diagonal matrix is both lower and upper triangular.

2. If $d_1$, $d_2$……$d_n$ are the diagonal elements in a diagonal matrix it can be represented as **diag** $[d_1, d_2, \ldots\ldots, d_n]$

Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

**Scalar Matrix:** A diagonal matrix whose leading diagonal elements are equal is called a "Scalar Matrix". Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**Unit/Identity Matrix:** If A = $\left[a_{ij}\right]_{nxn}$ such that $a_{ij}$=1 for i = j, and $a_{ij}$=0 for $i \neq j$ then A is called a "Identity Matrix" or Unit matrix. It is denoted by $I_n$

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Trace of Matrix:** The sum of all the diagonal elements of a square matrix A is called Trace of a matrix A, and is denoted by Trace A or tr A.

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then tr A = a+b+c

**Singular & Non Singular Matrices:** A square matrix A is said to be "Singular" if the determinant of $|A| = 0$**,** Otherwise A is said to be "Non-singular".

**Note:** 1. Only non-singular matrices possess inverse.

2. The product of non-singular matrices is also non-singular.

**Inverse of a Matrix:** Let A be a non-singular matrix of order n if there exist a matrix B such that AB=BA=I then B is called the inverse of A and is denoted by $A^{-1}$.

If inverse of a matrix exist, it is said to be invertible.

**Note: 1**. The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$.

**2** .Every Invertible matrix has unique inverse**.**

**3**. If A, B are two invertible square matrices then AB is also invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

**4.** $A^{-1} = \frac{AdjA}{\det A} \quad where\ detA \neq 0$,

**Theorem: The inverse of a Matrix if exists is Unique.**

**Note:** 1. $(A^{-1})^{-1} = A$ 2. $I^{-1} = I$

**Theorem: If A, B are invertible matrices of the same order, then**

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^{1})^{-1} = (A^{-1})^{1}$

**Sub Matrix: -** A matrix obtained by deleting some of the rows or columns or both from the given matrix is called a sub matrix of the given matrix.

Eg: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and $4^{th}$ column of A.

**Minor of a Matrix:** Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

**Note:** If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ be a 4x3 matrix

Here B = $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2

And C = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

det C = 2(7-12)-1(21-10)+(18-5) = -9

**Properties of trace of a matrix:** Let A and B be two square matrices and $\lambda$ be any scalar

1) tr ( $\lambda$ A) = $\lambda$ (tr A) ; 2) tr(A+B) = trA + trB ; 3) tr (AB) = tr(BA)

**Idempotent Matrix:** A square matrix A Such that $A^2$=A then A is called "Idempotent Matrix".

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Involutary Matrix:** A square matrix A such that $A^2$ = I then A is called an Involutary Matrix.

Eg: A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**Nilpotent Matrix:** A square matrix A is said to be Nilpotent if there exists a + ve integer n such that $A^n$ = 0 here the least n is called the Index of the Nilpotent Matrix.

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

**Transpose of a Matrix:** The matrix obtained by interchanging rows and columns of the given matrix A is called as transpose of the given matrix A. It is denoted by $A^T$ or $A^1$

Eg: A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then $A^T$ = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

**Properties of transpose of a matrix:** If A and B are two matrices and $A^T$ , $B^T$ are their transposes then

1) $\left(A^T\right)^T = A$ ; 2) $\left(A+B\right)^T = A^T + B^T$; 3) $\left(KA\right)^T = KA^T$ ; 4) $\left(AB\right)^T = B^T A^T$

**Symmetric Matrix:** A square matrix A is said to be symmetric if $A^T = A$

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]nxn$ where $a_{ij} = a_{ji}$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

**Skew-Symmetric Matrix:** A square matrix A is said to be Skew symmetric If $A^T = -A$.

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]_{nxn}$ *where* $a_{ij} = -a_{ji}$.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

**Note**: All the principle diagonal elements of a skew symmetric matrix are always zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

**Theorem: Every square matrix can be expressed uniquely as the sum of symmetric and skew symmetric matrices.**

**Proof:** Let A be a square matrix, A = $\frac{1}{2}(A+A)$ = $\frac{1}{2}\left(A+A^T+A-A^T\right)$ =

$\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$ = P + Q, where P =$\frac{1}{2}\left(A+A^T\right)$; Q = $\frac{1}{2}\left(A-A^T\right)$

Thus every square matrix can be expressed as a sum of two matrices.

Consider $P^T = \left[\frac{1}{2}\left(A+A^T\right)\right]^T = \frac{1}{2}\left(A+A^T\right)^T = \frac{1}{2}\left(A^T+\left(A^T\right)^T\right) = \frac{1}{2}\left(A+A^T\right)$=P, since $P^T = P$,

P is symmetric

Consider $Q^T = \left[\frac{1}{2}\left(A-A^T\right)\right]^T = \frac{1}{2}\left(A-A^T\right)^T = \frac{1}{2}\left(A^T-\left(A^T\right)^T\right) = -\frac{1}{2}\left(A-A^T\right)$ = - Q

Since $Q^T = -Q$, Q is Skew-symmetric.

**To prove the representation is unique:** Let A= R+S $\rightarrow (1)$ be the representation, where R is symmetric and S is skew symmetric. i.e. $R^T = R, S^T = -S$

Consider $A^T = (R+S)^T = R^T + S^T = R - S \rightarrow (2)$

$(1)-(2) \Rightarrow A - A^T = 2S \Rightarrow S = \frac{1}{2}\left(A-A^T\right) = Q$

Therefore every square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix

**Ex. Express the given matrix A as a sum of a symmetric and skew symmetric matrices**

**where A=**$\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$

**Solution:** $A^T = \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 3 & 11 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} \Rightarrow P = \frac{1}{2}\left(A + A^T\right) = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}; P \text{ is symmetric}$$

$$A - A^T = \begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} \Rightarrow Q = \frac{1}{2}\left(A - A^T\right) = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}; Q \text{ is skew-symmetric}$$

Now A =P+Q = $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

**Orthogonal Matrix:** A square matrix A is said to be an Orthogonal Matrix if $AA^T = A^TA = I$,

Similarly we can prove that $A = A^{-1}$; Hence A is an orthogonal matrix.

**Note:** 1. If A, B are orthogonal matrices, then AB and BA are orthogonal matrices.

2. Inverse and transpose of an orthogonal matrix is also an orthogonal matrix.

**Result:** If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

**Result:** The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal

**Solved Problems :**

**1. Show that A =** $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

**Sol:** Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

**2. Prove that the matrix** $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}$ Then $A^T = \frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}$

Consider $A.A^T = \frac{1}{9}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$

$\Rightarrow A.A^T = I$

Similarly $A^T.A = I$

Hence A is orthogonal matrix

**3. Determine the values of a, b, c when $\begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}$ is orthogonal.**

Sol: - For orthogonal matrix $AA^T = I$

So, $AA^T = \begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}\begin{bmatrix}0 & a & a\\ 2b & b & -b\\ c & -c & c\end{bmatrix} = I$

$$\begin{bmatrix}4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2\\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2\\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2\end{bmatrix} = I = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$   $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ (since $c^2=2b^2$) $\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}$;   $b = \pm\frac{1}{\sqrt{6}}$; $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

4. Is matrix $\begin{bmatrix}2 & -3 & 1\\ 4 & 3 & 1\\ -3 & 1 & 9\end{bmatrix}$ Orthogonal?

Sol:- Given A= $\begin{bmatrix}2 & -3 & 1\\ 4 & 3 & 1\\ -3 & 1 & 9\end{bmatrix} \Rightarrow A^T = \begin{bmatrix}2 & 4 & -3\\ -3 & 3 & 1\\ 1 & 1 & 9\end{bmatrix}$

$\Rightarrow AA^T = \begin{bmatrix}2 & -3 & 1\\ 4 & 3 & 1\\ -3 & 1 & 9\end{bmatrix}\begin{bmatrix}2 & 4 & -3\\ -3 & 3 & 1\\ 1 & 1 & 9\end{bmatrix} = \begin{bmatrix}14 & 0 & 0\\ 0 & 26 & 0\\ 0 & 0 & 91\end{bmatrix} \neq I_3$

$AA^T \neq A^TA \neq I_3$

$\therefore$ Matrix is not orthogonal.

**Complex matrix:** A matrix whose elements are complex numbers is called a complex matrix.

**Conjugate of a complex matrix:** A matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called conjugate of a complex matrix. It is denoted by $\overline{A}$

If $A = \left[a_{ij}\right]_{mxn}$ , $\overline{A} = \left[\overline{a_{ij}}\right]_{mxn}$, where $\overline{a_{ij}}$ is the conjugate of $a_{ij}$.

**Eg:** If A= $\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

**Note:** If $\overline{A}$ and $\overline{B}$ be the conjugate matrices of A and B respectively, then

(i) $\overline{\left(\overline{A}\right)}$=A (ii) $\overline{A+B} = \overline{A}+\overline{B}$ (iii) $\overline{(KA)} = \overline{K}\ \overline{A}$

**Transpose conjugate of a complex matrix:** Transpose of conjugate of complex matrix is called transposed conjugate of complex matrix. It is denoted by $A^{\theta}$ *or* $A^{*}$.

**Note:** If $A^{\theta}$ and $B^{\theta}$ be the transposed conjugates of A and B respectively, then

(i) $\left(A^{\theta}\right)^{\theta}$ = A (ii) $\left(A \pm B\right)^{\theta} = A^{\theta} \pm B^{\theta}$

(iii) $\left(KA\right)^{\theta} = \overline{K}A^{\theta}$ (iv) $\left(AB\right)^{\theta} = A^{\theta}B^{\theta}$

**Hermitian Matrix:** A square matrix A is said to be Hermitian Matrix iff $A^{\theta} = A$.

Eg: A= $\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $A^{\theta}$= $\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

**Note:** 1. In Hermitian matrix the principal diagonal elements are real.

2. The Hermitian matrix over the field of Real numbers is nothing but real symmetric matrix.

3. In Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij} = \overline{a_{ji}}\ \forall i, j$.

**Skew-Hermitian Matrix:** A square matrix A is said to be Skew-Hermitian Matrix iff $A^{\theta} = -A$.

Eg: Let A= $\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\overline{A}\right)^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore$ $\left(\overline{A}\right)^T$=-A $\therefore$ A is skew-Hermitian matrix.

