MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: DYNAMICS

SUBJECT CODE: TAM5C

SEMESTER: V

PREPARED BY: PROF.S.AROCKIYA PRINCY

DYNAMICS

YEAR: III **SEMESTER: V**

Learning outcomes:
Students will acquire knowledge of

The motion of bodies under the influence of forces.
Rectilinear motion of particles, Projectiles, Impact and Moment of Inertia of Particles.

Unit -I

Basic units – velocity – acceleration- coplanar motion – rectilinear motion under constant forces – acceleration and retardation – thrust on a plane – motion along a vertical line under gravity – line of quickest descent - motion along an inclined plane – motion of connected particles.

Unit – 2

Work, Energy and power – work – conservative field of force – power – Rectilinear motion under varying Force simple harmonic motion (S.H.M.) – S.H.M. along a horizontal line- S.H.M. along a vertical line – motion under gravity in a resisting medium.

Unit – 3

Forces on a projectile- projectile projected on an inclined plane- Enveloping parabola or bounding parabola – impact – impulse force - impact of sphere - impact of two smooth spheres – impact of a smooth sphere on a plane – oblique impact of two smooth spheres

Unit – 4

Circular motion – Conical pendulum – motion of a cyclist on a circular path – circular motion on a vertical plane – relative rest in a revolving cone – simple pendulum – central orbits - general orbits - central orbits- conic as centered orbit.

Unit – 5

Moment of inertia. Two dimensional motion of a rigid body –equations of motion for two dimensional motion – theory of dimensions- definition of dimensions.

UNIT I
KINEMATICS

The branch of mechanics which deals with the motion of object is called dynamics. It is divided into two branches:

(i) ***Kinematics*** ***(ii)*** ***Kinetics***

Kinematics:

The branch of dynamics which deals with geometry of motion of a body without any reference of the force acting on the body is called kinematics.

KINETICS:

The branch of dynamics which deals with geometry of motion of a body with reference to the force causing motion is called kinetics.

POINTS TO BE REMEMBER

(i) The position of a particle can be specified by a vector $\vec{r}$ whose initial point is at the origin of some fixed coordinate system and the terminal point is at the particle. This vector is called **position vector**. If the particle is moving, the vector $\vec{r}$ changes with time. i.e. it is a function of time.

(ii) The curve traced by a moving particle is called the **trajectory** or the path of the particle.

(iii) The path of the particle can be specified by the vector equation

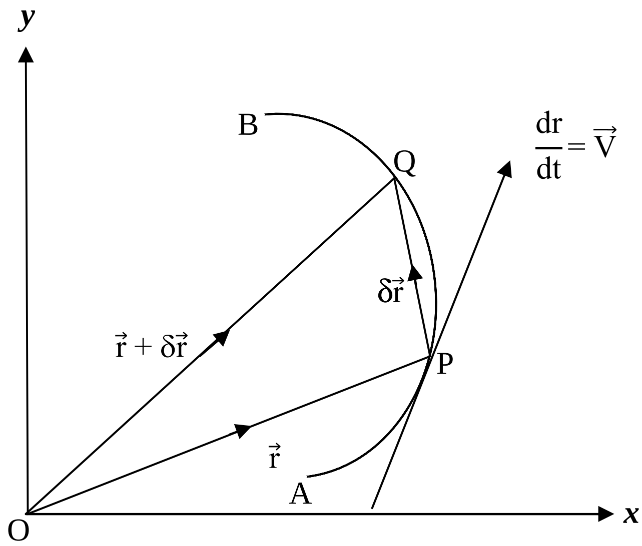
$$\vec{r} = \vec{r}\,(t) \qquad \text{\_\_\_\_\_\_\_\_}\ (i)$$

The path of the particle can also be specified by three scalar equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad \text{\_\_\_\_\_\_\_\_\_\_}\ (ii)$$

These equations are obtained by equating the components of vectors on two sides of the equation (i). Equation gives the coordinates of the points of the path for different value s of t. We call these as **parametric equations of the path.**

CARTESIAN COMPONENTS OF VELOCITY & ACCELERATION



Let AB be a part of the trajectory of the particle as shown in figure. Let the particle at time t be at the point P whose position vector is $\vec{r}$. After a small time δt, let the particle reach the point Q whose position vector is $\vec{r} + \delta\vec{r}$. The $\overrightarrow{PQ} = \delta\vec{r}$ is the displacement of the particle from the point P in the small time interval δt. The quotient

$$\frac{\delta\vec{r}}{\delta t}$$

gives the average rate of change of displacement of the particle in the interval δt. If we start decreasing the time interval δt, the displacement $\delta\vec{r}$ will go on deceasing and the point Q gets nearer and nearer to P. Thus

$$\lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}$$

can be considered as the instantaneous rate of change of displacement. This is defined as the instantaneous velocity or the simply velocity $\vec{v}$ of the particle at point P.

Thus,

$$\vec{v} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

Proceeding in similar way we can see that the acceleration $\vec{a}$ (the instantaneous rate of change of velocity) at time t is given by

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

In Cartesian coordinates, we can write

$$\vec{r} = x\hat{i} + y\hat{i}$$

Then

$$\vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{i}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{a} = \frac{d^2}{dt^2}\left(x\hat{i} + y\hat{i}\right) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

Thus

$$v_x = x\text{- component of velocity} = \frac{dx}{dt}$$

$$v_y = y\text{- component of velocity} = \frac{dy}{dt}$$

$$a_x = x\text{- component of acceleration} = \frac{d^2x}{dt^2}$$

$$a_y = y\text{- component of acceleration} = \frac{d^2y}{dt^2}$$

QUESTION 1

A particle is moving in such a way that it position at any time t is specified by

$$\vec{r} = \left(t^3 + t^2\right)\hat{i} + \left(\cos t + \sin^2 t\right)\hat{j} + \left(e^t + \log t\right)\hat{k}$$

Find the velocity and acceleration.

SOLUTION

If $\vec{v}$ and $\vec{a}$ are velocity and acceleration of particle respectively. Then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left((t^3 + t^2)\hat{i} + \left(\cos t + \sin^2 t\right)\hat{j} + (e^t + \log t)\hat{k}\right)$$

$$= (3t^2 + 2t)\hat{i} + (-\sin t + 2\sin t\cos t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

$$= (3t^2 + 2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

and $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left((3t^2 + 2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}\right)$$

$$= (6t + 2)\hat{i} + (2\cos 2t - \cos t)\hat{j} + \left(e^t - \frac{1}{t^2}\right)\hat{k}$$

QUESTION 2

A particle P start from O at t = 0. Find tits velocity and acceleration of particle at any time t if its position at that time is given by

$$\vec{r} = at^2\hat{i} + 4at\hat{j}$$

SOLUTION

If $\vec{v}$ and $\vec{a}$ are velocity and acceleration of particle respectively. Then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(at^2\hat{i} + 4at\hat{j}\right) = 2at\hat{i} + 4a\hat{j}$$

and $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(2at\hat{i} + 4a\hat{j}\right) = 2a\hat{i}$$

QUESTION 3

At any time t, the position of a particle moving in a plane can be specified by (acoswt, asinwt) where a and w are constants. Find the component of its velocity and acceleration along the coordinates axis.

SOLUTION

Let $\vec{r} = a\cos wt\,\hat{i} + a\sin wt\,\hat{j}$

Differentiate w.r.t "t", we get

$$\vec{v} = -aw\sin wt\,\hat{i} + aw\cos wt\,\hat{j}$$

Differentiate again w.r.t "t", we get

$$\vec{a} = -aw^2\cos wt\,\hat{i} - aw^2\sin wt\,\hat{j}$$

Thus the component of velocity and acceleration are

$$v_x = -aw\sin wt \quad , \quad v_y = aw\cos wt$$

$$a_x = -aw^2\cos wt \quad , \quad a_y = -aw^2\sin wt$$

QUESTION 4

The position of particle moving along an ellipse is given by $\vec{r} = a\cos t\,\hat{i} + b\sin t\,\hat{j}$ If a > b, find the position of the particle where velocity has maximum and minimum magnitude.

SOLUTION

As $\vec{r} = a\cos t\,\hat{i} + b\sin t\,\hat{j}$

Differentiate w.r.t "t", we get

$$\vec{v} = -a\sin t\,\hat{i} + b\cos t\,\hat{j}$$

$$\Rightarrow \quad v = \sqrt{(-a\sin t)^2 + (b\cos t)^2}$$

$$= \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$= \sqrt{a^2\sin^2 t + b^2\left(1 - \sin^2 t\right)}$$

$$= \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t}$$

$$= \sqrt{\sin^2 t\left(a^2 - b^2\right) + b^2}$$

v is maximum when $\sin^2 t$ is maximum. i.e. $\sin^2 t = 1 \Rightarrow \sin t = \pm 1 \Rightarrow t = 90, 270$

For t = 90

$$\vec{r} = a\cos 90\,\hat{i} + b\sin 90\,\hat{j} = b\hat{j}$$

For t = 270

$$\vec{r} = a\cos 270\,\hat{i} + b\sin 270\,\hat{j} = -\,b\hat{j}$$

So the position of the particle when velocity has maximum magnitude is $\pm\, b\hat{j}$.

v is minimum when $\sin^2 t$ is minimum. i.e. $\sin^2 t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, 180$

For t = 0

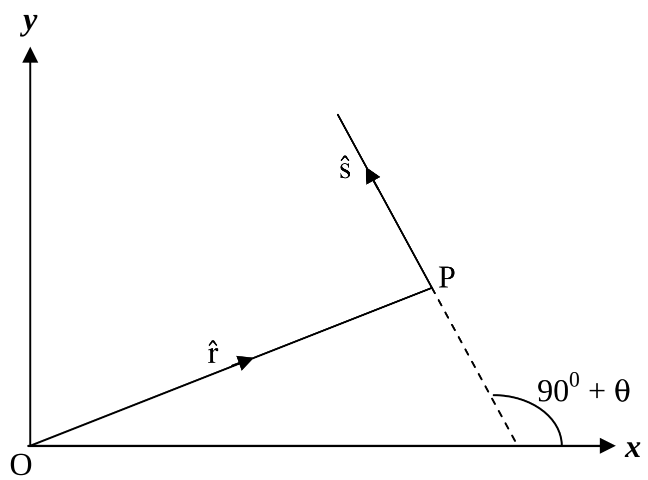
$$\vec{r} = a\cos 0\,\hat{i} + b\sin 0\,\hat{j} = a\hat{i}$$

For t = 180

$$\vec{r} = a\cos 180\,\hat{i} + b\sin 180\,\hat{j} = -\,a\hat{i}$$

So the position of the particle when velocity has minimum magnitude is $\pm\, a\hat{i}$.

