MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

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Affiliated to the University of Madras
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DEPARTMENT OF MATHEMATICS

SUBJECT NAME: REAL ANALYSIS-I

SUBJECT CODE: BMA-CSC10

SEMESTER: V

PREPARED BY: PROF.S.C.PREMILA

UNIVERSITY OF MADRAS

B.Sc. DEGREE COURSE IN MATHEMATICS

SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC10

CORE-X: REAL ANALYSIS-I

(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 6 **YEAR: III**

Credits : 4 **SEMESTER: V**

Learning outcomes:

Students will acquire knowledge to

- Apply Mathematical concepts and Principles to perform numerical and symbolic computations.
- Understand and perform simple proofs.
- Know how abstract ideas and rigorous methods in Mathematical Analysis can be applied to practical problems.

UNIT I

Sets and Functions:Sets and elements- Operations on sets- functions- real valued functions-equivalence- countability - real numbers- least upper bounds.

Chapter 1 Section 1. 1 to 1.7

UNIT II

Sequences of Real Numbers:Definition of a sequence and subsequence- limit of a sequence-convergent sequences- divergent sequences- bounded sequences- monotone sequences-

Chapter 2 Section 2.1 to 2.6

UNIT III

Operations on convergent sequences- operations on divergent sequences- limit superior and limit inferior- Cauchy sequences.

Chapter 2 Section 2.7 to 2.10

UNIT IV

Series of Real Numbers: Convergence and divergence- series with non-negative terms-alternating series- conditional convergence and absolute convergence- tests for absolute convergence- series whose terms form a non-increasing sequence- the class l^2

Chapter 3 Section 3.1 to 3.4, 3.6, 3.7 and 3.10

UNIT V

Limits and Metric Spaces:Limit of a function on a real line-. Metric spaces - Limits in metric spaces.

Continuous Functions on Metric Spaces: Function continuous at a point on the real line-Reformulation- Function continuous on a metric space.

Chapter 4 Section 4.1 to 4.3 Chapter 5 Section 5.1-5.3

Contents and Treatment as in
"Methods of Real Analysis" : Richard R. Goldberg (Oxford and IBH Publishing Co.).

Reference:
1. Principles of Mathematical Analysis by Walter Rudin,TataMcGrawHill.
2. Mathematical Analysis Tom M Apostol, Narosa Publishing House.

e-Resources:
1. https://mathcs.org/analysis/reals/numseq/sequence.html.
2. http://www-groups.mcs.st-andrews.ac.uk/~john/analysis/index.html
3. http://www.phengkimving.com.

1

SETS AND FUNCTIONS

1.1. SETS AND FUNCTIONS.

1.SET

A set is a collection of well-defined object.

Notation: sets are usually denoted by capital letter A, B, C…

The elements of a set is denoted by a, b, c…

2. Subset and Super set

We say that A is a subset of B if $x \in A \Rightarrow x \in B$.

Notation: $A \subseteq B$

Note: Here B is called super set A. [Or] $B \supseteq A$ i.e., B contains A.

3. Proper subset

A is said to be proper subset of B, If (1) $A \subset B$ (2) $A \neq B$

4. Equality of sets

Two sets A and B are said to be equal (i.e., A = B) if f

They contains the same elements.

i.e., A=B <=> if $x \in A$ <=> $x \in B$

i.e., A=B iff (1). $A \subseteq B$ (2) $B \subseteq A$.

5. Power set

The set of all subset of a set A is called the power set of A.

Notation: Power set of A = P(A)

Example: Let A = {1, 2, 3}

The P(A) = {{},{1},{2}.{3}.{1, 2},{1, 3},{2, 3}.{1, 2, 3}}

The number of elts in P(A)=2^n. if n(A)=n

6. Define mapping (OR) function.

A mapping f: A $\rightarrow$ B is a rule which associate every elements of A,

There exists a unique elt y in B,s.t f(x)=y.

Results

R1. The Range of f = f(B) = Image of A under

$f = \{y \in B / y = f(x)\ \forall x \in A\}$

R2. A is called domain set

R3. B is called co-domain.

R4. f is said to be a map if every element as unique image.

7. Define onto map.[OR] Surjection map

We say that f is onto map, if for every elt $y \in$B, $\exists$ $x \in$ A. such that $y = f(x)$.

[OR]

f is onto map if f(A) = B

(i.e., Range of $f - f$(A) = B)

[OR]

f is onto if every elts of B, there is a pre image in A.

8. Constant mapping [OR] many one map

A mapping f: A $\rightarrow$ B is said to be many one function,

If every element in A is mapped into one element in B.

i.e., $\forall\ x \in$ A, $\exists$ unique $y \in$B, s.t $y = f(x)$.

9. One-One map:[OR] Injection map

A mapping f: A $\rightarrow$ B is said to be 1-1 map,

If different elt of A have different image in B.

[if $a \neq b$ then $f(a) \neq f(b)$]

[OR]

Equal image in B have equal elt in A. [if $f(a) = f(b)$ then $a = b$]

10. Define 1-1 Correspondence [OR] Bijection map

A mapping f: A $\rightarrow$ B is said to be 1-1 correspondence, If f is 1-1 and onto.

11. Define composition of mapping

Let f: A $\rightarrow$ B and g:B $\rightarrow$ C, then g of: A $\rightarrow$ C

12. Define characteristic function

[A15 N13

Let A $\subset$B then the function $\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$

Is called characteristic function of A.

Properties. if A, B $\subseteq$ S.

P1. $\chi_{A \cup B}(x)$=max$\left(\chi_A, \chi_B\right)$ P.T--A13

P2. $\chi_{A \cap B}(x)$=min$\left(\chi_A, \chi_B\right)$

P3. $\chi_{A-B}(x) = \chi_A(x) - \chi_B(x)$

P4. $\chi_{A'}(x) = 1 - \chi_A(x)$.

P5. $\chi_{\phi}(x) = 0$. [Define char fun of empty set-A15]

P6. $\chi_S(x) = 1$.

13.Define the real valued function

A mapping f is said to be real valued function

If the range of f is a subset of R.

Example1. Let f: A $\rightarrow$ R is real valued function,

Example2. Let f: A $\rightarrow$ C is complex valued function.

Let f: A $\rightarrow$ R and g: A $\rightarrow$ R be two real valued function, then

1. $(f+g)(x) = f(x) + g(x)$
2. $(fg)(x) = f(x).g(x)$
3. $(f/g)(x) = f(x)/g(x)$ for $g(x) \neq 0$.
4. $\max(f, g) = \max(f(x), g(x))$
5. min(f, g)=min(f(x) , g(x)).
6. Max $(a, b) = \dfrac{(a+b)+|a-b|}{2}$;

   Min $(a, b) = \dfrac{(a+b)-|a-b|}{2}$ true for $a = f$ and $b = g$.

Theorem:1

If f is a function f: A $\rightarrow$ B and X, Y $\subseteq$ B.

Then f^{-1}(X $\cup$ Y) = f^{-1}(X) $\cup$ f^{-1}(Y)

[OR]

P.T the inverse image of union of two sets = the union of their inverse image.

Proof: Let '*a*' be an arbitrary element of $f^{-1}(X \cup Y)$

i.e., $a \in f^{-1}(X \cup Y) \Leftrightarrow f(a) \in X \cup Y.$

$\Leftrightarrow f(a) \in X$ or $f(a) \in Y$

$\Leftrightarrow a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$

i.e., $a \in f^{-1}(X \cup Y) \Leftrightarrow a \in f^{-1}(X) \cup f^{-1}(Y)$

Hence $f^{-1}(X \cup Y) \subseteq f^{-1}(X) \cup f^{-1}(Y)$ and

$f^{-1}(X) \cup f^{-1}(Y) \subseteq f^{-1}(X \cup Y).$

$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$

Theorem: 2

Let $f: A \to B$, X, Y subset of A, Then $f(X \cup Y) = f(X) \cup f(Y)$.

[OR]

P.T the image of union of two sets is the union of their images.

Proof: Let b be any arbitrary elt in $f(X \cup Y)$,

Then $\exists\ a$ in $X \cup Y$. s.t $f(a) = b$.

Since $a \in X \cup Y. \Rightarrow a \in X$ or $a \in Y$.

$\Rightarrow f(a) \in f(X)$ or $f(a) \in f(Y) \Rightarrow f(a) \in f(X) \cup f(Y).$

$\therefore \quad f(X \cup Y) \subseteq f(X) \cup f(Y)$ … (1.1)

Similarly, $f(X) \cup f(Y) \subseteq f(X \cup Y)$ … (1.2)

From (1.1) & (1.2) $f(X \cup Y) = f(X) \cup f(Y)$

Hence the proof.

Theorem: 3

Let f: A $\to$ B, X, Y $\subseteq$ A, then f(X $\cap$ Y) = f(X) $\cap$ f(Y) is true?

Justify your answer.

Proof:

This is not equal

For example. Let X = {0, –1, –2, –3 …..} and

Y = {0, 1, 2, 3, …..}

Let f: A $\to$ B is defined by $f(x) = x^2$

Here

$$X \cap Y = \{0\} \Rightarrow f(X \cap Y) = \{0\} \quad \ldots (1.3)$$

But

f(X) = {0, 1, 2, 3, ….} and

f(Y) = {0, 1, 2, 3, …..}

$$f(X) \cap f(Y) = \{0, 1, 2, 3 \ldots..\} \quad \ldots (1.4)$$

From (1.3) & (1.4) $f(X \cap Y) \neq f(X) \cap f(Y)$

Hence the proof.

PROBLEMS BASED ON FUNCTIONS

Problem 1.1 ***Consider the function defined by $f(x) = \sin x$, $-\infty < x < \infty$***

(i) What is range of f?

(ii) Find the domain of f?

(iii) what is the image of $\frac{\pi}{2}$ under f.

(iv) Find $f^{-1}(1)$

(v) Find $f\left(\left[0,\dfrac{\pi}{6}\right]\right)$, $f\left(\left[\dfrac{\pi}{6},\dfrac{\pi}{2}\right]\right)$

(vi) Let A=$\left[0,\dfrac{\pi}{6}\right]$***, B=***$\left[\dfrac{5\pi}{6},\pi\right]$ ***Does $f(A\cap B) = f(A) \cap f(B)$?***

☺***Solution:***

(i). The range of f is [0, 1] (since sin(0) = 0 and sin $\dfrac{\pi}{2}$=1)

(ii)The domain of f is R = $(-\infty,\infty)$ is a real line.

(iii) $f(\dfrac{\pi}{2}) = \sin(\dfrac{\pi}{2}) = 1$.

(iv) $f^{-1}(1) = \dfrac{\pi}{2}$.

(v) $f\left(\left[0,\dfrac{\pi}{6}\right]\right)$ = [sin o, sin $\dfrac{\pi}{6}$] = [o, ½]

(vi)Given A = $\left[0,\dfrac{\pi}{6}\right]$, B = $\left[\dfrac{5\pi}{6},\pi\right]$

$$\text{Then } A\cap B = \{0\} \Rightarrow f(A\cap B) = \sin 0 = 0 \qquad \ldots (1.5)$$

$$f(A) = f\left(\left[0,\frac{\pi}{6}\right]\right) = [\text{sino}, \sin\frac{\pi}{6}] = [\text{o}, ½]$$

$$f(B) = f\left(\left[\frac{5\pi}{6},\pi\right]\right) = \left[\sin\frac{5\pi}{6}, ,\sin\pi\right]$$

$$= [1/2, 0] = [0,1/2]$$

$$f(A)\cap f(B) = [0,1/2] \qquad \ldots (1.6)$$

From (1.5) &(1.6) $\Rightarrow$ $f(A\cap B) \neq f(A) \cap f(B)$?

Problem 1.2 ***Let $f(x) = x^2$, $-\infty < x < \infty$***

(i) What is the domain of f.

(ii) What is the range of f.

(iii) Find the image of 2 under f.

(iv) Find $f^{-1}(16)$

(v) Find $f^{-1}(-7)$?

(vi) Find $f[0, 3]$

☺***Solution:***

(i) The domain of f is R = $(-\infty, \infty)$ is a real line.

(ii) The range of f is $[0, \infty]$.

(iii) $f(2) = 2^2 = 4$

(iv) $f^{-1}(16) = 4$

(v) $f^{-1}(-7)$ there is no number -7 in $[0, \infty)$.

(vi) $f[0, 3] = [0^2, 3^2] = [0,9]$.

Problem 1.3 ***If $f(x) = \arcsin x$, $-1 \le x \le 1$.***

$$g(x) = \tan x,\ -\infty < x < \infty$$

Then $h = g \circ f$. Write a simple formula for h?

What are the domain of domain and the range of h?

☺***Solution:***

Given $f(x) = \sin^{-1} x$, $-1 \le x \le 1$.

$g(x) = \tan x, -\infty < x < \infty$

Then $h(x) = (g \circ f)(x) = g(f(x) = g(\sin^{-1}(x)) = \tan(\sin^{-1}(x))$.

1. The domain of h is $-1 \le x \le 1 = [-1,1]$

2. The range of h is $[\tan(\sin^{-1}(-1), \tan(\sin^{-1}(1)]$

$= [\tan(-\frac{\pi}{2}), \tan(\sin^{-1}(1)] = [-\infty, \infty]$

Problem 1.4 ***If $f(x) = 1 + \sin x,\ -\infty < x < \infty$***

$g(x) = x^2$, find gof and fog?

☺***Solution:***

(1) $(gof)(x) = g(f(x)) = g(1 + \sin x) = (1 + \sin x)^2$.

(2) $(fog)(x) = f(g(x)) = f(x^2) = 1 + \sin(x^2)$.

Hint composition of a mapping is not commutative. i.e., $(fog) \neq (gof)$.

Problem 1.5 ***Let $f(x) = 2x,\ -\infty < x < \infty$ can you think of function goh which satisfy the two equations gof = 2gh and hof = $h^2 - g^2$?***

☺***Solution:***

Given
$$f(x) = 2x,\ -\infty < x < \infty$$
$$gof = 2gh \quad \dots (1.7)$$
$$hof = h^2 - g^2 \quad \dots (1.8)$$
$$(gof)(x) = (2gh)(x)$$
$$\Rightarrow \quad g(f(x)) = 2g(x)\, h(x)$$
$$\Rightarrow \quad g(2x) = 2g(x)\, h(x)$$
$$\Rightarrow \quad 2g(x) = 2g(x)\, h(x)$$
$$h(x) = \mathrm{I}(x) \text{ [Multiply } g^{-1}] \quad \dots (1.9)$$

Also,
$$(hof)(x) = (h^2 - g^2)(x)$$
$$\Rightarrow \quad h(f(x)) = h^2(x) - g^2(x)$$
$$\Rightarrow \quad h(2x) = h^2(x) - g^2(x)$$
$$\Rightarrow \quad 2h(x) = h^2(x) - g^2(x)$$

Sub $h = \mathrm{I}$, we get, $2\mathrm{I}(x) = \mathrm{I}^2(x) - g^2(x)$

$$\Rightarrow \quad 2I(x) = I(x) - g^2(x)$$

$$\Rightarrow \quad I(x) = -g^2(x)$$

$$\Rightarrow \quad g(x) = \sqrt{-I(x)} \text{ is not defined.}$$

Hence g & h are not satisfied the given equations (1.7) & (1.81.8).

Problem 1.6 ***Let f (x) = 2x, $-\infty < x < \infty$ .Find two functions g & h which satisfy the two equations gof = 2g h and hof = h² – g²?***

☺***Solution:***

Given $f(x) = 2x$, $-\infty < x < \infty$.

$$gof = 2gh \qquad \ldots (1.10)$$

$$hof = h^2 - g^2 \qquad \ldots (1.11)$$

Let $g(x) = \sin$, $h(x) = \cos x$

Sub in (1.10) $\Rightarrow (gof)(x) = (2gh)(x)$

$$\Rightarrow \quad g(f(x)) = 2g(x)\,h(x)$$

$$\Rightarrow \quad g(2x) = 2g(x)\,h(x)$$

$$\Rightarrow \quad \sin 2x = 2.\sin x.\cos x \text{ is true.} \qquad \ldots. (1.12)$$

Also, $(hof)(x) = (h^2 - g^2)(x)$

$$\Rightarrow \quad h(f(x)) = h^2(x) - g^2(x)$$

$$\Rightarrow \quad h(2x) = h^2(x) - g^2(x)$$

$$\Rightarrow \quad \cos 2x = \cos^2 x - \sin^2 x \text{ is true} \qquad \ldots. (1.13)$$

Hence Let $g(x) = \sin$, $h(x) = \cos x$ are satisfies the given two equations (1.10) & (1.11).

Problem 1.7 ***If f is a function $f: A \rightarrow B$ & is the characteristic of $E \subset B$ of what subset of A is of the characteristic function .Ans: $f^{-1}(E)$.***

Problem 1.8 ***If A and B are subsets of S then –A13. Prove that (i) $(A \cup B)' = A' \cap B'$. and (ii) $(A \cap B)' = A' \cup B'$ [De Morgans' Laws]***

Problem 1.9 ***Define functions $f + g$ if $f: A \rightarrow R$ & $g: A \rightarrow R$. [A16]***

Problem 1.10 ***Give an example of onto functions. A15***

Problem 1.11 ***When do you say that the functions f is 1-1. A16***

Problem 1.12 ***When are the two functions f & g are equals. A13***

Problem 1.13 ***P.T $f(x) = \cos x$, $0 \le x \le \pi$, is 1-1.-N14.***

Problem 1.14 ***if $f(x) = x^2$, $-\infty < x < \infty$, find (i) $f^{-1}(-8)$ -A13 (ii) $f^{-1}(4)$ –N13.***

1.2. COUNTABLE SETS.

1. ***Define Countable set (Denumerable set). Give an example.***

   A set A is said to be countable set,

   if A is equivalent to set I (the set of all +ve integers)

   Ex1. Z-The set of all integer is countable.

   Ex2. Q-The set of all rational number is countable.

   Note. Equivalent of two sets is there is a 1–1 correspondence between them.

2. Define uncountable set. Give an Example?

A set which is not countable is called uncountable.

Ex1.R-the set of all real number is uncountable

Ex2.The set [0, 1] is uncountable

Ex3.Q^+: The set of all irrational number is uncountable.

3. Define cantor set, give an example?

[A N13,14

The cantor set K is the set of all numbers [0, 1] which have a ternary expansion without digit one.

Note: The cantor set K is uncountable. [N13]

4. Explain the construction of Cantor set?

The Cantor set K is obtained in the following way:

Step1.From [0.1] remove the open middle third leaving $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},\frac{2}{3}\right]$.

Step2.From each of $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},\frac{2}{3}\right]$ remove the open middle third leaving $\left[0,\frac{1}{9}\right]$, $\left[\frac{2}{9},\frac{3}{9}\right]$, $\left[\frac{6}{9},\frac{7}{9}\right]$ $\left[\frac{8}{9},\frac{9}{9}\right]$

Proceeding s in this way after the nth steps the open middle third is removed from each of 2^{n-1}intervals is of length 3^{-n+1}.

The Total lengths removed at the nth step is 2^{n-1}. $\frac{1}{3}$ 3^{-n+1}= $\frac{2^{n-1}}{3^n}$.

Then there remains 2^n intervals each of length 3^{-n} is clear that what remains of [0, 1] after this process is continued and infinitely is the set K.

P1. Is the Cantor set is countable? N13. Ans no, it is uncountable.

Thereom:1

P.T. the set of all integer is countable.

Proof:

Let Z be the set of all integer.

i.e., Z = {…..–3, –2, –1, 0, 1, 2, 3…..}

We define f: N $\rightarrow$ Z [Here N = I = {1, 2, 3 ….}

$$\text{By } f(n) = \begin{cases} \dfrac{n-1}{2} & if\ n = 1,3,5\ldots \\ -\dfrac{n}{2} & if\ n = 2,4,6\ldots. \end{cases}$$

Clearly f is 1-1 & onto

$\therefore$ N & Z are equivalent sets

Hence, Z –The set of all integer is countable

Theorem: 2

Prove that Countable union of countable set is countable.

[OR]

If $A_1, A_2, \ldots A_n$ are countable sets , Then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof:

Since $A_1, A_2, \ldots. A_n$ are the countable sets.

We can write

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots.\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots.\}$$

$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots.\}$$

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots.\}$$

We define height of $a_{ij} = i + j$

We can arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ according to the elts height as follows.

$$a_{11} : \text{ of height 2.}$$

$$a_{12}, a_{21} : \text{ of height 3.}$$

$$a_{13}, a_{22}, a_{31} : \text{ of height 4.}$$

Omitting the element a_{ij} which have been already counted.

$$\begin{array}{llll} a_{11} & a_{12} & a_{13} & a_{14} \ldots\ldots. \\ a_{21} & a_{22} & a_{23} & a_{24} \ldots\ldots. \\ a_{31} & a_{32} & a_{33} & a_{34}\ldots.. \\ a_{41} & a_{42} & a_{43} & a_{44} \ldots\ldots \end{array}$$

Hence $\bigcup_{n=1}^{\infty} A_n$ is countable.

Theorem: 3

P.T the set of all rational number is countable.

Proof:

Let $A_n = \{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots.\}$ for $n = 1, 2, 3, \ldots$

$A_n = Q^+$ (The set of all +ve rational number)

Clearly each A_n is countable.

$\therefore Q^+ = \bigcup_{n=1}^{\infty} A_n$ is countable.

Similarly, Q^- = The set of all negative rational number is countable.

$\Rightarrow$ $Q = Q^+ \cup Q^- \cup \{0\}$ is countable.

Hence Q = The set of all rational number is countable.

Theorem: 4

Prove that the set $[0,1] = \{x: 0 \le x \le 1\}$ is uncountable.

[A15,16 N13

Proof:

Let us assume that [0,1] is countable.

Then $[0, 1] = \{x_1, x_2, x_3, \ldots\}$

Where each number in [0, 1] occurs among any x_i's.

We write each x_i has an infinite decimals as follows

$x_1 = 0.a_{11}a_{12}a_{13}\ldots$

$x_2 = 0.a_{21}a_{22}a_{23}\ldots$

$x_3 = 0.a_{31}a_{32}a_{33}\ldots$

……

$x_n = 0.a_{n1}a_{n2}a_{n3}\ldots$

Let b_1 be any integer from 0 to 8, Such that, $b_1 \ne a_{11}$

Let b_2 be any integer from 0 to 8, Such that, $b_2 \ne a_{22}$

Let b_3 be any integer from 0 to 8, Such that, $b_3 \ne a_{33}$

……..

Let b_n be any integer from 0 to 8, Such that, $b_n \neq a_{nn}$

In general, for each n = 1, 2, 3…

Let $y = 0.b_1 b_2 b_3$……

Then for any n

The decimal expansion of y defer from the decimal expansion of x_n $[\because b_n \neq a_{nn}]$

Also, y is unique, (since nor b_n=9)
Hence $y \neq x_n$ $\forall n$. & $(0 \leq y \leq 1)$

Which is a contradiction

Hence the set [0, 1] is uncountable.

Theorem: 5

P.T the set of all real number R is uncountable. [A14]

Proof:

<u>[For 10 marks write above]</u>

We know that every subset of countable set is countable.

Suppose R is countable set.

Then [0, 1] which is a subset of R must also a countable set.

which is a contradiction to the set [0,1] is uncountable.

Hence R is uncountable.

Theorem: 6

P.T the set of all irrational number is uncountable.

Proof:

Since R-the set of real number is uncountable.

Also, Q = The set of rational number is countable.

$\therefore$ R – Q = The set of all irrational number is uncountable.

1.3. UPPER BOUND AND LOWER BOUND.

1. Define upper bound and lower bound of a set.

Upper bound:

A subset A $\subset$ R is said to be bounded above,

If $\exists$ a number M $\in$R, s.t $x \le$M , $\forall x \in$ A.

Then M is called upper bound of A.

Lower bound:

A subset A $\subset$ R is said to be bounded below,

If $\exists$ a number L $\in$R, s.t $x \ge$L, $\forall$ $x \in$ A.

Then L is called lower bound of A.

2. Define least upper bound. [l.u.b or supremum]

The number M is called the l.u.b for A (or) supremum of A.

If (1) M is an upper bound for A.

(2) No number less than M is an upper bound for A.

3. Define greatest lower bound. [g.l.b or infimum]

The number L is called the g.l.b for A (or) infimum of A.

If (1) L is a lower bound for A.

(2) No number greater than L is an lower bound for A.

4. Define bounded set. [A16]

A subset A $\subset$ R is said to be bounded,

If If $\exists$ a numbers L & M $\in$ R, s.t L $\leq x \leq$M, $\forall\ x \in$ A.

[OR]

A is bounded if it has bounded blow and bounded above.

Note: A subset which is not bounded is called unbounded set.

5. State the least upper bound axiom.

Every non-empty subset A of R which is bounded above has a l.u.b in R.

6. State the properties of Supremum and Infimum.

(1) Sup (A + B) = Sup(a) + Sup(B)

(2) Sup (kA) = k.Sup(A)

(3) Sup(–A) = – inf(A).

Similarly, for infimum.

PROBLEMS BASED ON L.U.B AND G.L.B.

P1. For S = [0, 1] here l.u.b = 1 & g.l.b = 0.

P2. For S = $\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$ find l.u.b & g.l.b?

Solution:

For the set n^{th} term is $s_n = \dfrac{2^n - 1}{2^n}$

here $a_n = \dfrac{2^n - 1}{2^n}$, nth term

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^n\left(1 - \frac{1}{2^n}\right)}{2^n} = (1 - 0)$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{2}$$

$\therefore$ g.l.b of A=1/2 & l.u.b of A =1.

$$s_\infty = \frac{2^n\left(1 - \frac{1}{2^n}\right)}{2^n} = 1. \text{ And g.l.b} = s_1 = \frac{1}{2}.$$

P3. Find g.l.b and l.u.b for the set N of all natural numbers.

Solution:

The set N of all natural number integers = {,,,,– 2, –1, 0, 1, 2 ,,,,}

Here glb = 1 and there is no lub.

Since N is not bounded above.

P4. S = (7, 8) here lub is 8 and glb is 7.

But both are not a member of S.

P5. Find lub and glb for the set {*x* in R/0 $\leq$ *x* $\leq$2}.Ans: glb = 0 and lub = 2.

P6. Write a lower bound for the set

$$\left\{\left(1+\frac{1}{n}\right)^n, n = 1,2,3\ldots or\ n \in N\right\}$$ **N15.**

Solution:

glb= 2 and lub = e.

P7. Find lub and glb of the following sets

(i) $\{\pi+1, \pi+2, \pi+3, \ldots.\}$ (ii) $\left\{\pi+1, \pi+\frac{1}{2}, \pi+\frac{1}{3}, \ldots.\right\}$

Solution:

(i) glb = $\pi+1$ and there is no lub [since it is not bounded above]

Glb = $\pi+1$ and lub = π. (since $\frac{1}{\infty}$=0)

P8. Give an example of a countable subset of odd whose g.l.b & l.u.b are both in R – A?

Solution:

Let A = The set of all rational number in ($\sqrt{2}, \sqrt{3}$).

Here g.l.b of A = $\sqrt{2}$ and l.u.b of A = $\sqrt{3}$ both are in R – A.

P9. Find the g.l.b for the following set

(a) (7, 8)

(b){ π +1, π +2, π +3,…….}

(c) { π +1, π +1/2, π + 1/3.,,,,}

Solution:

(a) Let A = (7,8); g.l.b of A = 7 & l.u.b of A = 8.

(b) Let A = { π +1, π +2, π +3,…….}

Here $a_n = \pi + n$, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (\pi + n) = \infty$

$\therefore$ g.l.b of A = π +1. But no l.u.b

(c) Let A={ π +1, π +1/2, π +1/3.,,,,} Here $a_n = \pi + n$, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (\pi + \frac{1}{n}) = \pi$.

$\therefore$ g.l.b of A = π & l.u.b of A = π + 1.

For the singleton set {5}

The g.l.b = l.u.b = 5.

Let A = $\{x/x$ is irrational $1 < 1 + x^3 \le 3\}$

P10. Find l.u.b & g.l.b?

Solution:

Given A = $\{x/x$ is irrational, $1 < 1 + x^3 \le 3\}$

= $\{x/x$ is irrational, $0 < x^3 \le 2\}$

(subtract 1 on both side)

A = $\{x/x$ is irrational, $0 < x \le \sqrt[3]{3}\ \}$

$\therefore$ g.l.b of A = 0 (not in the set) & l.u.b of A = $\sqrt[3]{3}$.