**Note:** 1. In Skew-Hermitian matrix the principal diagonal elements are either Zero or Purely Imaginary.

2. The Skew- Hermitian matrix over the field of Real numbers is nothing but real Skew - Symmetric matrix.

3. In Skew-Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij} = -\overline{a_{ji}}\ \forall i, j$ .

**Unitary Matrix:** A Square matrix A is said to be unitary matrix iff

$AA^{\theta} = A^{\theta}A = I \text{ or } A^{\theta} = A^{-1}$

Eg: $B = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

**Theorem1: Every square matrix can be uniquely expressed as a sum of Hermitian and skew – Hermitian Matrices.**

**Proof: -** Let A be a square matrix write

$$A = \frac{1}{2}(2A) = \frac{1}{2}(A+A) = \frac{1}{2}(A + A^{\theta} + A - A^{\theta})$$

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta}) i.e A = P + Q$$

Let $P = \frac{1}{2}\left(A + A^{\theta}\right); Q = \frac{1}{2}\left(A - A^{\theta}\right)$

Consider $P^{\theta} = \left[\frac{1}{2}(A + A^{\theta})^{\theta}\right] = \frac{1}{2}(A + A^{\theta})^{\theta} = (A + A^{\theta}) = P$

I.e. $P^{\theta} = P, P$ is Hermitian matrix.

$$Q^{\theta} = \left[\frac{1}{2}(A - A^{\theta})\right]^{\theta} = \frac{1}{2}\left(A^{\theta} - A\right) = -\frac{1}{2}(A - A^{\theta}) = -Q$$

Ie $Q^{\theta} = -Q, Q$ is skew – Hermitian matrix.

Thus every square matrix can be expressed as a sum of Hermitian & Skew Hermitian matrices.

To prove such representation is unique:

Let A = R+S------- (1) be another representation of A where R is Hermitian matrix & S is skew – Hermitian matrix.

$\therefore R = R^{\theta}; S^{\theta} = -S$

Consider $A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S -$. Ie $A^{\theta} = R - A - - - - - (2)$

$$(1)+(2) \Rightarrow A + A^{\theta} = 2R \; ie \; R = \frac{1}{2}\left(A + A^{\theta}\right) = P$$

$$(1)-(2) \Rightarrow A - A^{\theta} = 2S \; ie \; S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$$

Thus every square matrix can be uniquely expressed as a sum of Hermitian & skew Hermitian matrices.

**Solved Problems :**

**1) If A=** $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that A is Hermitian and iA is skew-Hermitian.**

**Sol:** Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\overline{(A)}^{T} = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = \left(\overline{A}\right)^{T}$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^{T} = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^{T}$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

**2). If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.**

**Sol:** Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^{T} = A$ And $\left(\overline{B}\right)^{T} = B$ ------------- (1)

$$\text{Now } \overline{(AB-BA)}^T = \left(\overline{AB}-\overline{BA}\right)^T$$

$$= \left(\overline{A}\,\overline{B}-\overline{B}\,\overline{A}\right)^T$$

$$= \left(\overline{A}\,\overline{B}\right)^T - \left(\overline{B}\,\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right)^T\left(\overline{B}\right)^T$$

$$= BA - AB \text{ (By (1))}$$

$$= -\left(AB-BA\right)$$

Hence AB-BA is a skew-Hemitian matrix.

**3). Show that A=** $\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ **is unitary if and only if** $a^2+b^2+c^2+d^2=1$

**Sol:** Given A= $\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^\theta = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

**4)Given that A=** $\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$**, show that** $(I-A)(I+A)^{-1}$ **is a unitary matrix.**

**Sol:** we have $I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-\left(4i^2-1\right)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B=(I-A)(I+A)^{-1}$

$$B=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\overline{(B)}^T=\frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e. B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

**5) Show that the inverse of a unitary matrix is unitary.**

**Sol:** Let A be a unitary matrix. Then $AA^{\theta}=I$

i.e $\left(AA^{\theta}\right)^{-1}=I^{-1}$

$$\Rightarrow \left(A^{\theta}\right)^{-1} A^{-1} = I$$

$$\Rightarrow \left(A^{-1}\right)^{\theta} A^{-1} = I$$

Thus $A^{-1}$ is unitary.

**Eigen Values and Eigen vectors:**

Let A= $\left[a_{ij}\right]_{nxn}$ be a square Matrix. Suppose the linear transformation Y = AX transforms X into a scalar multiple of itself i.e. AX = Y = $\lambda$ X, Then the unknown scalar $\lambda$ is known as an "Eigen value" of the Matrix A and the corresponding non-zero vector X is known as "Eigen Vector" of A. Corresponding to Eigen value $\lambda$. Thus the Eigen values (or) characteristic values (or) proper values (or) latent roots are scalars $\lambda$ which satisfy the equation.

AX = $\lambda$ X for X $\neq$ 0, $AX - \lambda IX = 0 \Rightarrow (A - \lambda I)X = 0$

Which represents a system of 'n' homogeneous equations in 'n' variables $x_1, x_2$, ----, $x_n$ this system of equations has non-trivial solutions If the coefficient matrix (A- $\lambda$ I) is singular i.e.

$$|A - \lambda I| = 0 \quad \Rightarrow \begin{vmatrix} a_{11-\lambda} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22-\lambda} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn-\lambda} \end{vmatrix} = 0$$

Expansion of the determinant is $(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + ----+K_n$ is the $n^{th}$ degree of a polynomial $P_n(\lambda)$ which is known as **"Characteristic Polynomial"**. Of A

$(-1)^n \lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + - - - - + K_n = 0$ is known as **"Characteristic Equation"**. Thus the Eigen values of a square Matrix A are the roots of the characteristic equation.

**Eg:** Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

**Eigen Value:** The roots of the characteristic equation are called Eigen values or characteristic roots or latent roots or proper values.

**Eigen Vector:** Let A=$\left[a_{ij}\right]_{nxn}$ be a Matrix of order n. A non-zero vector X is said to be a characteristic vector (or) Eigen vector of A if there exists a scalar $\lambda$ such that AX = $\lambda$ X.

**Method of finding the Eigen vectors of a matrix.**

Let A=[$a_{ij}$] be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition AX = λX.

⇒ AX = λIX

⇒ AX –λIX = 0

⇒ (A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns.

Will have a non-zero solution X if and only |A-λI| = 0

- A-λI is called characteristic matrix of A
- |A-λI| is a polynomial in λ of degree n and is called the characteristic polynomial of A
- |A-λI|=0 is called the characteristic equation
- Solving characteristic equation of A, we get the roots , $\lambda_{1,}\lambda_{2,}\lambda_{3,} \dots \dots . \lambda_{n,}$ These are called the characteristic roots or eigen values of the matrix.
- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

**Procedure to find Eigen values and Eigen vectors**

Let A = $\begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots . . & a_{2n} \\ \dots \dots . & \dots \dots . & \dots \dots . \\ a_{n1} & a_{n2} \dots \dots . & a_{nn} \end{bmatrix}$ *be a given matrix*

*Characteristic matrix of A is* $A - \lambda I$

$$i.e., A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A-\lambda I|$

$$say\, \phi(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A-\lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these eigen values say λ their eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

**Solved Problems**

1. **Find the eigen values and the corresponding eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

**Sol:** $Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic matrix = $[A-\lambda I] = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(2-\lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\,\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

**Eigen vector corresponding to $\lambda = 4$**

Put $\lambda = 4$ in the above system, we get $\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 4x_1 - 4x_2 = 0 --- (1) \qquad 2x_1 - 2x_2 = 0 --- (2)$

$from\ (1) and\ (2) we\ have\ x_1 = x_2$

Let $x_1 = \alpha$

$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $is\ a\ Eigen\ vector\ of\ matrix\ A, corresponding\ eigen\ value\ \lambda = 4$

**Eigen Vector corresponding to $\lambda = 6$**

$put$ $\lambda = 6$ in the above system, we get $\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 2x_1 - 4x_2 = 0 --- (1) \qquad 2x_1 - 4x_2 = 0 --- (2)$

from (1) and (2) we have $x_1 = 2x_2$

Let $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $is\ eigen\ vector\ of\ matrix\ A\ corresponding\ eigen\ value\ \lambda = 6$

2. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

**Sol:** Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Rightarrow \lambda - 2\ [-(\lambda-2)^2 - 1] = 0$$

$$\Rightarrow \lambda - 2\ [-\lambda^2 + 4\lambda - 3] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ $\lambda$=1, 2, 3

The eigen values of A is 1, 2, 3.

$For\ finding\ eigen\ vector\ the\ system\ is\ (A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 1}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$Let\ x_3 = \alpha$

$\Rightarrow x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $is\ Eigen\ vector$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 2}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Here\ x_1 = 0\ and\ x_3 = 0\ and\ we\ can\ take\ any\ arbitary\ value\ x_2\ i.e\ x_2 = \alpha\ (say)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$Eigen\ vector\ is$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 3}$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

$here\ by\ solving\ we\ get\ x_1 = x_3, x_2 = 0\ say\ x_3 = \propto$

$x_1 = \propto \quad , \quad x_2 = 0 \quad , x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$Eigen\ vector\ is\ \begin{bmatrix}1\\0\\1\end{bmatrix}$

3. Find the Eigen values and Eigen vectors of the matrix is $\begin{bmatrix}3 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}$

Sol: Let $A=\begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}$

Consider characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}8-\lambda & -6 & 2\\-6 & 7-\lambda & -4\\2 & -4 & 3-\lambda\end{vmatrix}=0$

$$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda)-(16)\right]+6\left[(-6)(3-\lambda)+8\right]+2\left[24-2(7-\lambda)\right]=0$$
$$\Rightarrow (8-\lambda)\left[21-7\lambda-3\lambda+\lambda^2-16\right]+6\left[-18+6\lambda+8\right]+2\left[24-14+2\lambda\right]=0$$
$$\Rightarrow (8-\lambda)\left[\lambda^2-10\lambda-5\right]+6\left[6\lambda-10\right]+2\left[10+2\lambda\right]=0$$
$$\Rightarrow 8\lambda^2-80\lambda-40-\lambda^3+10\lambda^3+5\lambda+36\lambda-60+20+4\lambda=0$$
$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$
$$\Rightarrow \lambda\left[-\lambda^2+18\lambda-45\right]=0$$
$$\Rightarrow \lambda=0 \quad (OR) \quad -\lambda^2+18\lambda-45=0$$
$$\Rightarrow \lambda=0, \quad \lambda=3, \quad \lambda=15$$