RADIAL & TRANSVERSE COMPONENTS OF VELOCITY & ACCELERATION



In polar coordinates, the position of a particle is specified by a radius vector r and the polar angle θ which are related to x and y through the relations

$$x = r\cos\theta$$
$$y = r\sin\theta$$

provided the two coordinates frames have the same origin and the x–axis and the initial line coincide. The direction of radius vector is known as **radial direction** and that perpendicular to it in the direction of increasing θ is called **transverse direction.**

Let $\hat{r}$ and $\hat{s}$ be units vectors in the radial and transverse direction respectively as shown in figure. Then

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (i)}$$

$$\hat{s} = \cos(90^0 + \theta)\hat{i} + \sin(90^0 + \theta)\hat{j} = -\sin\theta\hat{i} + \cos\theta\hat{j} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (ii)}$$

Differentiating (i) w.r.t "t"

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$= \left(-\sin\theta\,\hat{i}\left(\frac{d\theta}{dt}\right) + \cos\theta\hat{j}\left(\frac{d\theta}{dt}\right)\right)$$

$$= \frac{d\theta}{dt}\left(-\sin\theta\,\hat{i} + \cos\theta\hat{j}\right)$$

$$= \frac{d\theta}{dt}\hat{s} \qquad \text{By (ii)} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (iii)}$$

Differentiating (ii) w.r.t "t"

$$\frac{d\hat{s}}{dt} = \frac{d}{dt}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$

$$= \left(-\cos\theta\,\hat{i}\left(\frac{d\theta}{dt}\right) - \sin\theta\hat{j}\left(\frac{d\theta}{dt}\right)\right)$$

$$= -\frac{d\theta}{dt}\left(\sin\theta\,\hat{i} + \cos\theta\hat{j}\right)$$

$$= -\frac{d\theta}{dt}\hat{r} \qquad \text{By (i)} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (iv)}$$

We know that

$$\hat{r} = \frac{\vec{r}}{r} \quad \Rightarrow \quad \vec{r} = r\,\hat{r}$$

Now $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\theta}{dt}\hat{s}$$

Thus,

v_r = Radial component of velocity = $\boldsymbol{\dfrac{dr}{dt} = \dot{r}}$

v_θ = Transverse component of velocity = $\boldsymbol{r\dfrac{d\theta}{dt} = r\dot{\theta}}$

Where dot denotes the differentiation with respect to time "t".

Let $\vec{a}$ be the acceleration Then

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}.\hat{r} + r\frac{d\theta}{dt}\hat{s}\right)$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}.\hat{r}\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\hat{s}\right)$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right)\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{s}\right) + \frac{d}{dt}\left(\frac{d\theta}{dt}\right) r\hat{s} + \frac{d\hat{s}}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\hat{s} + \frac{d\hat{s}}{dt}\left(\frac{d\theta}{dt}\right) r$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{s}\right) + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\,\hat{s} + \left(-\frac{d\theta}{dt}\hat{r}\right)\left(\frac{d\theta}{dt}\right) r \qquad \text{By (iii)\& (iv)}$$

$$= \frac{d^2r}{dt^2}\hat{r} - r\left(\frac{d\theta}{dt}\right)^2\hat{r} + 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\,\hat{s}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d^2\theta}{dt^2} r\right]\hat{s}$$

Thus,

$$a_r = \textit{Radial component of acceleration} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r(\dot{\theta})^2$$

$$a_\theta = \textit{Transverse component of acceleration} = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

QUESTION 5

A particle P moves in a plane in such away that at any time t, its distance from point O is $r = at + bt^2$ and the line connecting O and P makes an angle $\theta = ct^{3/2}$ with a fixed line OA. Find the radial and transverse components of velocity and acceleration of particle at t = 1

SOLUTION

Given that

$$r = at + bt^2 \quad \text{and} \quad \theta = ct^{3/2}$$

Differentiate w.r.t "t", we get

$$\frac{dr}{dt} = a + 2bt \quad \text{and} \quad \frac{d\theta}{dt} = \frac{3}{2}ct^{1/2}$$

Differentiate again w.r.t "t", we get

$$\frac{d^2r}{dt^2} = 2b \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \frac{3}{4}ct^{-1/2}$$

At t = 1

$r = a + b$ and $\theta = c$

$\frac{dr}{dt} = a + 2b$, $\frac{d\theta}{dt} = \frac{3}{2}c$, $\frac{d^2r}{dt^2} = 2b$ and $\frac{d^2\theta}{dt^2} = \frac{3}{4}c$

Radial component of velocity = $v_r = \frac{dr}{dt} = a + 2b$

Transverse component of velocity = $v_\theta = r\frac{d\theta}{dt} = \frac{3}{2}c(a+b)$

Radial component of acceleration = $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= 2b - (a+b)\left(\frac{3}{2}c\right)^2$$

$$= 2b - \frac{9}{4}c^2(a+b)$$

$$= \frac{1}{4}\left(8b - 9c^2(a+b)\right)$$

Transverse component of acceleration = $a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}$

$$= 2(a+2b)\left(\frac{3}{2}c\right) + (a+b)\left(\frac{3}{4}c\right)$$

$$= \frac{3}{4}c(5a+9b)$$

QUESTION 6

Find the radial and transverse components of velocity of a particle moving along the curve

$$\mathbf{ax^2 + by^2 = 1}$$

at any time t if the polar angle is $\theta = ct^2$

SOLUTION

Given that

$\theta = ct^2$

Differentiate w.r.t "t", we get

$\frac{d\theta}{dt} = 2ct$

Also given that

$ax^2 + by^2 = 1$

First we change this into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$

$$ar^2\cos^2\theta + br^2\sin^2\theta = 1$$

$$\Rightarrow \quad r^2(a\cos^2\theta + b\sin^2\theta) = 1$$

$$\Rightarrow \quad r\sqrt{a\cos^2\theta + b\sin^2\theta} = 1$$

$$\Rightarrow \quad r = (a\cos^2\theta + b\sin^2\theta)^{-\frac{1}{2}}$$

Differentiate w.r.t "t", we get

$$\frac{dr}{dt} = -\frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}\left(-a2\cos\theta\sin\theta\frac{d\theta}{dt} + b2\sin\theta\cos\theta\frac{d\theta}{dt}\right)$$

$$= \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a - b)\sin 2\theta\frac{d\theta}{dt}$$

$$= \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a - b)\sin 2\theta . 2ct$$

$$= \frac{ct(a - b)\sin 2\theta}{(a\cos^2\theta + b\sin^2\theta)^{\frac{3}{2}}}$$

Radial component of velocity $= \dfrac{dr}{dt} = \dfrac{ct(a - b)\sin 2\theta}{(a\cos^2\theta + b\sin^2\theta)^{\frac{3}{2}}}$

Transverse component of velocity $= r\dfrac{d\theta}{dt} = \dfrac{2ct}{(a\cos^2\theta + b\sin^2\theta)^{\frac{1}{2}}}$

QUESTION 7

Find the radial and transverse components of acceleration of a particle moving along the circle $\mathbf{x^2 + y^2 = a^2}$ with constant velocity c.

SOLUTION

Given that

$$\frac{d\theta}{dt} = c$$

Differentiate w.r.t "t", we get

$$\frac{d^2\theta}{dt^2} = 0$$

Also given that

$$x^2 + y^2 = a^2$$

First we change this into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = a^2$$

$$\Rightarrow \quad r^2 = a^2$$

$\Rightarrow \quad r = a$

$\Rightarrow \quad \frac{dr}{dt} = 0 \;\; \Rightarrow \;\; \frac{d^2r}{dt^2} = 0$

Radial component of acceleration = $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

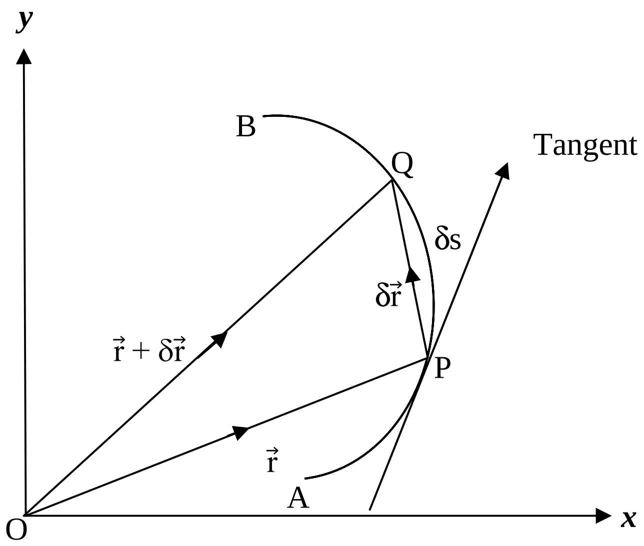
$= 0 - ac^2$

$= - ac^2$

Transverse component of acceleration = $a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}$

$= 0$

TANGENTIAL & NORMAL COMPONENTS OF VELOCITY & ACCELERATION



Let AB be a part of the trajectory of the particle as shown in figure. Let the particle at time t be at the point P whose position vector is $\vec{r}$. After a small time δt, let the particle reach the point Q whose position vector is $\vec{r} + \delta\vec{r}$. Then $\overrightarrow{PQ} = \delta\vec{r}$ and arcPQ = δs

Now $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} = v.\frac{d\vec{r}}{ds}$ ______________(i)

Here $\frac{d\vec{r}}{ds}$ is a unit tangent at point P.