$\therefore$ A is unbounded set.

[OR]

A is bounded above, but not bounded below.

P11. Find g.l.b & l.u.b (a) $\left\{1 - \frac{1}{n}, n \in N\right\}$ **(b)** $\left\{\frac{3n+2}{2n+1}, n \in N\right\}$

(c) $\{x/-5 < x < 3\}$ (d) $\{x: x = (-1)^n n, n \in \mathrm{N}\}$ (e) $\{x: x = 2^n, n \in \mathrm{N}\}$

[Note: N = The set of natural no/- = {1, 2, 3, ….}

Solution:

(a) Let A= $\left\{1 - \frac{1}{n}, n \in N\right\}$

= {0, 1/2, 1/3, 1/4, ……$(n$-1)/n, ….}

Here $a_n = \left(1 - \frac{1}{n}\right)$, $\lim_{n \to \infty} a_n = \lim_{n \to \infty}\left(1 - \frac{1}{n}\right) = 1$

$\therefore$ g.l.b of A =0 & l.u.b of A = 1.

(b) Let A= $\left\{\frac{3n+2}{2n+1}, n \in N\right\}$ = { $\left\{\frac{5}{3}, \frac{8}{5}, \frac{11}{7}, ...\frac{3n+2}{2n+1}, ...\frac{3}{2}\right\}$

Here $a_n = \left(\frac{3n+2}{2n+1}\right)$,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty}\left(\frac{3n+2}{2n+1}\right) = \lim_{n \to \infty} \frac{n}{n}\left(\frac{3 + \frac{2}{n}}{2 + \frac{1}{n}}\right) = \frac{3}{2}$$

$\therefore$ g.l.b of A = $\frac{3}{2}$ & l.u.b of A = $\frac{5}{3}$.

(c) Let A = $\{x/-5 < x < 3\}$, $\therefore$ g.l.b of A = –5 & l.u.b of A = 3.

(d) Let A = $\{x: x = (-1)^n n, n \in \mathrm{N}\} = \{-1, 2, -3, 4, -5, \ldots\ldots\ldots\}$

$= \{-5, -3, -1, 2, 4, 6, \ldots\ldots\}$

$\therefore$ g.l.b of A = –5 & l.u.b of A does not exists

Hence A is unbounded set.

(e) Let A = $\{x: x = 2^n, n \in \mathrm{N}\} = \{2, 2^2, 2^3, \ldots..\}$

$\therefore$ g.l.b of A = 2 & l.u.b of A does not exists.

Hence A is unbounded set.

2

SEQUENCE OF REAL NUMBERS

2.1. SEQUENCE OF REAL NUMBERS

1. Define sequence

A sequence S = $\{s_n\}_{n=1}$ of real numbers is a function from I into R.

2. Define subsequence

A sequence is said to be a subsequence of S, if it contains least one less than S.

Example 1: Let S=$\{n\}_{n=1}$ = {1, 2, 3,…..} is a sequence

Also, S′= $\{n + 1\}_{n=1}$ = {2, 3, 4, 5,…..} is a subsequence of S.

And S′= $\{2n\}_{n=1}$ ={2, 4, 6,….} is a subsequence of S.

3. Define limit of a sequence

Let $\{s_n\}_{n=1}$ be a sequence of real numbers.

We say that s_n approaches to the limit L as n approaches to ∞.

If $\forall \in > 0$, $\exists$ a +ve integer N,

Solve that $|s_n - L| < \in$, $\forall\, n \geq N$,

[OR]

$\lim\limits_{n\to\infty} s_n = L$.

4. Define convergent sequence

A sequence of real number $\{s_n\}_{n=1}$ is said to convergent to L,

If the sequence $\{s_n\}_{n=1}$ has a limit L.

[OR]

$\lim\limits_{n \to \infty} s_n = L$ exists finitely.

Example 1: The sequence {1, 1, 1…..} changes to 1.

Example 2: The sequence {1, 1/2, 1/3…..} changes to 0

(Since $\lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$)

5. Define divergent sequence

A sequence of real numbers $\{s_n\}_{n=1}$ is said to be divergent,

If the sequence. $\{s_n\}_{n=1}$ does not have a limit.

i.e., $\lim\limits_{n \to \infty} s_n \neq L$.(not finite).

Example 1: The sequence $\{n\}_{n=1}$ is diverges.

Since $\lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} n = \infty$.

6. Define divergent to minus infinity

Define divergent to minus infinity. [OR] Converges to minus infinite.

A sequence of real number $-\{s_n\}_{n=1}$ is said to be divergent to infinite.

If for all real number- M > 0, $\exists$ a +ve integer N,

Solve that $s_n \leq -M, \forall n \geq N$.

[OR]

$\lim\limits_{n \to \infty} s_n = -\infty$.

Example 1: The sequence $\left\{\log \frac{1}{n}\right\}_{n=1}$ diverges to $-\infty$.

7. Define oscillating sequence of real numbers

A sequence sequence $\{s_n\}$ of real numbers is said to be oscillating sequence.

If the sequence $\{s_n\}$ diverges but not diverges to ∞ and $-\infty$.

Example 1: The sequence $\{(-1)^n\}_{n=1}$ is not diverges to both ∞ and $-\infty$.

8. Define bounded sequence

A sequence $\{s_n\}_{n=1}$ is said to be bounded sequence

if $\exists$ a M $\in$R, solve that $|s_n| < M$, $\forall\, n \in$ I.

[OR]

We say that the sequence $\{s_n\}$ is said to be bounded sequence, if it is both bounded above and below.

Note:

(i) We say that the sequence $\{s_n\}$ is said to be bounded above, if the range of $\{s_n\}$ is bounded above.

(ii) Similarly, We say that the sequence $\{s_n\}$ is said to be bounded below, if the range of $\{s_n\}$ is bounded below.

Examples

Example 1: The oscillating sequence $\{(-1)^n\}$ is bounded . [Since its range set is $\{-1, 1\}$]

Example 2: The sequence {1, 2, 1, 3, 1, 4….} is an oscillating sequence is not bounded sequence.

[Since it is bounded below by 1, but it has not bounded above.]

9. Define Monotone sequence

Let $\{s_n\}_{n=1}$ be a sequence of real number

If $s_1 \leq s_2 \leq s_3 \leq \ldots\ldots \leq s_n \leq s_{n+1} \leq \ldots$

Then $\{s_n\}_{n=1}$ is called non-decreasing sequence.

If $s_1 \geq s_2 \geq \ldots.\ s_n \geq s_{n+1} \geq \ldots..$

Then $\{s_n\}_{n=1}$ is called non-increasing sequence.

A monotone sequence is a sequence which is either non-decreasing or non-increasing

10. Define limit superior of a sequence $\{s_n\}$ of all real numbers.

Let $\{s_n\}_{n=1}$ be a sequence of real number – that is bounded above.

Let M_n = l.u.b $\{s_n, s_{n+1}, s_{n+2}, \ldots\ldots\}$

If $\{M_n\}_{n=1}$ is convergent, then $\lim\limits_{n\to\infty} \text{Sup } s_n = \lim\limits_{n\to\infty} M_n$

If $\{M_n\}_{n=1}$ is divergent to $-\infty$, then $\lim\limits_{n\to\infty} \text{Sup } s_n = \infty$.

11. Define limit Inferior of a sequence $\{s_n\}$ of all real numbers.

Let $\{s_n\}_{n=1}$ be a sequence of real number- that is bounded below,

Let m_n = g.l.b $\{s_n, s_{n+1}, s_{n+2}, \ldots\ldots\}$

(a) If $\{m_n\}_{n=1}$ is convergent, then $\lim\limits_{n\to\infty} \inf s_n = \lim\limits_{n\to\infty} m_n$

(b) If $\{m_n\}_{n=1}$ is divergent to ∞, then $\lim\limits_{n\to\infty} \inf s_n =.= \infty$

12. Define Cauchy sequence

Let $\{s_n\}_{n=1}$ be sequence of real number- is Cauchy sequence,

If Given $\in\ > 0$, $\exists$ a +ve integer N,

solve that $\left|s_m - s_n\right| < \in$, $\forall\, m, n \geq$ N,

Example 1: The sequence $\{1/n\}_{n=1}$ is a Cauchy sequence.

Most Important Results in real analysis

Result 1: A non-decreasing sequence which is bounded above is convergent.

Result 2: A non-increasing sequence which is bounded below is convergent.

Result 3: The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}$ is converges to e.

Result 4: prove that every subsequence of a convergent sequence converges to the same limit.

Result 5: (a) If $0 < x < 1$, then the sequence $\{x^n\}$ converges to 0.

(b) If $x \geq 1$, then the sequence $\{x^n\}$ diverges to infinity.

Example:

(i) For $x = 1/2$, the sequence $\{x^n\}$, $\lim\limits_{n \to \infty} \frac{1}{2^n} = 0$.

(ii) For $x = 3$, The sequence $\{x^n\}$, $\lim\limits_{n \to \infty} 3^n = \infty$, it is divergent sequence.

2.2. THEOREMS ON LIMITS

Theorem: 1

If $\{s_n\}_{n=1}$ is a sequence of non-negative real number and if $\lim_{n \to \infty} s_n = L$, then $L \geq 0$.

Proof:

Given $\lim_{n \to \infty} s_n = L$ … (2.1)

To prove that, $L \geq 0$

Let us assume that $L < 0$.

By definition, for consider $\in = -\frac{L}{2}$

$\therefore \ |s_n - L| < \in, \ \forall\, n \geq N.$ … (2.2) by (2.1)

$\Rightarrow \therefore \ |s_n - L| < -\frac{1}{2}, \ \forall\, n \geq N.$

$\Rightarrow -\left(-\frac{L}{2}\right) < s_n - L < \left(-\frac{L}{2}\right), \ \forall\, n \geq N$

[Since $|x| \leq a \Rightarrow -a \leq x \leq a$]

Add L, on both sides, $\frac{3L}{2} < s_n < \frac{L}{2}, \forall\, n \geq N.$

i.e. $s_n < \frac{L}{2}, \ \forall\, n \geq N$

Which is a contradiction to $\{s_n\}$ is non-negative sequence. Hence, $L \geq 0$.

Theorem: 2 (Uniqueness of limits)

Prove that the limit of the sequence is unique.	If $\lim\limits_{n \to \infty} s_n$ = L, and $\lim\limits_{n \to \infty} s_n$ = M, then L = M.

Proof:

To prove that, if L and M are two limits of a convergent sequence $\{s_n\}_{n=1}$, then L = M.

We assume that $L \neq M$. Let $|M - L| > 0$,

Let $\in = \dfrac{|M - L|}{2} > 0$ … (2.3)

Since $\lim\limits_{n \to \infty} s_n$ = L, and $\lim\limits_{n \to \infty} s_n$ = M,

$\therefore \quad \exists$ a $+^{ve}$ integer N_1, N_2.

Solve that $|s_n - L| < \in, \ \forall \ n \geq N_1$ … (2.4)

$|s_n - M| < \in, \ \forall \ n \geq N_2$ … (2.5)

Choose N = Max (N_1, N_2)

Then,

$|M - L| = |(s_n - L) + (s_n - M)| \leq |s_n - L| + |s_n - M| < \in + \in = 2 \in$

i.e., $|M - L| < 2 \in$

i.e., $|M - L| < |M - L|$ by (2.3) – which is a contradiction.

$\Rightarrow$ L = M.

Hence the limit of the sequence is unique.

Theorem: 3

If the sequence $\{s_n\}_{n=1}$ converges to L, then prove that every subsequence of $\{s_n\}_{n=1}$ is also converges to L.

[OR]

Prove that every subsequence of a convergent sequence converges to the same limit.

Proof

Let $\{s_n\}_{n=1}$ be a convergent sequence, then $\lim\limits_{n \to \infty} s_n = L$.

Let $\{s_n\}_{i=1}$ be a subsequence of $\{s_n\}_{n=1}$

Since $\lim\limits_{n \to \infty} s_n = L$.

By definition, given $\in > 0$, $\exists$ a +ve integer N,

solve that $|s_n - L| < \in$, $\forall$ $n \geq N$,

$\Rightarrow |s_{ni} - L| < \in$, $\forall$ $n_i \geq N$,

$\Rightarrow \lim\limits_{n \to \infty} s_{ni} = L$.

Hence every subsequence of $\{s_n\}$ is also converges to L.

Theorem: 4

Prove that every convergent sequence is bounded.

Proof:

Let $\{s_n\}_{n=1}$ be a convergent sequence then $\lim\limits_{n \to \infty} s_n = L$.

By definition, given $\in = 1$, $\exists$ a $N \in I$,

Solve that $|s_n - L| < \in$, $\forall$ $n \geq N$,

Now $|s_n| = |(s_n - L) + L| \leq |s_n - L| + |L|$ $\forall$ $n \geq N$,

$\Rightarrow |s_n| \leq 1 + |L|$, $\forall\ n \geq$ N,

Choose M = Max $\{|s_1|, |s_2|, |s_3|, \ldots.. |s_{n-1}|\}$

$\therefore |s_n| \leq M$, $\forall\ n \geq$ N, $\{s_n\}_{n=1}$ is bounded. Hence the proof,

Result: Bounded sequence need not be convergent.

Example: Consider the sequence {1, –1, 1, –1, ……} it is a bounded sequence, with rang set {–1, 1}.But it is a oscillating sequence.

THEOREMS ON MONOTONIC SEQUENCE

Theorem: 5

Prove that a non-decreasing sequence which is bounded above is convergent. Give an example.

Proof:

Let $\{s_n\}$ be a non-decreasing sequence which is bounded above.

Let A = $\{s_1, s_2, s_3. ,,,,\}$ is a non-empty set which is bounded above.

$\therefore$ A has l.u.b say M (by axiom of l.u.b}

i.e., M = l.u.b $\{s_1, s_2, s_3.,,,,\}$ … (2.6)

To prove that $\lim\limits_{n \to \infty} s_{ni}$ = M.

By definition of l.u.b

Given $\in > 0$

M – $\in$ is not an u.b for A.

$\therefore$ There exists an integer N > 0, solve that $\underline{s_n} > M - \in$ … (2.7)

Since s_n is a non-decreasing sequence

(2.7) $\Rightarrow s_n > M - \in$, $\forall\ n \geq N$ … (2.8)

Since M is an u.b for A.

$S_n < M + \in\ \forall\, n = 1, 2, 3\ldots$ … (2.9)

$\therefore$ From (2.8) & (2.9)

$M - \in < s_n < M + \in, \ \forall\ n \geq N,$

Sub M, $- \in < s_n - M < \in, \ \forall\ n \geq N,$

$\Rightarrow |s_n - M| < \in, \ \forall\ n \geq N,$

$\therefore \lim\limits_{in \to \infty} s_{ni} = M.$

Hence a non-decreasing sequence which is bounded above is convergent.

Theorem: 6

Prove that a non-increasing sequence of real number which is bounded below is convergent.

Proof:

Let $\{s_n\}$ be a non-increasing sequence which is bounded below.

Let A = $\{s_1, s_2, s_{3,,,,,,}\}$ is a non-empty set which is bounded below.

$\therefore$ A has g.l.b say L [By axiom of g.l.b]

i.e., L = g.l.b $\{s_1, s_2, s_{3,,,,}\}$ … (2.10)

To prove that $\lim\limits_{in \to \infty} s_{ni} = L.$

By definition of g.l.b

Given $\in > 0$, $L + \in$ is not an l.b for A.

$\therefore$ There exists an integer $N > 0$, solve that $s_n < L + \in$. … (2.11)

Since s_n is a non-increasing sequence

(2.11) $\Rightarrow s_n < L + \in, \ \forall \ n \geq N$ … (2.12)

Since L is an l.b for A.

$S_n > L - \in \ \forall n = 1, 2, 3\ldots$ … (2.13)

$\therefore$ From (2.12) and (2.13)

$L - \in < s_n < L + \in, \ \forall \ n \geq N,$

Sub L, $- \in \ < s_n - L < \in, \ \forall \ n \geq N,$

$\Rightarrow |s_n - L| < \in, \ \forall \ n \geq N,$

$\therefore \lim\limits_{in \to \infty} s_{ni} = L.$

Hence a non-increasing sequence which is bounded below is convergent.

Theorem: 7

Prove that a non-decreasing sequence which is not bounded above is divergent to infinity.

Proof:

Let $\{s_n\}$ be a non-decreasing sequence which is not bounded above.

Given M > 0. We can find $N \in I$, solve that $s_n > M, \ \forall \ n \geq N$,… (2.14)

Since M is not an upper bound for the sequence $\{s_n\}$,

There must be $N \in I$, solve that $s_n > M$

For this N, (2.14) follows from the hypothesis that $\{s_n\}$ be a non-decreasing sequence.

Hence the proof.

Theorem: 8

Prove that a non-increasing sequence which is not bounded below is divergent to minus infinity.

Proof:

Let $\{s_n\}$ be a non-increasing sequence which is not bounded above.

Given M > 0.

We can find N $\in$ I, solve that $s_n < M$, $\forall\ n \geq N$, … (2.15)

Since M is not an upper bound for the sequence $\{s_n\}$, There must be N$\in$I, solve that $s_N < M$

For this N, (2.15) follows from the hypothesis that $\{s_n\}$ be a non-decreasing seq.

Hence the proof.

PROBLEMS BASED ON CONVERGENT SEQUENCE.

Problem 2.1 ***Write formula for s_n for each of the following sequence.***

(i) {2,1, 4, 3, 6, 5, 8, 7, ……}

Ans: $s_n = n + 1$ if n is odd, $s_n = n - 1$, if n is even.

(ii) {1, – 1, 1, – 1,…..}

Ans: $s_n = 1$ if n is odd, $s_n = -1$, if n is even.

(iii) {1, 0, 1, 0, 1…….}

Ans: $s_n = n + 1$ if n is odd, $s_n = n - 1$, if n is even.

(iv) {1, 3, 6, 10, 15, ……}

Ans: $s_n = n(n + 1)/2$.

(v) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots.\}$

Ans: $s_n = n / (n + 1)$

Problem 2.2 ***Test either that the sequence has a limit or not?***

(i) $\left\{\dfrac{n^2}{n+5}\right\}_{n=1}$ **(ii)** $\left\{\dfrac{3n}{n+7n^{1/2}}\right\}_{n=1}$ **(iii)** $\left\{\dfrac{3n}{n+7n^2}\right\}_{n=1}$ **(iv)** $\left\{n-\dfrac{1}{n}\right\}_{n=1}$

☺***Solution:***

(i) $\lim\limits_{n\to\infty} \dfrac{n^2}{n+5} = \lim\limits_{n\to\infty} \dfrac{n^2(1)}{n\left(1+\dfrac{5}{n}\right)} = \dfrac{n}{\left(1+\dfrac{5}{n}\right)} = \infty$

Hence the sequence $\left\{\dfrac{n^2}{n+5}\right\}_{n=1}$ is divergent sequence.

(ii) $\lim\limits_{n\to\infty} \dfrac{3n}{n+7n^{\frac{1}{2}}} = \lim\limits_{n\to\infty} \dfrac{n(3)}{n\left(1+\dfrac{7}{n^{1/2}}\right)} = \lim\limits_{n\to\infty} \dfrac{3}{\left(1+\dfrac{7}{n^{1/2}}\right)} = \dfrac{3}{(1+0)} = 3$

Hence the sequence $\left\{\dfrac{3n}{n+7n^{1/2}}\right\}_{n=1}$ changes to 3.

(iii) $\lim\limits_{n\to\infty} \dfrac{3n}{n+7n^2} = \lim\limits_{n\to\infty} \dfrac{n(3)}{n^2\left(7+\dfrac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{3}{n\left(7+\dfrac{1}{n}\right)} = 0$

The sequence $\left\{\dfrac{3n}{n+7n^2}\right\}_{n=1}$ changes to 0.

(iv) $\lim\limits_{n\to\infty} \left(n-\dfrac{1}{n}\right) = \infty - 0 = \infty$. Hence the sequence $\left\{n-\dfrac{1}{n}\right\}_{n=1}$ is divergent seq.

Home work

1. Show that the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}$ changes to 1.

2. Show that the sequence $\left\{\dfrac{n^2}{2n^2+1}\right\}_{n=1}$ changes to 1/2.

3. (i) Prove that the sequence $\{10^7/n\}_{n=1}$ has a limit 0.

   (ii) Prove that the sequence $\{n/10^7\}_{n=1}$ does not have a limit.

Problem 2.3 ***Solve That the sequence*** $\left\{2-\dfrac{1}{2^{n-1}}\right\}_{n=1}$ ***changes to 2.***

☺***Solution:***

Let $s_n = 2-\dfrac{1}{2^{n-1}} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(2-\dfrac{1}{2^{n-1}}\right) = 2 - 0 = 2.$

Hence the sequence $\left\{2-\dfrac{1}{2^{n-1}}\right\}_{n=1}$ changes to 2.

Problem 2.4 ***Prove that*** $\lim\limits_{n\to\infty} s_n = 0$ ***if*** $\{s_n\} = \{1/n\}$ ***[OR] Test for changes of {1/n}***

☺***Solution:***

By definition, given $\in > 0$. We must find N $\in$ I, 0

Solve that $|s_n - L| < \in$, $\forall\ n \geq$ N.

In this case $\left|\dfrac{1}{n}-0\right| < \in, \forall\ n \geq$ N.

$\therefore$ If we choose N, solve that $\dfrac{1}{N} < \in$

i.e., $\frac{1}{n} \leq \frac{1}{N} < \in, \forall\ n \geq$ N.

$\therefore$ If $N > \frac{1}{\in}$ for N $\in$ I,

$\therefore$ (1) holds.

Hence the sequence $\{s_n\}$ changes to 0.

Problem 2.5 ***Using definition of limit SOLVE THAT the sequence*** $\{s_n\}$ ***where*** $s_n = \frac{3n}{n+5\sqrt{n}}$ ***has a lt 3.***

☺***Solution:***

Let $s_n = \frac{3n}{n+5\sqrt{n}}$

By definition, given $\in > 0$. We can find N $\in$ I, solve that $|s_n - L| < \in, \ \forall\ n \geq$ N

$$\Rightarrow \left| \frac{3n}{n+5\sqrt{n}} - 3 \right| < \in, \forall\ n \geq N. \qquad \text{… (2.16)}$$

To prove that (1) holds for $n \geq$ N.

For $\left| \frac{3n - 3n - 15\sqrt{n}}{n+5\sqrt{n}} - 3 \right| < \in, \forall\ n \geq$ N.

$$\Rightarrow \frac{15\sqrt{n}}{n+5\sqrt{n}} < \in, \forall\ n \geq N.$$

$$\Rightarrow \frac{15\sqrt{n}}{n+5\sqrt{n}} < \frac{15\sqrt{n}}{n} = \frac{15}{\sqrt{n}} < \in, \forall\ n \geq N.$$

$\Rightarrow \dfrac{225}{\in^2} < n, \forall\ n \geq \text{N}.$ … (2.17)

$\therefore$ We choose N, solve that $N \geq \dfrac{225}{\in^2}$

$\therefore$ (2.17) holds & consequently (2.16) holds.

Hence for any +ve integer $N \geq \dfrac{225}{\in^2}$.

$\therefore \lim\limits_{n \to \infty} s_n = 3.$

Problem 2.6 ***Prove that the sequence*** $\left\{\log \dfrac{1}{n}\right\}_{n=1}$ ***is divergent to*** $-\infty$.

☺***Solution:***

To prove that the sequence $\left\{\log \dfrac{1}{n}\right\}_{n=1}$ is divergent to $-\infty$.

i.e. To prove that for given $\in > 0$. We can find N $\in$ I,

Solve that $\log \dfrac{1}{n} < -\text{M},\ \forall\ n \geq \text{N}$ … (2.18)

$\Rightarrow -\log n < -\text{M},\ \forall\ n \geq \text{N}$

$\Rightarrow \log n > \text{M},\ \forall\ n \geq \text{N}$

$\Rightarrow n > e^{\text{M}},\ \forall\ n \geq \text{N}$ … (2.19)

We choose N > e^{M}

Then (2.19) holds and Consequently (2.18) holds.

$\therefore$ The sequence $\left\{\log \dfrac{1}{n}\right\}_{n=1}$ divergent to ∞.

[OR]

$$\lim_{n\to\infty} \log\frac{1}{n} = \lim_{n\to\infty}(-\log n) = \lim_{n\to\infty} -\log n = -\infty.$$

Problem 2.7 ***Give an example of a sequence $\{s_n\}$ which is not bounded for which $\lim_{n\to\infty}\frac{s_n}{n} = 0$.***

☺***Solution:***

Let $s_n = \sqrt{n}$,

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sqrt{n} = \infty$

Hence the sequence $\{s_n\} = \{\sqrt{n}\}$ is bounded sequence.

But $\lim_{n\to\infty}\frac{s_n}{n} = \lim_{n\to\infty}\frac{\sqrt{n}}{n} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$

3

OPERATION ON CONVERGENT SEQUENCE

3.1. OPERATION ON CONVERGENT SEQUENCE

Theorem: 1

If $\{s_n\}$ and $\{t_n\}$ are sequence of real numbers converges to L & M respectively. Then the sequence $\{s_n + t_n\}$ converges to L + M.	If $\lim_{in \to \infty} s_n = L$ and $\lim_{in \to \infty} t_n = M$ Then $\lim_{in \to \infty} (s_n + t_n) = L + M$.

Proof:

Since $\lim_{in \to \infty} s_{ni} = L$ and $\lim_{in \to \infty} s_{ni} = M$

By definition, for given $\in > 0$. $\exists$ a +ve integers N_1, N_2

$|s_n - L| < \frac{\in}{2}, \ \forall \ n \geq N_1,$

$|s_n - M| < \frac{\in}{2}, \ \forall \ n \geq N_1,$

Choose $N = Max (N_1, N_2)$

For $n \geq N$,

$|(s_n + t_n) - (L + M)| = |(s_n - L) + (s_n - M)|, \ \forall \ n \geq N,$

$\leq |s_n - L| + |s_n - M| < \frac{\in}{2} + \frac{\in}{2} = \in$

$\Rightarrow |(s_n + t_n) - (L + M)| < \in \ \forall \ n \geq$ N.

$\Rightarrow \lim\limits_{in \to \infty} (s_{ni} + t_n) =$ L + M.

Hence the sequence $\{s_n + t_n\}$ is converges to L + M.

Theorem: 2

If $\{s_n\}$ be a sequence of real numbers converges to L, Then the sequence $\{c.\ s_n\}$ converges to c.L.	If $c \in$ R and $\lim\limits_{in \to \infty} s_{ni} =$ L Then $\lim\limits_{in \to \infty} c.s_{ni} = c$.L.

Proof:

Case (1): If $c = 0$, then the theorem is obvious.

Case (2): If $c \neq 0$.

Since $\lim\limits_{in \to \infty} s_{ni} =$ L

By definition, given $\in > 0$. $\exists$ N $\in$ I,

Solve that $|s_n - L| < \dfrac{\in}{c}$, $\forall \ n \geq$N,

$\therefore |cs_n - cL| = |c||s_n - L| \leq c..\dfrac{\in}{c}$, $\forall \ n \geq$ N,

$\Rightarrow |cs_n - cL| < \in \forall \ n \geq$ N,

$\Rightarrow \lim\limits_{in \to \infty} c.s_{ni} = c$.L.

Hence the the sequence $\{c.s_n\}$ converges to c.L

Corollary

If $\{s_n\}$ and $\{t_n\}$ are the sequence of real numbers converges to L & M respectively. Then the sequence $\{s_n - t_n\}$ converges to L – M.	If $\lim\limits_{in \to \infty} s_n = L$ and $\lim\limits_{in \to \infty} t_n = M$ Then $\lim\limits_{in \to \infty} (s_n - t_n) = L - M$.

Proof:

Since $\lim\limits_{in \to \infty} s_n = L$ and $\lim\limits_{in \to \infty} t_n = M$

$\therefore \lim\limits_{in \to \infty} (-t_n) = -M$ (by above th}

$\therefore \lim\limits_{in \to \infty} (s_n - t_n) = \lim\limits_{in \to \infty} [s_n + (-t_n)] = L - M.$

Theorem: 3

If $\{s_n\}$ and $\{t_n\}$ are the sequence of real numbers converges to L and M respectively. Then prove that the sequence $\{s_n\ t_n\}$ converges to L. M.	If = L and $\lim\limits_{in \to \infty} t_n = M$, Then prove that $\lim\limits_{in \to \infty} s_n\ t_n = L\ .\ M$.