Eigen Values $\lambda = 0, 3, 15$

Case (i): If $\lambda = 0$

$$\begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}X = 0$$

$$\begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow 8x_1-6x_2+2x_3=0----(1)$$
$$-6x_1+7x_2-4x_3=0----(2)$$
$$2x_1-4x_2+3x_3=0----(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{matrix}$$

$$\Rightarrow \frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14} = k$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10} = k$$

$$\Rightarrow x_1 = k, \quad x_2 = 2k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii): If $\lambda = 3$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0 ------(1)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 ------(2)$$

$$2x_1 - 4x_2 + 0 = 0 --------(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{matrix}$$

$$\Rightarrow \frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8} = k$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} = k$$

$$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{1} = \frac{x_3}{2} = k$$

$$\Rightarrow \frac{x_1}{-2} = k, \quad -x_2 = k, \quad x_3 = 2k$$

$$\Rightarrow x_1 = -2k, \quad x_2 = -k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix} = k\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii): If $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + (-6x_2) + 2x_3 = 0 ----(1)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 -----(2)$$

$$2x_1 - 4x_2 - 12x_3 = 0 -----(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{matrix}$$

$$\Rightarrow \frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16} = k$$

$$\Rightarrow \frac{x_1}{80} = \frac{-x_2}{80} = \frac{x_3}{40} = k$$

$$\Rightarrow \frac{x_1}{2} = k, \quad \frac{x_2}{2} = k, \quad \frac{x_3}{1} = k$$

$$\Rightarrow x_1 = 2k, \quad x_2 = 2k, \quad x_3 = k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} k$

**4. Find the Eigen values and the corresponding Eigen vectors of the matrix.**

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & -0 \end{bmatrix}$$

Sol: Let $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & -0\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0$

$$\Rightarrow(-2-\lambda)\left[-\lambda(1-\lambda)-12\right]-2[-2\lambda-6]-3\left[2(-2)+(1-\lambda)\right]=0$$

$$\Rightarrow\lambda^3+\lambda^2-21\lambda-45=0$$
$$\Rightarrow(\lambda+3)(\lambda+3)(\lambda-5)=0$$
$$\Rightarrow\lambda=-3,-3,5$$

The Eigen values are -3,-3, and 5

Case (i): If $\lambda=-3$

We get $\begin{bmatrix}-2+3 & 2 & -3\\ 2 & 1+3 & -6\\ -1 & -2 & 0+3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

The augment matrix of the system is $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 2 & 4 & -6 & 0\\ -1 & -2 & 3 & 0\end{array}\right]$

Performing $R_2-2R_1, R_3+R_1$, we get $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

Hence we have $x_1+2x_2-3xy=0\Rightarrow x_1=-2x_2+3x_3$

Thus taking $x_2=k_1$ *and* $x_3=k_2$, we get $x_1=-2k_1+3k_2; x_2=k_1; x_3=k_2$

Hence $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=k_1\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+k_2\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

So $\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$ and $\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$ are the Eigen vectors corresponding to $\lambda=-3$

Case (ii): If $\lambda=5$

We get $\begin{pmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0 ----(1)$$
$$2x_1 - 4x_2 - 6x_3 = 0 ----(2)$$
$$-x_1 - 2x_2 - 5x_3 = 0 ----(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{matrix}$$

$$\Rightarrow \frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4} = k_3$$
$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-16} = \frac{-x_3}{-8} = k_3$$
$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{-x_3}{-1} = k_3$$

$$Eigen\ vector\ is \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} k_3$$

**5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse where**

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of "A" is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$
$$\Rightarrow \lambda = 1,2,3 \quad i.e.\ Eigen\,Values\,are\ 1,2,3$$

Note: In upper $\Delta^{le}$ (or) Lower $\Delta^{lar}$ of a square matrix the Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Case (i): If $\lambda = 1$

$$\therefore (A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2 + 4x_3 = 0; x_2 + 5x_3 = 0; 2x_3 = 0 \quad \Rightarrow x_1 = k_1; x_2 = 0; x_3 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k_1$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \begin{bmatrix} +1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0; 5x_2 = 0; x_3 = 0$$

$$\Rightarrow -x_1 + 3k + 4(0) = 0 \Rightarrow -x_1 + 3k = 0 \Rightarrow x_1 = 3k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0; -x_2 + 5x_3 = 0; x_3 = 0$$

$$Let \quad x_3 = k$$

$$\Rightarrow -x_2 + 5x_3 = 0 \Rightarrow x_2 = 5k$$

$$and - 2x_1 + 3x_2 + 4k = 0 \Rightarrow -2x_1 + 15k + 4k = 0$$

$$\Rightarrow -2x_1 + 19x = 0 \Rightarrow x_1 = \frac{19}{2}k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = k \begin{bmatrix} \frac{19}{2} \\ 5 \\ 1 \end{bmatrix}$$

Note: Eigen Values of $A^{-1}$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ $i.e 1, \frac{1}{2}, \frac{1}{3}$ and the Eigen vectors of $A^{-1}$ are same as Eigen vectors of the matrix A

**6. Determine the Eigen values and Eigen vectors of**

$$B = 2A^2 - \frac{1}{2}A + 3I \text{ where } A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

**Sol** $:-$ *Given that* $B = 2A - \frac{1}{2}A + 3 \quad \Rightarrow A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$\text{we have } A^2 = A.A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}$$

$$B = 2A^2 - \tfrac{1}{2}A + 3I$$

$$= 2\begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

Characteristic equation of B is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

i.e. $\Rightarrow \lambda^2 + 105\lambda - 2376 = 0$

$$\Rightarrow (\lambda - 33)(\lambda - 72) = 0$$

$$\Rightarrow \lambda = 33 \quad or \quad 72$$

Eigen Values of B are 33 and 72.

Case (i): If $\lambda = 33$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Case (ii): If $\lambda = 72$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 111-72 & -78 \\ 39 & -6-72 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

**Properties of Eigen Values:**

**Theorem 1:** The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

**Proof:** Characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$ *expanding this we get*

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}$ (*a polynomial of degree n – 2*)+ $a_{13}$ (*a polynomial of degree n -2*) + … = 0

$\Rightarrow (-1)^n\left[\lambda^n-(a_{11}+a_{22}+....+a_{nn})\lambda^{n-1}+a\ polynomial\ of\ \deg ree\,(n-2)\right]=0$

$(-1)^n\lambda^n+(-1)^{n+1}(Trace\ A)\lambda^{n-1}+a\ polynomial\ of\ \deg ree\,(n-2)\ in\ \lambda=0$

$If\ \lambda_1,\lambda_2\ .....\lambda_n\ are\ the\ roots\ of\ this\ equation$

$sum\ of\ the\ roots=\frac{(-1)^{n+1}Tr(A)}{(-1)^n}=Tr(A)$s

$Further\ |A-\lambda I|=(-1)^n\lambda^n+\cdots.+a_0$

$put\ \lambda=0\ then\ |A|=a_0$

$(-1)^n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+\ ....+a_0=0$

$Product\ of\ the\ roots=\frac{(-1)^n a_0}{(-1)^n}=a_0$

$but\ a_0=\ |A|=\det A$

*Hence the result*

**Theorem 2**: If $\lambda$ is an eigen value of A corresponding to the eigen vector X, then $\lambda^n$ is eigen value $A^n$ corresponding to the eigen vector X.

**Proof:** Since $\lambda$ is an eigen value of A corresponding to the eigen value X, we have

AX= $\lambda X$ ----------(1)

Pre multiply (1) by A, A(AX) = A($\lambda$X)

(AA)X = $\lambda$(AX)

$A^2X=\lambda(\lambda X)$

$A^2X=\lambda^2X$

$\lambda^2$ is eigen value of $A^2$ with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

Let we assume it is true for n = k

i.e. $A^KX=\lambda^KX$------------(2)

Premultiplying (2) by A, we get

$A(A^kX)=A(\lambda^KX)$

$(AA^K)X=\lambda^K(AX)=\lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

$\lambda^{K+1}$ is eigen value of $A^{K+1}$ with X itself as the corresponding eigen vector.

Thus, by Mathematical induction. $\lambda^n$ is an eigen value of $A^n$.

**Theorem 3:** A Square matrix A and its transpose $A^T$ have the same eigen values.

**Proof:** We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I|$ $\left[\because |A^T| = |A|\right]$

$|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

Hence the theorem.

**Theorem 4:** If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and $BA^{-1}$ have same eigen values.

**Proof:** Given A is invertble i.e, $A^{-1}$ exist

We know that if A and P are the square matrices of order n such that P is non-singular then A and $P^{-1}AP$ have the same eigen values.

Taking $A = BA^{-1}$ and P=A, we have

$BA^{-1}$ and $A^{-1}(BA^{-1})A$ have the same eigen values

ie., $BA^{-1}$ and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

ie., $BA^{-1}$ and $(A^{-1}B)I$ have the same eigen values

ie., $BA^{-1}$ and $A^{-1}B$ have the same eigen values

**Theorem 5:** If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, \ldots.. k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

**Proof:** Let A be a square matrix of order n. Then $|KA-\lambda KI| = |K(A-\lambda I)| = K^n|A-\lambda I|$

Since $K \neq 0$, therefore $|KA-\lambda KI| = 0$ if and only if $|A-\lambda I| = 0$

$i.e., K\lambda$ *is an eigen value of* $KA \Leftrightarrow$ *if* $\lambda$ *is an eigen value of* $A$

*Thus* $k\lambda_1, k\lambda_2 \ldots k\lambda_n$ *are the eigen values of the matrix KA if*

$\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of the matrix A

**Theorem 6**: If $\lambda$ is an eigen values of the matrix A then $\lambda+k$ is an eigen value of the matrix A+KI

**Proof:** Let $\lambda$ be an eigen value of A and X the corresponding eigen vector. Then by definition

$AX = \lambda X$

Now $(A+KI)X$

$= AX + IKX = \lambda X + KX$

$= (\lambda+K) X$

$\lambda+k$ is an eigen value of the matrix A+KI.

**Theorem 7:** If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, then $\lambda_1 - K, \lambda_2 - K, \ldots \lambda_n - K$, *are the eigen values of the matrix* $(A - KI)$, *where K is a non – zero scalar*

**Proof:** Since $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$-----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k+\lambda)I|$

$= [\lambda_1 - (\lambda+K)][\lambda_2 - (\lambda+K)]$...............................$[\lambda_n - (\lambda+K)]$.

$= [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda]$...............................$[(\lambda_n - K) - \lambda]$.