Let $\hat{t}$ be a unit vector along the tangent at P and $\hat{n}$ unit vector along normal at the point P. Then

$$\frac{d\vec{r}}{ds} = \hat{t}$$

Using this in (i), we get

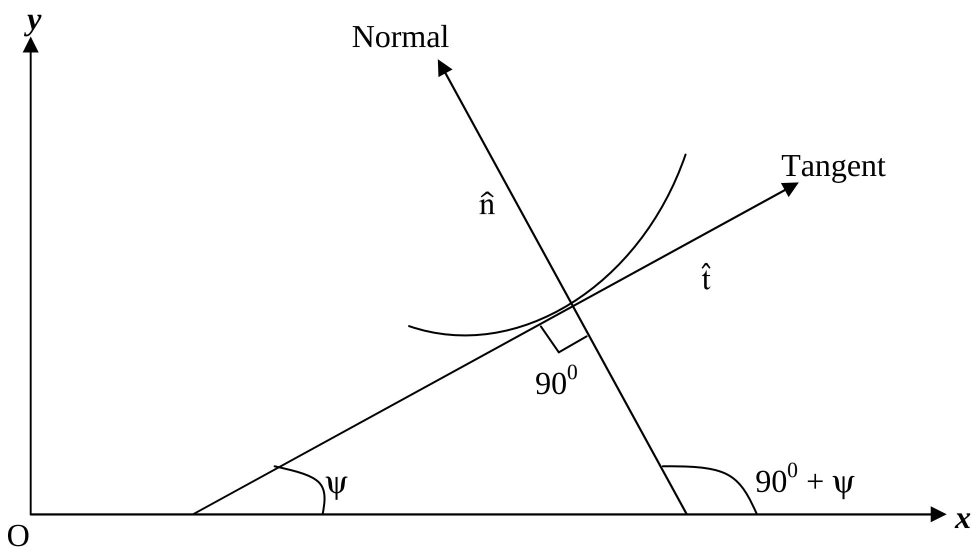
$\vec{v} = v\,\hat{t} + 0.\hat{n}$

Thus,

v_t = Tangential component of velocity = v

v_n = Normal component of velocity = 0

Hence the velocity is along the tangent to the path.



Let $\vec{a}$ be the acceleration. Then

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(v\,\hat{t})$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{dt}$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}\frac{d\psi}{ds}\frac{ds}{dt}$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}(Kv) \qquad \because \quad \frac{ds}{dt} = v$$

Where $\frac{d\psi}{ds} = K$ is called curvature and $K = \frac{1}{\rho}$

So $\quad \vec{a} = \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}.\frac{v}{\rho}$

$$= \frac{dv}{dt}\,\hat{t} + \frac{v^2}{\rho}\frac{d\hat{t}}{d\psi}$$

Since $\hat{t}$ and $\hat{n}$ are unit vectors along tangent and normal at P Therefore

$\hat{t} = \cos\psi\hat{i} + \sin\psi\hat{j}$

$\hat{n} = \cos(90^0 + \psi)\hat{i} + \sin(90^0 + \psi)\hat{j} = -\sin\psi\hat{i} + \cos\psi\hat{j}$

Now $\quad \frac{d\hat{t}}{d\psi} = \frac{d}{d\psi}\left(\cos\psi\hat{i} + \sin\psi\hat{j}\right)$

$$= \left(-\sin\psi\,\hat{i} + \cos\psi\,\hat{j}\right)$$

$$= \hat{n}$$

So $\quad \vec{a} = \frac{dv}{dt}\,\hat{t} + \frac{v^2}{\rho}\hat{n}$

Thus,

Tangential component of acceleration $= a_t = \frac{dv}{dt}$

Normal component of acceleration $= a_n = \frac{v^2}{\rho}$

Where

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

QUESTION 8

A particle is moving along the parabola $x^2 = 4ay$ with constant speed. Determine tangential and normal components of its acceleration when it reaches the point whose abscissa is $\sqrt{5}a$.

SOLUTION

Given that

$$x^2 = 4ay$$

Differentiate w.r.t "x", we get

$$2x = 4a\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{2a}$$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

Given that $x = \sqrt{5}a$ therefore

$$\frac{dy}{dx} = \frac{\sqrt{5}a}{2a} = \frac{\sqrt{5}}{2}$$

We know that

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1 + \left(\frac{\sqrt{5}}{2}\right)^2\right]^{3/2}}{\frac{1}{2a}} = 2a\left[1 + \frac{5}{4}\right]^{3/2} = 2a\left[\frac{9}{4}\right]^{3/2} = 2a\left[\frac{3}{2}\right]^3 = \frac{27a}{4}$$

Since the particle is moving with constant speed therefore

$$\frac{dv}{dt} = 0$$

Tangential component of acceleration = $a_t = \frac{dv}{dt} = 0$

Normal component of acceleration = $a_n = \frac{v^2}{\rho} = \frac{v^2}{\frac{27a}{4}} = \frac{4v^2}{27a}$

QUESTION 9

Find the tangential and normal component of acceleration of a point describing ellipse

$$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

With uniform speed v when the particle is at (0, b).

SOLUTION

Given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \quad x^2b^2 + y^2a^2 = a^2b^2$

Differentiate w.r.t "x", we get

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$= -\frac{b^2}{a^2}\left(\frac{y - x\left(-\frac{b^2x}{a^2y}\right)}{y^2}\right)$$

$$= -\frac{b^2}{a^2}\left(\frac{1}{y} + \frac{x^2b^2}{a^2y^3}\right)$$

At (0, b)

$$\frac{dy}{dx} = -\frac{b^2 0}{a^2b} = 0$$

and $\quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{1}{b} + \frac{0.b^2}{a^2b^3}\right) = -\frac{b}{a^2}$

We know that

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{[1 + (0)^2]^{3/2}}{\left|-\frac{b}{a^2}\right|} = \frac{a^2}{b}$$

Since the particle is moving with uniform speed therefore

$$\frac{dv}{dt} = 0$$

Thus, Tangential component of acceleration = $a_t = \frac{dv}{dt} = 0$

Normal component of acceleration = $a_n = \frac{v^2}{\rho} = \frac{v^2}{\frac{a^2}{b}} = \frac{bv^2}{a^2}$\

QUESTION 10

A particle is moving with uniform speed along the curve

$$\mathbf{x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)}$$

Show that acceleration has maximum value $\mathbf{\frac{10v^2}{9a}}$

SOLUTION

Given that

$$x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad y = a + \frac{a^3}{\sqrt{5}}x^{-2}$$

Differentiate w.r.t "x", we get

$$\frac{dy}{dx} = -\frac{2a^3}{\sqrt{5}}x^{-3}$$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = \frac{6a^3}{\sqrt{5}}x^{-4}$$

We know that

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{\left[1 + \left(-\frac{2a^3}{\sqrt{5}}x^{-3}\right)^2\right]^{3/2}}{\frac{6a^3}{\sqrt{5}}x^{-4}} = \frac{\left[1 + \frac{4a^6}{5x^6}\right]^{3/2}}{\frac{6a^3}{\sqrt{5}x^4}} = \frac{\sqrt{5}x^4}{6a^3}\left[\frac{5x^6 + 4a^6}{5x^6}\right]^{3/2} = \frac{[5x^6 + 4a^6]^{3/2}}{30a^3x^5} \quad \text{\_\_\_\_\_(i)}$$

We know that

$$\vec{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

Since the particle is moving with constant speed therefore

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \quad \vec{a} = \frac{v^2}{\rho}\hat{n}$$

$$\Rightarrow \quad |\vec{a}| = \frac{v^2}{\rho}|\hat{n}| = \frac{v^2}{\rho} \qquad \because |\hat{n}| = 1$$

$|\vec{a}|$ will maximum when ρ is minimum.

Differentiate (i) w.r.t "x", we get

$$\frac{d\rho}{dx} = \frac{30a^3x^5\left[\frac{3}{2}(5x^6 + 4a^6)^{1/2}30x^5\right] - [5x^6 + 4a^6]^{3/2}(150a^3x^4)}{(30a^3x^5)^2}$$

$$= \frac{(5x^6 + 4a^6)^{1/2}}{(30)^2a^6x^{10}}[30a^3x^5[45x^5] - [5x^6 + 4a^6](150a^3x^4)]$$

$$= \frac{(5x^6 + 4a^6)^{1/2}}{30a^3x^6}[45x^6 - 5(5x^6 + 4a^6)]$$

$$= \frac{(5x^6 + 4a^6)^{1/2}}{30a^3x^6}[45x^6 - 25x^6 - 20a^6]$$

$$= \frac{(5x^6 + 4a^6)^{1/2}}{30a^3x^6}[20x^6 - 20a^6]$$

$$= \frac{20(5x^6 + 4a^6)^{1/2}}{30a^3x^6}[x^6 - a^6]$$

$$= \frac{20(5x^6 + 4a^6)^{1/2}}{30a^3x^6}(x^2 - a^2)(x^4 + x^2a^2 + a^4)$$

Putting $\frac{d\rho}{dx} = 0$, we get

$$x = \pm a$$

Since

$$\frac{d\rho}{dx} < 0 \quad \text{before } x = a \qquad \text{and} \quad \frac{d\rho}{dx} > 0 \quad \text{after } x = a$$

Therefore ρ is minimum when $x = a$

Thus

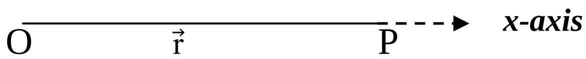
$$\rho_{min} = \frac{[5a^6 + 4a^6]^{3/2}}{30a^3a^5} = \frac{[9a^6]^{3/2}}{30a^8} = \frac{27a}{30} = \frac{9}{10}a$$

Maximum value of acceleration $= \dfrac{v^2}{\rho_{min}} = \dfrac{v^2}{\frac{9}{10}a} = \dfrac{10v^2}{9a}$

RECTILINEAR MOTION

INTRODUCTION

The motion of a particle along a straight line is called rectilinear motion. Let the particle start from O along a line. We take line along x-axis. Let after time ‘t’ particle be at a point P at a distance ‘x’ from O.



Let $\vec{r}$ be the position vector of the point P w.r.t origin O. Then

$$\vec{r} = \overline{OP} = x\,\hat{i}$$

Now $\quad \vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} \quad$ and $\quad \vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2x}{dt^2}\hat{i}$

Let $\quad |\vec{v}| = v \quad$ and $\quad |\vec{a}| = a$

Then $\quad v = \dfrac{dx}{dt} \quad$ and $\quad a = \dfrac{d^2x}{dt^2}$

Also $\quad a = \dfrac{dv}{dt} = \dfrac{dv}{dx}.\dfrac{dx}{dt}$

$$= \frac{dv}{dx}.v$$

$\Rightarrow \qquad a = v.\dfrac{dv}{dx}$

MOTION WITH CONSTANT ACCELERATION

Let the particle start from O with velocity u at time t = 0 with constant acceleration.. Let after time ‘t’ particle be at a point P at a distance ‘x’ from O. Then

$$a = \frac{dv}{dt} \quad \Rightarrow \quad adt = dv$$

On integrating we get

$$v = at + A \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of acceleration.