Proof:

Since $\lim\limits_{in \to \infty} s_n = L$ and $\lim\limits_{in \to \infty} t_n = M$

$\therefore \lim\limits_{in \to \infty} (s_n + t_n) = L + M. \Rightarrow \lim\limits_{in \to \infty} (s_n + t_n)^2 = (L + M)^2$

And $\lim\limits_{in \to \infty} (s_n - t_n) = L - M. \Rightarrow \lim\limits_{in \to \infty} (s_n - t_n)^2 = (L - M)^2$

$\therefore s_n\ t_n = \frac{1}{4}\left[(s_n + t_n)^2 - (s_n - t_n)^2\right]$

$\therefore \lim\limits_{in \to \infty} s_n \cdot t_n = \frac{1}{4}\left[\lim\limits_{n \to \infty}(s_n + t_n)^2 - \lim\limits_{n \to \infty}(s_n - t_n)^2\right]$

$= \frac{1}{4}\left[(L+M)^2 - (L-M)^2\right] = \frac{1}{4}[4LM]$

$\therefore \lim\limits_{in \to \infty} s_n \cdot t_n) = LM.$

Hence the sequence $\{s_n\ t_n\}$ converges to L.M

Theorem: 4

If $\{s_n\}$ and $\{t_n\}$ are the sequence of real numbers converges to L & M respectively. Then prove that the sequence $\left\{\frac{s_n}{t_n}\right\}_{n=1}$ converges to $\frac{L}{M}$	If $\lim\limits_{in \to \infty} s_n = L$ and $\lim\limits_{in \to \infty} t_n = M$, Then prove that $\lim\limits_{in \to \infty} \frac{s_n}{t_n} = \frac{L}{M}$.

Proof:

(d) Division Rule.

To prove that $\lim\limits_{in \to \infty} \frac{s_n}{t_n} = \frac{L.}{M}$ where $M \neq 0$

We first prove that $\lim\limits_{in \to \infty} \frac{1}{t_n} = \frac{1}{M}$ where $M \neq 0$

i.e. To prove that given $\in > 0$, $\exists\, N \in I$, 0

Solve that $\left|\frac{1}{t_n} - \frac{1}{M}\right| < \in, \forall\ n \geq N$,

Since $\lim\limits_{in \to \infty} t_n = M$

For, $\in > 0$, $\exists \, \mathrm{N} \in \mathrm{I}$, 0 s.t $|t_n - M| < \in, \forall \; n \geq \mathrm{N}$, … (3.1)

$\therefore |M| = |M - t_n + t_n| \leq |M - t_n| + |t_n|$

$\Rightarrow |M| < \in + |t_n|$ by (3.1)

$\Rightarrow |t_n| > |M| -- \in$, $\forall \; n \geq$ N,

$$\Rightarrow \frac{1}{|t_n|} < \frac{1}{|M| - \in} \;, \; \forall \; n \geq \mathrm{N} \qquad \text{… (3.2)}$$

Given $\in' > 0$, $\exists \, \mathrm{N} \in \mathrm{I}$

$$\text{,s.t} \left| \frac{1}{t_n} - \frac{1}{M} \right| < \left| \frac{t_n - M}{t_n . M} \right| \leq \frac{|t_n - M|}{|t_n| |M||} < \frac{\in}{(M - \in)} = \in' \text{ (say)}$$

$$\Rightarrow \left| \frac{1}{t_n} - \frac{1}{M} \right| < \in, \forall \; n \geq \mathrm{N},$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{t_n} = \frac{1}{M} \text{ where } \mathrm{M} \neq 0$$

$$\therefore \lim_{n \to \infty} \frac{s_n .}{t_n} = \lim_{n \to \infty} s_n \lim_{n \to \infty} \frac{1}{t_n} = \mathrm{L}. \; \frac{1}{M}$$

$$\Rightarrow \lim_{n \to \infty} \frac{s_n .}{t_n} == \frac{L}{M} \text{. Hence proved.}$$

Theorem: 5

If $\{s_n\}$ be a sequence of real number .converges to L, Then prove that the sequence $\{|s_n|\}_{n=1}$ converges to $|L|$	If $\lim_{n \to \infty} s_n$ = L then prove that $\lim_{n \to \infty} |s_n| = |L|$

Proof:

Since $\{s_n\}$ converges to L i.e. $\lim\limits_{in \to \infty} s_n = L$

By definition, given $\in > 0$. $\exists$ N $\in$ I,

s.t $|s_n - L| < \in$, $\forall$ n $\geq$ N

W.K.T $||a| - |b|| \leq |a - b|$,

$\Rightarrow ||s_n| - |L|| \leq |s_n - L| < \in \ \forall \ n \geq$ N ,

$\Rightarrow ||s_n| - |L|| < \in$, $\forall$ $n \geq$ N,

$\therefore \lim\limits_{in \to \infty} c.s_{ni} =$ c.L.

Hence the sequence $\{|s_n|\}_{n=1}$ converges to $|L|$

Result: But converse is not true.

i.e. if $\{|s_n|\}_{n=1}$ converges to $|L|$ then need not implies that $\{s_n\}$ converges to L.

Example 1: Consider the sequence, $\{s_n\} = \{1, -1, 1, -1, \ldots..\}$

Here $\{|s_n|\}_{n=1} = \{1, 1, 1, \ldots..\}$

But $\{s_n\}$ converges to 1. Hence the proof.

Example 2: Prove that if $\{|s_n|\}_{n=1}$ converges to 0, then the sequence. $\{s_n\}$ converges to 0.

Proof:

Given $\{|s_n|\}_{n=1}$ converges to 0 i.e. $\lim\limits_{in \to \infty} |s_n| = 0$.

By definition, given $\in > 0$. $\exists$ N $\in$ I, s.t $\left||s_n| - 0\right| < \in, \ \forall \ n \geq \text{N}$

$\Rightarrow |s_n| < \in \ \forall \ n \geq \text{N}, \Rightarrow |s_n - 0| < \in, \ \forall \ n \geq \text{N},$

$\therefore \lim\limits_{in \to \infty} = 0$. Hence then the sequence $\{s_n\}$ converges to 0.

Theorem : 6

If $\{s_n\}$ and $\{t_n\}$ are non-decreasing sequence of real numbers converges to L and M respectively and if $s_n \leq t_n$ $\forall \ n$. Then prove that L $\leq$ M.	If $\lim\limits_{in \to \infty} s_n$ = L and $\lim\limits_{in \to \infty} t_n$ = M, and if $s_n \leq t_n \ \forall \ n$. Then prove that L $\leq$ M.

Proof:

Since $\lim\limits_{in \to \infty} s_n$ = L and $\lim\limits_{in \to \infty} t_n$ = M,

By given if $s_n \leq t_n \ \forall \ n$

$\therefore \ t_n - s_n \geq 0. \Rightarrow \{\therefore t_n - s_n\}$ is a non-negative sequence of real numbers.

$\Rightarrow \lim\limits_{in \to \infty} (s_n - t_n) \geq 0.$

$\Rightarrow$ (M – L) $\geq$0. [By Th]

$\Rightarrow$M $\geq$ L

$\Rightarrow$ Or L $\leq$ M. Hence the theorem.

Theorem: 7

If the sequence $\{s_n\}$ converges to L, then prove that the sequence $\{s_n^2\}$ converges to L^2.	If $\lim\limits_{in \to \infty} s_n$ = L then prove that $\lim\limits_{in \to \infty} s_n^2 = \text{L}^2$.

Proof:

Since $\{s_n\}$ converges to L i.e. $\lim\limits_{in \to \infty} s_n = L$

W.K.T every cgt sequence is bounded.

For M > 0, $\exists$ N $\in$ I, $\therefore |s_n| < M, \forall\ n \geq N$

Since $\{s_n\}$ converges to L.

By definition, given $\in > 0$. $\exists$ N $\in$ I, s.t $|s_n - L| < \in, \forall\ n \geq N$

$$\text{For } n \geq N, \Rightarrow |s_n{}^2 - L^2| = |(s_n - L)(s_n + L)|, \forall\ n \geq N$$

$$\Rightarrow |s_n{}^2 - L^2| < \in (M + |L|) = \in' =, \forall\ n \geq N$$

$$\Rightarrow |s_n{}^2 - L^2| < \in', \forall\ n \geq N$$

$$\therefore \lim_{in \to \infty} s_n{}^2 = L^2.$$

Hence the sequence. $\{s_n{}^2\}$ converges to L^2.

Theorem: 8

(1) If $0 < x < 1$, then prove that the sequence $\{x^n\}$ converges to 0.

(2) If $x > 1$, then prove that the sequence $\{x^n\}$ diverges to ∞.

Proof:

Part (1): If $0 < x < 1$, then $x^{n+1} = x^n . x < x^n$. $[\because \frac{1}{2^2} < \frac{1}{2}]$

i.e. $x^{n+1} < x^n, \forall\ n \in Z$.

Hence, $\{x^n\}$ is an non-increasing sequence.

Since $x^n > 0, \forall\ n \in Z$.

$\therefore$ $\{x^n\}$ is a bounded below by zero.

By known result, [W.K.T, Every sequence which is bonded below is convergent.]

The sequence $\{x^n\}$ converges to 0.

$\therefore \lim\limits_{in \to \infty} x^n = \text{L (say)}$ … (3.3)

$\lim\limits_{in \to \infty} x^{n+1} = \lim\limits_{in \to \infty} x^n.x = x \lim\limits_{in \to \infty} x^n = \text{L}x$ by (3.3)

$\therefore \lim\limits_{in \to \infty} x^{n+1} = \text{L}x.$

Hence the sequence $\{x^{n+1}\}$ converges to Lx.

But, $\{x^{n+1}\}$ is a subsequence of $\{x^n\}$

$\therefore$ Lx = L [Every sequence and its subsequence converges to same limit.]

$\Rightarrow (x - 1) = 0$

$\Rightarrow \text{L} = 0$ ($\because x \neq 1$)

Hence the sequence $\{x^n\}$ converges to 0 if $0 < x < 1$.

Part (2): If $1 < x < \infty$, then $x^n < x^{n+1}$, $\forall\ n \in \text{Z}$.

[$\because$ for $x = 3$, $3^2 < 3^3$]

$\therefore \{x^n\}$ is a non-decreasing sequence.

Also, $x^n > 1$, $\forall\ n \in \text{Z}$.

Now Let us assume that $\{x^n\}$ is bounded above sequence.

By known result,

$\{x^n\}$ converges to L. i.e. $\therefore \lim\limits_{in \to \infty} x^n = \text{L (say)}$ … (3.4)

$\lim\limits_{in \to \infty} x^{n+1} = \lim\limits_{in \to \infty} x^n.x = x \lim\limits_{in \to \infty} .x^n = \text{L}x$ by (3.4)

$\therefore \lim\limits_{in \to \infty} x^{n+1} = Lx.$

Hence the sequence $\{x^{n+1}\}$ converges to Lx.

But, $\{x^{n+1}\}$ is a subsequence of $\{x^n\}$

$\therefore$ Lx = L [Every sequence and its subsequence converges to same limit.]

$\Rightarrow L(x - 1) = 0$

$\Rightarrow L = 0 \ (\because x \neq 1)$

Which is a contradiction to $\{x^n\}$ is not bounded above.

Hence the sequence $\{x^n\}$ diverges to ∞, for $1 < x < \infty$.

3.2. LIMIT SUPERIMUM AND LIMIT INFIMUM.

1. Define limit superimum of sequence $\{s_n\}$ of real numbers.

Let $\{s_n\}$ is a sequence of real numbers that is bounded above.

Let M_n = l.u.b $\{s_n, s_{n+1}, \ldots\}$

(a) If $\{M_n\}$ converges and we define $\lim\limits_{n \to \infty} \sup s_n = \lim\limits_{n \to \infty} M_n$

(b) If $\{M_n\}$ diverges to minus infinity and we define $\lim\limits_{n \to \infty} \sup s_n = -\infty$.

2. Define limit infimum of of sequence $\{s_n\}$ of real numbers.

Let $\{s_n\}$ is a sequence of real numbers that is bounded below.

Let m_n = g.l.b $\{s_n, s_{n+1}, \ldots\}$

(a) If $\{m_n\}$ converges and we define $\lim\limits_{n \to \infty} \inf s_n = \lim\limits_{n \to \infty} m_n$

(b) If $\{m_n\}$ diverges to infinity and we define $\lim\limits_{n \to \infty} \inf s_n = \infty$.

Theorem: 1

If $\{s_n\}$ is a convergent sequence of real number. Then prove that

$\lim\limits_{in \to \infty} \sup s_n = \lim\limits_{in \to \infty} \inf s_n = \lim\limits_{in \to \infty} s_n$

Proof:

Let $\{s_n\}$ be a convergent sequence of real number

Let $\lim\limits_{n \to \infty} s_n = \text{L}$

By definition, given $\in > 0$. $\exists$ a N $\in$ I, s.t $|s_n - L| < \in$, $\forall\ n \geq$ N

$\Rightarrow -\in < s_n - L < \in$, $\forall\ n \geq$ N

Add L, $\Rightarrow L - \in < s_n < L + \in$, $\forall\ n \geq$ N … (3.5)

$\Rightarrow$L + $\in$ is an upper bound for $\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$

But L $-\in$ is not an upper bound .

$\Rightarrow$L $-$ $\in$ < M_n = l.u.b $\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$< L + $\in$, $\forall\ n \geq$N

Apply limit, $\Rightarrow$ L- $\in < \lim\limits_{n \to \infty} M_n <$ L + $\in$, $\forall\ n \geq$N.

Add (–L),$\Rightarrow$ $-$ $\in$ < $\sup s_n$ $-$ L < $\in$, $\forall\ n \geq$ N

$\Rightarrow \lim\limits_{n \to \infty} |\sup s_n - L| < \in$, $\forall\ n \geq$ N

$\Rightarrow \lim\limits_{n \to \infty} \sup s_n$ = L, for arbitrary $\in$. … (3.6)

Similarly, by (3.5) ,

$\Rightarrow$L + $\in$ is an lower bound for $\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$

But L $-$ $\in$ is not an lower bound.

$\Rightarrow$L $-$ $\in$ < m_n = g.l.b$\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$< L + $\in$, $\forall\ n \geq$ N

Apply limit, $\Rightarrow$L $-$ $\in$ < $\lim\limits_{n \to \infty} m_n <$ L + $\in$,$\forall\ n \geq$ N

Add (– L), $\Rightarrow - \in \; < \inf s_n - L < \in, \forall \; n \geq N$

$\Rightarrow |\inf s_n - L| < \in, \; \forall \; n \geq N$

$\Rightarrow \lim\limits_{n \to \infty} \inf s_n = L$, for arbitrary $\in$. … (3.7)

From (3.5) & (3.6) $\Rightarrow \lim\limits_{in \to \infty} \sup s_n = \lim\limits_{in \to \infty} \inf s_n = \lim\limits_{in \to \infty} s_n$

Hence the proof.

Converse of the above theorem.

Theorem: 2

If $\{s_n\}$ is a sequence of real number and If

$\lim\limits_{in \to \infty} \sup s_n = \lim\limits_{in \to \infty} \inf s_n = L$

Then $\{s_n\}$ is converges to L.

Proof:

Since $\lim\limits_{in \to \infty} \sup s_n = L$

By definition, given $\in > 0$. $\exists \; N_1 \in I$, s.t $|\sup s_n - L| < \in, \; \forall \; n \geq$ N

$\Rightarrow - \in < \sup s_n - L < \in, \; \forall \; n \geq N$

Add L, $\Rightarrow L - \in < l.u.b\{s_n, s_{n+1}, s_{n+2}, ...\} < L + \in, \; \forall \; n \geq N$

$\Rightarrow s_n < L + \in$ … (3.8)

Also, $\lim\limits_{in \to \infty} \inf s_n = L$

By definition, given $\in > 0$. $\exists \; N_2 \in I$, s.t $|\inf s_n - L| < \in, \; \forall \; n \geq$ N_1

$\Rightarrow - \in < \sup s_n - L < \in, \; \forall \; n \geq N$

Add L, $\Rightarrow L - \in < g.l.b\{s_n, s_{n+1}, s_{n+2}, ...\} < L + \in, \ \forall \ n \geq N$

$\Rightarrow L - \in < s_n$ … (3.9)

Choose N = Max (N_1, N_2)

For $n \geq N$,From (3.8) & (3.9) $\Rightarrow L - \in < s_n < L + \in$

Add (–L), $\Rightarrow - \in < s_n - L < \in, \forall \ n \geq N$

$\Rightarrow |s_n - L| < \in, \ \forall \ n \geq N$

$\Rightarrow \lim_{n \to \infty} s_n = L$

Hence the sequence $\{s_n\}$ converges to L.

Theorem: 3

If $\{s_n\}$ is a bounded sequence of real numbers, then prove that $\lim_{in \to \infty} \inf \ s_n \leq \lim_{in \to \infty} \sup s_n$.

Proof:

Let $\{s_n\}$ be a bounded sequence of real number

m_n = g.l.b $\{s_n, s_{n+1}, s_{n+2}, \ldots\} \leq$ l.u.b $\{s_n, s_{n+1}, s_{n+2}, \ldots\} = M_n$.

$\Rightarrow m_n \leq M_n , \ \forall \ n$.

Apply limit, $\Rightarrow \lim_{n \to \infty} m_n \leq \lim_{n \to \infty} M_n \Rightarrow \lim_{n \to \infty} \inf s_n \leq \lim_{n \to \infty} \sup s_n$

Hence the proof.

Theorem: 4

If $\{s_n\}$ & $\{t_n\}$ be the bounded sequence of real numbers. And if $s_n \leq t_n$, $\forall \ n$.

Then prove that

$\lim_{n \to \infty} \sup s_n \leq \lim_{n \to \infty} \sup t_n$ and $\lim_{n \to \infty} \inf s_n \geq \lim_{n \to \infty} \inf t_n$

Proof:

Since $\{s_n\}$ and $\{t_n\}$ are bounded sequence of real numbers.

Also, $s_n \leq t_n, \ \forall \ n$.

$\Rightarrow$l.u.b$\{s_n, s_{n+1}, s_{n+2}, \ldots..\} \leq$ l.u.b $\{t_n, t_{n+1}, t_{n+2}, \ldots..\}$

$\Rightarrow \mathrm{M}_n \leq \mathrm{T}_n, \ \forall \ n$.

Apply limit, $\Rightarrow \lim\limits_{n\to\infty} M_n \leq \lim\limits_{n\to\infty} T_n$

$\Rightarrow \lim\limits_{n\to\infty} \sup s_n \leq \lim\limits_{n\to\infty} \sup t_n$

Also, $s_n \leq t_n, \ \forall \ n$.

$\Rightarrow$g.l.b$\{s_n, s_{n+1}, s_{n+2}, \ldots..\} \geq$ g.l.b $\{t_n, t_{n+1}, t_{n+2}, \ldots..\}$

$\Rightarrow m_n \leq p_n, \forall \ n$.

Apply limit, $\Rightarrow \lim\limits_{n\to\infty} m_n \geq \lim\limits_{n\to\infty} p_n$

$\Rightarrow \lim\limits_{n\to\infty} \inf s_n \geq \lim\limits_{n\to\infty} \inf t_n$. Hence the theorem.

Theorem: 5

If $\{s_n\}$ and $\{t_n\}$ be the bounded sequence of real numbers.

Then

(1) $\lim\limits_{n\to\infty} \sup(s_n + t_n) \leq \lim\limits_{n\to\infty} \sup s_n + \lim\limits_{n\to\infty} \sup t_n$

(2) $\lim\limits_{n\to\infty} \inf(s_n + t_n) \geq \lim\limits_{n\to\infty} \inf s_n + \lim\limits_{n\to\infty} \inf t_n$

Proof:

Part (1): Let $\{s_n\}$ and $\{t_n\}$ be the bounded sequence of real numbers.

$\therefore$ l.u.b$\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$ exists and

l.u.b$\{t_n, t_{n+1}, t_{n+2}, \ldots..\}$ exists.

Let M_n = l.u.b$\{s_n, s_{n+1}, s_{n+2},\ldots..\}$

T_n = l.u.b$\{t_n, t_{n+1}, t_{n+2},\ldots..\}$.

$\Rightarrow s_k \leq M_n \ \forall\, k \geq n.$

And $t_k \leq T_n \ \forall\, k \geq n.$

$\therefore \underline{s_k} + t_k \leq M_n + T_n, \ \forall\, k. \geq n.$

$\Rightarrow M_n + T_n$ is an upper bound for $\{(s_n + t_n), (s_{n+1} + t_{n+1}),,\ldots\ldots..\}$

$\Rightarrow$l.u.b $\{(s_n + t_n), (s_{n+1} + t_{n+1}),\ldots\ldots..\}. \leq M_n + T_n$

Apply limit,

$$\Rightarrow \lim_{n\to\infty} l.u.b\{(s_n + t_n),(s_n + t_n).......\} \leq \lim_{n\to\infty} (M_n + T_n)$$

$$\Rightarrow \lim_{n\to\infty} \sup(s_n + t_n) \leq \lim_{n\to\infty} M_n + \lim_{n\to\infty} T_n$$

$$\Rightarrow \lim_{n\to\infty} \sup(s_n + t_n) \leq \lim_{n\to\infty} \sup s_n + \lim_{n\to\infty} \sup t_n$$

Hence part (1) is proved.

Part (2):

Let $\{s_n\}$ & $\{t_n\}$ be the bounded sequence of real no/-s.

$\therefore$ g.l.b $\{s_n, s_{n+1}, s_{n+2},\ldots..\}$ exists and

g.l.b $\{t_n, t_{n+1}, t_{n+2},\ldots..\}$ exists.

Let m_n = g.l.b $\{s_n, s_{n+1}, s_{n+2},\ldots..\}$

p_n = g.l.b$\{t_n, t_{n+1}, t_{n+2},\ldots..\}$.

$\Rightarrow s_k \geq m_n \ \forall$ k.$\geq n.$

And $t_k \geq p_n \ \forall\, k. \geq n.$

$\therefore s_k + t_k \geq m_n + p_n, \ \forall\, k. \geq n.$

$\Rightarrow m_n + p_n$ is an lower bound for $\{(s_n + t_n),(s_{n+1} + t_{n+1}),\ldots\ldots..\}$.

$\Rightarrow$g.l.b $\{(s_n + t_n), (s_{n+1} + t_{n+1})\ldots\ldots\}. \geq m_n + p_n$

Apply limit,

$$\Rightarrow \lim_{n \to \infty} g.l.b\{(s_n + t_n), (s_n + t_n)\ldots\ldots\} \geq \lim_{n \to \infty} (m_n + p_n)$$

$$\Rightarrow \lim_{n \to \infty} \inf(s_n + t_n) \geq \lim_{n \to \infty} m_n + \lim_{n \to \infty} p_n$$

Hence part (2) is proved.

Theorem 6: (without proof)

Let $\{s_n\}$ be a bounded sequence of real numbers.

If Lt sup s_n = M.

Then for any $\in > 0$,

$s_n <$ M + $\in$, $\forall$, but finite number of values of n.

$s_n >$ M – $\in$, for infinitely many values of n.

Similarly. for if Lt inf s_n = m.

Then for any $\in > 0$,

a) $s_n >$ m + $\in$, $\forall$, but finite numbers of values of n.

b) $s_n <$ M – $\in$, for infinitely many values of n.

Theorem: 7

Prove that any bounded sequence of real number has a convergent subsequence.

Proof:

Let $\{s_n\}$ be a bounded sequence of real numbers.

To prove that, we have to construct a convergent subsequence $\{s_n\}$.

For, Let M = Lt sup s_n,

For every $\in > 0$,

Then there are infinitely many values of n, s.t $s_n > M - 1$.

Let n_1 be one such value.

i.e. $n_1 \in I$, and $s_n > M - 1$.

Similarly, there are infinitely many values of n

,s.t $s_n > M - \frac{1}{2}$.

Choose $n_2 > n$, $\underline{s_n} > M - \frac{1}{2}$.

Continuing in this way

$$s_n \geq M - \frac{1}{k}, \ \forall\, n_k > n, \qquad \ldots (3.10)$$

For $\in > 0$, $\exists$ $N \in I$,

s.t $s_n < M + \in$, $\forall$, but finite number of values of n.

i.e. $s_n < M + \in, \forall$, but finite number of values of n. … (3.11)

For $k > N$, $M - \in < M - \frac{1}{k}$

$\Rightarrow M - \in < s_n < M + \in$, $\forall$, but finite number of values of n.

$\Rightarrow - \in < s_n - M < + \in$, $\forall\, n_k > n$.

$\Rightarrow \left|s_n - M\right| < \in$, $\forall$ $n_k \geq n$

$\Rightarrow \lim\limits_{n \to \infty} s_n = M$. Hence the sequence $\{s_n\}$ converges to M.

Theorem: 8

Prove that every convergent sequence of real number is a Cauchy sequence.

Proof:

Let $\{s_n\}$ be a convergent sequence of real numbers.

Let $\lim\limits_{n \to \infty} s_n = L$,

By definition, given $\in > 0$. $\exists$ N $\in$ I, s.t $|s_n - L| < \in$, $\forall$ $n \geq$ N.

Choose m, n > N.

$|s_n - L| < \in$, $\forall$ $n \geq$ N.

$|s_m - L| < \in$, $\forall$ $m \geq$ N.

For m, n > N.

$|s_m - s_m| = |(s_m - L) - (s_n - L)|$

$\leq |s_m - L| + |s_n - L| < \in + \in = 2\in = \in$'

$\Rightarrow |s_m - s_m| < \in'$, $\forall$ $m, n \geq$ N.

$\Rightarrow \{s_n\}$ is a Cauchy sequence of real numbers.

Note: Every Cauchy sequence need not be convergent.

****[Every Cauchy sequence is bdd]****

Theorem: 9.

If $\{s_n\}$ be a Cauchy sequence of real numbers, Then prove that $\{s_n\}$ is a bounded sequence.

Proof:

Let $\{s_n\}$ be a Cauchy sequence of real numbers.

By definition, given $\in > 0$. $\exists$ N $\in$ I, s.t $|s_m - s_n| < \in$, $\forall$ $m, n \geq$ N.

$\Rightarrow |s_m - s_n| < 1$, $\forall$ $m, n \geq$ N

If $m > N$. $|s_m| = |s_m - s_n + s_n| \leq |s_m - s_n| + |s_n|$

$|s_m| < 1 + |s_n|$, $\forall$ m>N, n=N

Let M = max $\{|s_1|, |s_2|, |s_3|, ...|s_{n-1}|\}$

$\Rightarrow |s_m| < M + 1 + 1 + |s_N|$ = k (say)

$=> |s_m| < k$, $\forall$ k $\in$ I.

Hence $\{s_n\}$ is a bounded sequence.

Theorem: 10

Prove that if $\{s_n\}$ is a Cauchy sequence of real numbers, Then prove that $\{s_n\}$ is a convergent sequence.

Proof:

Let $\{s_n\}$ be a Cauchy sequence of real numbers,

Then $\{s_n\}$ is a bounded sequence [By previous Theorem 9]

$\lim\limits_{in \to \infty} \sup s_n$ and $\lim\limits_{in \to \infty} \inf s_n$ exists.

To prove that $\{s_n\}$ is a convergent sequence.

i.e. to prove that, $\lim\limits_{in \to \infty} \sup s_n = \lim\limits_{in \to \infty} \inf s_n$

For Clearly, $\lim\limits_{in \to \infty} \inf s_n \leq \lim\limits_{in \to \infty} \sup s_n$ … (3.12)

To Claim: $\lim\limits_{in \to \infty} \sup s_n \leq \lim\limits_{in \to \infty} \inf s_n$.

For since $\{s_n\}$ is a Cauchy sequence

By definition, given $\in$ > 0. $\exists$ N $\in$I, s.t $|s_m - s_n| < \frac{\in}{2}$, $\forall$ $m, n \geq$ N.

$\Rightarrow |s_N - s_n| < \frac{\in}{2}, \forall\ n \geq$ N.

$\Rightarrow s_N - \frac{\in}{2} < s_n < s_N + \frac{\in}{2}, \forall\ n \geq$ N.

$\Rightarrow s_N + \frac{\in}{2}$ is an upper bound for the set $\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$

And $s_N - \frac{\in}{2}$ is an lower bound for the set $\{s_n, s_{n+1}, s_{n+2}, \ldots..\}$

$\therefore s_N - \frac{\in}{2} <$ g.l.b $\{s_n, s_{n+1}, s_{n+2}, \ldots..\} \leq$ l.u.b $\{s_n, s_{n+1}, s_{n+2}, \ldots..\} < s_N + \frac{\in}{2}, \forall\ n \geq$ N.