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \ldots \ldots . \lambda_n - K$

**Theorem 8:** If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$

**Proof:** First we will find the eigen values of the matrix A- $\lambda$I

Since $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A - \lambda I| = (\lambda_1 - K)(\lambda_2 - K) \ldots . (\lambda_n - K) \quad ------(1)$ *where K is scalar*

The characteristic polynomial of the matrix (A- $\lambda$I) is

$|A - \lambda I - KI| = |A - (\lambda+K)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots [\lambda_n - (\lambda+K)]$

$= [(\lambda_1 - \lambda) - K)]\,[(\lambda_2 - \lambda) - K] \ldots [(\lambda_n - \lambda) - K)]$

Which shows that eigen values of (A- $\lambda$I) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \ldots . \lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$ then the eigen values of $A^2$ are $\lambda_1^2, \lambda_2^2 \ldots . \lambda_n^2$ Thus eigen values of $(A - \lambda I)^2$ *are* $(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \ldots . (\lambda_n - \lambda)^2$

**Theorem 9:** If $\lambda$ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then $\lambda^{-1}$ *is an eigen value of* $A^{-1}$ *and corresponding eigen vector X itself.*

**Proof:** Since A is non-singular and product of the eigen values is equal to |A|, it follows that none of the eigen values of A is 0.

If $\lambda$ is an eigen value of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and $AX = \lambda X$.

premultiplying this with $A^{-1}$, we get $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X \ (\lambda \neq 0)$

Hence $\lambda^{-1}$ is an eigen value of $A^{-1}$

**Theorem 10:** If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix AdjA.

**Proof:** Since λ is an eigen value of a non-singular matrix, therefore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that

$AX = \lambda X$ -----(1)

$\Longrightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Longrightarrow [(adj\ A)A]X = \lambda(\text{adj A})X$

$\Longrightarrow |A|IX = \lambda\ (\text{adj A})X \ [\because (adjA)A = |A|I]$

$\Rightarrow \frac{|A|}{\lambda}X = (adj\ A)\,X \ or\ (adj\ A)\,X = \frac{|A|}{\lambda}X$

Since X is a non – zero vector, therefore the relation (1)

it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

**Theorem 11**: If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an Eigen value $A$

**Proof:** We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of $A^{-1}$

Since A is an orthogonal matrix, therefore $A^{-1} = A^{1}$

$\therefore$ $\frac{1}{\lambda}$ is an eigen value of $A^{1}$

But the matrices A and $A^{1}$ have the same eigen values, since the determinants $|A- \lambda I|$ and $|A^{1} - \lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

**Theorem 12:** If λ is eigen value of A then prove that the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

**Proof:** If X be the eigen vector corresponding to the eigen valueλ, then $AX = \lambda X$ --- (1)

Premultiplying by A on both sides

$\Longrightarrow A(AX) = A(\lambda X)$

$\Longrightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

This shows that $\lambda^2$ is an eigen value of $A^2$

We have $B = a_0A^2+a_1A+a_2I$

$\therefore BX = a_0A^2+a_1A+a_2I) X$

$= a_0A^2 X+a_1AX+a_2 X$

$= a_0\lambda^2 X+a_1 \lambda X+a_2X = (a_0\lambda^2 +a_1 \lambda+a_2 ) X$

$(a_0\lambda^2 +a_1 \lambda+a_2 )$ is an eigen value of B and the corresponding eigen vector of B is X.

**Theorem 13:** Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

**Proof:** Consider the characteristic equation of $P^{-1}AP$

It is $| ( P^{-1}AP)-\lambda I| = | P^{-1}AP-\lambda P^{-1}IP|$ $(\because I = P^{-1}P)$

$= | P^{-1} (A-\lambda I) P| = | P^{-1} | |A-\lambda I| |P|$

$= |A-\lambda I|$ since $|P^{-1} | |P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

**Corollary 1:** If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and $BA^{-1}$ have the same eigen values.

**Corollary 2**: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

**Theorem 14:** The eigen values of a triangular matrix are just the diagonal elements of the matrix.

**Proof:** Let A = $\begin{vmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ 0 & a_{22}\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0\ldots\ldots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A- \lambda I|=0$

i.e., $\begin{vmatrix} a_{11-\lambda} & a_{12}\cdots\cdots & a_{1n} \\ 0 & a_{22-\lambda}\cdots\cdot & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0\ldots\ldots & a_{nn-\lambda} \end{vmatrix} = 0$

i.e, $(a_{11}- \lambda) (a_{22}- \lambda) \ldots\ldots (a_{nn}- \lambda)=0$

$\Longrightarrow \lambda = a_{11} , a_{22} ,\ldots . a_{nn}$

Hence the eigen values of A are $a_{11} , a_{22} ,\ldots . a_{nn}$ which are just the diagonal elements of A.

**Note:** Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

**Theorem 15**: The eigen values of a real symmetric matrix are always real.

**Proof:** Let $\lambda$ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then AX= $\lambda X----(1)$

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A$ *and* $A^T = A$, *we have* $\bar{X}^T\ A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T\ AX = \bar{\lambda}\,\bar{X}^T X$------- (2)

Premultiplying (1) with $\bar{X}^T$ , we get $\overline{X}^T AX = \lambda\overline{X}^T X$ ------ (3)

(1) $-$ (3) gives $(\lambda - \bar{\lambda})\overline{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \bar{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

**Theorem 16**: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

**Proof:** Let $\lambda_1$, $\lambda_2$ be eigen values of a symmetric matrix A and let $X_1$, $X_2$ be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$. We want to show that $X_1$ is orthogonal to $X_2$ (i.e., $X_1^T X_2 = 0$)

Sice $X_1$, $X_2$ are eigen values of A corresponding to the eigen values $\lambda_1$, $\lambda_2$ we have

$AX_1 = \lambda_1 X_1$ ----- (1) $\quad AX_2 = \lambda_2 X_2$ ------- (2)

Premultiply (1) by $X_2^T$

$\Longrightarrow\ X_2^T A X_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Rightarrow X_2^T A^T (X_2^T)^T = \lambda_1 X_1^T (X_2^T)^T$

$i.e, X_1^T A X_2 = \lambda_1 X_1^T X_2$-------- (3) s

Premultiplying (2) by $X_1^T$, *we get* $X_1^T A X_2 = \lambda_2 X_1^T X_2 - - - - - (4)$

Hence from (3) and (4) we get

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Longrightarrow\ X_1^T X_2 = 0$

$(\because \lambda_1 \neq \lambda_2)$

$X_1$ *is orthogonal to* $X_2$

**Note:** If $\lambda$ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f($\lambda$) .

**Theorem17: The Eigen values of a Hermitian matrix are real.**

**Proof:** Let A be Hermitian matrix. If X be the Eigen vector corresponding to the eigen value $\lambda$ of A, then AX = $\lambda$ X-------------------- (1)

Pre multiplying both sides of (1) by $X^\theta$ ,we get

$X^\theta AX = \lambda X^\theta X$ ----------------------- (2)

Taking conjugate transpose of both sides of (2)

We get $\left(X^\theta AX\right)^\theta = \left(\lambda X^\theta X\right)^\theta$

i.e $X^\theta A^\theta \left(X^\theta\right)^\theta = \bar{\lambda} X^\theta \left(X^\theta\right)^\theta \left[\because (ABC)^\theta = C^\theta B^\theta A^\theta \text{ and } (KA)^\theta = \bar{K} A^\theta\right]$

(or) $X^\theta A^\theta X = \bar{\lambda} X^\theta X \left[\because \left(X^\theta\right)^\theta = X, \left(A^\theta\right)^\theta = A\right]$-------------------- (3)

From (2) and (3), we have

$\lambda X^\theta X = \bar{\lambda} X^\theta X$

i.e $\left(\lambda - \bar{\lambda}\right) X^\theta X = 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda} \left(\because X^\theta X \neq 0\right)$

$\therefore$ Hence $\lambda$ is real.

**Note:** The Eigen values of a real symmetric are all real

**Corollary**: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

**Proof**: Let A be the skew-Hermitian matrix

If X be the Eigen vector corresponding to the Eigen value $\lambda$ of A, then

$AX = \lambda X (or) (iA) X = (i\lambda) X$

From this it follows that $i\lambda$ is an Eigen value of iA

Which is Hermitian (since A is skew-hermitian)

$\therefore A^\theta = -A$

Now $(iA)^\theta = \bar{i} A^\theta = -iA^\theta = -i(-A) = iA$

Hence $i\lambda$ is real. Therefore $\lambda$ must be either

Zero or purely imaginary.

Hence the Eigen values of skew-Hermitian matrix are purely imaginary or zero

**Theorem 18:** The Eigen values of an unitary matrix have absolute value 1.

Proof: Let A be a square unitary matrix whose Eigen value is $\lambda$ with corresponding eigen vector X

$\Rightarrow \quad AX = \lambda X \rightarrow (1)$

$\Rightarrow \quad \overline{A}\overline{X} = \bar{\lambda}\overline{X} \Rightarrow \overline{X}^T \overline{A}^T = \bar{\lambda} X^T \rightarrow (2)$

Since A is unitary, we have $\left(\overline{A}\right)^T A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^T \overline{A}^T \left(AX\right) = \lambda\overline{\lambda}\overline{X}^T X$

i.e $\overline{X}^T X = \lambda\overline{\lambda}\overline{X}^T X$ From (3)

$\Rightarrow \overline{X}^T X\left(1-\lambda\overline{\lambda}\right) = 0$

Since $\overline{X}^T X \neq 0$ ,we must have $1-\lambda\overline{\lambda} = 0$

$$\Rightarrow \lambda\overline{\lambda} = 1$$

Since $|\lambda| = \overline{|\lambda|}$

We must have $|\lambda|$=1

**Note** 1: From the above theorem, we have "The characteristic root of an orthogonal matrix is of unit modulus".

2. The only real eigen values of unitary matrix and orthogonal matrix can be $\pm$ 1

**Theorem 19:** Prove that transpose of a unitary matrix is unitary.

**Proof:** Let A be a unitary matrix, then $A.A^\theta = A^\theta.A = I$

where $A^\theta$ is the transposed conjugate of A.

$$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = \left(I\right)^T$$

$$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = \left(I\right)^T$$

$$\Rightarrow \left(A^\theta\right)^T A^T = A^T\left(A^\theta\right)^T = I$$

$$\Rightarrow \left(A^T\right)^\theta A^T = A^T\left(A^T\right)^\theta = I$$

Hence $A^T$ is a unitary matrix.

**Solved Problems:**

1. For the matrix $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$ find the Eigen values of $3A^3+5A^2-6A+2I$

Sol: The Characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}1-\lambda & 2 & -3\\ 0 & 3-\lambda & 2\\ 0 & 0 & -2-\lambda\end{vmatrix}=0$

$\Rightarrow(1-\lambda)(3-\lambda)(-2-\lambda)=0$

$\therefore$ Eigen values are 1, 3,-2.