At t = 0, v = u

Using this in (i), we get

$$A = v$$

Using value of A in (i), we get

$$\mathbf{v = u + at} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(ii)}$$

Which is 1st equation of motion.

As we know that

$$v = \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = u + at \qquad \text{By (ii)}$$

$$\Rightarrow \quad dx = (u + at)dt$$

On integrating we get

$$x = ut + \frac{1}{2}at^2 + B \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(iii)}$$

At t = 0, x = 0

Using this in (ii), we get B = 0

Using value of B in (ii), we get

$$\mathbf{x = ut + \frac{1}{2}at^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(iv)}$$

Which is 2nd equation of motion.

As $\quad a = v.\frac{dv}{dx} \quad \Rightarrow \quad a \,.\, dx = v.\, dv$

On integrating, we get

$$ax + C = \frac{v^2}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(v)}$$

At t = 0, x = 0, v = u

Using these values in(v), we get

$$C = \frac{u^2}{2}$$

Using value of C in (v), we get

$$ax + \frac{u^2}{2} = \frac{v^2}{2} \quad \Rightarrow \quad 2ax + u^2 = v^2$$

$$\Rightarrow \quad \mathbf{2ax = v^2 - u^2}$$

Which is 3rd equation of motion.

If a particle is moving with constant retardation then a = − a

DISTANCE TRAVELLED IN NTH SECOND

Let x_1 and x_2 be the distances traveled in n and n – 1 seconds respectively. Then by 2nd equation of motion we have

$$x_1 = un + \frac{1}{2}an^2$$

and $$x_2 = u(n-1) + \frac{1}{2}a(n-1)^2$$

Distance traveled in nth second = $x_1 - x_2$

$$= un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$

$$= un + \frac{1}{2}an^2 - un + u - \frac{1}{2}a(n^2 - 2n + 1)$$

$$= \frac{1}{2}an^2 + u - \frac{1}{2}an^2 + \frac{1}{2}a(2n - 1)$$

$$= u + \frac{1}{2}a(2n - 1)$$

QUESTION 1

A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity 60 miles per hours in 4 seconds. Finds the acceleration of motion and distance travelled by the particle in the last three seconds.

SOLUTION

Given that

Initial velocity = u = 0

Time = t = 4sec

Final velocity = v = 60 miles/h

$$= \frac{60 \times 1760 \times 3}{3600} = 88 \text{ ft/sec}$$

We know that

$$v = u + at$$

$$\Rightarrow \quad a = \frac{v - u}{t} = \frac{88 - 0}{4} = 22 \text{ ft/sec}^2$$

Now

x_1 = Distance covered in 1st second

$$= ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2}(22)(1)^2 = 11\text{ft}$$

x_2 = Distance covered in 4 seconds

$$= ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2}(22)(4)^2 = 176\text{ft}$$

Distance covered in last 3 seconds = $x_2 - x_1$

$$= 176 - 11 = 165\text{ft}.$$

QUESTION 2

Find the distance travelled and velocity attained by a particle moving on a straight line at any timre t. If it starts from rest at t = 0 and subject to an acceleration $\mathbf{t^2 + sint + e^t}$

SOLUTION

Given that

$$a = t^2 + \sin t + e^t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = t^2 + \sin t + e^t$$

On integrating, we get

$$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t + A$$

Where A is constant of integration

When t = 0 then $\frac{dx}{dt} = 0$

$\Rightarrow$ A = 0

Hence velocity is:

$$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t$$

On integrating again, we get

$$x = \frac{t^4}{12} - \sin t + e^t + B$$

Where B is constant of integration

When t = 0 then x = 0

$\Rightarrow$ B = -1

Hence the distance travelled is given by

$$x = \frac{t^4}{12} - \sin t + e^t - 1$$

QUESTION 3

Discuss the motion of a particle moving in a straight line if it starts from rest at $t = 0$ and its acceleration is equal to **(i)** $\mathbf{t^n}$ **(ii)** **acost + bsint** **(iii)** $\mathbf{-n^2x}$

SOLUTION

(i)

Given that

$$a = t^n$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = t^n$$

On integrating, we get

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1} + A$$

Where A is constant of integration

When $t = 0$ then $\frac{dx}{dt} = 0$

$\Rightarrow \quad A = 0$

Hence velocity is:

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1}$$

On integrating again, we get

$$x = \frac{t^{n+2}}{(n+1)(n+2)} + B$$

Where B is constant of integration

When $t = 0$ then $x = 0$

$\Rightarrow \quad B = 0$

Hence the distance travelled is given by

$$x = \frac{t^{n+2}}{(n+1)(n+2)}$$

(ii)

Given that

$$a = a\cos t + b\sin t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = a\cos t + b\sin t$$

On integrating, we get

$$\frac{dx}{dt} = a\sin t - b\cos t + A$$

Where A is constant of integration

When $t = 0$ then $\frac{dx}{dt} = 0$

$\Rightarrow \quad A = b$

Hence velocity is

$$\frac{dx}{dt} = a\sin t - b\cos t + b$$

On integrating again, we get

$$x = -a\cos t - b\sin t + bt + B$$

Where B is constant of integration

When $t = 0$ then $x = 0$

$\Rightarrow \quad B = a$

Hence the distance travelled is given by

$$x = -a\cos t - b\sin t + bt + a$$

$$= a(1 - \cos t) + b(t - \sin t)$$

(iii)

Given that

$$a = -n^2x$$

$\Rightarrow \quad v\frac{dv}{dx} = -n^2x \qquad \because \; a = v\frac{dv}{dx}$

$\Rightarrow \quad v\,dv = -n^2x\,dx$

On integrating, we get

$$\frac{v^2}{2} = -n^2\frac{x^2}{2} + A$$

Where A is constant of integration.

$\Rightarrow \quad v^2 = 2A - n^2x^2$

$\Rightarrow \quad v^2 = B - n^2x^2$

$\Rightarrow \quad v = \sqrt{B - n^2x^2}$

Which is the velocity of the particle.

$\Rightarrow \quad \frac{dx}{dt} = \sqrt{B - n^2x^2} \qquad \because \; v = \frac{dx}{dt}$

$\Rightarrow \quad \frac{dx}{\sqrt{B - n^2x^2}} = dt$

On integrating again, we get

$$\frac{1}{n}\sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = t + B$$

Where B is constant of integration.

$$\frac{1}{n}\sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = t + B$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = nt + nB$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = nt + C$$

$$\Rightarrow \quad x = \frac{\sqrt{B}}{n}\sin(nt + C)$$

QUESTION 4

A particle moves in a straight line with an acceleration kv^3. If its initial velocity is u, then find the velocity and the time spend when the particle has travelled a distance x.

SOLUTION

Given that

$$a = kv^3$$

$$\Rightarrow \quad v\frac{dv}{dx} = kv^3 \qquad \because \ a = v\frac{dv}{dx}$$

$$\Rightarrow \quad v^{-2}dv = kdx$$

On integrating, we get

$$-v^{-1} = kx + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$$\Rightarrow \quad A = -u^{-1}$$

Using value of A in (i), we get

$$-v^{-1} = kx - u^{-1}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{u} - kx = \frac{1 - kxu}{u}$$

$$\Rightarrow \quad v = \frac{u}{1 - kux}$$

Which is the velocity of the particle.

$$\Rightarrow \quad \frac{dx}{dt} = \frac{u}{1 - kxu} \qquad \because v = \frac{dx}{dt}$$

$\Rightarrow \quad (1 - kxu)dx = udt$

On integrating again, we get

$$x - ku\frac{x^2}{2} = ut + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, v = u, x = 0 and t = 0

$\Rightarrow \quad B = 0$

Using value of B in (ii), we get

$$x - ku\frac{x^2}{2} = ut$$

$$\Rightarrow \quad ut = \frac{x}{2}(2 - kux)$$

$$\Rightarrow \quad t = \frac{x}{2u}(2 - kux)$$

Which is required time spend when the particle has travelled a distance x.

QUESTION 5

A particle moving in a straight line starts with a velocity u and has acceleration v^3, where v is the velocity of the particle at time t. Find the velocity and the time as functions of the distance travelled by the particle

SOLUTION

Given that

$$a = v^3$$

$$\Rightarrow \quad v\frac{dv}{dx} = v^3 \qquad \because \; a = v\frac{dv}{dx}$$

$$\Rightarrow \quad v^{-2}dv = dx$$

On integrating, we get

$$-v^{-1} = x + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$$\Rightarrow \quad A = -u^{-1}$$

Using value of A in (i), we get

$$-v^{-1} = x - u^{-1}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{u} - x = \frac{1 - xu}{u}$$

$$\Rightarrow \quad v = \frac{u}{1 - ux}$$

Which is the velocity of the particle.

$$\Rightarrow \quad \frac{dx}{dt} = \frac{u}{1 - xu} \qquad \because v = \frac{dx}{dt}$$

$$\Rightarrow \quad (1 - xu)dx = udt$$

On integrating again, we get

$$x - u\frac{x^2}{2} = ut + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, v = u, x = 0 and t = 0

$$\Rightarrow \quad B = 0$$

Using value of B in (ii), we get

$$x - u\frac{x^2}{2} = ut$$

$$\Rightarrow \quad ut = \frac{x}{2}(2 - ux)$$

$$\Rightarrow \quad t = \frac{x}{2u}(2 - ux)$$

QUESTION 6

A particle starts with a velocity u and moves in a straight line. If it suffers a retardation equal to the square of the velocity. Find the distance travelled by the particle in a time t.

SOLUTION

Given that

Retardation = v^2

$$\Rightarrow \quad a = - v^2$$

$$\Rightarrow \quad v\frac{dv}{dx} = -v^2 \qquad \because a = v\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dv}{v} = - dx$$

On integrating, we get

$$\ln v = - x + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$$\Rightarrow \quad A = \ln u$$

Using value of A in (i), we get

$$\ln v = -x + \ln u$$

$\Rightarrow \quad x = \ln u - \ln v$

$\Rightarrow \quad x = \ln\left(\frac{u}{v}\right)$

$\Rightarrow \quad e^x = \frac{u}{v}$

$\Rightarrow \quad v = \frac{u}{e^x}$

Which is the velocity of the particle.