$\Rightarrow$l.u.b $\{s_n, s_{n+1}, s_{n+2}, \ldots..\} -$ g.l.b $\{s_n, s_{n+1}, s_{n+2}, \ldots..\} \leq (s_N + \frac{\in}{2}) - (s_N - \frac{\in}{2})$

$\Rightarrow M_n - m_n \leq \in, \forall\ n \geq$ N.

$\Rightarrow \lim\limits_{n \to \infty} (M_n - m_n) \leq \in$

$\Rightarrow \lim\limits_{n \to \infty} M_n - m_n \leq \in$

$\Rightarrow \lim\limits_{n \to \infty} M_n \leq \lim\limits_{n \to \infty} m_n + \in$

$\Rightarrow \lim\limits_{in \to \infty} \sup s_n \leq \lim\limits_{in \to \infty} \inf s_n$. For arbitrary $\in$. … (3.13)

From (3.12) & (3.13) $\Rightarrow \lim\limits_{in \to \infty} \sup s_n = \lim\limits_{in \to \infty} \inf s_n = \lim\limits_{n \to \infty} s_n =$ L,

$\therefore \{s_n\}$ is a convergent sequence. Hence the proof .

Theorem: 11

State and prove Nested interval theorem.

Statement

For each $n \in$ I,

Let $I_n = [a_n, b_n]$ be any non-empty closed bounded interval of real numbers.

$I_n \supset I_{n+1} \supset I_{n+2} \supset \ldots..$

$\lim\limits_{n \to \infty} (a_n - b_n) = \lim\limits_{n \to \infty} \textit{length of } I_n = 0.$

Then $\bigcap\limits_{n=1}^{\infty} I_n$ contains (exactly) precisely one point.

Proof:

By hypothesis (1), $.I_n \supset I_{n+1} \supset I_{n+2} \supset \ldots..$

$\Rightarrow a_n \le a_{n+1} \le b_{n+1} \le b_n \;\forall\; n.$

$\Rightarrow$ The sequence $\{a_n\}$ is an non-decreasing sequence and the sequence $\{b_n\}$ is an non-increasing sequence.

But all the points of the sequence $\{a_n\}$ and $\{b_n\}$ lies in the interval I_1.

The sequence $\{a_n\}$ and $\{b_n\}$ are bounded above and bounded below respectively.

The sequence $\{a_n\}$ and $\{b_n\}$ are convergent sequence [By theorem]

$\therefore \lim\limits_{n \to \infty} = \mathrm{x}, \lim\limits_{n \to \infty} b_n = \mathrm{y},$

then $a_n \le x, y \le b_n \;\forall\; n$

By hypothesis (2),

$y - x = \lim\limits_{n \to \infty} b_n - \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} (b_n - a_n) = 0$ (Given)

$\therefore y - x = 0 \Rightarrow y = x.$

$a_n \leq x \leq bn, \ \forall \ n \in$ I,

$x_n \in I_n, \ \forall \ n \in$ I,

$\Rightarrow x_n \in \bigcap_{n=1}^{\infty} I_n$

To prove the uniqueness part.

Let $z \neq x \in \bigcap_{n=1}^{\infty} I_n$

Then $|z - x| \neq 0$

$\Rightarrow |z - x| \leq |b_n - a_n|$

$\Rightarrow \lim_{n \to \infty} |z - x| \leq \lim_{n \to \infty} |b_n - a_n|$

$\Rightarrow \lim_{n \to \infty} |z - x| \leq 0$

$\Rightarrow |z - x| = 0$ -which is a contradiction.

$\therefore z = x.$

Hence $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

PROBLEMS BASED ON LIMITS OF SEQUENCE

P1. Evaluate:-

(i) $\lim_{n \to \infty} \dfrac{2n}{n+3}$ (ii) $\lim_{n \to \infty} \dfrac{2n^3 + 5n}{4n^3 + n^2}$ (iii) $\lim_{n \to \infty} \dfrac{n}{\sqrt{n^2 + 1}}$

(iv) $\lim_{n \to \infty} \dfrac{3n^2 - 6n}{5n^2 + 4}$ N13 (v) $\lim_{n \to \infty} \dfrac{2n^2 - 5n + 4}{3n^2 + 6n + 11}$

(vi) $\lim\limits_{n\to\infty} \dfrac{n^2}{(n-7)^2-6}$ (vii) $\lim\limits_{n\to\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$ A16.

(viii) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n}-n\right)$ (ix) $\lim\limits_{n\to\infty} \left(5+\dfrac{4}{n^2}\right)$ N15

(x) Prove that $\lim\limits_{n\to\infty} \dfrac{2n^3+5n}{8n^3-6}=\dfrac{1}{4}$ (xi) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

Solution

(i) $\lim\limits_{in\to\infty} \dfrac{2n}{n+3} = \lim\limits_{in\to\infty} \dfrac{n(2)}{n(1+3/n)} = \lim\limits_{in\to\infty} \dfrac{(2)}{(1+3/n)}$

$$= \frac{\lim\limits_{n\to\infty} 2}{\lim\limits_{n\to\infty} (1+3/n)} = \frac{2}{(1+0)} = 2.$$

(ii) $\lim\limits_{n\to\infty} \dfrac{2n^3+5n}{4n^3+n^2} = \lim\limits_{n\to\infty} \dfrac{n^3\left(2+\dfrac{5}{n^2}\right)}{n^3\left(4+\dfrac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{\left(2+\dfrac{5}{n^2}\right)}{\left(4+\dfrac{1}{n}\right)}$

$$= \frac{\lim\limits_{n\to\infty}\left(2+\dfrac{5}{n^2}\right)}{\lim\limits_{n\to\infty}\left(4+\dfrac{1}{n}\right)} = \frac{2+0}{4+0} = \frac{1}{2}.$$

(iii) $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \dfrac{n}{n\sqrt{1+\dfrac{1}{n^2}}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}} = \dfrac{1}{\lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n^2}}}$

$$= \frac{1}{\sqrt{1+0}} = 1.$$

[(iv),(v),(vi) Same as (ii)]

(vi) [For Root sums Multiply Nr & Dr by its conjugates]

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty} \sqrt{n}\,\frac{\left(\sqrt{n+1}-\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}\left(\sqrt{n+1}+\sqrt{n}\right)$$

$$= \lim_{n\to\infty} \sqrt{n}\,\frac{(n+1-n)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \lim_{n\to\infty} \frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= \frac{1}{\lim\limits_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{(\sqrt{1}+1)} = \frac{1}{2}.$$

(vii) $$\lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n}-n\right)}{\left(\sqrt{n^2+n}+n\right)}\left(\sqrt{n^2+n}+n\right)$$

$$= \lim_{n\to\infty} \frac{\left(n^2+n-n\right)}{\left(\sqrt{n^2+n}+n\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n\left(\sqrt{1+\frac{1}{n}}+1\right)} = \lim_{n\to\infty} \frac{n}{\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= \frac{\lim\limits_{n\to\infty} n}{\lim\limits_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \infty.$$

Therefore the sequence divergent to ∞.

P2. If P is a polynomial of degree two, then P.T $\lim\limits_{n\to\infty} \dfrac{P(n+1)}{P(n)}$ =1.

N16.

Solution:

Let P(x) = $ax^2 + bx$ + c, (a, b, c are real numbers) be a polynomial of degree two.

Then

$$\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \lim_{n\to\infty} \frac{a(n+1)^2 + b(n+1) + c}{an^2 + bn + c}$$

$$= \lim_{n\to\infty} \frac{an^2 + (2a+b)n + (a+b+c)}{an^2 + bn + c}$$

$$= \lim_{n\to\infty} \frac{n^2\left[a + (2a+b)\frac{1}{n} + (a+b+c)\frac{1}{n}\right]}{n^2\left[a + b\frac{1}{n} + c\frac{1}{n}\right]}$$

$$= \lim_{n\to\infty} \frac{\left[a + (2a+b)\frac{1}{n} + (a+b+c)\frac{1}{n}\right]}{\left[a + b\frac{1}{n} + c\frac{1}{n}\right]}$$

$$= \frac{\lim\limits_{n\to\infty}\left[a + (2a+b)\frac{1}{n} + (a+b+c)\frac{1}{n}\right]}{\lim\limits_{n\to\infty}\left[a + b\frac{1}{n} + c\frac{1}{n}\right]} = \frac{a+0+0}{a+0+0} = 1.$$

Hence $\lim\limits_{n\to\infty} \dfrac{P(n+1)}{P(n)}$ =1.

P3. Is a P(*x*) = *ax*^{3} + *bx*^{2} + *cx* + *d*, *a*, *b*, *c*, *d* are in R.P.T $\lim\limits_{n\to\infty}$ $\dfrac{P(n+1)}{P(n)}$ =1.

Solution:

Given P(x) = $ax^3 + bx^2 + cx + d$,

$$\lim_{n\to\infty}\frac{P(n+1)}{P(n)}=\lim_{n\to\infty}\frac{a(n+1)^3+b(n+1)^2+c(n+1)+d}{an^3+bn^2+cn+d}$$

$$=\lim_{n\to\infty}\frac{(n+1)^3\left[a+\dfrac{b}{(n+1)}+\dfrac{c}{(n+1)^2}+\dfrac{d}{(n+1)^3}\right]}{n^3\left[a+\dfrac{b}{n}+\dfrac{c}{n^2}+\dfrac{d}{n^3}\right]}$$

$$=\lim_{n\to\infty}\frac{n^3\left[1+\dfrac{1}{n}\right]^3\left[a+\dfrac{b}{(n+1)}+\dfrac{c}{(n+1)^2}+\dfrac{d}{(n+1)^3}\right]}{n^3\left[a+\dfrac{b}{n}+\dfrac{c}{n^2}+\dfrac{d}{n^3}\right]}$$

$$=\lim_{n\to\infty}\frac{\left[1+\dfrac{1}{n}\right]^3\left[a+\dfrac{b}{(n+1)}+\dfrac{c}{(n+1)^2}+\dfrac{d}{(n+1)^3}\right]}{\left[a+\dfrac{b}{n}+\dfrac{c}{n^2}+\dfrac{d}{n^3}\right]}$$

$$=\frac{[1+0]^3[a+0+0+0]}{[a+0+0+0]}=\frac{a}{a}=\text{a.}$$

$$\therefore\ \lim_{n\to\infty}\frac{P(n+1)}{P(n)}=\text{a.}$$

P4. (UQ)**S.T the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}$ is convergent.**

[OR] prove that $\lim\limits_{n \to \infty}\left(1+\frac{1}{n}\right)^n$ = e.

Solution:

Let $s_n = \left(1+\frac{1}{n}\right)^n =$

$$1+\frac{n}{1!}\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\ldots\ldots+\frac{n(n-1)\ldots..[n-(n-1)]}{k!}\left(\frac{1}{n}\right)^k+\ldots..$$

For $k = 1, 2, 3,\ldots n$

The $(k + 1)^{th}$ term is $\frac{n(n-1)\ldots..[n-(n-1)]}{k!}\left(\frac{1}{n}\right)^k$

$$= \frac{n^k}{k!.n^k}\left[(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots..\left[1-\frac{(k-1)}{n}\right]\right]$$

$$= \frac{1}{1.2.3\ldots k}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots..\left[1-\frac{(k-1)}{n}\right]\right]$$

We expand s_{n+1}, one more term than s_n .

For $k = 1, 2, 3,\ldots n$

The $(k + 2)^{th}$ term is

$$\frac{1}{1.2.3\ldots k}\left[\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots..\left[1-\frac{(k-2)}{n+1}\right]\right]$$

Clearly, $s_n \leq s_{n+1}$

The sequence $\{s_n\}$ is a non-decreasing sequence.

$\therefore$ s_n =

$$1+\frac{n}{1!}\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+......+\frac{n(n-1).....[n-(n-1)]}{k!}\left(\frac{1}{n}\right)^k+.....$$

$$\leq 1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+.....+\frac{1}{1.2.3...n}$$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+.....+\frac{1}{2^{n-1}}=1+\frac{1}{1-\frac{1}{2}}$$

[$s_n = 1 + a + a^2 +.....+ a^{n-1} = 1/1 - r$.

$$\leq 1+\frac{1}{\frac{1}{2}}=1+2=3.$$

$\therefore s_n \leq 3$.

$\therefore$ The sequence $\{s_n\}$ is bounded above by 3.

$\therefore$ $\{s_n\}$ is an increasing sequence which is bounded above by 3.

Hence $\{s_n\}$ is a convergent sequence.

$\therefore$ $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ = e [where 2< e <3 and e=2.7182….]

Alter Methods.

Let $s_n = \left(1+\frac{1}{n}\right)^n$

Then $\log s_n = \log\left(1+\frac{1}{n}\right)^n = n\log\left(1+\frac{1}{n}\right)$

$$= n\left[\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\frac{1}{3}\frac{1}{n^3}+.....\right]$$

$$\log s_n = \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\right]$$

$$s_n = e^{\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\right]} = e^1 . e^{\left[-\frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\right]} = \text{e.}$$

$$\left[1 + \frac{\left(-\frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ..\right)}{1!} + \frac{\left(-\frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ..\right)^2}{2!} + ...\right]$$

$$s_n = e\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - ....\right]$$

Apply limit, $\lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} e\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - .....\right]$

Hence $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. where $2 < e < 3$, $e = 2.7182\ldots.$

P5. Prove that

(i) $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} = e$ (ii) $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n+1}\right)^n = e$

(iii) $\lim\limits_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = e^2$.

Solution:

Given $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n . \left(1 + \frac{1}{n}\right)$

$= \lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right) = e.(1 + 0) = e$

Hence $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+1} = e.$

(ii) For $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)^{n}$

Put $x = n + 1$ as n $\to\infty$, $x\to\infty$

$$=\lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{x-1} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{x}\right)^{x}}{\left(1+\frac{1}{x}\right)} = \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{x}\right)^{x}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{x}\right)} = \frac{e}{1} = e.$$

(iii) $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{n} = e^2.$

Solution:

$$1+\frac{2}{n} = \left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)$$

Check $\left(\dfrac{n+1+1}{n+1}\right)\left(\dfrac{n+1}{n}\right) = \left(\dfrac{n+2}{n+1}\right)\left(\dfrac{n+1}{n}\right)$]

$$\therefore \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n}.\left(1+\frac{1}{n}\right)^{n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e.e$$

Hence $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{n} = e^2.$

P6 Prove that if $a_{n+1} = \sqrt{2+\sqrt{a_n}}$ for n = 0, 1, 2, 3….

Then $\{a_n\}$ is convergent and that $a_n < 2$, $\forall$ n.

Solution:

$$\text{Let } a_{n+1} = \sqrt{2+\sqrt{a_n}}\text{, for } n = 0, 1, 2,\ldots.$$

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2+\sqrt{a_1}} = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$$

$$\Rightarrow a_2 > a_1$$

$$a_3 = \sqrt{2+\sqrt{a_2}} > \sqrt{2+\sqrt{a_1}} = a_2 \; [\because\; a_2 > a_1]$$

$$\Rightarrow a_3 > a_2$$

$$a_{n+1} > a_n \text{ for } n = 0, 1, 2,\ldots.$$

$$\Rightarrow \{a_n\} \text{ is an non-decreasing sequence.}$$

To prove that $a_n < 2$. $\forall$ n = 1, 2,….

Suppose $a_n \geq 2$, $\forall$ n = 1, 2, … (3.14)

Then $a_n = \sqrt{2+\sqrt{a_{n-1}}} = \sqrt{2+\sqrt{2+\sqrt{a_{n-2}}}}$

$\Rightarrow a_n < \sqrt{2+\sqrt{2}}$ … (3.15)

From (3.14) and (3.15) $\Rightarrow 2 \leq 0a_n < \sqrt{2+\sqrt{2}}$

Which is a contradiction to $2 < \sqrt{2+\sqrt{2}}$

Hence $a_n < 2$. $\forall$ n = 1, 2, 3….

$\therefore$ $\{a_n\}$ is an non-decreasing sequence, which is bounded above by 2.

Hence the sequence $\{a_n\}$ is convergent.

P7. If $s_n = \sqrt{2}$, $s_{n+1} = \sqrt{2}\sqrt{s_n}$, $\forall$ n $\geq$ 2. Then prove that $\lim\limits_{n\to\infty} s_n$ = 2.

Solution:

Given $s_n = \sqrt{2}$, $s_{n+1} = \sqrt{2}\sqrt{s_n}$, $\forall\ n \geq 2$.

$s_1 = \sqrt{2}$, $s_2 = \sqrt{2}\sqrt{s_1} = \sqrt{2}\sqrt{\sqrt{2}}$ => $s_2 > s_1$.

Suppose $s_{n+1} > s_n$

$\Rightarrow \sqrt{2}\sqrt{s_{n+1}} > \sqrt{2}\sqrt{s_n}$

$\Rightarrow s_{n+2} > s_{n+1} > s_n$

$\Rightarrow s_{n+2} > s_n$, $\forall\ n = 1, 2, 3\ldots.$

$\Rightarrow \{s_n\}$ is an non-decreasing sequence.

Also, $s_1 = \sqrt{2} < 2$.

Suppose $s_2 < 2$.

Then $s_{n+1} = \sqrt{2}\sqrt{s_n}$

$\Rightarrow (s_{n+1})^2 = 2\ s_n \Rightarrow \lim\limits_{n\to\infty} (s_{n+1})^2 = 2\lim\limits_{n\to\infty} s_n$

$\Rightarrow L^2 = 2L$ [$\because$ $\{s_{n+1}\}$ is a subsequence of $\{s_n\}$ converges to same limit] –

$\Rightarrow L^2 - 2L = 0 \Rightarrow L(L-2)$

$\Rightarrow L = 2$,[$\because L \neq 0$ and $s_1 = \sqrt{2} > 0$]

$\therefore \lim\limits_{n\to\infty} s_n = 2$. Hence $\{s_n\}$ is convergent to 2.

PROBLEMS BASED ON LIMIT SUP & LIMIT INF

P8. let $s_n = (-1)^n$, n $\in$ I, find Lt sup & Lt inf.

Solution:

Given $s_n = (-1)^n$, $n \in$ I.

$S_n = \{-1, 1, -1, 1, -1 \ldots\ldots\}$

Here M_1 = l.u.b$\{-1, 1, -1, 1, -1, \ldots.\}$

M_2 = l.u.b $\{1, -1, 1, -1, \ldots..\}$

Clearly $M_1 = 1$, $M_2 = 1 \ldots\ldots$

$\therefore \{M_n\}$ is the sequence consist of 1.

$\therefore \lim\limits_{n \to \infty} M_n = \lim\limits_{n \to \infty} (1) = 1.$

Hence Lt sup s_n =1.

Also, m_1 = g.l.b $\{-1, 1, -1, 1, -1, \ldots.\}$

m_2 = g.l.b $\{1, -1, 1, -1, \ldots..\}$

Clearly $m_1 = -1$, $m_2 = -1$

$\therefore \{m_n\}$ is the sequence consist of -1.

$\therefore \lim\limits_{n \to \infty} M_n = \lim\limits_{n \to \infty} (-1) = -1.$

Hence Lt inf $s_n = -1$.

P9. Find the Lt sup & Lt inf for the following sequence.

1, –1, 1, –2, 1, –3, 1, –4,…..

1, 2, 3, 1, 2, 3, 1, 2, 3…..

$S_n = (-n)$, *n* in I.

$$\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=}$$

Solution:

(a) Given sequence {1, –1, 1, –2, 1, – 3, 1, –4….}

Here M_n = l.u.b {1, –1, 1, –2, 1, –3, 1, –4…}}= 1.

$\forall$ n = 1, 2, 3….

$$\therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty} (1) = 1.$$

Hence limit sup s_n = 1.a

4

SERIES OF REAL NUMBERS

4.1. CONVERGENT AND DIVERGENT SERIES

1. Define series of real numbers

The series $a_1 + a_2 + a_3 + \ldots.. + a_n + \ldots.$ is called an infinite series [or] series.

We denoted by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots.. + a_n + \ldots.$

Then $s_n = a_1 + a_2 + a_3 + \ldots.. + a_n$ is called the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$.

2. Define convergence of the series

A series $\sum_{n=1}^{\infty} a_n$ is said to be cges to A

If the seq $\{s_n\}$ be the n^{th} partial sum of the series cges to A.

i.e., If $\lim_{n \to \infty} s_n = A$.

3. Define divergence of the series

A series $\sum_{n=1}^{\infty} a_n$ is said to be dges to ∞

If the seq $\{s_n\}$ be the n^{th} partial sum of the series dges to ∞.

i.e., If $\lim_{n \to \infty} s_n = \infty$.

Notations:

(1) If a series $\sum_{n=1}^{\infty} a_n$ cges to A $\Rightarrow \lim_{n \to \infty} s_n$ = A. [Notations $\sum_{n=1}^{\infty} a_n$ = A

(2) If the series $\sum_{n=1}^{\infty} a_n$ is a cgt series of non-negative terms, Then $\sum_{n=1}^{\infty} a_n < +\infty$.

(3) If the series $\sum_{n=1}^{\infty} a_n$ is a dgt series of non-negative terms, Then $\sum_{n=1}^{\infty} a_n = \infty$.

Theorem: 1

If $\sum_{n=1}^{\infty} a_n$ cges to A & $\sum_{n=1}^{\infty} b_n$ cges to B, Then P.T

(a) $\sum_{n=1}^{\infty} (a_n + b_n)$ cges to (A + B); (b) if $c \in$R, then $\sum_{n=1}^{\infty} c.a_n$ cges to cA; (c) $\sum_{n=1}^{\infty} (a_n - b_n)$ cges to (A – B)

Proof:

Let $s_n = a_1 + a_2 + a_3 + \ldots.. + a_n$

$t_n = b_1 + b_2 + b_3 + \ldots.. + b_n$ are the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ respectively.

By hypothesis, $\sum_{n=1}^{\infty} a_n$ cges to A & $\sum_{n=1}^{\infty} b_n$ cges to B,

Then $\lim_{n\to\infty} s_n = A$, & $\lim_{n\to\infty} t_n = B$

Let $u_n = n^{th}$ partial sum of the series

$$\sum_{n=1}^{\infty}(a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) +$$

Then $u_n = (a_1 + a_2 + a_3 + \ldots.. + a_n) + (b_1 + b_2 + b_3 + \ldots.. + b_n)$

$$\Rightarrow \quad u_n = s_n + t_n$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = (A + B)$$

Hence the series $\sum_{n=1}^{\infty}(a_n + b_n)$ cges to (A + B).

Part(b). Let $c \in$ R,

then $p_n = n^{th}$ partial sum of $\sum_{n=1}^{\infty} c.a_n$.

$$= ca_1 + ca_2 + ca_3 + \ldots + ca_n$$

$$= c(a_1 + a_2 + a_3 + \ldots.. + a_n) = c.s_n$$

$$p_n = c.s_n$$

$$\therefore \quad \lim_{n\to\infty} p_n = \lim_{n\to\infty} \text{c.sn} = \text{c} \lim_{n\to\infty} s_n = \text{c.A}$$

$$\lim_{n\to\infty} p_n = \text{c.A}$$

Hence the series $\sum_{n=1}^{\infty} c.a_n$ cges to c.A.

(3) By using (a) & (b)

Put $c = -1$ in (a)

$\sum_{n=1}^{\infty} [a_n + (-b_n)]$ cges to A + (–B) = A – B

Hence $\sum_{n=1}^{\infty} (a_n - b_n)$ cges to A – B.

Theorem: 2

The necessary condition for the series to be cgt. If $\sum_{n=1}^{\infty} a_n$ is a cgt series then $\lim_{n \to \infty} a_n = 0$.

Proof:

Let $\sum_{n=1}^{\infty} a_n$ is a cges to A. i.e., $\sum_{n=1}^{\infty} a_n = A$.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ be the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$.

Also, $\qquad s_{n-1} = a_1 + a_2 + a_3 + \ldots + a_{n-1}$

Clearly $\lim_{n \to \infty} s_n = A$ & $\lim_{n \to \infty} s_{n-1} = A$

[$\because$ $\{s_{n+1}\}$ is a subseq of $\{s_n\}$ cges to same limit]

$\therefore s_n - s_{n-1} = (a_1 + a_2 + a_3 + \ldots + a_n) - (a_1 + a_2 + a_3 + \ldots + a_{n-1})$

$= a_n$

$\therefore \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} s_n - s_{n-1}$

$$= \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = \text{A} - \text{A} = 0$$

Hence $\lim_{n \to \infty} a_n = 0.$

Result

The converse is not true.

i.e., if $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ need not be cgt.

Proof:

Let us consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Here $a_n = \frac{1}{n}$

$$\therefore \qquad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

But the series $\sum_{n=1}^{\infty} \frac{1}{n}$ always dges.

PROBLEMS BASED ON CONVERGENT AND DIVERGENT SERIES

P1. P.T if $a_1 + a_2 + a_3 + \ldots\ldots$ cges to s. Then $a_2 + a_3 + \ldots\ldots$ also cges to $s - a_1$.

Solution:

Given $\sum_{n=1}^{\infty} a_n = s$, $\lim_{n \to \infty} a_n = s$

Let $\qquad s_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$

Let $\sum_{n=2}^{\infty} a_n = p$, then $t_n = a_2 + a_3 + \ldots.. + a_n$,

$\therefore \lim_{n\to\infty} t_n = p,$

$s_n - t_n = (a_1 + a_2 + a_3 + \ldots.. + a_n) - (a_2 + a_3 + \ldots.. + a_n) = a_1.$

$\therefore \lim_{n\to\infty} s_n - t_n = \lim_{n\to\infty} a_1 = a_1$

$\Rightarrow \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n = a_1$

$\Rightarrow s - p = a_1$ or $p = s - a_1$

Hence the series $\sum_{n=2}^{\infty} a_n$ cges to $s - a_1$

P2. For what value of x does the series $(1 - x) + (x - x^2) + (x^2 - x^3) + (x^3 - x^4) + \ldots.$ Cges?

Solution:

Given series $\sum_{n=1}^{\infty} (x^{n-1} - x^n)$

Then the n^{th} partial sum

$s_n = ((1 - x) + (x - x^2) + (x^2 - x^3) + (x^3 - x^4) + \ldots.+(x^{n-1} - x^n).$

$= 1 - x^n$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - x^n)$

$= 1 - \lim_{n\to\infty} x^n = 1 - 0 = 1 \; [0 \leq < x < 1)$

The seq $\{s_n\}$ cges to 1.

Hence the series $\sum_{n=1}^{\infty}(x^{n-1}-x^{n})$ cges to 1, if $0 \le < x < 1$.

P3. P.T if **a**$_1$ **+ a**$_2$ **+ a**$_3$ **+…..**∞ ***cges to A. Then*** $\frac{1}{2}$***(a***$_1$ ***+ a***$_2$***) +*** $\frac{1}{2}$***(a***$_2$ ***+ a***$_3$***) +*** $\frac{1}{2}$***(a***$_3$ ***+ a***$_4$***) +…. Cges. What is the sum of the 2***nd ***series.***

Solution:

Given $\sum_{n=2}^{\infty} a_n$ sges to A.

Then the n^{th} partial sum $s_n = a_1 + a_2 + a_3 + \ldots.. + a_n$.

Given 2 $^{\text{nd}}$ series is $\sum_{n=1}^{\infty}\frac{1}{2}(a_n + a_{n+1})$.