If $\lambda$ is the Eigen value of A. and F (A) is the polynomial in A then the Eigen value of f (A) is f ( $\lambda$ )

Let $f(A)=3A^3+5A^2-6A+2I$

$\therefore$ Eigen Value of f (A) are f (1), f (-2), f (3)

f (1) = 3+5-6+2 = 4

f (-2) = 3(-8)+5(4)-6(-2)+2 = -24+20+12+2 = 10

f (3) = 3(27)+5(9)+6(3)+2 = 81+45-18+2 = 110

The Eigen values of f (a) are f ( $\lambda$ ) = 4, 10,110

**2. Find the eigen values and eigen vectors of the matrix A and its inverse, where**

**A** = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

**Sol:** Given A = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$\Rightarrow \begin{vmatrix}1-\lambda & 3 & 4\\ 0 & 2-\lambda & 5\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)]=0$

$\Rightarrow \lambda = 1, 2, 3$

Charecetstic roots are 1, 2, 3.

Case (i): If $\lambda = 1$

$$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where $\alpha$ is arbitrary constant

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Case (i): If $\lambda = 2$

$$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$

$Let\, x_2 = k$

$x_1 = 3k$

$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

is the solution where k is arbitrary constant

$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Case (iii): If $\lambda = 3$

$$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow\ -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

$Say\ x_3 = K \implies x_2 = 5K$

$x_1 = \frac{19}{2}K$

X $= \begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the solution, where k/2 is arbitrary constant.

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Eigen values of $A^{-1}$ are $1, \frac{1}{2}, \frac{1}{3}$.

We know Eigen vectors of $A^{-1}$ are same as eigen vectors of A.

**3. Find the eigen values of A=$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$**

**Sol:** we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow A^T = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

**4.Find the eigen values of** $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and\ \left(\overline{A}\right)^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

We can see that $\overline{A}^T.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix} = 0$$

Which gives $\lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2} \, and \, \frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and

Hence above $\lambda$ values are Eigen values of A.

**Cayley-Hamilton Theorem:** Every Square Matrix satisfies its own characteristic equation

**To find Inverse of matrix:** If A is non-singular Matrix, then $A^{-1}$ exists, Pre multiplying (1) above by $A^{-1}$ we have $a_0A^{n-1} + a_1A^{n-2} + --- + a_{n-1}I + a_nA^{-1} = 0$,

$$A^{-1} = \frac{1}{a_n}\left[a_0A^{n-1} + a_1A^{n-2} + --- a_{n-1}I\right]$$

**To find the powers of A: -** Let K be a +ve integer such that $K \geq n$

Pre multiplying (1) by $A^{k-n}$ we get $a_0A^K + a_1A^{K-1} + --- + a_nA^{k-n} = 0$,

$$A^K = \frac{-1}{a_0}\left[a_1A^{k-1} + a_2A^{K-1} --- + a_nA^{K-n}\right]$$

**Solved Problems :**

**1. S.T the matrix A =** $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ **satisfies its characteristic equation and hence find $A^{-1}$**

**Sol:** Characteristic equation of A is det $(A-\lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_2 \rightarrow C_2+C_3$ $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have $A^3-A^2+A-I = 0$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with $A^{-1}$ we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

**2. Using Cayley - Hamilton Theorem find the inverse and $A^4$ of the matrix**

**A** $= \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

**Sol:** Let A $= \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$ *i.e.*, $\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$…..(1)

Multiply with $A^{-1}$ we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

multiplying (1) with A,we get,

$A^4-5A^3+7A^2-3A= 0$

$A^4 = 5A^3-7A^2+3A$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

**3.** *If* $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ **Verify Cayley-Hamilton theorem hence find $A^{-1}$**

Sol: - Given that $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix}=0$

$$\Rightarrow (2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right]-1\left[-10-5\lambda+3\right]+2\left[0+(3-\lambda)\right]$$

$$\Rightarrow (2-\lambda)\left[\lambda^2-\lambda-6\right]-1\left[-5\lambda-7\right]+2\left[3-x\right]=0$$

$$\Rightarrow 2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+5\lambda+7+6-2\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0----(1)$$

According to Cayley Hamilton theorem. Square matrix ‘A’ satisfies equation (1)

Substitute A in place of $\lambda$

Now $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

Now $A^3-3A^2-7A-I=0$

$$\Rightarrow \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & 3 & -7 \end{bmatrix}+\begin{bmatrix} 21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix}+\begin{bmatrix} 14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix}+\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0$$

Cayley –Hamilton theorem is verified.

To find $A^{-1}$

$$\Rightarrow A^3-3A^2-7A-I=0$$

Multiply $A^{-1}$, we get

$$A^{-1}\left(A^3-3A^2-7A-I\right)=0$$

$$\Rightarrow A^2-3A-7I-A^{-1}=0$$

$$\Rightarrow A^{-1}=A^2-3A-7I$$

$$\therefore A^{-1}=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}+\begin{bmatrix}-6 & -3 & -6\\ 15 & -9 & 9\\ 3 & 0 & 6\end{bmatrix}+\begin{bmatrix}-7 & 0 & 0\\ 0 & -7 & 0\\ 0 & 0 & -7\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}$$

Check $A.A^{-1}= I$

$$A.A^{-1}=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

**4. Using Cayley – Hamilton theorem, find $A^8$, if** $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Sol: Given $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Characteristic equation of A is $\left|A-\lambda I\right|=0$

$$\Rightarrow(1-\lambda)(-1-\lambda)-4=0$$

$$\Rightarrow \lambda^2-5=0----(1)$$

Substitute A in place of $\lambda$

$$A^2-5I=0\Rightarrow A^2=5I$$

$$\textit{find } A^8$$

$$\therefore A^8=5A^6=5(A^2)(A^2)(A^2)$$

$$=5(5I)(5I)(5I)$$

$$=625I$$

$$\Rightarrow A^8=625I$$

**Diagonalization of a Matrix by similarity transformation:**

**Similar Matrix:** A matrix A is said to be similar to the Matrix B if there Exists a non-singular matrix P such that B= $P^{-1}AP$. This transformation of A to B is known as "Similarity Transformation"

**Diagonalization of a Matrix:**

Let A be a square Matrix. If there exists a non-singular Matrix P and a diagonal Matrix D such that $P^{-1}AP=D$, then the Matrix A is said to be diagonalizable and D is said to be "Diagonal" form (or) canonical diagonal form of the Matrix A

**Modal Matrix:**The modal matrix which diagonalizes A is called the modal Matrix of A and is obtained by grouping the Eigen vectors of A into a Square Matrix.

**Spectral Matrix:** The resulting diagonal Matrix D is known as Spectral Matrix.

In this spectral Matrix D whose principal diagonal elements are the Eigen values of the Matrix.

**Calculation of powers of a matrix:**

We can obtain the power of a matrx by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2= (P^{-1}AP)\,(P^{-1}AP)$

$= P^{-1}A\,(PP^{-1})\,AP$

$= P^{-1}A^2P \qquad$ (since $PP^{-1}=I$)

Simlarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain $A^n$, Premultiply (1) by P and post multiply by $P^{-1}$

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n\,(PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

Hence $A^n = P\begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix}P^{-1}$

**Diagonalization of a matrix:**

**Theorem**: If a square matrix A of order n has n linearly independent eigen vectors ($X_1,X_2\ldots X_n$) corresponding to the n eigen values $\lambda_1,\lambda_2\ldots\lambda_n$ respectively then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

**Note: 1.** If $X_1,X_2\ldots X_n$ are not linearly independent this result is not true.

**2.** Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1,\lambda_2\cdots\lambda_n$ then the corresponding eigen vectors $X_1,X_2\ldots X_n$ are pairwise orthogonal.

Hence if P = $(e_1, e_2 \dots e_n)$

Where $e_1 = (X_1 / ||X_1||)$, $e_2 = (X_2 / ||X_2||) \dots e_n = (X_n)/ ||X_n||$ then P will be an orthogonal matrix.

i.e, $P^T P = PP^T = I$

Hence $P^{-1} = P^T$

$\therefore P^{-1}AP = D$

**Solved Problems :**

**1. Determine the modal matrix P of A=** $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **. Verify that $P^{-1}AP$ is a diagonal matrix.**

**Sol:** The characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

When $\lambda=5 \implies \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value $\lambda = -3$ we can have two linearly independent eigen vectors

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $and\ X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

P = $[X_1\, X_2\, X_3]$

P = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = modal matrix of A

Now det P=1(-1)-2(2)+3(0-1) = -8

$$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag } [5,-3,-3].$$

$\therefore\ P^{-1}AP$= diag [5,-3,-3].

**2. Find a matrix P which transform the matrix A =** $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate $A^4$.

**Sol:** Characteristic equation of A is given by $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

=> $(1-\lambda)[(2-\lambda)(3-\lambda)-2]-0-[2-2(2-\lambda)]=0$

=> $9\,\lambda-1)(\lambda-209\lambda-30=0$

=> $\lambda=1, \lambda=2, \lambda=3$

Thus the eigen values of A are 1, 2, 3.