$\Rightarrow \quad \frac{dx}{dt} = \frac{u}{e^x} \qquad \because v = \frac{dx}{dt}$

$\Rightarrow \quad e^x dx = u dt$

On integrating again, we get

$$e^x$$

$$= ut + B \qquad \text{\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, v = u, x = 0 and t = 0

$\Rightarrow \quad B = 1$

Using value of B in (ii), we get

$$e^x = ut + 1 \qquad \Rightarrow \qquad x = \ln(1 + ut)$$

QUESTION 7

Discuss the motion of a particle moving in a straight line with an acceleration x^3 where x is the distance of the particle from a fixed point O on the line, if it starts at t = 0 from a point x = c with a velocity $\mathbf{c^2/\sqrt{2}}$

SOLUTION

Given that

$$a = x^3$$

$\Rightarrow \quad v\frac{dv}{dx} = x^3 \qquad \because a = v\frac{dv}{dx}$

$\Rightarrow \quad v\,dv = x^3 dx$

On integrating, we get

$$\frac{v^2}{2} = \frac{x^4}{4} + A \qquad \text{\_\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially, t = 0, x = c and v = $c^2/\sqrt{2}$

$\Rightarrow \quad A = 0$

Using value of A in (i), we get

$$\frac{v^2}{2}=\frac{x^4}{4}$$

$$\Rightarrow \quad v^2=\frac{x^4}{2}$$

$$\Rightarrow \quad v=\frac{x^2}{\sqrt{2}}$$

Which is the velocity of the particle.

$$\Rightarrow \quad \frac{dx}{dt}=\frac{x^2}{\sqrt{2}} \qquad \because v=\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{x^2}=\frac{dt}{\sqrt{2}}$$

$$\Rightarrow \quad x^{-2}dx=\frac{dt}{\sqrt{2}}$$

On integrating again, we get

$$-x^{-1}=\frac{t}{\sqrt{2}}+B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, x = c and t = 0

$\Rightarrow \quad B = -c^{-1}$

Using value of B in (ii), we get

$$-x^{-1}=\frac{t}{\sqrt{2}}-c^{-1}$$

$$\Rightarrow \quad c^{-1}-x^{-1}=\frac{t}{\sqrt{2}} \qquad \Rightarrow \quad t=\sqrt{2}(c^{-1}-x^{-1}) \qquad \Rightarrow \quad t=\sqrt{2}\left(\frac{1}{c}-\frac{1}{x}\right)$$

QUESTION 8

Discuss the motion of a particle moving in a straight line if it starts from the rest at a distance a from the point O and moves with an acceleration equal to μ times its distance from O.

SOLUTION

Let x be the distance of particle from O then

$$a=\mu x$$

$\Rightarrow \quad v\dfrac{dv}{dx} = \mu x \qquad \because\ a = v\dfrac{dv}{dx}$

$\Rightarrow \quad vdv = \mu x dx$

On integrating, we get

$$\frac{v^2}{2} = \frac{\mu x^2}{2} + A \qquad \text{\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially, v = 0, x = a and t = 0

$\Rightarrow \quad A = -\dfrac{\mu a^2}{2}$

Using value of A in (i), we get

$$\frac{v^2}{2} = \frac{\mu x^2}{2} - \frac{\mu a^2}{2}$$

$\Rightarrow \quad v^2 = \mu x^2 - \mu a^2$

$\Rightarrow \quad v = \sqrt{\mu(x^2 - a^2)}$

Which is the velocity of the particle.

$\Rightarrow \quad \dfrac{dx}{dt} = \sqrt{\mu(x^2 - a^2)} \qquad \because\ v = \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dx}{\sqrt{x^2 - a^2}} = \sqrt{\mu}dt$

On integrating again, we get

$$\cosh^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, x = a and t = 0

$\Rightarrow \quad B = \cosh^{-1} 1 = 0$

Using value of B in (ii), we get

$$\cosh^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t$$

$\Rightarrow \quad x = a\cosh\left(\sqrt{\mu}t\right)$

QUESTION 9

The acceleration of a particle falling freely under the gravitational pull is equal to $k/_{x^2}$, where x is the distance of particle from the centre of the earth. Find the velocity of the particle if it is let fall from an altitude R, on striking the surface of the earth if the radius of earth is r and the air offers no resistance to motion.

SOLUTION

Given that

$$a = -\frac{k}{x^2}$$

Here we measuring distance x from centre O of the earth. The distance and acceleration is in opposite direction. So we take –ive sign. Therefore

$$v\frac{dv}{dx} = -\frac{k}{x^2} \qquad \because \; a = v\frac{dv}{dx}$$

$$\Rightarrow \quad vdv = -\frac{k}{x^2}dx$$

On integrating, we get

$$\frac{v^2}{2} = \frac{k}{x} + A \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

When x = R then v = 0

$$\Rightarrow \quad A = -\frac{k}{R}$$

Using value of A in (i), we get

$$\frac{v^2}{2} = \frac{k}{x} - \frac{k}{R}$$

$$\Rightarrow \quad v^2 = 2k\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$\Rightarrow \quad v = \sqrt{2k\left(\frac{1}{x} - \frac{1}{R}\right)}$$

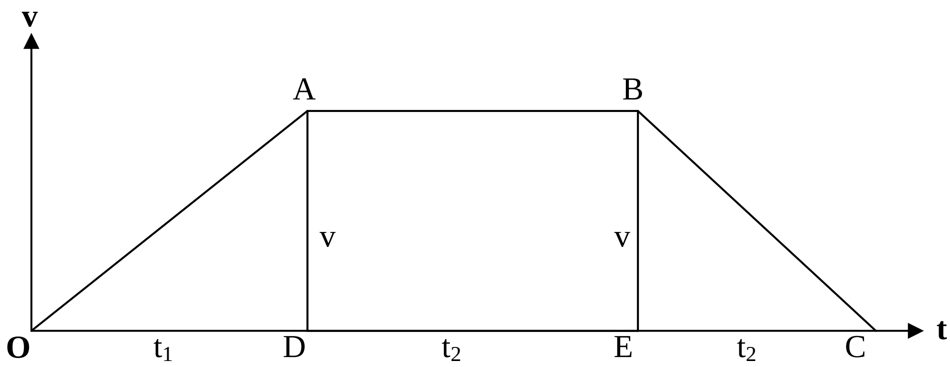
QUESTION 10

A particle starts from rest with a constant acceleration a. When its velocity acquires a certain value v, it moves uniformly and then its velocity starts decreasing with a constant retardation 2a till it comes to rest. Find the distance travelled by the particle, if the time taken from rest to rest is t.

SOLUTION

Let t_1, t_2 and t_3 be the times for acceleration, uniform motion and retardation motion respectively. Then

$$t = t_1 + t_2 + t_3 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$



Now

acceleration = slope of OA

$\Rightarrow \quad a = \dfrac{v}{t_1}$

$\Rightarrow \quad t_1 = \dfrac{v}{a}$

Similarly

retardation = slope of BC

$\Rightarrow \quad 2a = \dfrac{v}{t_3}$

$\Rightarrow \quad t_3 = \dfrac{v}{2a}$

From (i), we have

$$t_2 = t - t_1 - t_3$$

$$= t - \frac{v}{a} - \frac{v}{2a}$$

$$= t - \frac{3v}{2a}$$

Distance = Area under the velocity-time curve

= Area of trapezium OABC

$$= \frac{1}{2}(OC + AB)(AD)$$

$$= \frac{1}{2}(t_1 + t_2 + t_3 + t_2)v$$

$$= \frac{1}{2}(t + t_2)v$$

$$= \frac{1}{2}\left(t + t - \frac{3v}{2a}\right)v$$

$$= \frac{1}{2}v\left(2t - \frac{3v}{2a}\right)$$

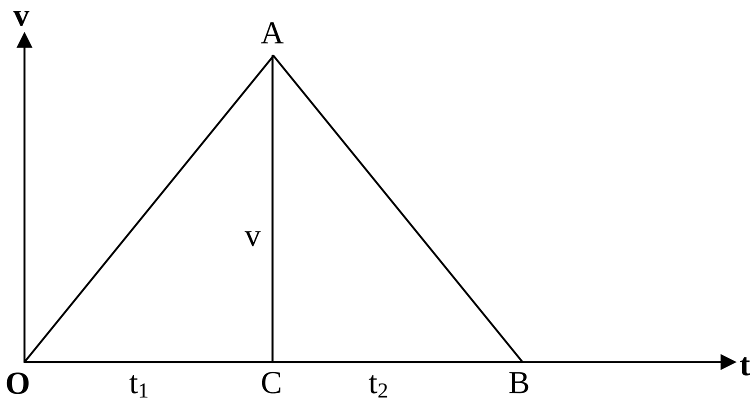
QUESTION 11

A particle moving along a straight line starts from rest and is accelerated uniformly until it attains a velocity v. The motion is then retarded and the particle comes to rest after traversing a total distance x. If acceleration is f, find the retardation and the total time taken by the particle from rest to rest.