Then the n^{th} partial sum is

$$p_n = \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_2 + a_3) + \frac{1}{2}(a_3 + a_4) + \ldots + \frac{1}{2}(a_{n-1} + a_n) + \frac{1}{2}(a_n + a_{n+1})$$

$$= \frac{1}{2}(a_1 + a_{n+1}) + (a_2 + a_2 + a_3 + \ldots.. + a_n)$$

$$= (a_1 + a_2 + a_3 + \ldots.. + a_n) - \frac{1}{2}(a_1 - a_{n+1})$$

$$p_n = s_n - \frac{1}{2}(a_1 - a_{n+1})$$

$$\therefore \quad \lim_{n\to\infty} p_n = \lim_{n\to\infty}\left[s_n - \frac{1}{2}(a_1 - a_{n+1})\right]$$

$$\lim_{n\to\infty} s_n - \lim_{n\to\infty} \frac{1}{2}(a_1 - a_{n+1})$$

$$\therefore \qquad \lim_{n\to\infty} p_n = A - \frac{1}{2}(a_1 - a_{n+1})$$

$$\Rightarrow \{p_n\} \text{ cges to } A - \frac{1}{2}(a_1 - a_{n+1})$$

Hence the 2nd series $\sum_{n=1}^{\infty}\frac{1}{2}(a_n + a_{n+1})$ cges to $A - \frac{1}{2}(a_1 - a_{n+1})$

[Or] $\sum_{n=1}^{\infty}\frac{1}{2}(a_n + a_{n+1}) = A - \frac{1}{2}(a_1 - a_{n+1})$

P4. Test for the series is cgt or dgt:- if cgt find value. (a) $\sum_{n=1}^{\infty}\frac{1-n}{1+2n}$

(A15) ***(b)*** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ ***(c)*** $\sum_{n=1}^{\infty}(-1)^n$ ***(d)*** $\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right)$

(e) $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots.$

Solution:

(a) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1-n}{1+2n}$ here $a_n = \frac{1-n}{1+2n}$ A15

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1-n}{1+2n}$$

$$= \lim_{n\to\infty}\frac{n\left(\frac{1}{n}-1\right)}{n\left(\frac{1}{n}+2\right)}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{n}-1\right)}{\left(\frac{1}{n}+2\right)}=\frac{(0-1)}{(0+2)}=\frac{-1}{2}\neq 0.$$

$\therefore \lim\limits_{n\to\infty} a_n \neq 0$. Hence the series $\sum\limits_{n=1}^{\infty}\frac{1-n}{1+2n}$ diverges.

(c) Prove that the series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges. Find its value.

Let $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$ here $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 0.$

$\therefore \lim\limits_{n\to\infty} a_n = 0$. Hence the series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$ cges

To find value.

The nth partial sum is $s_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\ldots\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1.$$

the series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges to 1.

i.e., $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$

Hence (c) Let $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n$, here $a_n=(-1)^n$

$\therefore \lim_{n\to\infty}a_n=\lim_{n\to\infty}(-1)^n$ limit does not exists.

Hence the series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

(d) Let $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right)$

here $a_n=\log\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\frac{1}{3}\frac{1}{n^3}-$

$\therefore \lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\frac{1}{3}\frac{1}{n^3}-\right)=0.$

Hence the series $\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right)$ cges.

(e) Given series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ Here $a_n=\frac{1}{2^{n-1}}$

$\therefore \lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(\frac{1}{2^{n-1}}\right)=0.$

Hence series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....$ is convergent.

To find the value.

The nth partial sum is $s_n=a_1+a_2+a_3+.......+a_n$

$$=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}=a\left(\frac{1-r^n}{1-r}\right)=a\left(\frac{1-r^n}{1-r}\right)$$

$$=2\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)=2\left(1-\frac{1}{2^n}\right)$$

$$\therefore \lim_{n\to\infty} s_n=\lim_{n\to\infty} 2\left(1-\frac{1}{2^2}\right) =2(1-0) = 2.$$

$\Rightarrow$ the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....$ converges to 2.

i.e., $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....=2.$

P5. What is the value of k where $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ ***= k? A13.***

Solution:

Let $\sum_{n=1}^{\infty} a_n=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ here $a_n=\frac{1}{2^n}$

$$\therefore \lim_{n\to\infty} a_n=\lim_{n\to\infty}\left(\frac{1}{2^n}\right)= 0.$$

$\therefore \lim_{n\to\infty} a_n=0$. Hence the series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ cges.

To find value.

The nth partial sum is $s_n = a_1+ a_2+ a_3 +.......+ a_n$

$$= 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+.....+\frac{1}{2^{n-1}}=a\left(\frac{1-r^n}{1-r}\right)=\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)$$

$$\therefore \lim_{n\to\infty} s_n=\lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2(1 = 0) = 2.$$

Hence the series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ converges to 2.

i.e., $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ =2. Hence k = 2.

P6. A13. Let $\sum_{n=1}^{\infty} a_n$ be a infinite series where $a_n=\frac{1}{n(n+1)}$ .if $s_n = a_1 + a_2 + a_3 +\ldots+ a_n$. Then find s_{100}.

Solution:

Given $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$S_{100} = a_1 + a_2 + a_3 +\ldots+ a_{99} + a_{100}$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\ldots\left(\frac{1}{99}-\frac{1}{100}\right)+\left(\frac{1}{100}-\frac{1}{101}\right)$$

$$=1-\frac{1}{101}=\frac{100}{101}.$$

4.2. SERIES WITH NON-NEGATIVE TERMS

Theorem: 1

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative numbers with $s_n = a_1 + a_2 + a_3 + \ldots.. + a_n$ Then P.T (a) $\sum_{n=1}^{\infty} a_n$ cges if seq $\{s_n\}$ is bounded.

If $\sum_{n=1}^{\infty} a_n$ dvges , if seq $\{s_n\}$ is unbounded.

Proof:

(a) Since $a_n \geq 0.\ \forall\, n.$

We have $s_{n+1} = a_1 + a_2 + a_3 + \ldots.. + a_n + a_{n+1} = s_n + a_{n+1} \geq s_n.$

$\Rightarrow s_{n+1} \geq s_n,\ \forall\, n.$

$\Rightarrow$ the seq $\{s_n\}$ is a non-decreasing seq & bounded.

$\Rightarrow \{s_n\}$ cges

Hence the series $\sum_{n=2}^{\infty} a_n$ is convergent.

If the seq $\{s_n\}$ is unbounded.

Then $\{s_n\}$ is not cgnt.

Hence $\sum_{n=2}^{\infty} a_n$ is divergent.

Theorem: 2

(a) If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ cges to $\frac{1}{1-x}$ (b)If $x \geq 1$, then $\sum_{n=0}^{\infty} x^n$ dges.

Proof:

(b) if If $x \geq 1$, then x^n cges ∞ as $n \to \infty$.

Hence the series $\sum_{n=0}^{\infty} x^n$ diverges. [Take $x = 3$, $3^n \to \infty$ as $n \to \infty$]

Part(a): Let $0 < x < 1$.

Let $s_n = 1 + x + x^2 + x^3 + \ldots + x^n$ be the n^{th} partial sum of $\sum_{n=0}^{\infty} x^n$.

$\therefore s_n = \frac{1-x^{n+1}}{1-x}$ by G.P.

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1-x^{n+1}}{1-x}\right) = \lim_{n\to\infty}\left(\frac{1}{1-x}\right) - \lim_{n\to\infty}\left(\frac{x^{n+1}}{1-x}\right)$$

$$= \frac{1}{1-x} - 0$$

[by known th if $0 < x < 1, \lim_{n\to\infty} x^n = 0$]

$$\therefore \lim_{n\to\infty} s_n = \frac{1}{1-x}.$$

Hence the series $\sum_{n=0}^{\infty} x^n$ cges to $\frac{1}{1-x}$ if $0 < x < 1$.

Hence the proof.

Define Alternating series

The alternating series is of the form

$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots$ is denoted by $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$.

Theorem: 3

State and Prove Leibnitz's Theorem. If $\{a_n\}$ is a seq of positive no/–s. Such that (a) $a_1 \geq a_2 \geq a_3 \geq . \geq a_n \geq a_{n+1} \geq$ (b) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is cges.

Proof:

Given series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ … (4.1)

Let $s_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n$ be the n^{th} partial sum of (1).

To P.T the seq $\{s_{2n}\}$ cges.

We have $s_{2n} = a_1 - a_2 + a_3 - a_4 + \ldots + a_{2n-1} - a_{2n}$

$\therefore\ s_{2n+2} = s_{2n} + a_{2n-1} - a_{2n-2}$

$\Rightarrow s_{2n+2} - s_{2n} = a_{2n-1} - a_{2n-2} \geq 0$ [by hypothesis (a).]

Also, we have

$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$.

[$\because$ By (a), $a_2 \geq a_3 \Rightarrow a_2 - a_3 \geq 0$, $a_4 - a_5 \geq 0$ and a_1 is max no/- $a > b \Rightarrow a - b \geq a$]

$\Rightarrow$ $s_{2n} < a_1$. $\forall\, n$,

$\Rightarrow$ the seq $\{s_{2n}\}$ is bounded seq.

$\Rightarrow$ the seq $\{s_{2n}\}$ is cgt.

$\therefore \lim\limits_{n\to\infty} s_{2n}$ =L (say).

We have $s_{2n+1} = s_{2n} + a_{2n+1}$.

$\Rightarrow \lim\limits_{n\to\infty} s_{2n+1} =. \lim\limits_{n\to\infty} s_{2n} + \lim\limits_{n\to\infty} a_{2n+1}$

= L + 0 [By hyp (b) $\lim\limits_{n\to\infty} a_n = 0$]

$\Rightarrow \lim\limits_{n\to\infty} s_{2n+1}$ =L

Since $\{s_{2n}\}$ & $\{s_{2n+1}\}$ cges to the same limit L.

$\Rightarrow \lim\limits_{n\to\infty} s_n$ = L.

$\Rightarrow \sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ = L.

Hence the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ cges.

Hence the theorem.

Problems based on alternating series.

Theorem: 4

S.T the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof:

Given series $\sum_{n=1}^{\infty}\frac{1}{n}$

Let $s_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+......+\frac{1}{n}$ be the n^{th} partial sum of $\sum_{n=1}^{\infty}\frac{1}{n}$

We now examine the subseq $s_1, s_2, s_4, s_8\ldots\ldots s_2$ of $\{s_n\}$.

We have $s_1 = 1$

$$s_2 = 1+\frac{1}{2} = \frac{3}{2}$$

$$s_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = s_2+\frac{1}{3}+\frac{1}{4} > \frac{3}{2}+\frac{1}{4}+\frac{1}{4} = 2.$$

$$s_8 = s_4+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > 2+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{2}.$$

In general, $s_2 > \frac{(n+2)}{2}$ for $n = 1, 2, 3\ldots.$

$\Rightarrow \{s_2\}$ diverges.

$\Rightarrow \{s_n\}$ also diverges. [Seq & subseq cges or dges simultaneously]

Hence the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

Theorem: 5

S.T the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ cges.

Solution:

Given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, Here $a_n = \frac{1}{n}$

(1) Clearly $a_n > a_{n+1}$ $\quad [\because \frac{1}{2} > \frac{1}{3}]$

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ Leibniz's test true.

Hence the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ cges.

Theorem: 6

Solve that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$ diverges.

Solution:

Given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$ where $a_n = \frac{n}{2n-1}$

$a_1 = 1$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{5}$,

(1) Clearly $a_1 > a_2 > a_3 > \ldots$ holds

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n}{n\left(2 - \frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(2 - \frac{1}{n}\right)}$

$= \frac{1}{(2-0)} = \frac{1}{2} \neq 0$. (fails)

Leibniz's Conditions (2) fails.

Hence the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$ diverges.

HOME WORK

1. Show that the following series do not converges.

(i) $1\frac{1}{2} - 1\frac{1}{4} + 1\frac{1}{8} - 1\frac{1}{16} . + ...$ (ii) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$

(iii) $\sum_{n=0}^{\infty} (-1)^n \frac{3n}{4n-1}$

2. Test the series $\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{n^3+1}$ ***for convergence. (Ans: cges)***

3. Prove that: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1}$ ***diverges.***

Theorem: 6

Prove that (a) $2 - 2^{\frac{1}{2}} + 2^{\frac{1}{3}} - 2^{\frac{1}{4}} + 2^{\frac{1}{5}} - ....$ diverges.

(b) $(1-2) - (1 - 2^{\frac{1}{2}}) - (1 - 2^{\frac{1}{3}}) - (1 - 2^{\frac{1}{4}})$ …… converges.

Solution:

(a) Given series $2 - 2^{\frac{1}{2}} + 2^{\frac{1}{3}} - 2^{\frac{1}{4}} + 2^{\frac{1}{5}} -....$

$= \sum_{n=1}^{\infty} (-1)^{n+1} 2^{\frac{1}{n}}$ here $a_n = 2^{\frac{1}{n}}$.

$a_1 = 2$, $a_2 = 2^{\frac{1}{2}}$, $a_3 = 2^{\frac{1}{3}}$

(1)Clearly $a_1 > a_2 > a_3 > \ldots$holds

(2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \neq 0$. (fails)

Leibniz's Conditions (2) fails.

Hence the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}.2^{\frac{1}{n}}$ diverges.

Proof(b) Given series $(1 - 2) - (1 - 2^{\frac{1}{2}}) - (1 - 2^{\frac{1}{3}}) - (1 - 2^{\frac{1}{4}})$

$= \sum\limits_{n=1}^{\infty}(-1)^{n+1}(1 - 2^{\frac{1}{n}})$

Here $a_n = (1 - 2^{\frac{1}{n}})$

$a_1 = (1-2)$, $a_2 = (1 - 2^{\frac{1}{2}})$, $a_3 = (1 - 2^{\frac{1}{3}})$ holds

(1) Clearly $a_1 > a_2 > a_3 > \ldots$

(2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (1 - 2^{\frac{1}{n}}) = (1 - 2^0) = (1 - 1) = 0$.

$\therefore$ Leibniz's test true.

Hence the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}.(1 - 2^{\frac{1}{n}})$ converges.

Theorem: 7

For what value of p does the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots\ldots$ cges?

Solution:

Given series $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+........=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$

Here $a_1=\frac{1}{1^p}=1$, $a_2=\frac{1}{2^p}$, $a_3=\frac{1}{3^p}$ ….. $a_n=\frac{1}{n^p}$

(1) Clearly $a_1 > a_2 > a_3 >$…. holds

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^p} = 0$ true if $p > 1$.

$\lim_{n\to\infty} a_n \neq 0$ if $p \leq 1$

Leibniz's Conditions

Hence the series $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+........$ cges if $p > 1$.

$\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+........$ dges if $p \leq 1$.

4.3. CONDITIONAL CONVERGENTS.

1. Define absolute convergence and conditional convergence.

Let $\sum_{n=1}^{\infty} a_n$ be a series of real no/s-.

(a) if $\sum_{n=1}^{\infty}|a_n|$ cges , then $\sum_{n=1}^{\infty} a_n$ is said to be absolute convergent.

(b) if $\sum_{n=1}^{\infty} a_n$ cges, but $\sum_{n=1}^{\infty}|a_n|$ dges, then $\sum_{n=1}^{\infty} a_n$ is said to be conditional convergence.

Ex1. Give an example of absolute cgt series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n} = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots..$$

Proof:

Given series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots..$

Take the absolute value of each term,

The series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots..$ is convergent.

Hence the series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots..$ cges absolutely.

Ex2. Give an example of conditional cgt. [A13,15, N14

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots..$$

Proof:

Given series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots..$ it is an alternating seq

Which is convergent series.

The absolute value of each term is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots..$ is divergent

Hence the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots..$ is conditionally convergent.

Define Positive and negative components of a series.

Let $\sum_{n=1}^{\infty} a_n$ be a series of real no/s-.

Let $p_n = \begin{cases} a_n & if\ a_n > 0 \\ 0 & if\ a_n \le 0 \end{cases}$.

Similarly. $q_n = \begin{cases} a_n & if\ a_n \le 0 \\ 0 & if\ a_n > 0 \end{cases}$

Then p_n & q_n are called positive and negative terms of the series.

Result

R1. $p_n = \max(a_n, 0)$; $q_n = \min(a_n, 0)$

R2. $\text{Max}(a, b) = \dfrac{(a+b)+|a-b|}{2}$; $\text{Min}(a, b) = \dfrac{(a+b)-|a-b|}{2}$;

Then $p_n = \dfrac{a_n + |a_n|}{2}$; $q_n = \dfrac{a_n - |a_n|}{2}$.

Theorem: 8

If $\sum_{n=1}^{\infty} a_n$ cges absolutely, then $\sum_{n=1}^{\infty} a_n$ cges.

[OR]

Solve that if $\sum_{n=1}^{\infty} |a_n|$ cges , then $\sum_{n=1}^{\infty} a_n$ cges.

Proof:

By hypothesis $\sum_{n=1}^{\infty} |a_n| < \infty$ i.e $\sum_{n=1}^{\infty} |a_n|$ cges.

By def, Let t_n = $|a_1| + |a_2| + |a_3| + \ldots$.be the n^{th} partial sum of $\sum_{n=1}^{\infty} |a_n|$

$\therefore$ the seq $\{t_n\}$ is cgt.

$\Rightarrow$ The seq $\{t_n\}$ is a Cauchy seq

[$\because$ every bdd cgt seq is Cauchy seq]

$\Rightarrow$ By def, Given $\in > 0$, $\exists$ a $N \in I$, S.t $|t_m - t_n| < \in, \forall$ m, $n > N$.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ be the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$

To Prove that: $\sum_{n=1}^{\infty} a_n$ cges,

i.e., To P.T: $\{s_n\}$ cgt.

i.e., To P.T: $\{s_n\}$ is a Cauchy seq.

[$\because$ Ever y Cauchy seq is cgt]

For Choose $m > n$.

$$|s_m - s_n| = |a_{n+1} + a_{n+2} + a_{n+3} + \ldots + a_m|$$

$$\leq |a_{n+1}| + |a_{n+2}| + |a_{n+2}| + \ldots + |a_m| = |t_m - t_n| < \in$$

$\Rightarrow$ $|s_m - s_n| < \in$, $\forall$ $m, n > N$.

$\Rightarrow$ the seq $\{s_n\}$ is a Cauchy seq. Hence the proof.

Result: Converse is not true. Justify your answer.

[OR]

If $\sum_{n=1}^{\infty} a_n$ cges then $\sum_{n=1}^{\infty} |a_n|$ need not be cgt.

Proof:

Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots..$

here $a_n = \frac{1}{n}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$.

$a_1 > a_2 > a_3 > \ldots\ldots$ holds.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. (holds)

$\therefore$ By Leibniz test true.

$\therefore$ the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ cges.

But $\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots..$here $\left|\frac{-1}{2}\right| = \frac{1}{2}$

$= \sum_{n=0}^{\infty} \frac{1}{n}$ is diverges. [By problem-]

$\therefore \sum_{n=1}^{\infty} |a_n|$ is dges.

i.e., $\sum_{n=1}^{\infty} a_n$ is not absolutely cgt.

Theorem: 9

(a) if $\sum_{n=1}^{\infty} a_n$ cges absolutely, Then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ cges

(b)if $\sum_{n=1}^{\infty} a_n$ cges conditionally, Then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ dges.

Proof:

(a) Let $\sum_{n=1}^{\infty} a_n$ cges absolutely, Then $\sum_{n=1}^{\infty} |a_n|$ cges.

Let $p_n = \max(a_n, 0)$; $q_n = \min(a_n, 0)$

Then $2p_n = a_n + |a_n|$; $2q_n = a_n - |a_n|$

Since $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} |a_n|$ cges

$\Rightarrow \sum_{n=1}^{\infty} (a_n + |a_n|)$ cges $\Rightarrow \sum_{n=1}^{\infty} 2p_n$ cges $\Rightarrow \sum_{n=1}^{\infty} p_n$ cges.

similarly, $\sum_{n=1}^{\infty} (a_n - |a_n|)$ cges $\Rightarrow \sum_{n=1}^{\infty} 2q_n$ cges $\Rightarrow \sum_{n=1}^{\infty} q_n$ cges.

(b)Let $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Then $\sum_{n=1}^{\infty} a_n$ cges, but $\sum_{n=1}^{\infty} |a_n|$ dges … (4.2)

Since $2p_n = a_n + |a_n| \Rightarrow |a_n| = 2p_n - a_n$

Suppose $\sum_{n=1}^{\infty} p_n$ converges.

$\Rightarrow \sum_{n=1}^{\infty} (2p_n - a_n)$ cges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ cges, $\Rightarrow$ to $\sum_{n=1}^{\infty} |a_n|$ dges.

Hence $\sum_{n=1}^{\infty} p_n$ diverges.

Also Since $2q_n = a_n - |a_n| \Rightarrow |a_n| = a_n - 2q_n$

Suppose $\sum_{n=1}^{\infty} q_n$ converges.

$\Rightarrow \sum_{n=1}^{\infty} (a_n - 2q_n)$ cges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ cges, =><= to $\sum_{n=1}^{\infty} |a_n|$ dges.

Hence $\sum_{n=1}^{\infty} q_n$ diverges.

[Here =><= means which is a contradiction]

PROBLEMS BASED ON CONDITIONAL CONVERGENCE.

P1. Classify as to divergent or conditionally cget or absolutely cgt?

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots..$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots..$$

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots..$$

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{2^3} + \frac{1}{4} - \frac{1}{2^4} + \ldots\ldots$$

4.4. TEST FOR ABSOLUTE CONVERGENCE OF THE SERIES.

1. Define dominance of a series.

We say that $\sum_{n=1}^{\infty} a_n$ is dominate the series $\sum_{n=1}^{\infty} b_n$

If $\exists$ N $\in$ I, Solve that $|a_n| \le |b_n|$, $\forall\, n$.

We shell denotes by $\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} b_n$, here is dominated by.

Theorem: 10

STATE AND PROVE COMPARISON TEST

Statement. (a) If $\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} b_n$ & if $\sum_{n=1}^{\infty} b_n$ cges absolutely

Then $\sum_{n=1}^{\infty} a_n$ cges absolutely.

(b) If $\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} b_n$ & $\sum_{n=1}^{\infty} b_n$ dges, Then $\sum_{n=1}^{\infty} a_n$ dges.

Proof:

(a) Let M = $\sum_{n=1}^{\infty} |b_n|$

Where $|a_n| \le |b_n|$, $\forall$ n $\ge$N.

If $s_n = |a_1| + |a_2| + |a_3| + ... + |a_N| + |a_{N+1}| + ...|a_n|$

$$\le |a_1| + |a_2| + |a_3| + ... + |a_N| + |b_{N+1}| + ...|b_n|$$

$$\le |a_1| + |a_2| + |a_3| + ... + |a_N| + M$$

The seq $\{s_n\}$ is bounded above

$\Rightarrow$ the seq $\{s_n\}$ cges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ cges $\Rightarrow \sum_{n=1}^{\infty} a_n$ cges absolutely.

(b) if $\sum_{n=1}^{\infty} |b_n|$ cges, then by comparison test (a)

$\sum_{n=1}^{\infty} |a_n|$ also cges $\Rightarrow$ <= to $\sum_{n=1}^{\infty} |b_n|$ dges.

For example

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n+3}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{3n}$

Clearly the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ is dominated by $\sum_{n=1}^{\infty} \frac{1}{3n}$

But $\sum_{n=1}^{\infty} \frac{1}{3n}$ dges => $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ dges.

Theorem: 11

STATE AND PROVE LIMIT TEST.

If $\sum_{n=1}^{\infty} b_n$ cges absolutely & $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists.

Then $\sum_{n=1}^{\infty} a_n$ cges absolutely.

$\sum_{n=1}^{\infty} |a_n| = \infty$ & $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists. Then $\sum_{n=1}^{\infty} |b_n| = \infty$ (diverges)

Proof:

(a) Since $\lim\limits_{n\to\infty}\dfrac{|a_n|}{|b_n|}$ exsist

$\therefore$ $\left\{\dfrac{|a_n|}{|b_n|}\right\}_{n=1}$ is cges. It is bounded.

Hence, $\exists$ a M > 0, Solve that $\dfrac{|a_n|}{|\mathrm{b_n}|}\le M$, $\forall$ n $\in$ I,

$\Rightarrow$ $|a_n|\le M|b_n|$, $\forall$ n $\in$ I,

$\Rightarrow$ $\sum\limits_{n=1}^{\infty}|a_n|\le M\sum\limits_{n=1}^{\infty}|b_n|$, $\forall$ n $\in$ I,

$\Rightarrow$ $\sum\limits_{n=1}^{\infty}|a_n|$ is dominated by M$\sum\limits_{n=1}^{\infty}|b_n|$

$\therefore$ By Comparison test

$\Rightarrow\sum\limits_{n=1}^{\infty}|a_n|$ converges [$\because$ $\sum\limits_{n=1}^{\infty}|b_n|$ cges]

$\Rightarrow\sum\limits_{n=1}^{\infty}a_n$ is converges absolutely.

Since $\therefore$ $\left\{\dfrac{|a_n|}{|b_n|}\right\}_{n=1}$ is cges. It is bounded.

$\Rightarrow\sum\limits_{n=1}^{\infty}|a_n|\le M\sum\limits_{n=1}^{\infty}|b_n|$, $\forall$ n $\in$ I,

$\Rightarrow\sum\limits_{n=1}^{\infty}|a_n|$ is dominated by M$\sum\limits_{n=1}^{\infty}|b_n|$

$\therefore$ By Comparison test

$\Rightarrow$ if $\sum_{n=1}^{\infty} |a_n|$ is diverges then $\sum_{n=1}^{\infty} |b_n|$ diverges.

Hence the proof.

Theorem: 12

STATE AND PROVE RATIO TEST.

Statement:

Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative real no/-s.

Let $a = \lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right|$ & A = $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$ so a < A.

A < 1, then $\sum_{n=1}^{\infty} |a_n| < \infty$ (cges)

If a > 1. Then $\sum_{n=1}^{\infty} a_n$ diverges.

If a $\leq$ 1 $\leq$ A, then the test fails.

Proof:

(a) Let A < 1. Choose B s.t A < B < 1 … (4.3)

Then B = A + $\in$ for some $\in$ > 0.

Since A = $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$

By def, limit sup, Given $\in$ >0, $\exists$ N $\in$ I, Solve that $\left|\dfrac{a_{n+1}}{a_n}\right| < A + \in$, $\forall$ $n \geq$ N,

For $n \geq$ N,

$$\left|\frac{a_{N+1}}{a_N}\right| \leq B,\ \left|\frac{a_{N+2}}{a_{N+1}}\right| \leq B. \Rightarrow \left|\frac{a_{N+1}}{a_N}\right|,.\left|\frac{a_{N+2}}{a_{N+1}}\right|. = \mathrm{B}^2 \Rightarrow \left|\frac{a_{N+2}}{a_{N+1}}\right| \leq B^2.$$

For k>0.

$$\left|\frac{a_{N+k}}{a_N}\right| \leq \left|\frac{a_{N+k}}{a_{N+k-1}}\right|.\left|\frac{a_{N+k-1}}{a_{N+k-2}}\right|......\left|\frac{a_{N+1}}{a_N}\right|$$

$$\Rightarrow \left|\frac{a_{N+k}}{a_N}\right| \leq B^k => |a_{N+k}| \leq B^k |a_N|. \Rightarrow \sum_{k=1}^{\infty} |a_{N+k}| \leq B^k \sum_{k=1}^{\infty} |a_N|.$$

Since $B^k \sum_{k=1}^{\infty} |a_N|.$cges [o < B < 1]

$\therefore$ By Comparison test, $\sum_{k=1}^{\infty} |a_n|.$ converges.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) Let a >1.

Choose B, s.t a > B > 1. Or 1 < B < a … (4.4)

Let B = a – $\in$ for some $\in >$ 0.

Since $a = \lim\limits_{n\to\infty} \inf \left|\dfrac{a_{n+1}}{a_n}\right|$

By def of limit inf, By def, Given $\in$ >0, $\exists$ N $\in$ I, Solve that $\left|\frac{a_{n+1}}{a_n}\right| > a - \in = B$, $\forall$ $n \geq$ N,

$\Rightarrow$ $|a_{n+1}| > B|a_n| > |a_n|$. [1 < B < a]

$\Rightarrow$ {a_n} is not cges to zero.

$\Rightarrow$ $\lim\limits_{n\to\infty} a_n \neq 0.$

Hence the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

(c) consider the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ here $a_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)} =$$

$$\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1/1 = 1 \neq 0.$$

$\therefore a$ = A = 1

Hence the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

If we consider the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n}\right)^2} =$$

$$\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^2} = 1/1 = 1 \neq 0.$$

But the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ cges.

Hence the ratio test fails.

Theorem: 13

STATE AND PROVE ROOT TEST.

[OR] CAUCHY ROOT TEST.

Let $\sum_{n=1}^{\infty} a_n$ be the series of real numbers.

Let A= $\lim_{n\to\infty} \sup|a_n|^{\frac{1}{n}}$

If A < 1, then $\sum_{n=1}^{\infty}|a_n|$ cges.

If A>1, then $\sum_{n=1}^{\infty}|a_n|$ dvges.

If A = 1, the test fails.