If $x_1$, $x_2$, $x_3$ be the components of an eigen vector corresponding to the eigen value λ, we have

$[A-\lambda I]\ X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case (i): If $\lambda = 1$

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ i.e, $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an eigen vector corresponding to λ=1

For λ=2, λ=3 we can obtain eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The Matrix P is called modal matrix of A

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$Now\, P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$A^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

**3. Determine the modal matrix P for $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence diagonalize A**

Sol: Given that $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda)-1]-1[(1-\lambda)-3]+3(1-3(5-\lambda))=0$$
$$\Rightarrow (1-\lambda)(5-5\lambda-\lambda+\lambda^2-1)-(-2-\lambda)+3(1-15+3\lambda)=0$$
$$\Rightarrow (1-\lambda)(4-6\lambda+\lambda^2)-(-2-\lambda)+3(-14+3\lambda)=0$$
$$\Rightarrow 4-6\lambda+\lambda^2-4\lambda+6\lambda^2-\lambda^3+2+\lambda-42+9\lambda=0$$
$$\Rightarrow \lambda^3-7\lambda^2-9\lambda+9\lambda-36=0$$
$$\Rightarrow \lambda^3-7\lambda^2-36=0$$
$$\Rightarrow \lambda=-2,3,6$$

The Eigen Values are -2, 3, and 6

Case (i): If $\lambda = -2$

⇨ $[A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1+x_2+3x_3=0----(1)$$
$$x_1+7x_2+x_3=0----(2)$$
$$3x_1+x_2+3x_3=0----(3)$$

From (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{1-21} = \frac{-x_2}{3-3} = \frac{x_3}{21-1} = k$$

$$\Rightarrow \frac{x_1}{-20} = \frac{-x_2}{0} = \frac{x_3}{20} = k$$

$$\Rightarrow x_1 = -20k, \quad x_2 = 0, \quad x_3 = 20k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20k \\ 0k \\ 20k \end{bmatrix} = 20k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii): If $\lambda = 3$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + 3x_3 = 0 ----(1)$$

$$x_1 + 2x_2 + x_3 = 0 -----(2)$$

$$3x_1 + x_2 - 2x_3 = 0 ----(3)$$

Consider (1) & (2)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -2 & 1 & 3 \\ 1 & 2 & 1 \end{array}$$

$$\Rightarrow \frac{x_1}{1-6} = \frac{-x_2}{-2-3} = \frac{x_3}{-4-1} = k$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5k \\ 5k \\ -5k \end{bmatrix} = -5k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii): If $\lambda = 6$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -11 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x_1 + x_2 + 3x_3 = 0 ----(1)$$

$$x_1 - x_2 + x_3 \quad = 0 ---(2)$$

$$3x_1 - x_2 - 5x_3 = 0 ---(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{matrix}$$

$$\Rightarrow \frac{x_1}{5-1} = \frac{-x_2}{-5-3} = \frac{x_3}{+1+3} = k$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{8} = \frac{x_3}{4} = k$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} k$$

$$\rho = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|\rho| = -1(-1-2) - 1(0-2) + 1(0+1)$$

$$|\rho| = (-1)(-3) - 1(-2) + 1 = 3 + 2 + 1 = 6$$

$$\rho = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$Adj(\rho) = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\rho^{-1} = \frac{Adj\,\rho}{|\rho|}$$

$$Cofactor\ of\ \rho = \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$D = \rho^{-1} A \rho$

$$D = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

**4.** $\boldsymbol{If\ A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ Find (a) $A^8$ (b) $A^4$

Sol: Given that A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-4\right]-1\left[0+4\right]+1\left[0+4(2-\lambda)\right]=0$$
$$\Rightarrow (1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-4\right]-4+8-4x=0$$
$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+2\right]+4-4\lambda=0$$
$$\Rightarrow \lambda^2-5\lambda+2-\lambda^3+5\lambda^2-2\lambda+4-4\lambda=0$$
$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$
$$\Rightarrow \lambda = 1,2,3$$

The Eigen values are 1, 2, and 3

Case (i): If $\lambda = 1$

$$[A - \lambda I]X = 0$$

=> $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$

=> $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1+x_2=0$, $x_1+x_2=0$, $-4x_1+4x_2+2x_3=0$

Let $x_3 = k$, $x_2 + k=0$, $x_2 = -k$

$$\Rightarrow -4x_1 + 4(-k) + 2K = 0 \Rightarrow -4x_1 - 2k = 0 \Rightarrow -4x_1 = 2k \Rightarrow x_1 = \frac{-k}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-k}{2} \\ -k \\ k \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} - k = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \frac{-k}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \quad [A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 + x_3 = 0 ----(1)$$

$$x_3 = 0 ---(2)$$

$$-4x_1 + 4x_2 + x_3 = 0 ---(3)$$

Consider (1) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -1 & 1 & 1 \\ -4 & 4 & 1 \end{matrix}$$

$$\Rightarrow \frac{x_1}{1-4} = \frac{-x_2}{-1+4} = \frac{x_3}{-4+4} = k$$

$$\Rightarrow \frac{x_1}{-3} = \frac{-x_2}{3} = \frac{x_3}{0} = k$$

$$\Rightarrow x_1 = -k; \quad x_2 = -k$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} (-k) \qquad \therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow \quad [A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-4x_1 + 4x_2 = 0$$

$Let\ x_1 = k \quad and \quad x_3 = k$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0 \Rightarrow -2k + x_2 + k = 0 \Rightarrow x_2 = k$$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

$$\therefore p = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\rho^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$D = \rho^{-1} AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$$

$(a).\quad A^8 = PD^8P^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

$$(b).\quad D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1+64-162 & 1-48+162 & 0-16+81 \\ -2+64-162 & 2-48+162 & 0-16+81 \\ 2+0-162 & -2-0+162 & 0-0+81 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$$

# UNIT - V

# Number Theory

**Number theory** is a branch of mathematics which helps to study the set of positive whole numbers, say 1, 2, 3, 4, 5, 6,. . . , which are also called the set of natural numbers and sometimes called "higher arithmetic".

Number theory helps to study the relationships between different sorts of numbers. Natural numbers are separated into a variety of times. Here are some of the familiar and unfamiliar examples with quick number theory introduction.

## Applications of Number Theory

Here are some of the most important number theory applications. Number theory is used to find some of the important divisibility tests, whether a given integer m divides the integer n. Number theory have countless applications in mathematics as well in practical applications such as

- Security System like in banking securities
- E-commerce websites
- Coding theory
- Barcodes
- Making of modular designs
- Memory management system
- Authentication system

It is also defined in hash functions, linear congruences, Pseudo random numbers and fast arithmetic operations.

## Problems

1. **Find the Greatest Common Divisor(G.C.D ) of a number 30 and 52**

**Solution:**

Divisors of 30 are 1, 2, 3, 5, 6, 10, 15, 30

Divisors of 52 are 1, 2, 4, 13, 26, 52

The common divisors in 30 and 52 is 2

Therefore, the G. C.D of 30 and 52 is 2

g.c.d (30,52)= 2

**2. Find the common factors of 10 and 16**

**Solution:**

Factors of 10 are:

2 x 5 =10

1 x 10 = 10

Therefore, the factors are 1, 2, 5 and 10

Factors of 16 are

4 x 4 = 16

1 x 16 = 16

2 x 8 = 16

Therefore, the factors of 16 are as follows: 1, 2, 4, 8, 16

Then, the common factors are 1 and 2.

**3. Show that the greatest factor of a number is the number itsel**f.

**Solution:**

Assume the number 24

The factors of 24 are

1 x 24 =24

12 x 2 = 24

8 x 3 = 24

6 x 4 = 24

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24

From this, we can say that 24 is the greatest factor of a number 24.

Hence proved

## 1. Prime Numbers

A prime number is the one which has exactly two factors, which means, it can be divided by only “1” and itself. But “1” is not a prime number.

**Example of Prime Number**

3 is a prime number because 3 can be divided by only two number’s i.e. 1 and 3 itself.

3/1 = 3

3/3 = 1

In the same way, 2, 5, 7, 11, 13, 17 are prime numbers.

## Composite Numbers

A composite number has more than two factors, which means apart from getting divided by number 1 and itself, it can also be divided by at least one integer or number. We don’t consider ‘1’ as a composite number.

**Example of Composite Number**

12 is a composite number because it can be divided by 1,2,3,4,6 and 12. So, the number ‘12’ has 6 factors.

12/1 = 12

12/2 =6

12/3 =4

12/4 =3

12/6 =2

12/12 =1

## Coprime Numbers

The coprime-numbers or mutually primes or relatively primes are the two numbers which have only one common factor, which is 1. Let us understand the concept with an example.

Suppose there are two numbers, 14 and 15. Find whether both are coprime or not.

The factors of 14 are 1, 2 and 7

The factors of 15 are 1, 3 and 5.

For both the numbers, we can see, the common factor is 1. Therefore, 14 and 15 are coprime numbers. But if we consider another number say, 21, whose factors are 1, 3 and 7. Then, 21 is neither a coprime for 14 nor for 15.

Problems:

**1:** Check whether 13 and 31 are co-prime.

**Solution:**

13 and 31 are two prime numbers; therefore, they are co-prime to each other. (Property 2)

The factors of 13 are 1, 13 and the factors of 31 are 1, 31.

They have only 1 as their common factor. So, they are coprime numbers.

**2:** Check whether 150 and 295 are coprimes.

**Solution:**

Given two number are: 150 and 295
150 and 295 are divisible by 5.
From the properties of coprime numbers, 150 and 295 are not coprime.
Alternatively,
$150 = 2 \times 3 \times 5 \times 5$
$295 = 5 \times 59$
$HCF(150, 295) = 5 \neq 1$
Therefore, 150 and 295 are not coprime.

## Sieve of Eratosthenes

The ancient Greek mathematician, poet, and scientist Eratosthenes (third century BCE) suggested a relatively method of determining all prime numbers up to a certain number. Eratosthenes was a chief librarian in the famous Library of Alexandria, and made scholarly contributions to several fields.

Among his other contributions, he is known for having been the first person to calculate the circumference of the Earth.

To find all prime numbers up to a certain number, Eratosthenes developed what later became known as the Sieve of Eratosthenes. To help illustrate his method,

2 3 4 5 6 7 8 9 10

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |

We consider how we might find all prime numbers up to 30. For our purposes, we ignore the number 1, for a reason that will become momentarily clear. We begin by circling 2 and then crossing off all subsequent numbers that are multiples of 2. We then find the next smallest number that is not crossed out, which in the case is 3. Since 3 is not crossed out, it must not be a multiple and continue the process till 30.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |

## Divisibility

A basic concept that arises in studying numbers, especially in studying prime and composite numbers, is that of divisibility. The numbers 18 and 24 can be "evenly divided" by 2 and 3, but not by 5 or 7. The following definition makes this idea precise

Definition 12.

We say that a divides b if there exists some integer k∈Z such that b=k$x$a. We write a/b to indicate that a divides b; we write a∤b if a does not divide b

## Divisor of a given number N

Let $N = p^a q^b c^r$ ... p,q,r be primes and a,b,c be integers . Then t

the number of divisors of N = $(a+1)(b+1)(c+1)$ [the divisor include 1 and the number itself].

Sum of all divisor is $S = \frac{p^{a+1}-1}{p-1} . \frac{q^{b+1}-1}{q-1} . \frac{r^{c+1}-1}{r-1}$.

**Problems:**

1. Find the number of divisor of 720

   **Solution:**

   Now, **720** = $2^4 \times 3^2 \times 5^1$

   So, the **number** of integral **divisors of 720** are.

$(4 + 1) \times (2 + 1) \times (1 + 1) = 5 \times 3 \times 2 = 30$.

2. Find the number of divisor of 480

   **Solution:**

   Now, **480** = $2^5 \times 3^1 \times 5^1$

   So, the **number** of integral **divisors of 480** are.