SOLUTION

Let t_1 and t_2 be the times for acceleration and retardation respectively. Then

$$t = t_1 + t_2 \qquad \text{\_\_\_\_\_\_\_(i)}$$



Now

acceleration = slope of OA

$\Rightarrow \quad f = \dfrac{v}{t_1}$

$\Rightarrow \quad t_1 = \dfrac{v}{f}$

Let g be the retardation. Then

retardation = slope of BC

$\Rightarrow \quad g = \dfrac{v}{t_2}$

$\Rightarrow \quad t_2 = \dfrac{v}{g}$

Distance = Area under the velocity-time curve

$\Rightarrow \quad x = \text{Area of } \Delta ABC$

$$= \frac{1}{2}(OB)(AC)$$

$$= \frac{1}{2}(t_1 + t_2)v \qquad \text{\_\_\_\_\_\_(ii)}$$

$$= \frac{1}{2}tv$$

$$\Rightarrow \quad t = \frac{2x}{v}$$

Thus

$$\text{Total time} = \frac{2x}{v}$$

From (ii), we have

$$x = \frac{1}{2}(t_1 + t_2)v$$

$$= \frac{1}{2}\left(\frac{v}{f} + \frac{v}{g}\right)v$$

$$= \frac{v^2}{2}\left(\frac{1}{f} + \frac{1}{g}\right)$$

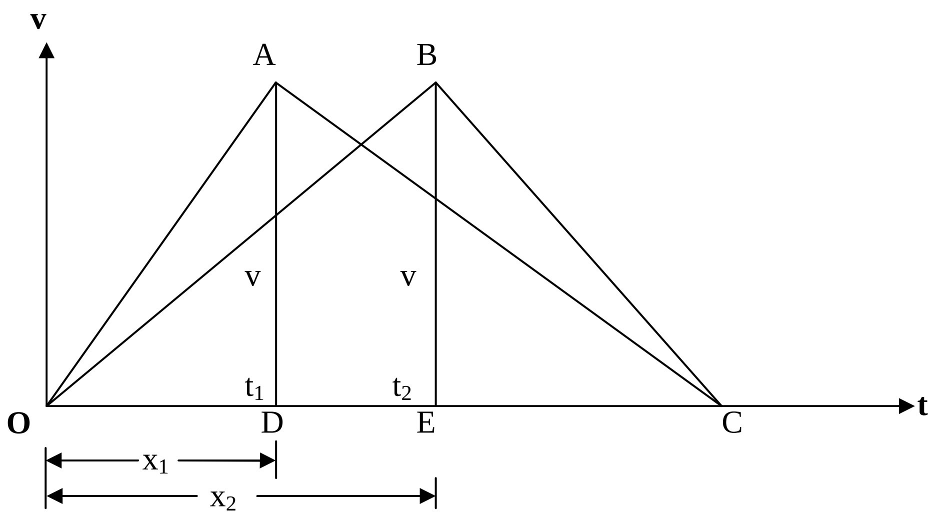
$$\Rightarrow \quad \frac{2x}{v^2} = \frac{1}{f} + \frac{1}{g} \quad \Rightarrow \quad \frac{1}{g} = \frac{2x}{v^2} - \frac{1}{f} \quad \Rightarrow \quad \frac{1}{g} = \frac{2xf - v^2}{fv^2}$$

$$\Rightarrow \quad g = \frac{fv^2}{2xf - v^2}$$

QUESTION 12

Two particles travel along a straight line. Both start at the same time and are accelerated uniformly at different rates. The motion is such that when a particle attains the maximum velocity v, its motion is retarded uniformly. Two particles come to rest simultaneously at a distance x from the starting point. If the acceleration of the first is a and that of second is $\frac{1}{2}a$. Find the distance between the point where the two particles attain their maximum velocities.

SOLUTION



Let both particle attain maximum velocity at t_1 and t_2 respectively. Then

For 1st Particle

Acceleration = slope of OA

$\Rightarrow \quad a = \dfrac{v}{t_1} \;\Rightarrow\; t_1 = \dfrac{v}{a}$

For 2nd Particle

Acceleration = slope of OB

$\Rightarrow \quad \dfrac{1}{2}a = \dfrac{v}{t_2} \;\Rightarrow\; t_2 = \dfrac{2v}{a}$

Let x_1 and x_2 be distances covered by the 1st and 2nd particles to attain velocity v. Then

x_1 = Area of ΔOAD

$$= \frac{1}{2}(OD)(AD)$$

$$= \frac{1}{2}vt_1 = \frac{1}{2}v\left(\frac{v}{a}\right) = \frac{v^2}{2a}$$

Similarly

x_2 = Area of ΔOBE

$$= \frac{1}{2}(OE)(BE)$$

$$= \frac{1}{2}vt_2 = \frac{1}{2}v\left(\frac{2v}{a}\right) = \frac{v^2}{a}$$

Required Distance = $x_2 - x_1$

$$= \frac{v^2}{a} - \frac{v^2}{2a} = \frac{v^2}{2a}$$

QUESTION 13

Two particles start simultaneously from point O and move in a straight line one with velocity of 45 mile/h and an acceleration $2ft/sec^2$ and other with a velocity of 90mile/h and a retardation of $8ft/sec^2$. Find the time after which the velocities of particles are same and the distance of O from the point where they meet again.

SOLUTION

For 1st Particle

Given that

u = 45 mile/h

$$= \frac{45 \times 1760 \times 30}{60 \times 60} = 66ft/sec$$

$a = 2ft/sec^2$

We know that

$$v = u + at$$

$$= 66 + 2t \qquad \text{\_\_\_\_\_(i)}$$

For 2nd Particle

Given that

$$u = 90 \text{ mile/h}$$

$$= \frac{90 \times 1760 \times 30}{60 \times 60} = 132\text{ft/sec}$$

$$a = -8\text{ft/sec}^2$$

We know that

$$v = u + at$$

$$= 132 - 8t \qquad \text{\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$66 + 2t = 132 - 8t$$

$$\Rightarrow \quad 10t = 66$$

$$\Rightarrow \quad t = 6.6\text{sec}$$

So after 6.6sec velocities of particles will same. Let both particle meet after a distance x.

Then

For 1st Particle

$$x = ut + \frac{1}{2}at^2$$

$$= 66t + \frac{1}{2}(2)t^2$$

$$= 66t + t^2$$

______(iii)

For 2nd Particle

$$x = ut + \frac{1}{2}at^2$$

$$= 132t + \frac{1}{2}(-8)t^2$$

$$= 132t - 4t^2 \qquad \text{\_\_\_\_\_\_(iv)}$$

From (iii) and (iv), we get

$$66t + t^2 = 132t - 4t^2$$

$$\Rightarrow \quad 5t^2 = 66t$$

$$\Rightarrow \quad t = 13.2$$

Putting value of t in (iii), we get

$$x = 10.4544\text{ft}$$

VERTICAL MOTION UNDER GRAVITY

For a falling body, the acceleration is constant. It is called acceleration due to gravity and is denoted by "g".

In FPS system value of g is 32ft/sec^2

In CGS system value of g is 981cm/sec^2

In MKS system value of g is 9.81m/sec^2

If the body is projected vertically upward then g = − g. For a falling body equations of motion are

$$v = u + gt$$

$$x = ut + \frac{1}{2}gt^2$$

$$2gx = v^2 - u^2$$

Note:

If $ax^2 + bx + c = 0$ be a quadratic equation and α, β be the roots of this equation. Then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

QUESTION 14

A particle is projected vertically upward at t = 0 with a velocity u, passes a point at a height h at $t = t_1$ and $t = t_2$. Show that

$$\mathbf{t_1 + t_2 = \frac{2u}{g} \text{ and } t_1 t_2 = \frac{2h}{g}}$$

SOLUTION

The distance travelled by the particle in time t is given by

$$x = ut - \frac{1}{2}gt^2$$

Put x = h

$$h = ut - \frac{1}{2}gt^2$$

$$\Rightarrow \quad 2h = 2ut - gt^2$$

$$\Rightarrow \quad gt^2 - 2ut + 2h = 0$$

The time t_1 and t_2 when the particle is at a height h from the point of projection, are roots of the quadratic equation

$$gt^2 - 2ut + 2h = 0$$

We know that

$$\text{Sum of the roots} = -\frac{\text{coefficient of t}}{\text{coefficient of t}^2}\ , \quad \text{Product of the roots} = \frac{\text{coefficient of t}^0}{\text{coefficient of t}^2}$$

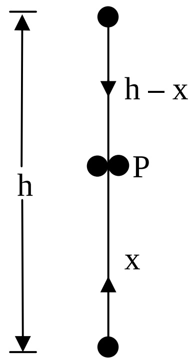
$$\Rightarrow \quad t_1 + t_2 = \frac{2u}{g} \quad \text{and} \quad t_1 t_2 = \frac{2h}{g}$$

QUESTION 15

A particle is projected vertically upward with a velocity $\sqrt{2gh}$ and another is let fall from a height h at the same time. Find the height of the point where they meet each other.

SOLUTION

Let both particles meet at point P at height x. Then



For 1st Particle

$$x = ut - \frac{1}{2}gt^2 \qquad \text{\_\_\_\_\_\_(i)}$$

Put $u = \sqrt{2gh}$

$$x = \sqrt{2gh}\, t - \frac{1}{2}gt^2$$

For 2nd Particle

$$x = ut + \frac{1}{2}gt^2$$

Put $u = 0$ and $x = h - x$

$$h - x = \frac{1}{2}gt^2$$

$$x = h - \frac{1}{2}gt^2 \qquad \text{\_\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$h - \frac{1}{2}gt^2 = \sqrt{2gh}\, t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad h = \sqrt{2gh}\, t \quad \Rightarrow \quad t = \frac{h}{\sqrt{2gh}}$$

Using value of t in (i), we get

$$x = \sqrt{2gh}\,\frac{h}{\sqrt{2gh}} - \frac{1}{2}g\left(\frac{h}{\sqrt{2gh}}\right)^2 = h - \frac{1}{2}g\left(\frac{h^2}{2gh}\right) = h - \frac{h}{4} = \frac{3h}{4}$$

QUESTION 16

A particle is projected vertically upwards. After a time t, another particle is sent up from the same point with the same velocity and meets the first at height h during the downward flight of the first. Find the velocity of the projection.

SOLUTION

Let u be the velocity of projection and v be the velocity at height h. Then

$v^2 - u^2 = -2gh$

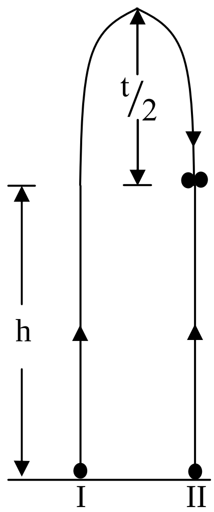
$\Rightarrow \quad v^2 = u^2 - 2gh$

$\Rightarrow \quad v = \sqrt{u^2 - 2gh}$ ________(i)

Since time taken by 1st particle from height h to the maximum point and back to height h is t therefore time taken from the height h to the heights point is t/2. Velocity at the highest point is zero and at the height h the velocity is v.