Proof:

(a) Let A= $\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$ Let A < 1,

Choose A < B < 1 … (4.5)

Take B = A + $\in$ for some $\in$ > 0 … (4.6)

Since A = $\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$

By def of limit sup, Given $\in$ >0, $\exists$ N $\in$ I,

Solve that $|a_n|^{\frac{1}{n}} < A + \in = B$, $\forall$ $n \geq$ N,

$\Rightarrow |a_n| < B^n$, $\forall$ $n \geq$ N,

$\Rightarrow \sum_{n=1}^{\infty} |a_n| < \sum_{n=1}^{\infty} B^n$

Now $\sum_{n=1}^{\infty} B^n$ converges [since B < 1]

$\therefore$ By Comparison test, $\sum_{k=1}^{\infty} |a_n|$. converges.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) Let A > 1. S.t 1 < B < A.

Choose B = A – $\in$ for some $\in$ > 0.

But A = $\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$

By def of limit sup, $\Rightarrow |a_n|^{\frac{1}{n}} > A - \in$, for sufficiently many value of n,

$\Rightarrow |a_n|^{\frac{1}{n}} > B > 1$, for sufficiently many value of n,

$\Rightarrow |a_n| > 1$

$\therefore$ $\{a_n\}$ is not cges to zero.

$\Rightarrow \sum_{k=1}^{\infty} a_n.$ diverges $\Rightarrow \sum_{k=1}^{\infty} a_n. = \infty$.

(c) Consider the two series $\sum_{k=1}^{\infty} a_n. = \sum_{k=1}^{\infty} \frac{1}{n}$. the series diverges.

$\sum_{k=1}^{\infty} b_n. = \sum_{k=1}^{\infty} \frac{1}{n^2}$. the series converges.

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1.$

Also, $\lim_{n\to\infty} |b_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1.$

$\therefore$ Test fails when A = 1.

Theorem: 14

STATE AND PROVE POWER TEST.

Let $\{a_n\}$ be a seq of real numbers.

(a) If $\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$, then $\sum_{k=1}^{\infty} a_n x^n$.cges absolutely, $\forall$ real x,

(b) If $\lim\limits_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$ =L>0. Then $\sum\limits_{k=1}^{\infty} a_n x^n$.cges absolutely for $|x| < \dfrac{1}{L}$ & diverges for $|x| > \dfrac{1}{L}$

(c) If $\lim\limits_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$ =. ∞.then $\sum\limits_{k=1}^{\infty} a_n x^n$. cges only x = 0 & div $\forall$ other x.

Note: Here L is called Radius of convergence.

Proof:

Given $\lim\limits_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$ =0 … (4.7)

$$\therefore \lim_{n\to\infty} \sup |a_n . x^n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} . |x| = 0.|x| = 0 < 1$$

[By Root test]

$\therefore \sum\limits_{k=1}^{\infty} a_n x^n$.cges absolutely, $\forall$ real x,

(b) Let $\lim\limits_{n\to\infty} \sup |a_n . x^n|^{\frac{1}{n}}$ =L > 0 … (4.8)

$$\therefore \lim_{n\to\infty} \sup |a_n . x^n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} . |x| = L.|x|$$

$\therefore$ By Root Test,

$\sum\limits_{k=1}^{\infty} a_n x^n$.cges absolutely, if $L.|x|$ <1 i.e $|x| < \dfrac{1}{L}$

$\sum_{k=1}^{\infty} a_n x^n$.diverges if $L.|x| > 1$ i.e $|x| > \frac{1}{L}$.

(c) Let $\lim_{n\to\infty} \sup |a_n . x^n|^{\frac{1}{n}} = \infty$.

$\therefore \lim_{n\to\infty} \sup |a_n . x^n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} . |x| = \infty . |x| = \infty > 1$ if $.|x| \neq 0$.

$\therefore$ By Root Test,

$\sum_{k=1}^{\infty} a_n x^n$.diverges if $x \neq 0$.

if $x = 0$, $\sum_{k=1}^{\infty} a_n x^n . = a_0 + a_1 x + a_2 x^2 + \ldots.. = a_0 + 0 + 0 + \ldots.$

[$\because x = 0$]

SERIES WHOSE TERMS FORM A NON-INCREASING SEQUENCE.

Theorem: 14

STATE AND PROVE CAUCHY CONDENSATION TEST.

If $\{a_n\}$ is a non-increasing seq of +ve numbers & if $\sum_{k=1}^{\infty} 2^n a_{2^n}$.cges.

Then $\sum_{n=1}^{\infty} a_n$ cges. [A16 N13

Proof:

We have $a_1 \leq a_2$.

$a_2 + a_3 \leq a_2 + a_2 \leq 2a_2$. [$\because a_3 \leq a_2$]

$a_4 + a_5 + a_6 + a_7 \leq 4a_4 \leq 2^2 a_4$

$a_2 + a_{2+1} + \ldots.. + a_{2-1} \leq 2^n a_2$

i.e., $\sum_{k=1}^{2^{n+1}-1} a_k \leq \sum_{k=0}^{n} 2^k a_{2^k} \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$. [Sum LHS &RHS]

By hypothesis, $\sum_{k=0}^{\infty} 2^k a_{2^k}$ cges.

$\therefore$ By Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

Converse of the above theorem.

Theorem: 15

If $\{a_n\}$ is a non-decreasing seq of +ve numbers & If $\sum_{k=1}^{n} 2^k a_{2^k}$ diverges.

Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof:

Given $a_1 \geq a_2 \geq a_3 \geq \ldots. \geq a_n \geq$

[A N13

We have $a_3 + a_4 \geq 2a_2$ [$a_3 \geq a_2$]

And $a_5 + a_6 + a_7 + a_8 \geq 4a_2$ [$a_5, a_6, a_7 \geq a_8$]

In general, $a_{2+1} + a_{2+3} + a_{2+3} + \ldots. + a_2 \geq 2^n.a_2 = \frac{1}{2}[2^{n+1}.a_2 \quad]$

Sum LHS & RHS,

$$\Rightarrow \sum_{k=1}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=1}^{n} 2^{k+1} a_{2^{k+1}} = \sum_{k=2}^{n} 2^k a_{2^1}$$

$\therefore$ By Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

$\sum_{k=1}^{n} 2^k a_{2^1}$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Hence the theorem.

Theorem: 16

If $\{a_n\}$ is a non-increasing seq of +ve numbers.

If $\sum_{n=1}^{\infty} a_n$ cges, then $\lim_{n\to\infty} n.a_n = 0$.

Proof:

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ be the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$

Let $\sum_{n=1}^{\infty} a_n$ =A ($\sum_{n=1}^{\infty} a_n$ cges to A.)

Then $\lim_{n\to\infty} s_n$ =A.= $\lim_{n\to\infty} s_{2n}$. [Seq & subseq cges to same limit]

$\Rightarrow \lim_{n\to\infty} (s_{2n} - s_n) = 0.$

$\Rightarrow$ $s_{2n} - s_n = a_{n+1} + a_{n+2} + \ldots + a_{2n} \geq a_{2n} + a_{2n} + \ldots \geq 0$. [$\{a_n\}$ is non-increasing}

$\Rightarrow$ $0 \leq \lim_{n\to\infty} n.a_{2n} \leq \lim_{n\to\infty} (s_{2n} - s_n) = 0.$

$$\Rightarrow \ 0 \le \lim_{n\to\infty} 2n.a_{2n} \le 0$$

$$\Rightarrow \ \lim_{n\to\infty} 2n.a_{2n} = 0 \qquad \dots (4.9)$$

But $a_{2n+1} \le a_{2n}$, then $(2n+1)a_{2n+1} \le \left(\frac{2n+1}{2n}\right)2n.a_{2n}$.

$$\text{By}(1) \Rightarrow \lim_{n\to\infty} (2n+1).a_{2n+1} = 0 \qquad \dots (4.10)$$

From (1) & (2), we get, $\sum_{n=1}^{\infty} n.a_n = 0$. For all n.

PROBLEMS BASED ON TEST FOR CONVERGECNE OF THE SERIES

Formula for limits theorem.

Type-I Comparison Test

If $\sum_{n=1}^{\infty}|b_n|$ cges & $\lim_{n\to\infty}\frac{|a_n|}{|b_n|}$ exsist, Then $\sum_{n=1}^{\infty}|a_n|$ cges.

Type-II Ratio Test

[If Factorial is Given Use Ratio Test]

If $a = \lim_{n\to\infty}\inf\left|\frac{a_{n+1}}{a_n}\right|$ & $A = \lim_{n\to\infty}\sup\left|\frac{a_{n+1}}{a_n}\right|$ so $a < A$.

A<1, then $\sum_{n=1}^{\infty}|a_n|$ (cges)

If A >1. Then $\sum_{n=1}^{\infty}a_n$ diverges.

If A=1, then the test fails.

Type-III Cauchy Root Test

[If Power is N Use Root Test]

Let A = $\lim\limits_{n\to\infty} \sup|a_n|^{\frac{1}{n}}$

If A < 1, then $\sum\limits_{n=1}^{\infty}|a_n|$ cges.

If A > 1, then $\sum\limits_{n=1}^{\infty}|a_n|$ dvges.

If A = 1, the test fails.

Type-IV Cauchy condensation Test

If $\{a_n\}$ is a non-increasing seq of +ve numbers & if $\sum\limits_{k=1}^{\infty} 2^n a_{2^n}$. cges.

Then $\sum\limits_{n=1}^{\infty} a_n$ cges.

TYPE-V Rabee's Test

If $\lim\limits_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]$ = L

Then (a) $\sum\limits_{n=1}^{\infty} a_n$ cges if L > 1.

(b) $\sum\limits_{n=1}^{\infty} a_n$ dges if L < 1.

(c) Test fails if L = 1.

Standard limits formulas.

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

2. $\lim\limits_{n\to 0}(1+n)^{\frac{1}{n}}$ =e or $\lim\limits_{n\to 0}\left(1+\dfrac{1}{n}\right)^{n} = e$

3. $\lim\limits_{n\to\infty}\left(1+\dfrac{a}{n}\right)^{\frac{1}{n}} = e^{a}$.

4. $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

5. $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = n.a^{n-1}$.

6. $\lim\limits_{x\to 0}\dfrac{a^x - b^x}{x}$ =log (a/b).

7. $\lim\limits_{n\to 0} n^n$ =1.

8. $\lim\limits_{n\to\infty} x^n$ =0 if $x < 1$,

   $=\infty$ if $x \geq 1$.

TYPE I COMPARISON TEST

P1. Examine the convergence of the series.

(i) $\sum\limits_{n=0}^{\infty}\dfrac{n}{n^2+n+6}$ (ii) $\sum\limits_{n=1}^{\infty}\dfrac{1}{1+n^2}$ (iii) P.T $\sum\limits_{n=1}^{\infty}\dfrac{1}{n(n+1)}$ cges.

(iv) $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n(1+n^2)}}$ (v) $\sum\limits_{n=1}^{\infty}\dfrac{1+n}{1+n^2}$ (Div) (vi) $\sum\limits_{n=1}^{\infty}\dfrac{2n}{n^2-4n+7}$ (div)

Solution:

(i) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n}{n^2+n+6}$,

Here $a_n = \frac{n}{n^2+n+6} = \frac{n}{n^2\left(1+\frac{1}{n}+\frac{6}{n^2}\right)} = \frac{1}{n\left(1+\frac{1}{n}+\frac{6}{n^2}\right)}$

Take $b_n = \frac{1}{n}$

$$\therefore \frac{a_n}{b_n} = \frac{1}{n\left(1+\frac{1}{n}+\frac{6}{n^2}\right)} \times \frac{n}{1}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}+\frac{6}{n^2}\right)} = \frac{1}{(1+0+0)} = 1 \text{ exists.}$$

$\therefore$ By Comparison Test, $\sum b_n = \sum \frac{1}{n}$ diverges.

$\Rightarrow \sum a_n = \sum_{n=0}^{\infty} \frac{n}{n^2+n+6}$ diverges.

P2. Test for convergence of the series

(i) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ...\infty$

(ii) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$ [Hint $a_n = \frac{2n-1}{n(n+1)(n+2)}$

(iii) $\frac{1}{2.3.4} + \frac{1}{4.5.6} + \frac{1}{6.7.8} + ...\infty$ [Hint $a_n = \frac{1}{2n(2n+1)(2n+2)}$]

(iv) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+.....\infty$ [Hint:$a_n=\frac{1}{n(n+3)}$

(v) $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+.....\infty\ (x>0)$

(vi) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+.....+\frac{1}{\sqrt{n+1}}+....$[Ans:Div]

(vii) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^2}{4^4}+$.....Hint : $a_n=\frac{n^n}{(n+1)^{n+1}}$ [On omitting first term] Take $b_n=\frac{1}{n}$ Ans:-div

Solution:

(1). Given series. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+.....\infty$

$$=\sum_{n=0}^{\infty}\frac{1}{n(n+1)(n+2)}$$

[Hint:- Use $t_n = a + (n-1)d$ [a = First Term , d = Common Difference]

1, 2, 3, … = 1 + (n – 1)1 = 1+n – 1 = n

2, 3, 4, …. $t_n = 2 + (n-1)1 = 2 + n - 1 = n + 1$

3, 4, 5, …. $t_n = 3 + (n-1)1 = 3 + n - 1 = n + 2$]

Let $\sum_{n=1}^{\infty}a_n=\sum_{n=0}^{\infty}\frac{1}{n(n+1)(n+2)}$

Here $a_n = \dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

Take $b_n = \dfrac{1}{n^3}$

$$\therefore \frac{a_n}{b_n} = \frac{1}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \times \frac{n^3}{1}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{1}{(1+0)(1+0)} = 1 \text{ exists.}$$

$\therefore$ By Comparison Test, $\sum b_n = \sum \dfrac{1}{n^3}$ converges

Hence the series $\sum a_n = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)}$ also converges.

P3. Discus whether the series cges or div?

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ (ii) $\displaystyle\sum_{n=0}^{\infty}\left(\sqrt{n^2+1}-n\right)$ (Ans:-div)

(iii) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^4+n}-n^2\right)$ (iv) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

(v) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+n^2}{1+n^3}\right)$ (vi) $1 - - \dfrac{1}{2!} + \dfrac{1}{4!} - - \dfrac{1}{6!} + \ldots..$

(vii) Show that the series $\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{n^2+1}-n\right)$ is conditionally convergence.

Solution:

(i) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

here

a_n

$$=\left(\sqrt{n+1}-\sqrt{n}\right)=\left(\sqrt{n+1}-\sqrt{n}\right)\text{x}\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}=\frac{(n+1-n)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$=\frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$a_n=\frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$ Take $b_n=\dfrac{1}{\sqrt{n}}$

$$\therefore \frac{a_n}{b_n}=\frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}\ \frac{\sqrt{n}}{1}=\frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}=\frac{1}{(1+1)}=\frac{1}{2}\text{ exists.}$$

$\therefore$ By Comparison Test, $\sum b_n=\sum\dfrac{1}{\sqrt{n}}$ diverges.

$\therefore$ Where $p = ½ < 1$.

Hence the series $\sum a_n = \sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ also diverges.

Solution:

(vii).Let $\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n\left(\sqrt{n^2+1}-n\right)$ be an alternating series.

here $a_n = \left(\sqrt{n^2+1}-n\right) = \left(\sqrt{n^2+1}-n\right) \text{x} \frac{\left(\sqrt{n^2+1}+n\right)}{\left(\sqrt{n^2+1}+n\right)} = \frac{\left(n^2+1-n^2\right)}{\left(\sqrt{n^2+1}+n\right)}$

$$= \frac{1}{\left(\sqrt{n^2+1}+n\right)}$$

$a_n = \frac{1}{\left(\sqrt{n^2+1}+n\right)}$ and $a_{n+1} = \frac{1}{\left(\sqrt{(n+1)^2+1}+(n+1)\right)}$

Clearly $a_{n+1} > a_n$ for all n.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}+n} = 0.$

$\therefore$ By Leibnitz test, the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.

Let us consider

$$\sum_{n=0}^{\infty}\left|(-1)^n a_n\right| = \sum_{n=0}^{\infty}\left|\frac{(-1)^n}{\left(\sqrt{n^2+1}+n\right)}\right| = \sum_{n=0}^{\infty}\frac{1}{\left(\sqrt{n^2+1}+n\right)} = \sum_{n=0}^{\infty}u_n$$

Take $v_n = \dfrac{1}{n}$

$$\therefore \frac{a_n}{b_n} = \frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}\frac{n}{1} = \frac{1}{\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{n^2}}+1\right)} = \frac{1}{(1+1)} = \frac{1}{2} \text{ exists.}$$

$\therefore$ By Comparison Test, $\sum b_n = \sum \dfrac{1}{n}$ diverges.

$\Rightarrow$ the series $\sum u_n = \sum_{n=0}^{\infty}\left|(-1)^n\left(\sqrt{n^2+1}-n\right)\right|$ is diverges.

Hence the given series converges conditionally.

P4. Test for convergence of the series $\sum_{n=0}^{\infty} n^{-1-\frac{1}{n}}$**.**

Solution:

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} n^{-1-\frac{1}{n}}$, Here $a_n = n^{-1-\frac{1}{n}} = \dfrac{1}{n^{1+\frac{1}{n}}} = \dfrac{1}{n.n^{\frac{1}{n}}}$,

Take $b_n = \dfrac{1}{n}$

$$\therefore \frac{a_n}{b_n} = \frac{1}{n.n^{\frac{1}{n}}} \times \frac{n}{1}$$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}}}=1 \text{ exists.}$$

$\therefore$ By Comparison Test, $\sum b_n = \sum \frac{1}{n}$ diverges.

$$\Rightarrow \sum a_n = \sum_{n=0}^{\infty} n^{-1-\frac{1}{n}} \text{ diverges.}$$

P5. Test for convergent (i) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\left(\frac{1}{n}\right)$ ***(ii)*** $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$

(iii) $\sum_{n=1}^{\infty}\frac{\log(n)}{n}$

Solution:

(i): Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\left(\frac{1}{n}\right)$

Here $a_n = \frac{1}{\sqrt{n}}\sin\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\times\frac{1}{n} = \frac{1}{(n)^{\frac{3}{2}}}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

Take $b_n = \frac{1}{(n)^{\frac{3}{2}}}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\times\frac{(n)^{\frac{3}{2}}}{1}=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1 \text{exists.}$$

$\therefore$ By Comparison Test, $\sum b_n = \sum \frac{1}{n^{\frac{3}{2}}}$ converges.

Where $p = \frac{3}{2} < 1$.

Hence the series $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{1}{n}\right)$ also converges.

TYPE-II RATIO TEST

[IF FACTORIAL IS GIVEN USE RATIO TEST]

P1. Using Ratio test discuss the convergence of the series.

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $(x > 0)$ (ii) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ $(x > 0)$ [A14] (iii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(iv) $\sum_{n=1}^{\infty} \frac{n^4}{n!}$ (v) $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}.x^n$

(vi) $(3-e)(3-e^{1/2})(3-e^{1/3})(3-e^{1/4})\ldots\ldots(3-e^{1/n})$

Discuss the cges of the series $1+\frac{(\angle 1)^2}{\angle 2}x+\frac{(\angle 2)^2}{\angle 4}x^2+\frac{(\angle 3)^2}{\angle 6}x^3+\ldots\ldots$

Solution:

(i) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n}$,

Here $a_n = \frac{x^n}{n}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)} \times \frac{n}{x^n} = x.\frac{n}{n\left(1+\frac{1}{n}\right)}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x.\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = x.1 = x$$

$\therefore$ By Ratio Test,

If $x < 1$, Hence the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

If $x > 1$, Hence the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ diverges.

If $x = 1$, The test fails.

(ii) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, if $x > 0$.

Here $a_n = \frac{x^n}{n!}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = x.\frac{n!}{(n+1)n!} = x.\frac{1}{(n+1)}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x.\lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n}\right)} = x.0 = 0. = A < 1.$$

$\therefore$ By Ratio Test,

Hence the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges.

(iii) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$,

Here $a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{(n+1).n!}{(n+1)(n+1)^n} \times \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}$$

$$= \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ =A<1. (since $2 < e < 3$)

$\therefore$ By Ratio Test, Hence the series $\sum a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

(iv) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^4}{n!}$,   Here $a_n = \frac{n^4}{n!}$, $a_{n+1} = \frac{(n+1)^4}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{(n+1)!} \times \frac{n!}{n^4} = \frac{(n+1)^4}{(n+1).n!} \times \frac{n!}{n^4} = \frac{(n+1)^3}{n^4} = \frac{n^3\left(1+\frac{1}{n}\right)^3}{n^4}$$

$$= \frac{\left(1+\frac{1}{n}\right)^3}{n}$$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^3}{n}=\frac{1}{\infty}$$ = 0 = A < 1 [Check------]

$\therefore$ By Ratio Test, Hence the series $\sum a_n=\sum_{n=1}^{\infty}\frac{n^4}{n!}$ converges.

(v) Let $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}.x^n$,

Here $a_n = \sqrt{\frac{n}{n+1}}\,x^n$, $a_{n+1} = \sqrt{\frac{n+1}{n+2}}\,x^{n+1}$,

$$\frac{a_{n+1}}{a_n}=\sqrt{\frac{n+1}{n+2}}\quad x^{n+1}\quad \sqrt{\frac{n+1}{n}}\,\frac{1}{x^n}\,x^n \quad =\frac{(n+1)}{\sqrt{n}.\sqrt{n+2}}.x$$

$$=\frac{n\left(1+\frac{1}{n}\right)}{n.\sqrt{\left(1+\frac{2}{n}\right)}}.x=\frac{\left(1+\frac{1}{n}\right)}{\sqrt{\left(1+\frac{2}{n}\right)}}.x$$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\sqrt{\left(1+\frac{2}{n}\right)}}.x=\frac{1.x}{1}=x=\text{A}.$$

$\therefore$ By Ratio Test,

If $x < 1$, $\sum a_n$ converges

If $x > 1$, $\sum a_n$ diverges.

If $x = 0$, the test fails.

(vi) Given series $(3 - e)(3 - e^{1/2})(3 - e^{1/3)}(3 - e^{1/4})\ldots\ldots.(3 - e^{1/n})$

Here $a_n = (3 - e)(3 - e^{1/2})(3 - e^{1/3)}(3 - e^{1/4})\ldots\ldots.(3 - e^{1/n})$

$a_{n+1} = (3 - e)(3 - e^{1/2})(3 - e^{1/3)}(3 - e^{1/4})\ldots\ldots.\ (3 - e^{1/n}).(3 - e^{1/(n+1)})$

$$\frac{a_{n+1}}{a_n} = 3 - e^{\frac{1}{n+1}}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (3 - e^{\frac{1}{n+1}}) = (3 - 1) = 3 = A > 1.$$

$\therefore$ By Ratio Test,

the series $(3 - e)(3 - e^{\ 1/2})(3 - e^{1/3)}(3 - e^{1/4})\ldots\ldots.(3 - e^{1/n})$ diverges.

(vii) Hint. $a_n = \frac{(\angle n)^2}{\angle(2n)} x^n$, [on omitting the first term.]

$$\frac{a_{n+1}}{a_n} = \frac{(\angle(n+1))^2 x^{n+1}}{\angle(2n+2)} \qquad \frac{(\angle(2n))}{(\angle n)^2 x^n} = \frac{(n+1)^2(\angle 2n)}{(2n+2)(2n+1)(\angle 2n)} x \times$$

$$= \frac{x}{2} \frac{n+1}{(2n+1)} .\text{x} = \frac{x}{2} \frac{1+\frac{1}{n}}{(2+\frac{1}{n})}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{x}{2} \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)} = \frac{x}{4} = \text{A} < 1.$$

$\therefore$ By Ratio Test, Hence

(i) the series $\sum_{n=0}^{\infty} \frac{(\angle n)^2}{\angle(2n)} x^n$ converges if $\frac{x}{4} < 1$ i.e., $x < 4$

(ii) the series $\sum_{n=0}^{\infty} \frac{(\angle n)^2}{\angle(2n)} x^n$ diverges if $\frac{x}{4} > 1$ i.e., $x > 4$

(iii) the test fails if $x = 4$.

P2. For any x > 0. P.T, the series

(i) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + ...\infty$

(ii) $\frac{x}{1.2} + + \frac{x^2}{2.3} + \frac{x^3}{3.4} + ... + \infty$.

(iii) $1 + \frac{x}{2} + \frac{x^2}{5} + ... + \frac{x^n}{n^2+1} + ... \infty$ $(x > 0)$.

(iv) $1 + \frac{x}{3} + \frac{x^2}{5} + ... + \infty$.

Solution:

(i). Given series. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + ...\infty = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

[Hint:- Use $t_n = a + (n - 1)d$ [a = First Term, d = Common Difference]

1, 2, 3, … = 1 + $(n - 1)1 = 1 + n - 1 = n$

2, 3, 4, …. $t_n = 2 + (n - 1)1 = 2 + n - 1 = n + 1$]

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

Here $a_n = \dfrac{x^n}{n(n+1)}$ and $a_{n+1} = \dfrac{x^{n+1}}{(n+1)(n+2)}$

Take $b_n = \dfrac{1}{n^3}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)(n+2)} \times \frac{n(n+1)}{x^n} = \text{x.} \times \frac{n}{(n+2)}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \text{x.} \lim_{n\to\infty} \frac{n}{n\left(1+\frac{2}{n}\right)} = \text{x.} \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)} = \text{x..1=x.}$$

$\therefore$ By Ratio Test,

If $x < 1$, then the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n(n+1)}$ convergence.

If $x > 1$, then the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n(n+1)}$ divergence.

If $x = 1$, then the test fails.

P3. Prove that the Exponential series

(i) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n!}$ cges absolutely.

(ii) $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \infty$.converges absolutely for all values of x.

(iii) $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots\ldots + \infty$ cges.

(iv) For $x > 0$, P.T. the series $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\ldots+\infty$ is cges absolutely.

(v) For $x > 0$, P.T the series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\ldots+\infty$. is cges absolutely.

(vi) For what value of x does $1 + 2x + 3x^2+\ldots\ldots$ cges?

Solution:

(i) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$, Here $a_n = \frac{(-1)^{n+1}}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(\angle n+1)}\cdot\frac{\angle n}{(-1)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-1)}{(n+1)}\right| = \lim_{n\to\infty}\frac{1}{(n+1)} = 0 < 1.$$

$\Rightarrow \sum_{n=1}^{\infty}|a_n|$ Converges.=> $\sum_{n=1}^{\infty} a_n$ cges absolutely.

$\therefore$ By Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ converges absolutely.

(vi) Given series $1 + 2x + 3x^2+\ldots\ldots$ cges?

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(n+1)x^n$, Here $a_n = (n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)+1)x^{n+1}}{(n+1)x^n}.\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n\left(1+\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right| = |x|\lim_{n\to\infty}\left|\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\right| = |x.|$$

$\therefore$ By Ratio Test, the series $\sum_{n=1}^{\infty}(n+1)x^n$ converges if <1. & div if $|x.|>1$.

P4. (a)Does the ratio test gives any information about the series

$$\left(\frac{1}{2}\right)^0+\left(\frac{1}{4}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{2}\right)^4+.....$$

(b) Does the series converges?

Solution:

Let $\sum_{n=1}^{\infty}a_n = \left(\frac{1}{2}\right)^0+\left(\frac{1}{4}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{2}\right)^4+.....,$

Case(i) When n is odd:- here $a_n = \left(\frac{1}{4}\right)^n$ and $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{4}\right)^n}.\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(\frac{1}{2}\right)^n\frac{1}{2}}{\left(\frac{1}{2}\right)^n\cdot\left(\frac{1}{2}\right)^n}\right| = \lim_{n\to\infty}\frac{2^n}{2} = \lim_{n\to\infty}2^{n-1} = \infty > 1 \qquad \ldots (4.11)$$

Case(ii) When n is even: here $a_n = \left(\frac{1}{2}\right)^n$ and $a_{n+1} = \left(\frac{1}{4}\right)^{n+1}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{4}\right)^{n+1}}{\left(\frac{1}{2}\right)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(\frac{1}{4}\right)^n\frac{1}{4}}{\left(\frac{1}{2}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n\frac{1}{4}}{\left(\frac{1}{2}\right)^n}\right|$$

$$=\lim_{n\to\infty}\frac{1}{2^n}\frac{1}{4} = 0 < 1 \qquad \ldots (4.12)$$

From (i) & (ii)

We see that by Ratio Test, does not give any information

i.e., Ratio Test fails.