   $(5 + 1) \times (1 + 1) \times (1 + 1) = 6 \times 2 \times 2 = 24$.

3. Find the number of divisor and sum of all the divisors of 360.

   **Solution:**

   Now, **360** = $2^3 \times 3^2 \times 5^1$

   So, the **number** of integral **divisors of 360** are.

   $(3 + 1) \times (2 + 1) \times (1 + 1) = 4 \times 3 \times 2 = 24$.

   Sum of all divisor is $S = \frac{2^{3+1}-1}{2-1} . \frac{3^{2+1}-1}{3-1} . \frac{5^{1+1}-1}{5-1}$.

$$= \frac{15}{1} . \frac{26}{2} . \frac{24}{4} = 1170 ,$$

## 2 . Euler's Function

**Definition:**

The Euler's Function is the number of positive integers less than N and prime to it.it is denoted by$\varphi(N)$.

In general $\varphi(N) = N\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)$ ...

Notes:

1. $\varphi(N) = \varphi(a)\varphi(b)$ $if$ $N = ab$, a and b are prime to one another.
2. $If\ p\ is\ prime\ then\ \varphi(p^r) = p^r\left(1 - \frac{1}{p}\right)$.

**Problems:**

1. Find the number of integer lessthan 210 and prime to it.

   **Solution:**

   **210** = 2 × 3 × 5× 7.

   We know that $\varphi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)\ldots$

   $$\varphi(210) = 210\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 48.$$

2. Find the number of integer lessthan 210 and prime to it.

   **Solution:**

   **720** = 2 × 3 × 5

   We know that $\varphi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)\ldots$

   $$\varphi(720) = 720\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 192.$$

**Integral part of a real number**

1. $\frac{7}{2} = 3 + \frac{1}{2}$ , 3 is the integral part, $\frac{1}{2}$ is the fractional part.
2. $\frac{9}{2} = 4 + \frac{1}{2}$ , 4 is the integral part, $\frac{1}{2}$ is the fractional part.

**Highest power of a prime number p contained in $n$!.**

Highest power of p is $n$! is $=\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^{k-1}}\right]$.

**Problems:**

1. Find the highest power of 7 is 1000!

   **Solution:**

   Now $\left[\frac{1000}{7}\right] = 142$

   $$\left[\frac{142}{7}\right] = 20$$

$$\left[\frac{20}{7}\right] = 2$$

Highest power of p is $n!$ is $=\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^{k-1}}\right]$.

$$= 142 + 20 + 2 = 164$$

$7^{164}$ is the highest power of 7 in 1000!.

2. With how many zero does 61! end?

**Solution:**

Let us find the highest power of 2 and 5 in 61!

$$\left[\frac{61}{2}\right] = 30$$

$$\left[\frac{30}{2}\right] = 15$$

$$\left[\frac{15}{2}\right] = 7$$

$$\left[\frac{7}{2}\right] = 3$$

$$\left[\frac{7}{2}\right] = 1$$

The highest power of 2 in 61! $= 30 + 15 + 7 + 1 = 56$.

$$\left[\frac{61}{5}\right] = 12$$

$$\left[\frac{12}{5}\right] = 2$$

The highest power of 5 in 61!=12+2=14.

The highest power of 10 in 61!=14.

61! Will end in 14 zeros.

# 3. Congruence

## 3.1 Basic properties of congruence

**Definition 3.1.1.** Let $n$ be a fixed positive integer. Two integers $a$ and $b$ are said to be *congruent modulo* $n$, symbolized by

$$a \equiv b(mod\ n)$$

if $n$ divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer $k$.

**Theorem 3.1.2.** *For arbitrary integers $a$ and $b$, $a \equiv b(mod\ n)$ if and only if $a$ and $b$ leave the same nonnegative remainder when divided by $n$.*

**Proof.** First take $a \equiv b(mod\ n)$, so that $a = b + kn$ for some integer $k$. Upon division by $n$, $b$ leaves a certain remainder $r$; that is, $b = qn + r$, where $0 \leq r < n$. Therefore,

$$a = b + kn = (qn + r) + kn = (q + k)n + r$$

which indicates that $a$ has the same remainder as $b$.

On the other hand, suppose we can write $a = q_1 n + r$ and $b = q_2 n + r$, with the same remainder $r$ $(0 \leq r < n)$. Then

$$a - b = (q_1 n + r) - (q_2 n + r) = (q_1 - q_2)n$$

whence $n|a-b$. In the language of congruences, we have $a \equiv b(mod\ n)$. □

**Example 3.1.3.** Because the integers $-56$ and $-11$ can be expressed in the form

$$-56 = (-7)9 + 7 \qquad -11 = (-2)9 + 7$$

with the same remainder 7, Theorem 3.1.2 tells us that $-56 \equiv -11(mod\ 9)$. Going in the other direction, the congruence $-31 \equiv 11(mod\ 7)$ implies that $-31$ and 11 have the same remainder when divided by 7; this is clear from the relations

$$-31 = (-5)7 + 4 \qquad 11 = 17 + 4$$

**Theorem 3.1.4.** *Let $n > 1$ be fixed and $a, b, c, d$ be arbitrary integers. Then the following properties hold:*

*(a) $a \equiv a(mod\ n)$.*

*(b) If $a \equiv b(mod\ n)$, then $b \equiv a(mod\ n)$.*

*(c) If $a \equiv b(mod\ n)$ and $b \equiv c(mod\ n)$, then $a \equiv c(mod\ n)$.*

*(d) If $a \equiv b(mod\ n)$ and $c \equiv d(mod\ n)$, then $a + c \equiv b + d(mod\ n)$ and $ac \equiv bd(mod\ n)$.*

*(e) If $a \equiv b(mod\ n)$, then $a + c \equiv b + c(mod\ n)$ and $ac \equiv be(mod\ n)$.*

*(f) If $a \equiv b(mod\ n)$, then $ak \equiv bk(mod\ n)$ for any positive integer $k$.*

**Proof.** For any integer $a$, we have $a - a = 0 \cdot n$, so that $a \equiv a(mod\ n)$. Now if $a \equiv b(mod\ n)$, then $a - b = kn$ for some integer $k$. Hence, $b - a = -(kn) = (-k)n$ and because $-k$ is an integer, this yields property (b).

Property (c) is slightly less obvious: Suppose that $a \equiv b(mod\ n)$ and also $b \equiv c(mod\ n)$. Then there exist integers $h$ and $k$ satisfying $a - b = hn$ and $b - c = kn$. It follows that

$$a - c = (a - b) + (b - c) = hn + kn = (h + k)n$$

which is $a \equiv c(mod\ n)$ in congruence notation.

In the same vein, if $a \equiv b(mod\ n)$ and $c \equiv d(mod\ n)$, then we are assured that $a - b = k_1 n$ and $c - d = k_2 n$ for some choice of $k_1$ and $k_2$. Adding these equations, we obtain

$$\begin{aligned}(a + c) - (b + d) &= (a - b) + (c - d) \\ &= k_1 n + k_2 n = (k_1 + k_2)n\end{aligned}$$

or, as a congruence statement, $a + c \equiv b + d(mod\ n)$. As regards the second assertion of property (d), note that

$$ac = (b + k_1 n)(d + k_2 n) = bd + (bk_2 + dk_1 + k_1 k_2 n)n$$

Because $bk_2 + dk_1 + k_1 k_2 n$ is an integer, this says that $ac - bd$ is divisible by $n$, whence $ac \equiv bd(mod\ n)$.

The proof of property (e) is covered by (d) and the fact that $c \equiv c(mod\ n)$. Finally, we obtain property (f) by making an induction argument. The statement certainly holds for $k = 1$, and we will assume it is true for some fixed $k$. From (d), we know that $a \equiv b(mod\ n)$ and $a^k \equiv b^k(mod\ n)$ together imply that $aa^k \equiv bb^k(mod\ n)$, or equivalently $a^{k+1} \equiv b^{k+1}(mod\ n)$. This is the form the statement should take for $k + 1$, and so the induction step is complete. □

**Example 3.1.5.** Show that 41 divides $220 - 1$. We begin by noting that $2^5 \equiv -9(mod\ 41)$, whence $(2^5)^4 \equiv (-9)^4(mod\ 41)$ by Theorem 3.1.4(f); in other words, $2^{20} \equiv 81 \cdot 81(mod\ 41)$. But $81 \equiv -1(mod\ 41)$, and so $81 \cdot 81 \equiv 1(mod\ 41)$. Using parts (b) and (e) of Theorem 3.1.4, we finally arrive at

$$2^{20} - 1 \equiv 81 \cdot 81 - 1 \equiv 1 - 1 \equiv 0(mod\ 41)$$

Thus, $41|2^{20} - 1$, as desired.

**Example 3.1.6.** For another example in the same spirit, suppose that we are asked to find the remainder obtained upon dividing the sum

$$1! + 2! + 3! + 4! + \cdots + 99! + 100!$$

by 12. Without the aid of congruences this would be an awesome calculation. The observation that starts us off is that $4! \equiv 24 \equiv 0(mod 12)$; thus, for $k \geq 4$,

$$k! \equiv 4! \cdot 5 \cdot 6 \cdots k \equiv 0 \cdot 5 \cdot 6 \cdots k \equiv 0(mod\ 12)$$

In this way, we find that

$$1! + 2! + 3! + 4! + \cdots + 100! \equiv 1! + 2! + 3! + 0 + \cdots + 0 \equiv 9(mod\ 12)$$

Accordingly, the sum in question leaves a remainder of 9 when divided by 12.

**Theorem 3.1.7.** *If* $ca \equiv cb(mod\ n)$*, then* $a \equiv b(mod\ n/d)$*, where* $d = gcd(c, n)$*.*

**Proof.** By hypothesis, we can write

$$c(a - b) = ca - cb = kn$$

for some integer $k$. Knowing that $gcd(c, n) = d$, there exist relatively prime integers $r$ and $s$ satisfying $c = dr$, $n = ds$. When these values are substituted in the displayed equation and the common factor $d$ canceled, the net result is

$$r(a - b) = ks$$

Hence, $s|r(a - b)$ and $gcd(r, s) = 1$. Euclid's lemma yields $s|a - b$, which may be recast as $a \equiv b(mod\ s)$; in other words, $a \equiv b(mod\ n/d)$. □

**Corollary 3.1.8.** *If* $ca \equiv cb(mod\ n)$ *and* $gcd(c, n) = 1$*, then* $a \equiv b(mod n)$*.*

**Corollary 3.1.9.** *If* $ca \equiv cb(mod\ p)$ *and* $p \nmid c$*, where* $p$ *is a prime number, then* $a \equiv b(mod\ p)$*.*

**Proof.** The conditions $p \nmid c$ and $p$ a prime imply that $gcd(c, p) = 1$. □

**Example 3.1.10.** Consider the congruence $33 \equiv 15(mod\ 9)$ or, if one prefers, $3 \cdot 11 \equiv 3 \cdot 5(mod\ 9)$. Because $gcd(3, 9) = 3$, Theorem 3.1.7 leads to the conclusion that $11 \equiv 5(mod\ 3)$.
A further illustration is given by the congruence $-35 \equiv 45(mod\ 8)$, which is the same as $5 \cdot (-7) \equiv 5 \cdot 9(mod\ 8)$. The integers 5 and 8 being relatively prime, we may cancel the factor 5 to obtain a correct congruence $-7 \equiv 9(mod\ 8)$.