We know that

$v = u - gt$



Since the velocity at the highest point is zero and at the height h the velocity is v. therefore

Put $v = 0$, $u = v$ and $t = t/2$

$0 = v - \frac{gt}{2}$

$\Rightarrow \quad v = \frac{gt}{2}$ ________(ii)

From (i) and (ii), we get

$\frac{gt}{2} = \sqrt{u^2 - 2gh}$

$\Rightarrow \quad \frac{g^2t^2}{4} = u^2 - 2gh \quad \Rightarrow \quad g^2t^2 = 4u^2 - 8gh$

$\Rightarrow \quad 4u^2 = g^2t^2 + 8gh \quad \Rightarrow \quad 2u = \sqrt{g^2t^2 + 8gh} \quad \Rightarrow \quad u = \frac{\sqrt{g^2t^2 + 8gh}}{2}$

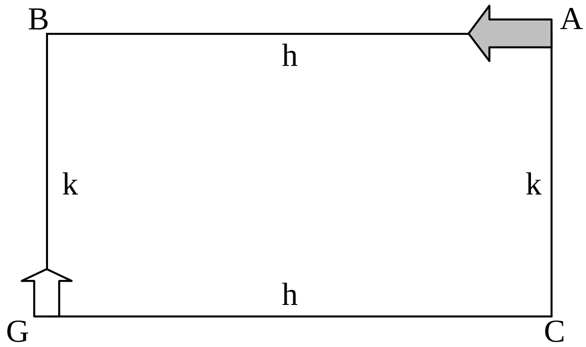
QUESTION 17

A gunner detects a plane at t = 0 approaching him with a velocity v, the horizontal and the vertical distances of the plane being h and k respectively. His gun can fire a shell vertically upwards with an initial velocity u. Find the time when he should fire the gun and the condition on u so that he may be able to hit the plane if it continuous its flight in the same horizontal line.

SOLUTION

Let G be a gun and A be the position of plane at t = 0. Let gun hits the plane at point B and AB = h. Let time taken by plane from A to B is t_1. Then

$$t_1 = \frac{\text{Distance}}{\text{Velocity}} = \frac{h}{v}$$



Let t_2. be time taken by shell to reach at point B.
We know that

$$x = ut - \frac{1}{2}gt^2$$

Putting x = k and t = t_2, we get

$$k = ut_2 - \frac{1}{2}gt_2^2$$

$\Rightarrow$ $2k = 2ut_2 - gt_2^2$

$\Rightarrow$ $gt_2^2 - 2ut_2 + 2k = 0$

$\Rightarrow$ $t_2 = \dfrac{2u \pm \sqrt{4u^2 - 8gk}}{2g} = \dfrac{u \pm \sqrt{u^2 - 2gk}}{g}$

Let T be the time after which gun should be fired. Then

$$T = t_1 - t_2$$

$$= \frac{h}{v} - \frac{u \pm \sqrt{u^2 - 2gk}}{g}$$

For a shell to reach at B, the maximum velocity at B is zero.
Since

$$v^2 - u^2 = 2ax$$

Putting v = 0, a = $-$g and x = k, we get

$$-u^2 = -2gk \quad \Rightarrow \quad u^2 = 2gk$$

Which gives the least value of u. Hence $u^2 > 2gk$

QUESTION 18

Two particles are projected simultaneously in the vertically upward direction with velocities $\sqrt{2gh}$ and $\sqrt{2gk}$ (k > h). After time t, when the two particles are still in flight, another particle is projected upwards with velocity u. Fin the condition so that the third particle may meet the first two during their upward flight.

SOLUTION

For 1st particle

$$v^2 - u^2 = 2ax$$

For maximum height put v = 0, a = −g and u = $\sqrt{2gh}$

$$2gh = 2gx$$

$\Rightarrow$ a = h

Thus maximum height attained by 1st particle is h. Similarly maximum height attained by 2nd particle is k.

Let t_1 be time take by the 1st particle to attain the maximum height h then

$$v = u + at$$

Put v = 0, u = $\sqrt{2gh}$, a = −g and t = t_1

$$0 = \sqrt{2gh} - gt_1$$

$$\Rightarrow \quad t_1 = \frac{\sqrt{2gh}}{g}$$

$$\Rightarrow \quad t_1 = \sqrt{\frac{2h}{g}}$$

Similarly time t_2 taken by the 2nd particle to attain the maximum height k is

$$t_2 = \sqrt{\frac{2k}{g}}$$

Since k > h therefore $t_2 > t_1$

Thus the 1st particle reach the maximum height earlier then 2nd.

If the 3rd particle is projected after time t then t must be less than t_1 in order to meet the 1st two particles during their upward flight. i.e. $t < t_1$

or $$t < \sqrt{\frac{2h}{g}}$$

Now time left with 3rd particle is

$$\sqrt{\frac{2h}{g}} - t$$

and during this time it has to meet both the particles. i.e. It may have to cover a distance k.

Since

$$x = ut - \frac{1}{2}gt^2$$

When x = k, time = $\sqrt{\frac{2h}{g}} - t$ Then

$$k = u\left(\sqrt{\frac{2h}{g}} - t\right) - \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)^2$$

$$\Rightarrow \quad k + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)^2 = u\left(\sqrt{\frac{2h}{g}} - t\right)$$

$$\Rightarrow \quad u = \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)$$

$$\Rightarrow \quad u = \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}\left(\sqrt{2hg} - t\right)$$

Thus the third particle meet the tow 1^{st} particles if

$$u > \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}\left(\sqrt{2hg} - t\right)$$

UNIT II
WORK ENERGY

Work Energy and Conservative Force

Work and energy are the same thing. Energy cant be created or destroyed, it can only be changed from one type into another type. When a force is applied on an object and it moves a distance we say that work has been done and energy has been transformed (changed from one type to another type).

1 Work

Consider a regular trihedral system with O as origin. Let a particle of mass m is moving under a force $\vec{F}$ along a curve C. Let at time t it be at point P, with position vector $\vec{r}$. After a very small time interval Δt it moved an infinitesimal displacement $\vec{dr}$ and is at point Q as shown in Fig. 1. Then the work done by a force $\vec{F}$ in taking the particle from point P to point Q along the curve C in an infinitesimal displacement $\vec{dr}$ is the dot product of $\vec{F}$ and $\vec{dr}$. Hence

$$dW = \vec{F} \cdot \vec{dr} \qquad (1.1)$$

Also the total work done in moving from A to B is

$$W = \int_A^B \vec{F} \cdot \vec{dr} \qquad (1.2)$$

If θ is an angle between $\vec{F}$ and $\vec{dr}$, then (8.1.1) can be written as

$$dW = F dr \cos\theta$$

is the general expression for work done by a force. The expression may be rearranged as

$$dW = F \cos\theta dr \qquad (1.3)$$

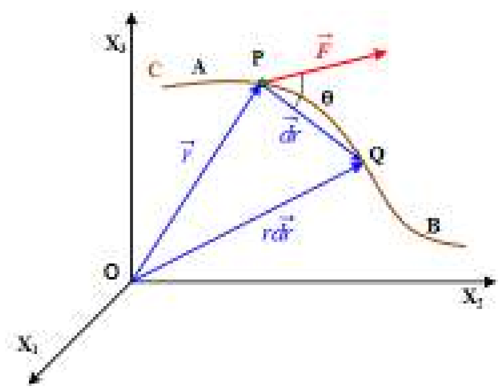


Figure 1: Work done

gives a new definition of work done. The work done by a force is defined as the product of component of force in the direction of motion and the distance moved.

1.1 Work done by a Constant Force

If $\vec{F}$ is constant and $\vec{AB} = \vec{r}_B - \vec{r}_A = \vec{r}$

$$\begin{aligned} W &= \vec{F} \cdot \int_{r_A}^{r_B} \vec{dr} \\ &= \vec{F} \cdot (\vec{r}_A - \vec{r}_B) \\ &= \vec{F}.\vec{r} \\ Net\,Work &= Net\,Force \cdot displacement \end{aligned}$$

If θ is an angle between $\vec{F}$ and $\vec{r}$ then the work done is

$$W = Fr\cos\theta$$

or

$$W = F\cos\theta r$$

Its unit in SI is Joule (J) or $N.m$. Note the work is done only if an object moves in the direction of F.

Example 1.1. *A crate is pulled for a distance of* 6 m *along a floor with a horizontal force of* 5 N. *Find the work done by the force.*

Solution The given data is:

$$\begin{aligned} F &= 5\ N \\ d &= 6\ m \end{aligned}$$

Here the force and the displacement are in the same direction, so the angle between them is $\theta = 0$. Hence the work done is just the product of force and distance. *i.e*

$$\begin{aligned} W &= Fd \\ &= 5(6) = 30\ N \cdot m \end{aligned}$$

2 Energy

Energy is defined as the capacity to do work. It is non-material property capable of causing changes in matter. In dynamics, we deal with mechanical energy which is of two types, namely kinetic and potential energy.

2.1 Kinetic Energy

Energy of an object due to its motion is called kinetic energy. It is the amount of work done by a force in bringing a moving particle to rest from its existing position. It is denoted by T.
Consider a regular trihedral system with O as origin. Let a particle of mass m is moving with velocity $\vec{v}$, under the a force $\vec{F}$ along a curve C. Let at time t it be at point P, with position vector $\vec{r}$. Then the work done by a force $\vec{F}$ in taking the particle from point P to Q (rest) along the curve C is:

$$T = W = \int_P^Q \vec{F} \cdot d\vec{r}$$

If the applied force $\vec{F} = m\vec{a}$, then

$$T = \int_P^Q m\vec{a} \cdot d\vec{r} \qquad (2.1)$$

The acceleration is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} \qquad (2.2)$$

and the velocity of the particle is

$$\vec{v} = \frac{d\vec{r}}{dt}$$

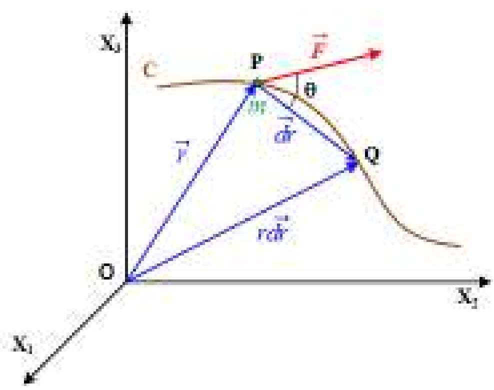


Figure 2: Work done

or

$$d\vec{r} = \vec{v}dt \quad (2.3)$$

Using (2.2) and (2.3), (2.1) becomes

$$\begin{aligned} T &= \int_P^Q m\frac{d\vec{v}}{dt}\cdot\vec{v}dt \\ &= \int_P^Q \frac{1}{2}m\frac{dv^2}{dt}dt \\ &= \int_P^Q \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt \\ &= \frac{1}{2}mv^2 \quad (2.4) \end{aligned}$$

(2.4) is an expression for kinetic energy. In SI, it is measured in Joules (J).