(b) Let $\sum_{n=1}^{\infty} a_n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots..,$

And Let $\sum_{n=1}^{\infty} b_n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots..,$

Clearly $a_n \le b_n, \ \forall\, n \in \mathrm{I}$

$\therefore \sum_{n=1}^{\infty} a_n << \sum_{n=1}^{\infty} b_n$

But $\sum_{n=1}^{\infty} b_n$ converges. [It is GP with d = 1/2 < 1]

Hence $\sum_{n=1}^{\infty} a_n$ converges absolutely.

P5. P.T the Binomial series

(i) $1 + nx + \frac{n.x}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$ ***cge absol for*** $|x.| < 1$.

(ii)Examine the test for cges for the series.

$x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.3}\frac{x^3}{5} + \frac{1.3.5}{2.4.6}\frac{x^5}{7} + ...$ cge absolutely for $|x.| < 1$.

(iii) $\frac{2}{3} + \frac{2.3}{3.5} + \frac{2.3.4}{3.5.7} + ...$.

(iv) 1+ $\frac{1}{2}\frac{1}{3} + \frac{1.3}{2.4}\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + ...$

(v) Examine the cges of $\frac{3}{4}\frac{x}{5} + \frac{3.6}{4.7}\frac{x^2}{8} + \frac{3.6.9}{4.7.10}\frac{x^3}{11} + ...$.

Solution:

(i) Given series

$1 + nx + \frac{n.x}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$ for $|x.| < 1$.

Here $a_{r+1} = \dfrac{n(n-1)(n-2).....(n-(r-1))}{1.2.3.....r} x^r$.

and $a_r = \dfrac{n(n-1)(n-2).....(n-r)}{1.2.3.....(r-1)}$

$$\therefore \lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = \lim_{r\to\infty}\left|\frac{n-r+1}{r}.x\right| = |x|\lim_{r\to\infty}\frac{r}{r}\left(\frac{n+1}{r}-1\right)$$

$$= |x|\lim_{r\to\infty}\left(\frac{n+1}{r}-1\right) = |x| < 1 \text{ (Given)}$$

$\Rightarrow \sum_{n=1}^{\infty}|a_n|$ Converges $\Rightarrow$ $\sum_{n=1}^{\infty}a_n$ cges absolutely.

$\therefore$ By Ratio Test, the series $\sum_{r=0}^{\infty}\dfrac{n(n-1)....(n-r+1)}{r!}.x^r$ converges absolutely for $|x.| < 1$.

P6. Examine the convergence or divergence of

(i) $\sum_{n=1}^{\infty}\dfrac{3}{4+2^n}$ (ii) $\dfrac{1}{1+x}+\dfrac{1}{1+2x^2}+\dfrac{1}{1+3x^3}+.......$

(iii) $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+3^3}+.......$(iv) $\sum_{0}^{\infty}\dfrac{n^3+1}{2^n+1}$

Let $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\dfrac{3}{4+2^n}$, Here $a_n = \dfrac{3}{4+2^n}$, $a_{n+1} = \dfrac{3}{4+2^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{3}{4+2^{n+1}}.\frac{4+2^n}{3} = \frac{2^n\left(\frac{4}{2^n}+1\right)}{2^n\left(\frac{4}{2^n}+2\right)}. = \frac{\left(\frac{4}{2^n}+1\right)}{\left(\frac{4}{2^n}+2\right)}$$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(\frac{4}{2^n}+1\right)}{\left(\frac{4}{2^n}+2\right)} = \frac{(0+1)}{(0+2)} = \frac{1}{2} = \text{A} < 1.$$

$\therefore$ By Ratio Test, Hence the series $\sum_{n=1}^{\infty}\frac{3}{4+2^n}$ converges.

TYPE III. CAUCHY ROOT TEST

[IF POWER IS N USE ROOT TEST]

P1. Using Root test for the convergence of the following series.

(i)S.T $\sum_{n=1}^{\infty}\frac{x^n}{n^n}$ cges for all x in R.

(ii) Solve that if $|x|<1$,t hen $\sum_{n=1}^{\infty} n^{10000}x^n$ cges absolutely?

(iii) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ (iv) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ (v) $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$ (vi) $\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$

(vii) $\sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

(viii) $\sum_{n=1}^{\infty}\frac{x^n}{e^{\sqrt{n}}}$,$x > 0$.

(ix) Show that the series $\sum_{n=1}^{\infty}\frac{(nx)^n}{\angle n}$ cges if $x < \frac{1}{e}$ and div if $x > \frac{1}{e}e$.

(x) Test for cges of the series $\sum_{n=1}^{\infty} p^n n^p$, $p > 1$.

Solution:

(i) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^n}$, Here $a_n = \frac{x^n}{n^n}$, $|a_n|^{\frac{1}{n}} = \left(\frac{x^n}{n^n}\right)^{\frac{1}{n}} = \frac{x}{n}$

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{x}{n}\right| = 0 = A < 1.$

$\therefore$ By Cauchy Root Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges.

(ii)Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^{10000} x^n$, Here $a_n = n^{10000} x^n$,

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |n^{10000} x^n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|n^{\frac{1}{n}}\right|^{10000} |x| = 1.\ |x| = A < 1.$

$\therefore$ By Cauchy Root Test, the series $\sum_{n=1}^{\infty} n^{10000} x^n$ converges.

(iii) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n}$, Here $a_n = \frac{x^n}{n}$,

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{x^n}{n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{|n|^{\frac{1}{n}}} = 1.\ \frac{|x|}{1} = |x| = A < 1.$

By Cauchy Root Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

(iv) Given series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$, Here $a_n = \frac{n!}{n^n} = \frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}....\frac{n}{n}$

$$|a_n|^{\frac{1}{n}} = \left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}....\frac{n}{n}\right)^{\frac{1}{n}} = \frac{x}{n}$$

$$\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}....\frac{n}{n}\right)^{\frac{1}{n}} = \text{A (say)}$$

Then

$$\log(\text{L}) = \log\left(\lim_{n\to\infty}\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}....\frac{n}{n}\right)^{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\log\frac{1}{n}+\log\frac{2}{n}+\log\frac{3}{n}+.....+\log\frac{n}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)\right] = \lim_{n\to\infty}\frac{1}{n}\left[\sum_{r=1}^{n}f\left(\frac{r}{n}\right)\right] = \int_0^1 f(x)dx$$

[by summation integration formula]

$$= \int_0^1 \log x dx = \left[x(\log x - 1)\right]_{x=0} \boldsymbol{=-1—0= --1.}$$

Log A= – 1 $\Rightarrow$ A = $e^{-1} = \frac{1}{e}$.

By Cauchy Root Test, the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

(v) Let $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\left(\frac{1}{\log n}\right)^n$, Here $a_n = \left(\frac{1}{\log n}\right)^n$, $|a_n|^{\frac{1}{n}} = \left[\left(\frac{1}{\log n}\right)^n\right]^{\frac{1}{n}} = \frac{1}{\log n}$

$$\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\left(\frac{1}{\log n}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\left(\frac{1}{\log n}\right)\right| = \frac{1}{\infty} = 0 = A < 1.$$

$\therefore$ By Cauchy Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{1}{\log n}\right)^n$ converges.

(vi) Given series $\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$ Here $a_n = \frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$,

$$|a_n|^{\frac{1}{n}} = \left[\frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}\right]^{\frac{1}{n}} = \frac{\left(1+\frac{1}{n}\right)^{2}}{e} = \frac{1}{e}\left[1+\frac{1}{n^2}+\frac{2}{n}\right]$$

$$\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{e}\left[1+\frac{1}{n^2}+\frac{2}{n}\right] = \frac{1}{e}[1+0+0] = \frac{1}{e} = A < 1.$$

$\therefore$ By Cauchy Root Test, the series $\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$ converges.

(vii) Let $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$ Here $a_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$,

$$|a_n|^{\frac{1}{n}} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}\right]^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} = \text{A} \ <1.$$

$\therefore$ By Cauchy Root Test, the series $\sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$ converges.

(viii) Given series $\sum_{n=1}^{\infty}\frac{x^n}{e^{\sqrt{n}}}$, here $a_n = \frac{x^n}{e^{\sqrt{n}}}$,

$$|a_n|^{\frac{1}{n}} = \left(\frac{x^n}{e^{\sqrt{n}}}\right)^{\frac{1}{n}} = \frac{x}{e^{\frac{1}{\sqrt{n}}}}$$

$$\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{x}{e^{\frac{1}{\sqrt{n}}}} = \frac{x}{1} = x = \text{A}$$

$\therefore$ By Cauchy Root Test,

If $x < 1$, the series $\sum_{n=1}^{\infty}\frac{x^n}{e^{\sqrt{n}}}$ converges.

If $x > 1$, the series $\sum_{n=1}^{\infty}\frac{x^n}{e^{\sqrt{n}}}$ diverges.

If $x = 0$, the Test fails.

(x) Test for cges of the series $\sum_{n=1}^{\infty} p^n n^p$, $p > 1$.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p^n n^p$ here $a_n = p^n.n^p$

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |p^n n^p|^{\frac{1}{n}}$ =p. $\lim_{n\to\infty} \left|n^{\frac{1}{n}}\right|^p$ =p.1=p.

$\therefore$ By Cauchy Root Test,

the series $\sum_{n=1}^{\infty} p^n n^p$ converges if $p < 1$. & div if $p > 1$.

When p=1, $\sum_{n=1}^{\infty} n$ which is diverges.

TYPE-V CAUCHY CONDENSATION TEST

P1. For what value of p , the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ***converges & diverges.***

Solution:

Given series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, here $a_n = \frac{1}{n^p}$, $a_2 = \frac{1}{(2^n)^p}$

Clearly a_n is an non-incerasing seq of +ve terms.

$$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^{p-1}\right]^n$$

Which is converges if $\left(\frac{1}{2}\right)^{p-1}$ <1 i.e., if $(2)^{1-p} < 1$

i.e., if $\log(2)^{1-p} < \log(1)$ i.e., if $(1 - p)\log(2) < 0$.

i.e., if $(1-p) < 0$ [log2 = 0]

i.e., if $1 < p$ i.e., if $p > 1$.

The series diverges if $\left(\frac{1}{2}\right)^{p-1} \geq 1$

i.e., if $(2)^{1-p} \geq 1$

i.e., if $\log(2)^{1-p} \geq \log(1)$ i.e., if $(1-p).\log(2) \geq 0$.

i.e., if $(1-p) \geq 0$ i.e., if $1 \geq p$ i.e., $p \leq 1$.

$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty$ (cges) if $p > 1$.

And $\sum_{n=1}^{\infty} 2^n a_{2^n} = \infty$.

$\therefore$ By Cauchy condensation test,

Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ cges if $p > 1$. Div if $p \leq 1$.

P2. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ***cges.***

Solution:

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Here $a_1 = \frac{1}{1^2}$, $a_2 = \frac{1}{2^2}$, $a_3 = \frac{1}{3^2}$ …… $a_n = \frac{1}{n^2}$

Clearly $a_1 \geq a_2 \geq a_3 \geq \ldots.$ is a $\downarrow$ and +ve numbers.

$\therefore$ By Cauchy condensation test,

We have $\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{(2^n)} < \infty$ (cges)

[$\because x = 1/2,\ 0 < x < 1$]

$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty$, Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

P3. Show that $\sum_{n=1}^{\infty} \frac{1}{(n.\log n)}$ ***diverges.***

Solution:

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(n.\log n)}$ here

$a_n = \frac{1}{n \log n}$ & $a_{2^n} = \frac{1}{2^n \log 2^n} = \frac{1}{n.2^n \log 2}$

$\therefore$ By Cauchy condensation test, We have

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{n.2^n.\log 2} = \frac{1}{\log 2} \sum_{n=1}^{\infty} \frac{1}{n.} = \infty \text{ (diverges)}$$

$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} = \infty$ (diverges)

Hence the series $\sum_{n=1}^{\infty} \frac{1}{(n.\log n)}$ diverges.

P4. Solve that $\sum_{n=1}^{\infty} \frac{1}{(n.\log n)^2}$ ***converges.***

Solution:

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(n.\log n)^2}$ here

$a_n = \frac{1}{(n\log n)^2}$, $a_{2^n} = \frac{1}{(2^n \log 2^n)^2} = \frac{1}{.(2^n)^2 (n.\log 2)^2}$

$\therefore$ By Cauchy condensation test,

We have

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{.(2^n)^2.(n\log 2)^2}$$

$$= \left(\frac{1}{\log 2}\right)^2 \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{n^2.} < \infty \text{ \textbf{\textit{(converges)}}}$$

$$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty \text{ (converges)}$$

Hence the series $\sum_{n=1}^{\infty} \frac{1}{(n.\log n)^2}$ converges.

A16.For what value of x does the series $(1 - x) + (x - x^2) + (x^2 - x^3) + \ldots$cges?

5

CLASS l^2

1. Define class l^2

The class l^2 is the class of all sequences s $=\{s_n\}_{n=1}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ (i.e., dges)

Note. The elements of l^2 are sequences.

Ex1.The sequence 0, 0, 0….is clearly an elements of l^2.

Ex2.the seq $\left\{\frac{1}{n^n}\right\}_{n=1}$ is an element of l^2 since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ cges.

2. Define norm in class l^2.

If s $=\{s_n\}_{n=1}$ is an element of l^2 we define $\|s\|_2$ called the norm of s as,

$$\|s\|_2 = \left(\sum_{n=1}^{\infty} s_n{}^2\right)^{\frac{1}{2}}$$

Theorem: 1

State and Prove Schewarz Inequality.

If s= $\{s_n\}_{n=1}$ and t= $\{t_n\}_{n=1}$ are in l^2 ,then $\sum_{n=1}^{\infty} s_n t_n$ is cges absolutely and $\left|\sum_{n=1}^{\infty} s_n t_n\right| \le \left(\sum_{n=1}^{\infty} s_n{}^2\right)^{\frac{1}{2}} . \left(\sum_{n=1}^{\infty} t_n{}^2\right)^{\frac{1}{2}}$

Proof:

Let us assume that at least one s_n say $s_N \neq 0$, otherwise the theorem is trivial.

For fixed $n \geq$ N and any $x \in$ R, we have $\sum_{k=1}^{n} (xs_k + t_k)^2 \ge 0$.

$\Rightarrow \; x^2 \sum_{k=1}^{n} s_k{}^2 + 2x \sum_{k=1}^{n} s_k t_k + \sum_{k=1}^{n} t_k{}^2 \ge 0$

This is of the form, $Ax^2+Bx+C \ge 0$.

Where A= $\sum_{k=1}^{n} s_k{}^2 \ge 0$,B= $\sum_{k=1}^{n} s_k t_k$, C= $\sum_{k=1}^{n} t_k{}^2$

From the Calculus, the minimum value of $Ax^2 + Bx + C$ ($A \geq 0$) occur

when $x = -\frac{B}{2A}$

Setting $x = \frac{B}{2A}$, we get,

$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C \geq 0 \Rightarrow \frac{B^2}{4A} - \frac{B^2}{2A} + C \ge 0$

X4A, $\Rightarrow B^2 - 2B^2 + 4AC \geq 0$

$\Rightarrow - B^2 \geq - 4AC \Rightarrow B^2 \leq 4AC$

$$\Rightarrow \left(\sum_{k=1}^{n} s_k t_k\right)^2 \leq \left(\sum_{k=1}^{n} s_k{}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} t_k{}^2\right)^{\frac{1}{2}} \qquad \ldots (5.1)$$

Replacing s_k, t_k by $|s_k t_k|$ in (5.1)

we obtain, $\left|\sum_{n=1}^{\infty} s_n t_n\right| \leq \left(\sum_{n=1}^{\infty} s_n{}^2\right)^{\frac{1}{2}} . \left(\sum_{n=1}^{\infty} t_n{}^2\right)^{\frac{1}{2}}$

Thus, the seq of partial sums of $\sum_{k=1}^{\infty} |s_k t_k|$ is bounded.

Hence $\sum_{k=1}^{\infty} |s_k t_k| < \infty \Rightarrow \sum_{k=1}^{\infty} s_k t_k$ cges. Letting n to infinity in (2), We obtain (1).

Theorem: 2

State and Prove Minkowski Inequality.

If s= $\{s_n\}_{n=1}$ and t= $\{t_n\}_{n=1}$ are in l^2 ,then $s+t = \{s_n + t_n\}_{n=1}$ is in l^2 and $\left(\sum_{n=1}^{\infty} (s_n + t_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} s_n{}^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n{}^2\right)^{\frac{1}{2}}$

Proof:

Given s= $\{s_n\}_{n=1}$ and t= $\{t_n\}_{n=1}$ are in l^2

By def, $\sum_{n=1}^{\infty} s_n{}^2$ and $\sum_{n=1}^{\infty} t_n{}^2$ converges.

$\Rightarrow \sum_{n=1}^{\infty} s_n t_n$ converges (by Schewarz inequality)

$$\therefore \sum_{n=1}^{\infty}(s_n+t_n)^2=\sum_{n=1}^{\infty}s_n{}^2+2\sum_{n=1}^{\infty}s_nt_n+\sum_{n=1}^{\infty}t_n{}^2$$

$$\sum_{n=1}^{\infty}(s_n+t_n)^2\le\sum_{n=1}^{\infty}s_n{}^2+2\left(\sum_{n=1}^{\infty}s_n{}^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}t_n{}^2\right)^{\frac{1}{2}}+\sum_{n=1}^{\infty}t_n{}^2$$

By Schewarz inq

$$\sum_{n=1}^{\infty}(s_n+t_n)^2\le\left[\left(\sum_{n=1}^{\infty}s_n{}^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}t_n{}^2\right)^{\frac{1}{2}}\right]^2$$

$$\Rightarrow\left(\sum_{n=1}^{\infty}(s_n+t_n)^2\right)^{\frac{1}{2}}\le\left(\sum_{n=1}^{\infty}s_n{}^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}t_n{}^2\right)^{\frac{1}{2}}$$

[By taking square root on both sides]

5.1. LIMITS OF A FUNCTION ON REAL LINE.

What is difference between limit of a seq and limit of function?

Limit of Sequence in Real line R	Limit of a function in R	Limit of a function in Metric space
If Given $\in$>0, $\exists$ N $\in$ I, Solve that $\lvert s_n-L\rvert<\in$, $\forall\ n\ge$ N. *[OR]* $\lim\limits_{n\to\infty} s_n$ = L.	If Given $\in$>0, $\exists\ \delta$>0 Solve that $\lvert f(x)-L\rvert<\in$, $\forall\ 0<\lvert x-a\rvert<\delta$. $\lim\limits_{x\to a} f(x)$ = L.	If Given $\in$>0, $\exists\ \delta$>0 Solve that $\rho_2(f(x),L)<\in$, $\forall\ \rho_1(x,a)<\delta$. $\lim\limits_{x\to a} f(x)$ = L.

$S=\{s_n\}$	$f(x)$	
$\in$	$\in$	
N	δ	
N	x	
A	∞	

1. Define limit of a function on a real line

We say that f(x) approaches to L in R as x approaches a.

If Given $\in>0$, $\exists\ \delta>0$ Solve that $|f(x)-L|<\in, \forall\ 0<|x-a|<\delta$.

[OR]

$\lim\limits_{x \to a} f(x) = L$

2. Define right hand limit of a function

We say that $f(x) \to$ to L in R as $x \to$ a from right,

If Given $\in>0$, $\exists\ \delta>0$ s.t $|f(x)-L|<\in,\ \forall\ a<x<a+\delta$.

[OR]

RHL = $\lim\limits_{x \to a^+} f(x) = L$

Is called right hand limit of $f(x)$ at a.

3. Define left hand limit of a function

We say that $f(x) \to$ L in R as $x \to$ a from left,

If Given $\in>0$, $\exists\ \delta>0$ Solve that $|f(x)-L|<\in, \forall\ a-\delta<x<a$.

[OR]

LHL = $\lim\limits_{x \to a^+} f(x)$ =L.

Is called left hand limit of $f(x)$ at a.

In General: $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

4. Define Increasing function on J

If f is a real valued function in an interval J $\subset$R.

We say that

f is increasing on J, if $f(x) \leq f(y), x \leq y, \forall x, y \in$ J

f is strictly increasing on J, if $f(x) < f(y), x \leq y \ \forall x, y \in$ J

f is decreasing on J, f $f(x) \geq f(y), \forall x < y$.in J

f is strictly decreasing on J, if $f(x) > f(y), \forall x < y$ in J.

In our book –

Non-increasing means decreasing.

Non-decreasing means increasing.

Algebra of Limits

Theorem: 3

If $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$

Then (a) $\lim_{x \to a} [f(x) + g(x)] = L+M$.

(b) $\lim_{x \to a} [f(x) - g(x)] = L - M$.

(c) $\lim_{x \to a} [f(x) \cdot g(x)] = L.M$.

[P.T limit of the product is the product of limits] A13.

(d) $\lim_{x\to a} \frac{f(x).}{g(x)} = \frac{L.}{M}$ where M $\neq 0$

Proof:

Given $\lim\limits_{x\to a} f(x)$ =L. & $\lim\limits_{x\to a} g(x)$ =M.

By def, Given $\in$>0,

$\exists\ \delta_1$>0 Solve that $\left|f(x)-L\right|<\in, \forall\ 0<\left|x-a\right|<\delta_1$ … (5.2)

& Given $\in$>0, $\exists\ \delta_2$>0 Solve that

$\left|g(x)-M\right|<\in, \forall\ 0<\left|x-a\right|<\delta_2$ … (5.3)

Choose δ =min(δ_1, δ_2)

For $0<\left|x-a\right|<\delta$.

We have $\left|[f(x)+g(x)]-[L+M]\right|=\left|[f(x)-L]+[g(x)-M]\right|$

$\leq\left|f(x)-L\right|+\left|g(x)-M\right| <\in+\in=2\in=\in'$

$\Rightarrow\ \left|[f(x)+g(x)]-[L+M]\right|<\in'\ ,\ \forall\ 0<\left|x-a\right|<\delta$

$\Rightarrow\ \lim\limits_{x\to a}\left[f(x)+g(x)\right]$=L+M.

$\Rightarrow\ \lim\limits_{x\to a}\left[f(x)+g(x)\right]=\lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)$ .Hence the proof.

Similarly (b)

(c) Choose δ =min(δ_1, δ_2)

For $0<\left|x-a\right|<\delta$.

We have

$\left|[f(x).g(x)]-[L.M]\right|=\left|f(x)g(x)-g(x)L+g(x)L-LM\right|$

Add & Sub $g(x)$L.

$\leq\left|g(x)[f(x)-L]+L[g(x)-M]\right|$

$\leq |g(x)||f(x)-L|+|L||g(x)-M|$ … (5.4)

Since $\lim_{x\to a} g(x)$ =M

For, $\in$=1, $\exists\ \delta_3 > 0$ Solve that

$|g(x)-M| < \in = 1, \forall\ 0<|x-a|<\delta_3$ … (5.5)

Also $|g(x)| = |g(x)-M+M| \leq |g(x)-M|+|M|$ <1+$|M|$ by (4)s

$|g(x)| < 1+|M|$ … (5.6)

Choose δ =min($\delta_1, \delta_2, \delta_3$)

From (3),

$|[f(x).g(x)]-[L.M]|$

$\leq |g(x)||f(x)-L|+|L||g(x)-M| \leq [1+|M| \in +|L| \in$ by (4) & (5).

$\Rightarrow$ $|[f(x).g(x)]-[L.M]| \leq k \in = \in'$ where k = $1+|M|+|L|$

$\Rightarrow$ $\lim_{x\to a} f(x).g(x)$ =L.M.

$\Rightarrow$ $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$ Hence the proof.

(d) Division Rule. To P.T $\lim_{x\to a} \dfrac{f(x).}{g(x)} = \dfrac{L.}{M}$ where M$\neq$0

We first prove that: $\lim_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{M}$ where M$\neq$0

i.e., To Prove that

Given $\in$>0, $\exists\ \delta > 0$ s.t $\left|\dfrac{1}{g(x)} - \dfrac{1}{M}\right| < \in, \forall\ 0<|x-a|<\delta$.

Since $\lim\limits_{x \to a} g(x) =$M

For, $\in > 0$, $\exists\ \delta_3 > 0$ s.t $|g(x) - M| < \in, \forall\ 0 < |x-a| < \delta$ … (5.7)

$\therefore\ |\mathrm{M}| = |M - g(x) + \mathrm{g(x)}| \leq |g(x) - M| + |M|$

$\Rightarrow |M| < \in + |g(x)|$ by(1)

$\Rightarrow\ |g(x)| > |M| - \in\ ,\ \forall\ 0 < |x-a| < \delta$.

$$\Rightarrow \frac{1}{|g(x)|} < \frac{1}{M - \in}\ ,\ \forall\ 0 < |x-a| < \delta \qquad \ldots (5.8)$$

Given $\in' > 0$, $\exists\ \delta > 0$ s.t

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{g(x) - M}{g(x)\mathrm{M}}\right| \leq \frac{|g(x) - M|}{|g(x)||M|} < \frac{\in}{(M - \in)|M|} = \in' \text{ (Say)}$$

$$\Rightarrow\ \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \in', \forall\ 0 < |x-a| < \delta.$$

$$\Rightarrow\ \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M} \text{ where } \mathrm{M} \neq 0$$

$$\therefore \lim_{x \to a} \frac{f(x).}{g(x)} = \lim_{x \to a} f(x).\lim_{x \to a} \frac{1}{g(\mathrm{x})} = \mathrm{L}.\ \frac{1}{M}$$

$$\Rightarrow\ \lim_{x \to a} \frac{f(x).}{g(x)} == \frac{L}{M}.\ \text{ Hence Proved.}$$

Theorem: 4

(a)Let f be non-decreasing function on the bounded open interval (a, b), Then $\lim\limits_{x \to b^-} f(x)$ exists.

(b)If f is bounded below on (a, b), Then $\lim\limits_{x \to a^+} f(x)$ exists.

Proof:

(a) Given f is bounded & f is increasing on (a, b).

Let M =l.u.b $f(x)$, $x \in (a, b)$

By def of l.u.b,

Given $\in > 0$, M-$\in$ is not an upper bound of $f(x)$ in (a, b).

$\exists\ y \in (a, b)$, s.t $f(y) >$ M-$\in$

Let $y = b - \delta$ where $\delta > 0$.

$\Rightarrow$ M $- \in < f(b - \delta)$, $\forall$ b-$\delta < x <$b

$\Rightarrow$ M $- \in < f(b - \delta) < f(x)$, $\forall$ b-$\delta < x < b$

[$\because f$ is an increasing fun]

$\Rightarrow$ M $- \in < f(x)$, $\forall\ b - \delta < x < b$ … (5.9)

since M =l.u.b $f(x) \Rightarrow f(x) <$ M + $\in\ \forall\ b - \delta < x < b$ … (5.10)

From (1) & (2)

$\Rightarrow$ M $- \in < f(x) <$ M + $\in\ \forall\ b - \delta < x < b$

$\Rightarrow -\in < f(x) -$ M $< \in, \forall\ b - \delta < x < b$

$\Rightarrow |f(x) - M| < \in, \forall$ b-$\delta < x < b$

$\Rightarrow \lim\limits_{x \to b^-} f(x)$ =M. Hence (a) is proved.

For (b). if f is bounded below on (a, b)

And f is increasing on (a, b)

similarlly we can prove that $\lim\limits_{x \to a+} f(x) = m$ =g.l.b $f(x)$,

Let m = g.l.b $f(x)$, x in (a, b)

By def, g.l.b, Given $\in > 0$, M + $\in$ is not an upper bound of $f(x)$ in (a, b).

$\exists\ y \in (a, b)$,s.t $f(y) <$ M + $\in$

Let $y = b - \delta$ where $\delta > 0$.

$\Rightarrow f(a + \delta) <$ M + $\in$, $\forall\ a < x < a + \delta$

$\Rightarrow f(a) < f(x) < f(a + \delta) <$ M + $\in$,

$\forall\ a < x < a + \delta$ [$\because f$ is an increasing fun]

$\Rightarrow f(x) <$ M + $\in\ \forall\ a < x < a + \delta$ … (5.11)

since M =l.u.b f(x) $\Rightarrow f(x) >$ M – $\in$,

$\forall\ a < x < a + \delta$ … (5.12)

From (1) & (2)

$\Rightarrow$ M – $\in < f(x) <$ M + $\in$, $\forall\ a < x < a + \delta$

$\Rightarrow -\in < f(x) - \text{M} < \in$, $\forall\ a < x < a + \delta$

$\Rightarrow \left| f(x) - M \right| < \in, \forall\ a < x < a + \delta$

$\Rightarrow \lim\limits_{x \to a^+} f(x) = m$. Hence (b) is proved.