## 4.1 Fermat's Little Theorem and Pseudo primes

**Theorem 4.1.1** (Fermat's theorem)**.** *Let $p$ be a prime and suppose that $p|a$. Then $a^{p-1} \equiv 1(mod\ p)$.*

**Proof.** We begin by considering the first $p-1$ positive multiples of $a$; that is, the integers

$$a, 2a, 3a, \cdots, (p-1)a$$

None of these numbers is congruent modulo $p$ to any other, nor is any congruent to zero. Indeed, if it happened that

$$ra \equiv sa(mod\ p) \quad 1 \leq r < s \leq p-1$$

then a could be canceled to give $r \equiv s(mod\ p)$, which is impossible. Therefore, the previous set of integers must be congruent modulo $p$ to $1, 2, 3, \cdots, p-1$, taken in some order. Multiplying all these congruences together, we find that

$$a \cdot 2a \cdot 3a \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdots (p-1)(mod\ p)$$

whence

$$a^{p-1}(p-1)! \equiv (p-1)!(mod\ p)$$

Once $(p-1)!$ is canceled from both sides of the preceding congruence (this is possible because since $p|(p-1)!$), our line of reasoning culminates in the statement that $a^{p-l} \equiv 1(mod\ p)$, which is Fermat's theorem. □

**Corollary 4.1.2.** *If $p$ is a prime, then $a^p \equiv a(mod\ p)$ for any integer $a$.*

**Proof.** When $p|a$, the statement obviously holds; for, in this setting, $a^p \equiv 0 \equiv a(mod\ p)$. If $p \nmid a$, then according to Fermat's theorem, we have $a^{p-1} \equiv 1(mod\ p)$. When this congruence is multiplied by $a$, the conclusion $a^p \equiv a(mod\ p)$ follows. □

**Lemma 4.1.3.** *If $p$ and $q$ are distinct primes with $a^p \equiv a(mod\ q)$ and $a^q \equiv a(mod\ p)$, then $a^{pq} \equiv a(mod\ pq)$.*

**Proof.** The last corollary tells us that $(a^q)^p \equiv a^q(mod\ p)$, whereas $a^q \equiv a(mod\ p)$ holds by hypothesis. Combining these congruences, we obtain $a^{pq} \equiv a(mod\ p)$ or, in different terms, $p|a^{pq} - a$. In an entirely similar manner, $q|a^{pq} - a$. Corollary 2 to Theorem 1.4.8 now yields $pq|a^{pq} - a$, which can be recast as $a^{pq} \equiv a(mod\ pq)$. □

**Theorem 4.1.4.** *If $n$ is an odd pseudo prime, then $M_n = 2^n - 1$ is a larger one.*

**Proof.** Because $n$ is a composite number, we can write $n = rs$, with $1 < r \leq s < n$. Then, according to Problem 21, Section 2.3, $2^r - 1|2^n - 1$, or equivalently $2^r - 1|M_n$, making $M_n$ composite. By our hypotheses, $2^n \equiv 2(mod\ n)$; hence $2^n - 2 = kn$ for some integer $k$. It follows that

$$2^{M_n - 1} = 2^{2^n - 2} = 2^{kn}$$

This yields

$$\begin{aligned} 2^{M_n - 1} &= 2^{kn} - 1 \\ &= (2^n - 1)(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 2^n + 1) \\ &= M_n(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 2^n + 1) \\ &= 0(mod\ M_n) \end{aligned}$$

We see immediately that $2^{M_n} - 2 \equiv 0(mod\ M_n)$, in light of which $M_n$ is a pseudo prime. □

**Theorem 4.1.5.** *Let $n$ be a composite square-free integer, say, $n = p_1p_2 \cdots p_r$, where the $p_i$ are distinct primes. If $p_i - 1|n - 1$ for $i = 1, 2, \cdots, r$, then $n$ is an absolute pseudo prime.*

**Proof.** Suppose that $a$ is an integer satisfying $gcd(a, n) = 1$, so that $gcd(a, p_i) = 1$ for each $i$. Then Fermat's theorem yields $p_i|a^{p_i-l} - 1$. From the divisibility hypothesis $p_i - 1|n - 1$, we have $p_i|a^{n-1} - 1$, and therefore $p_i|a^n - a$ for all $a$ and $i = 1, 2, \cdots, r$. As a result of Corollary 2 to Theorem 1.4.8, we end up with $n|a^n - a$, which makes $n$ an absolute pseudo prime. □

## 4.2 Wilson's Theorem

**Theorem 4.2.1** (Wilson). *If $p$ is a prime, then $(p - 1)! \equiv -1(mod\ p)$.*

**Proof.** Dismissing the cases $p = 2$ and $p = 3$ as being evident, let us take $p > 3$. Suppose that $a$ is any one of the $p - 1$ positive integers

$$1, 2, 3, \cdots, p - 1$$

and consider the linear congruence $ax \equiv 1(mod\ p)$. Then $gcd(a, p) = 1$. By Theorem 3.3.1, this congruence admits a unique solution modulo $p$; hence, there is a unique integer $a'$, with $1 \leq a' \leq p - 1$, satisfying $aa' \equiv 1(mod\ p)$.

Because $p$ is prime, $a = a'$ if and only if $a = 1$ or $a = p - 1$. Indeed, the congruence $a^2 \equiv 1(mod\ p)$ is equivalent to $(a - 1) \cdot (a + 1) \equiv 0(mod\ p)$. Therefore, either $a - 1 \equiv 0(mod\ p)$, in which case $a = 1$, or $a + 1 \equiv 0(mod\ p)$, in which case $a = p - 1$.

If we omit the numbers 1 and $p - 1$, the effect is to group the remaining integers $2, 3, \cdots, p - 2$ into pairs $a, a'$, where $a \neq a'$, such that their product $aa' \equiv 1(mod\ p)$. When these $(p - 3)/2$ congruences are multiplied together and the factors rearranged, we get

$$2 \cdot 3 \cdots (p-2) \equiv 1 (mod\ p)$$

or rather

$$(p-2)! \equiv 1 (mod\ p)$$

Now multiply by $p-1$ to obtain the congruence

$$(p-1)! \equiv p-1 \equiv -1 (mod\ p)$$

as was to be proved. □

**Example 4.2.2.** A concrete example should help to clarify the proof of Wilson's theorem. Specifically, let us take $p = 13$. It is possible to divide the integers $2, 3, \cdots, 11$ into $(p-3)/2 = 5$ pairs, each product of which is congruent to 1 modulo 13. To write these congruences out explicitly:

$$\begin{aligned} 2 \cdot 7 &= 1 (mod\ 13) \\ 3 \cdot 9 &= 1 (mod\ 13) \\ 4 \cdot 10 &= 1 (mod\ 13) \\ 5 \cdot 8 &= 1 (mod\ 13) \\ 6 \cdot 11 &= 1 (mod\ 13) \end{aligned}$$

Multiplying these congruences gives the result

$$11! = (2 \cdot 7)(3 \cdot 9)(4 \cdot 10)(5 \cdot 8)(6 \cdot 11) \equiv 1 (mod\ 13)$$

and so

$$12! \equiv 12 \equiv -1 (mod\ 13)$$

Thus, $(p-1)! \equiv -1 (mod\ p)$, with $p = 13$.

**Theorem 4.2.3.** *The quadratic congruence* $x^2 + 1 \equiv 0 (mod\ p)$, *where* $p$ *is an odd prime, has a solution if and only if* $p \equiv 1 (mod\ 4)$.

**Proof.** Let $a$ be any solution of $x^2 + 1 \equiv 0(mod\ p)$, so that $a^2 \equiv -1(mod\ p)$. Because $p \nmid a$, the outcome of applying Fermat's theorem is

$$1 \equiv a^{p-1} \equiv (a^2)^{(p-1)/2} \equiv (-1)^{(p-1)/2}(mod\ p)$$

The possibility that $p = 4k + 3$ for some $k$ does not arise. If it did, we would have

$$(-1)^{(p-1)/2} = (-1)^{2k+1} = -1$$

hence, $1 \equiv -1(mod\ p)$. The net result of this is that $p|2$, which is patently false. Therefore, $p$ must be of the form $4k + 1$.

Now for the opposite direction. In the product

$$(p-1)! = 1 \cdot 2 \cdots \tfrac{p-1}{2} \cdot \tfrac{p+1}{2} \cdots (p-2)(p-1)$$

we have the congruences

$$\begin{aligned} p-1 &\equiv -1(mod\ p) \\ p-2 &\equiv -2(mod\ p) \\ &. \\ &. \\ &. \\ \frac{p+1}{2} &\equiv -\frac{p-1}{2}(mod\ p) \end{aligned}$$

Rearranging the factors produces

$$\begin{aligned} (p-1)! &\equiv 1 \cdot (-1) \cdot 2 \cdot (-2) \cdots \frac{p-1}{2} \cdot \left(-\frac{p-1}{2}\right)(mod\ p) \\ &\equiv (-1)^{(p-l)/2}\left(1 \cdot 2 \cdots \frac{p-1}{2}\right)^2 (mod\ p) \end{aligned}$$

because there are $(p-1)/2$ minus signs involved. It is at this point that Wilson's theorem can be brought to bear; for, $(p-1)! \equiv -1(mod\ p)$, whence

$$-1 \equiv (-1)^{(p-1)/2}\left[(\tfrac{p-1}{2})!\right]^2 (mod\ p)$$

If we assume that $p$ is of the form $4k+1$, then $(-1)^{(p-1)/2} = 1$, leaving us with the congruence

$$-1 \equiv \left[(\tfrac{p-1}{2})!\right]^2 (mod\ p)$$

The conclusion is that the integer $[(p-1)/2]!$ satisfies the quadratic congruence $x^2 + 1 \equiv 0(mod\ p)$. □