Theorem: 5

Let f be a non-increasing (dec) function on the bounded open interval (a, b)

If f is bounded above on (a, b), then $\lim\limits_{x \to b^-} f(x)$ exists.

If f is bounded below on (a, b), then $\lim\limits_{x \to a^+} f(x)$ exists.

Proof:

Take $f(x) = -g(x)$

Then $g(x)$ will be non-decreasing .

$\therefore$ By Previous theorem , applied to the function g.

We prove then to f.

Corollary:

If f is a monotonic function on the open interval (a, b)

If $c \in (a, b)$,then P.T $\lim\limits_{x \to c^+} f(x)$ and $\lim\limits_{x \to c^-} f(x)$ both exists.

Proof:

Case(i) f is increasing, choose $\delta > 0$.

s.t $(c - \delta, c + \delta) < (a, b)$

Since f is increasing, f is bounded above on $(c - \delta, c)$ by $f(c)$

By theorem, $\lim\limits_{x \to c^-} f(x)$ exists ... (5.13)

Since f is increasing, f is bounded below on $(c, c + \delta)$ by $f(c)$

By theorem, $\lim\limits_{x \to c^+} f(x)$ exists ... (5.14)

Case(ii) f is decreasing,

Since f decreasing, f is bounded below on $(c - \delta, c)$ by $f(c)$

$\therefore$ By the, $\lim\limits_{x \to c^-} f(x)$ exists ... (5.15)

Since f is decreasing, f is bounded above on $(c, c + \delta)$

$\therefore$ By the, $\lim\limits_{x \to c^+} f(x)$ exists.

PROBLEMS BASED ON LIMIT OF A FUNCTIONS

P1. Evaluate: $\lim\limits_{x\to 3}(x^2+2x)$ ***=15 using definition of limit.***

Solution:

Let $f(x) = x^2 + 2x$, L = 15, a = 3.

Given $\in$>0, To find δ>0.

Solve that $\left|(x^2+2x)-15\right| < \in$, $\forall\ 0<\left|x-3\right|<\delta$ … (5.16)

Take δ<1, $0<\left|x-3\right|<1 \Rightarrow -1 < x - 3 < 1 \Rightarrow 2 < x < 4 \Rightarrow x \in (2,4)$

$\Rightarrow x + 5 \in (7,9) \Rightarrow \left|x+5\right| < 9$

$\therefore \left|(x^2+2x)-15\right| = \left|(x+5)(x-3)\right| = \left|x+5\right|\left|x-3\right| < 9\delta$ if δ<1.

Choose δ=min (1, $\in$/9)

$\therefore \left|(x^2+2x)-15\right| < 9\delta < \in$, $\forall\ 0<\left|x-3\right|<\delta$.

Hence $\lim\limits_{x\to 3}(x^2+2x)$=15 is verified.

P2. Evaluate: $\lim\limits_{x\to 1}\sqrt{x+3}$ ***=2, using definition of limit.***

Solution:

Let f(x)= $\sqrt{x+3}$, L = 2, a = 1.

Given $\in$>0, To find δ>0.

s.t $\left|\sqrt{x+3}-2\right| < \in$, $\forall\ 0<\left|x-1\right|<\delta$.

i.e., $\dfrac{\left|\sqrt{x+3}-2\right|\left|\sqrt{x+3}+2\right|}{\left|\sqrt{x+3}+2\right|} < \in, \forall\ 0 < |x-1| < \delta$.

i.e., $\dfrac{|(x+3)-4|}{\left|\sqrt{x+3}+2\right|} < \in, \forall\ 0 < |x-1| < \delta$.

i.e., $\dfrac{|x-1|}{\left|\sqrt{x+3}+2\right|} < \in, \forall\ 0 < |x-1| < \delta$.

Take $\delta < 1$, $0 < |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$

$\Rightarrow\ x \in (0,2) \Rightarrow \sqrt{x+3}+2 > \sqrt{3}+2$

$\Rightarrow\ \dfrac{1}{\sqrt{x+3}+2} < \dfrac{1}{\sqrt{3}+2}$

$\Rightarrow\ \dfrac{|x-1|}{\sqrt{x+3}+2} < \dfrac{\delta}{\sqrt{3}+2}$

$\therefore\ |(x^2+2x)-15| = |(x+5)(x-3)| = |x+5||x-3| < 9\delta$ if $\delta < 1$.

Take $\delta = \in.(\sqrt{3}+2)$.

Choose $\delta = \min(1, \in.(\sqrt{3}+2))$

$\left|\sqrt{x+3}-2\right| < \in$, $\forall\ 0 < |x-1| < \delta$ is true.

$\lim\limits_{x\to 1}\sqrt{x+3} = 2.$

5.2. METRIC SPACE

1. Define metric space. Give an example?

[A13,14,15 N16

Let M be a non-empty set.

A function ρ : MxM-→[0, ∞] is called a metric space for m,

If the following conditions are satisfied.

M1. $\rho(x, y) = 0$ if $x = y$, $\forall x, y \in$ M.

M2. $\rho(x, y) > 0$ $\forall x, y \in$ M.

M3. $\rho(x, y) = \rho(y, x)$ [Symmetric axiom]

M4. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.[Triangular inequality]

Then the ordered pair < M, ρ > is called a metric space.

Give an Example of Metric space. A15 ,13.

1. The function ρ :R $\times$ R $\rightarrow$ [0, ∞] is defined by

$\rho(x, y) = |x - y|, \forall x, y \in$ R.

Then <R, ρ > is called absolute value metric space on the real line.

Proof:

To prove: < R, ρ > is a metric space.

For all $x, y \in$ R,

M1. $\rho(x, x) = |x - x| = 0$, $\forall x \in$ R.

M2. $\rho(x, y) = |x - y| > 0$, $\forall x, y \in$ R.

M3. $\rho(x, y) = |x - y| = |y - x| = \rho(y, x)$ [Symmetric is true]

M4. $\rho(x, y) = |x - y| \leq |x - z| + |z - y| \leq \rho(x, z) + \rho(z, y)$

[Triangular inequality]

Hence < R, ρ > is called a metric space.

Example2. Define discrete metric space R_d.

We define a function d:R × R → [0, ∞] by

$$d(x, y) = \begin{cases} 1 & if \ x \neq y \\ 0 & if \ x = y \end{cases}, \ \forall x, y \in R.$$

Then <R, d > =R_d is called the discrete metric space.

Proof:

To Prove: <R, d > is a metric space.

For all $x, y \in$ R,

M1. $d(x, x) = 0$ (given) is true.

M2. $d(x, y) = 1 > 0$ (given) is true.

M3.when $x = y$, $d(x, y) = 0 = d(y, x)$

when $x \neq y$, $d(x, y)$= 1= $d(y, x)$ [Symmetric is true]

M4. Case(i) when $x = y = z$.

Then $d(x, y) = 0$, $d(x, z) = 0$, $d(z, y) = 0$

Clearly $d(x, y) \leq d(x, z) + d(z, y)$ is true.

Case(ii) when $x = y \neq z$.

$d(x, y) = 0$,$d(x, z)$ =1, $d(z, y) = 1$,

Clearly $d(x, y) \leq d(x, z) + d(z, y)$ is true.

Case(iii) when $x \neq y \neq z$.

$d(x, y) = 1$, $d(x, z) = 1$, $d(z, y) = 1$, Clearly $d(x, y) \le d(x, z) + d(z, y)$ is true.

Hence <R, d > = R_d is a metric space.

Example3. Show that R^n is a Metric space.

[A14,15 N14

R^n = n-dimensional Euclidian Space

Let R^n=The set of all n-tuples of real numbers.

For $n \in I$, if $x = (x_1, x_2, \ldots . x_n)$ and $y = (y_1, y_2, \ldots . y_n)$ are two ordered n-tuples of real numbers.

We define $\rho(x, y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}}$

[For $n = 2$, $\rho(x, y)$ is distance formula in Cartesian plane]

(i) Clearly $\rho(x, y) = 0$ if $x = y$

(ii) $\rho(x, y) > 0$. since distance is always positive.

By def $\rho(x, y) = \rho(y, x)$ [Symmetric is true]

Triangular inequality.

Let $z = (z_1, z_2, \ldots . z_n)$

To Prove: $\rho(x, y) \le \rho(x, z) + \rho(z, y)$

Let $x_k - z_k = a_k$ and $y_k - z_k = b_k$ for $k = 1, 2, 3 \ldots n$.

Then $\rho(x, z) = \left[\sum_{k=1}^{n} (x_k - z_k)^2\right]^{\frac{1}{2}} = \left[\sum_{k=1}^{n} a_k{}^2\right]^{\frac{1}{2}}$

$$\rho(z, y) = \left[\sum_{k=1}^{n} (y_k - z_k)^2\right]^{\frac{1}{2}} = \left[\sum_{k=1}^{n} b_k{}^2\right]^{\frac{1}{2}}$$

$$\rho(x,y)=\left[\sum_{k=1}^{n}(x_k-y_k)^2\right]^{\frac{1}{2}}$$

$$=\left[\sum_{k=1}^{n}(a_k+b_k)^2\right]^{\frac{1}{2}}\le\left[\sum_{k=1}^{n}a_k^{\;2}\right]^{\frac{1}{2}}+\left[\sum_{k=1}^{n}b_k^{\;2}\right]^{\frac{1}{2}}$$

[Minkowski inequality]

$$=\left[\sum_{k=1}^{n}(x_k-z_k)^2\right]^{\frac{1}{2}}+\left[\sum_{k=1}^{n}(y_k-z_k)^2\right]^{\frac{1}{2}}$$

$\Rightarrow\ \rho(x,y)\le\rho(x,z)+\rho(z,y)$

$\Rightarrow\rho$ satisfies all conditions for metric space.

Hence R^n=n-tuples of Euclidean Space is a Metric space.

5.3. LIMITS IN METRIC SPACE

1. Define limit of a seq in Metric space

Let <M, ρ> , be a metric space. and Let $\{s_n\}$ be a seq of points M.

We say that s_n approaches L as n approaches infinity

If Given $\in$>0, $\exists$ N $\in$I, s.t ρ (s_n ,L)< $\in$, $\forall$ n $\ge$N.

[OR]

$\lim\limits_{n\to\infty} s_n$ =L.

2. Define convergence seq on a metric space

Let <M, ρ> , be a metric space and Let $\{s_n\}$ be a seq of points M.

We say that $\{s_n\}_{n=1}$ is said to be convergent to L, If the seq $\{s_n\}_{n=1}$ has a limit L.

[OR]

$\lim_{n\to\infty} s_n$ =L. exists finitely.

3. Define Cauchy seq in a Metric space

Let <M, ρ> , be a metric space. and Let $\{s_n\}$ be a seq of points M.

We say that $\{s_n\}$ is said to be Cauchy seq,

If Given $\in$>0, $\exists$N $\in$I, s.t ρ $(s_m, s_n)< \in, \forall$ $m, n \geq$N.

4. Define limit of function in a metric space.

Let <M_1, ρ_1> , <M_2, ρ_2> be two metric spaces.

If $f: M_1 \to M_2$.

We say that $f(x)$ approaches L (L in M_2) as x approaches a (a in M_1),

If given $\in > 0$, $\exists$ $\delta > 0$ s.t $\rho_2(f(x),L) < \in, \forall$ $0 < \rho_1(x,a) < \delta$.

[OR]

$\lim_{x\to a} f(x)$ =L.

[Cges seq $\Rightarrow$ Cauchy seq in metric space]

Theorem: 6

Let <M, ρ> be a metric space. If $\{s_n\}$ is a convergent seq of points in M, then Prove that $\{s_n\}$ is cauchy seq.

Proof:

Let $\{s_n\}$ be a convergence seq of points in M i.e., $\lim\limits_{x \to a} f(x) = L$ exists.

By def, By def,

Given $\in >0$, $\exists\ \delta_1 >0$ s.t $\rho(s_n, L) < \frac{\in}{2}, \forall\ n \geq .N$ … (5.17)

Hence if $m, n \geq N$

We have $\rho(s_m, s_n) \leq \rho(s_m, L) + \rho(L, s_n)$ [by Triangular Lamina]

$\rho(s_m, s_n) \leq \rho(s_m, L) + \rho(s_n, L)$ [By symmetric]

$\Rightarrow \rho(s_m, s_n) \leq \frac{\in}{2} + \frac{\in}{2} = \in\ \forall\ m, n \geq .N$

$\Rightarrow \{s_n\}$ is a cauchy seq of points in M.

Theorem: 7

If <M, ρ> be a metric space and let a be a point in M.

Let f and g be real –valued function whose domains are subsets of M.

If $\lim\limits_{x \to a} f(x)$ =L and $\lim\limits_{x \to a} g(x)$ =N

Then Prove that: (i) $\lim\limits_{x \to a} [f(x) + g(x)]$=L+N

(ii) $\lim\limits_{x \to a} [f(x) - g(x)]$=L--N

(iii) $\lim\limits_{x \to a} [f(x).g(x)]$=L.N

(iv) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{N}$ where $N \neq 0$.

Proof:

Given $\lim\limits_{x\to a} f(x)$ =L. & $\lim\limits_{x\to a} g(x)$ =M.

By def,

Given $\in$>0, $\exists\ \delta_1$>0 s.t$\left| f(x)-L\right| <\in, \forall\ 0<\rho(x,a)<\delta_1 \ldots$ (5.18)

& Given $\in$>0, $\exists\ \delta_2$>0 s.t $\left|g(x)-M\right| <\in, \forall\ 0<\rho(x,a)<\delta_2 \ldots$

(5.19)

(a)Choose $\delta = \min(\delta_1, \delta_2)$

For $0<\rho(x,a)<\delta$.

We have $\left|[f(x)+g(x)]-[L+M]\right| = \left|[f(x)-L]+[g(x)-M]\right|$

$\leq \left|f(x)-L\right| + \left|g(x)-M\right| < \in + \in = 2\in = \in'$

$\Rightarrow$ $\left|[f(x)+g(x)]-[L+M]\right| < \in'$, $\forall\ 0<\rho(x,a)<\delta$

$\Rightarrow$ $\lim\limits_{x\to a}\left[f(x)+g(x)\right]$=L+M.

$\Rightarrow$ $\lim\limits_{x\to a}\left[f(x)+g(x)\right]$=$\lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$ .Hence the proof.

Similarly (b)

(c) Choose δ =min(δ_1, δ_2)

For $0<\rho(x,a)<\delta$.

We have

$\left|[f(x).g(x)]-[L.M]\right| = \left|f(x)g(x)-g(x)L+g(x)L-LM\right|$

Add & Sub $g(x)$L.

$\leq \left|g(x)[f(x)-L]+L[g(x)-M]\right|$

$\leq |g(x)||f(x)-L| + |L||g(x)-M|$ … (5.20)

Since $\lim\limits_{x \to a} g(x) = \text{M}$

For, $\in = 1$, $\exists\ \delta_3 > 0$ s.t $|g(x)-M| < \in = 1$,

$\forall\ 0 < \rho(x,a) < \delta_3$ … (5.21)

Also $|g(x)| = |g(x)-M+M| \leq |g(x)-M| + |M| < 1 + |M|$ by (4)

$|g(x)| < 1 + |M|$ … (5.22)

Choose $\delta = \min(\delta_1, \delta_2, \delta_3)$

From (3),

$|[f(x).g(x)] - [L.M]|$

$\leq |g(x)||f(x)-L| + |L||g(x)-M| \leq [1+|M| \in + |L| \in$ by (4) & (5).

$\Rightarrow$ $|[f(x).g(x)] - [L.M]| \leq k \in = \in'$ where $k = 1 + |M| + |L|$

$\Rightarrow$ $\lim\limits_{x \to a} f(x).g(x) = \text{L.M.}$

$\Rightarrow$ $\lim\limits_{x \to a} f(x).g(x) = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} g(x)$ Hence the proof.

(d) Division Rule. To P.T $\lim\limits_{x \to a} \dfrac{f(x).}{g(x)} = \dfrac{L.}{M}$ where $\text{M} \neq 0$

We first prove that: $\lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{M}$ where $\text{M} \neq 0$

i.e., To P.T Given $\in > 0$, $\exists\ \delta > 0$

s.t $\left|\dfrac{1}{g(x)} - \dfrac{1}{M}\right| < \in, \forall\ 0 < \rho(x,a) < \delta$.

Since $\lim_{x \to a} g(x) = \text{M}$

For, $\in > 0$, $\exists\ \delta_3 > 0$ s.t $\left| g(x) - M \right| < \in$,

$\forall\ 0 < \rho(x,a) < \delta$ … (5.23)

$\therefore |\text{M}| = |M - g(x) + \text{g(x)}| \le |g(x) - M| + |M|$

$\Rightarrow |M| < \in + |g(x)|$ by(1)

$\Rightarrow |g(x)| > |M| - \in$, $\forall\ 0 < \rho(x,a) < \delta$.

$$\Rightarrow \frac{1}{|g(x)|} < \frac{1}{M - \in}\ ,\ \forall\ 0 < \rho(x,a) < \delta \qquad \text{… (5.24)}$$

Given $\in' > 0$, $\exists\ \delta > 0$

s.t $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{g(x) - M}{g(x)\text{M}} \right| \le \frac{|g(x) - M|}{|g(x)||M|} < \frac{\in}{(M - \in)|M|} = \in'$ (Say)

$$\Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \in', \forall\ 0 < \rho(x,a) < \delta.$$

$$\Rightarrow \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M} \text{ where M} \ne 0$$

$$\therefore \lim_{x \to a} \frac{f(x).}{g(x)} = \lim_{x \to a} f(x). \lim_{x \to a} \frac{1}{g(\text{x})} = \text{L}.\ \frac{1}{M}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x).}{g(x)} == \frac{L}{M}.\ \text{Hence Proved.}$$

5.4. CONTINUOUS FUNCTION ON A METRIC SPACE.

1. Define continuous function at a point on a real line

A real valued function f is continuous at a $\in$ R.

If Given $\in >0$, $\exists\ \delta>0$ s.t $\left|f(x)-f(a)\right|<\in, \forall\ 0<\rho(x,a)<\delta$

[OR]

$$\lim_{x\to a} f(x)=f(a)$$

2. Define continuous function on a metric space

Let $<M_1, \rho_1>$ and $<M_2, \rho_1>$ be two metric spaces.

Let $f: M_1 \to M_2$

We say that the function f is continuous at $a \in M$.

If Given $\in >0$, $\exists\ \delta>0$ s.t $\rho_2(f(x), f(a))<\in, \forall\ 0<\rho_1(x,a)<\delta$

[OR]

$\lim\limits_{x\to a} f(x)$ =f(a). in Metric space M.

Define Open ball in a real line

B[a, r] = $\{x\in R/\left|x-a\right|<r\}$ = the set of all x s.t open ball of radius r about a.

Define open ball in a metric space.

Let $<M, \rho>$ be a metric space. If $a\in M$ and $r>0$,

Then B[a, r] is defined to be the set of all points in M whose distance to a is less than r.

i.e., B[a, r] = $\{x\in M/\ \rho(x, a)<r\}$ is called a open ball of radius r about a.

Theorem: 8

If the real valued functions f and g are continuous at $a \in R$, then (i) $(f+g)$, $(f-g)$, (fg) and (f/g) $g\neq 0$ are also continuous at $a\in R$.

Proof:

Since f and g are continuous at $a \in$ R.

By def, $\lim\limits_{x \to a} f(x) = f(a)$ and $\lim\limits_{x \to a} g(x) = g(a)$.

$$\lim_{x \to a}[(f+g)(x)] = \lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) =$$

$f(a) + g(a) = (f + g)(a)$

$$\Rightarrow \lim_{x \to a}[(f+g)(x)] = (f+g)(a)$$

$\therefore$ $(f + g)$ is continuous at $a \in$ R.

Similarly (i)

$$\lim_{x \to a}[(f.g)(x)] = \lim_{x \to a}[f(x).g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) =$$

$f(a).g(a) = (f.g)(a)$

$$\Rightarrow \lim_{x \to a}[(f.g)(x)] = (f.g)(a)$$

$\therefore$ $(f.g)$ is continuous at $a \in$ R.

(iv) $$\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \lim_{x \to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

$$\Rightarrow \lim_{x \to a}\left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(a)$$

$\therefore (f/g)$ $g \neq 0$ is continuous at $a \in$ R.

Theorem: 9

If f is continuous at $a \in$ R. Then Prove that cf is continuous at $a \in$ R.

Proof:

Since f is continuous at $a \in$ R.

By def, $\lim\limits_{x\to a} f(x) = f(a)$

$$\lim_{x\to a}[(cf)(x)] = \lim_{x\to a}[cf(x)] = c.\lim_{x\to a} f(x) = c.f(a) = (cf)(a)$$

$$\Rightarrow \lim_{x\to a}[(cf)(x)] = (cf)(a)$$

$\therefore$ cf is continuous at $a \in$ R.

Theorem: 10

If f is continuous at $a \in$ R. Then Prove that $|f|$ is continuous at $a \in$ R.

Since f is continuous at $a \in$ R.

By def, If Given $\in >0$, $\exists\ \delta>0$

s.t $|f(x) - f(a)| < \in, \forall\ 0 < \rho(x,a) < \delta$

For $0 < \rho(x,a) < \delta$, We have

$$\big||f|(x) - |f|(a)\big| = \big||f(x)| - |f(a)|\big| < |f(x) - f(a)| < \in$$

$$\Rightarrow \big||f|(x) - |f|(a)\big| < \in,\ \forall\ 0 < \rho(x,a) < \delta$$

$\Rightarrow$ $|f|$ is continuous at $a \in$ R.

Theorem: 11

If f and g are real valued functions.

If f is continuous at a and g is continuous at $f(a)$, Then Prove that (gof) is continuous at $a \in$ R.

Proof:

Let $f(a) = b$. and $y = f(x)$.

Since g is continuous at b.

By def, Given $\in$>0, $\exists\ \eta$>0

s.t $|g(y) - g(a)| < \in, \forall\ 0 < |y - b| < \delta$ … (5.25)

Again, f is continuous at a.

By def, Given η>0, $\exists\ \delta$>0 such that

$\Rightarrow\ |f(x) - f(a)| < \eta, \forall\ 0 < |x - a| < \delta$

$\Rightarrow\ |f(x) - b| < \eta, \forall\ 0 < |x - a| < \delta$

$\Rightarrow\ |g(y) - g(b)| < \in, \forall\ 0 < |x - a| < \delta$

$\Rightarrow\ |g(f(x)) - g(f(a))| < \in, \forall\ 0 < |x - a| < \delta$ by (1)

$\Rightarrow\ |(gof)(x) - (gof)(a)| < \in, \forall\ 0 < |x - a| < \delta$

$\Rightarrow$ (gof) is continuous at a in R.

Theorem: 12

If f is continuous at a in R if Given $\in$>0, $\exists\ \delta$>0 s.t f^{-1}(B[$f(a), \in$]) $\supset$ B[a, δ]

[OR]

The real valued function f is continuous at $a\ \in$R if and only if the inverse image under f of any open ball B[$f(a)$, $\in$] contains an open ball

B[a, δ] about a.

Proof:

Let f be a continuous at $a \in \mathrm{R}$

Given $\in >0$, $\exists\ \delta >0$ s.t $|f(x) - f(a)| < \in, \forall\ 0 < \rho(x,a) < \delta$

$x \in \mathrm{B}[a,\ \delta\,] \Rightarrow f(x)\mathrm{B}[f(a),\ \in] \Rightarrow x \in f^{-1}(\mathrm{B}[f(a),\ \in]$

$f^{-1}(\mathrm{B}[f(a), \in]) \supset \mathrm{B}[a,\ \delta\,]$

Hence the proof.

Theorem: 13

Prove that f is conts at $a \in \mathrm{R}$ if $\lim\limits_{n\to\infty} x_n = a \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

[OR]

The real-valued function f is conts at $a \in \mathrm{R}$ if Whenever $\{x_n\}$ is a seq of real no/-s cges to a, then the seq $\{f(x_n)\}$ cges to $f(a)$

Proof:

Let us assume that f is conts at $a \in \mathrm{R}$ and $\{x_n\}$ be a seq cges to a.

Since f is conts at a.

By def, Given $\in >0$, $\exists\ \delta >0$, s.t $f^{-1}(\mathrm{B}[f(a), \in]) \supset \mathrm{B}[a,\ \delta\,]$

i.e., $f(\mathrm{B}[a,\ \delta\,]) \subset \mathrm{B}[f(a), \in]$ … (5.26)

also since $\{x_n\}$ cges to a.

By def, Given $\delta >0$, $\exists\, \mathrm{N} \in \mathrm{I}$, s.t $|x_n - a| < \delta$, $\forall\ n \geq \mathrm{N}$,

$\Rightarrow\ x_n \in \mathrm{B}[a,\ \delta\,]$, $\forall\ n \geq \mathrm{N}$,

$\Rightarrow\ f(x_n) \in f(\mathrm{B}[a,\ \delta\,] \subset \mathrm{B}[f(a), \in]$ by (1), $\forall\ n \geq \mathrm{N}$,

$\Rightarrow\ f(x_n) \in \mathrm{B}[f(a), \in], \forall\ n \geq \mathrm{N}$,

$\Rightarrow\ |f(x_n) - f(a)| < \in, \forall\ n \geq \mathrm{N}$,

$\Rightarrow$ $\{f(x_n)\}$ converges to $f(a)$

$\Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

Conversely.

We assume that if $\lim\limits_{n\to\infty} x_n = a$ and

$$\lim_{n\to\infty} f(x_n) = f(a) \qquad \dots (5.27)$$

Then To Prove that: f is continuous at $a \in$ R.

For Let us assume that the contrary.

We assume that for some $\in > 0$,

The inverse image under f of B = B[$f(a)$, $\in$] contains no open ball B[a, δ] at a.

In particular,

f^{-1}(B) does not contains B[$a, \frac{1}{n}$] for any n = 1, 2, 3, …

Then for such n

There is a point $x_n \in$ B[$a, \frac{1}{n}$], such that $f(x_n) \notin$ B[$f(a)$, $\in$].

$\Rightarrow |x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| > \in$

$\Rightarrow \lim\limits_{n\to\infty} x_n$ =a but $\lim\limits_{n\to\infty} f(x_n) \neq f(a)$

This is a contradiction to (2),

Hence f is continuous at $a \in$ R.

Hence the proof.

Easy proof of Theorem.4 using theorem5.

Theorem: 14

If f and g are real valued functions.

If f is continuous at a and g is continuous at $f(a)$, then Prove that (gof) is continuous at $a \in$ R.

Proof:

Let $\{x_n\}$ be a seq of real numbers

such that $\lim\limits_{n\to\infty} x_n = a \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

[by f is continuous at a]

$\Rightarrow \quad \underset{n\to\infty}{Lim}\, g\big(f(x_n)\big) = g(f(a))$

[By g is conts at $f(a)$]

$\Rightarrow \quad \underset{n\to\infty}{Lim}\big(gof\big)(x_n) =(\text{gof})(a)$

$\Rightarrow$ gof is continuous at a in M_1.

PROBLEMS ABED ON CONTINUOUS OF A FUNCTIONS

Hint. f is conts at a (i) $\underset{x\to a}{Lim}\, f(x)$ exists (ii) f(a) exists

(iii) $\underset{x\to a}{Lim}\, f(x) = f(a)$

If at least one is not true, then f is not conts at a.

P1. Check the continuity of the function?

(i) $f(x) = \dfrac{\sin x}{x}$, $x \in$ R, $x \neq 0$.

The function is not defined at $x = 0$. Hence f is not conts at $x = 0$.

(ii) $g(x) = \begin{cases} \dfrac{\sin x}{x} & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$

Clearly $\underset{x \to 0}{Lim} \dfrac{\sin x}{x} = 1 = g(0)$. Hence g is conts at $x = 0$.

(iii) $\chi(x) = 1$ if $x = 0$, $x \in$ R, x is rational

$= 0$ if $x \neq 0$. $x \in$ R, x is irrational

Here $\underset{x \to 0}{Lim}\, \chi(x)$ does not exists.

Hence $\chi(x)$ is not conts at $x = 0$.

A13. If $f(x) = \begin{cases} \dfrac{\sin x}{x} & if \;\; x \neq 0 \\ k & if \;\; x = 0 \end{cases}$ is conts, then find k?

$f(x) = x^2 + 2x$, $x \in$ R.

Clearly f is conts at $x = 3$.