2.2 Kinetic Energy in terms of Work

Kinetic energy is the amount of work done by a force in bringing a moving particle to rest from its existing position. If m is the mass of the particle, then by Newton's second law of motion, the applied force is $\vec{F} = m\vec{a}$. Using (2.2) and (2.3), (1.1) can be written as

$$dW = m\frac{dv}{dt}\cdot vdt$$

$$
\begin{aligned}
dW &= mvdv \\
&= d\left(\frac{1}{2}mv^2\right)
\end{aligned} \qquad (\;2.5)
$$

Now the total work done from A to B is

$$
\begin{aligned}
W_{AB} &= \int_A^B F dr \\
&= \int_A^B d\left(\frac{1}{2}mv^2\right) \\
&= \left(\frac{1}{2}mv^2\right)_A^B \\
&= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2
\end{aligned}
$$

The quantity $T = \frac{1}{2}mv^2$ is the kinetic energy. Hence the work done is

$$
\begin{aligned}
W_{AB} &= T_B - T_A \\
&= \Delta T
\end{aligned} \qquad (\;2.6)
$$

We can postulate some results as under:

(a) If $T_A > T_B$ then $W_{AB} < 0$
The work is done by the particle against the force and its kinetic energy has decreased.

(b) If $T_A < T_B$ then $W_{AB} > 0$
The work is done by the force on the particle and its kinetic energy has increased.

In any case the work done depends upon the difference in kinetic energies of the particle in the two positions. The work done against the dissipative force like the fractional force is always negative.

2.3 Potential Energy

Potential energy is energy of position. The amount of potential energy possessed by an object is proportional to how far it was displaced from its original position. If the displacement occurs vertically, raising an object off of the ground, is known as gravitational potential energy. It is denoted by U. If m is the mass of the object raised a height h from the ground as shown in Fig. 3, then gravitational potential energy of the object is

$$
\begin{aligned}
\text{gravitational potential energy} &= \text{weight} \times \text{height} \\
U &= mgh
\end{aligned}
$$

The concept of potential energy can be used when dealing with conservative force that will be discussed later.

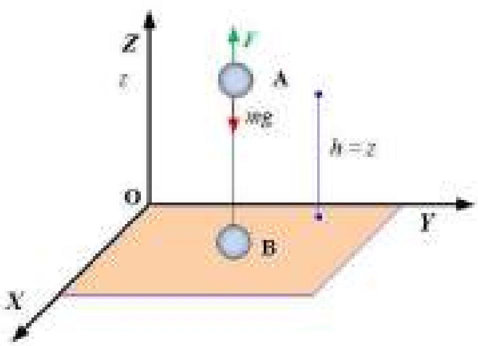


Figure 3: Potential energy

8.2.4 Potential Energy is converted to Kinetic Energy and vice-versa

Consider Fig. 1. Let $\vec{v}_i$ be the velocity of the particle at P and $\vec{v}_f$ be at P. Considering xy plane as the zero level and hight above it is the distance, that is $(d = h$ height $)$. Its equation of motion can be written as

$$\begin{aligned} v_f^2 - v_i^2 &= 2gh \\ \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= mgh \\ \Delta T &= \Delta U \end{aligned}$$

Note only changes in potential energy can be measured. Total amount of energy at any instant cannot be determined.
At ground level all energy is kinetic energy and at maximum height h all energy is potential energy.

3 Power

Rate of doing work by a force $\vec{F}$ is called power or activity.

$$\begin{aligned} dP &= \frac{dW}{dt} \\ &= \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \qquad (3.1) \end{aligned}$$

Also the power is defined as the rate at which energy is transferred by a force $\vec{F}$. Consider

$$\begin{aligned}\frac{dT}{dt} &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\\ &= \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = m\frac{d\vec{v}}{dt}\cdot\vec{v}\\ &= \vec{F}\cdot\vec{v}\\ &= P\end{aligned}$$

It is measured in Watts $(W)\rightarrow$ 1 Joule of energy transferred in 1 second
We usually measure it in kW (kilowatts)

3.1 Efficiency

Ratio of output work to input work of a machine

$$Efficiency = \frac{W_{output}}{W_{input}}\times 100$$

4 Work done by a Variable Force

Consider a body moves under the influence of a force $\vec{F}(t)$. Suppose that the body moves a displacement $d\vec{r}(t)$ between time t_1 and t_2. Then the work done by the force is

$$W = \int_C \vec{F}\cdot d\vec{r}$$

As F and r are functions of t, hence the work done is

$$W = \int_{t_1}^{t_2} \vec{F}\cdot d\vec{r}dt$$

That is, work is the path integral of the force along the trajectory. Work may be either positive or negative, where in the latter case we will say that it is the body that has performed work.

5 Conservative Force

If the force field acting on a physical body is such that the work done along a closed path is zero, the force is called conservative. In other words we can say that a vector field is

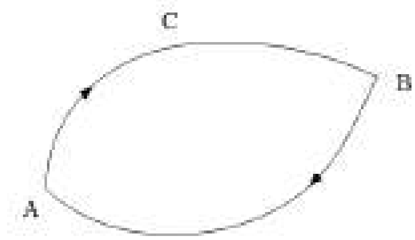


Figure 4: Work done

called conservative if integrals along paths only depend on the end points and not on the trajectory.

Consider a particle is moving along a curve C under the action of a variable force F. Let the particle moves from A to B and then from B to A, forming a closed path. If the total work done is zero, the acting force is conservative. *i.e.*

$$W = \int_C F.dr = 0 \quad (5.1)$$

If the force F is uniquely defined at every point of a region of space, the set of all such forces is a called a force field. If at every point of the space, the force F is conservative, then the force field is said to be conservative.
It is not always true that the work done by an external force is stored as some form of potential energy. This is only true if the force is conservative.
Examples: the force of gravity and the spring force are conservative forces.
For a non-conservative (or dissipative) force, the work done in going from A to B depends on the path taken.
Examples: friction and air resistance.

6 Examples of conservative and Non Conservative Force Field

In this section we will give some examples of conservative and non conservative systems.

6.1 The Earth's Gravitational Field is Conservative

The zero level of the potential energy is arbitrary; it can be assigned to any position. If the xy plane is chosen the zero level of potential energy , the potential energy at any point A is the work done by the force when the body moves from the point A to the point B with zero potential energy. So,

$$U(A) \quad = \quad W_{AB}$$

Consider a particle of mass m is initially at A, the gravitational force is $F = mg$. The work done by this force along the path AB is

$$W_{AB} \quad = \quad mg(h-0) = mgh$$

Near Earths surface, the work done by gravity on an object of mass m depends only on

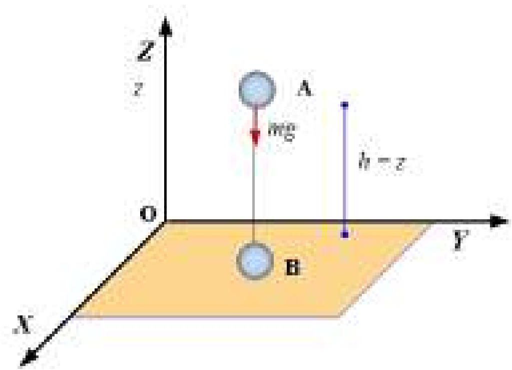


Figure 5: Work done

the change in the objects height h that depends on the end points of the path.
Vector Approach
Consider a particle of mass m is initially at A, the gravitational force always acts downward, having only z component, so can be written as

$$\vec{F} \quad = \quad mg\hat{k} = \langle 0,\ 0,\ mg\rangle$$

and dr can be written as

$$d\vec{r} \quad = \quad dx\hat{i} + dy\hat{j} + dz\hat{k} = \langle dx,\ dy,\ dz\rangle$$

The work done by this force along the path AB is

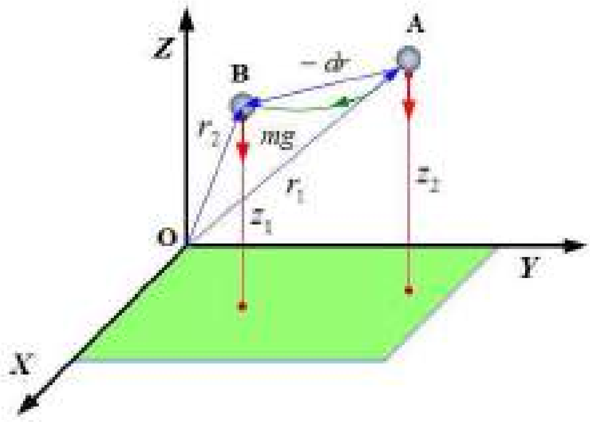


Figure 6: Work done

$$
\begin{aligned}
W &= -\int_A^B \vec{F}.d\vec{r} \\
&= \int_B^A \langle 0,\ 0,\ mg\rangle.\langle dx,\ dy,\ dz\rangle \\
&= mg\int_B^A dz \\
&= mg.z\Big|_B^A \\
&= mg.\left(z_A - z_B\right) \\
&= mg.\left(z_2 - z_1\right)
\end{aligned}
$$

Here the work done depends upon the initial and final positions of the particle, and is independent of the path. Hence the force is a conservative force and the earth's gradational field is conservative.

2.

$$\vec{F} = k\hat{s}$$

where k is some constant and $\hat{s}$ unit arc length. Also we can take $dr \equiv ds$. The work done by this force along the path AB is

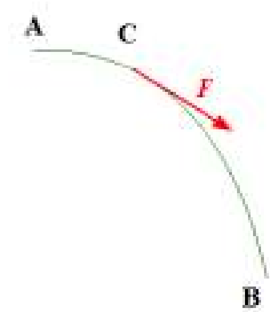


Figure 7: Work done

$$
\begin{aligned}
W &= \int_A^B k\hat{s}.dr = k\int_A^B \hat{s}.ds \\
&= k\int_A^B 1ds\cos\theta \\
&= k\cos\theta s\Big|_A^B \\
&= k\cos\theta\,(s_B - s_A)
\end{aligned}
$$

Here the work done depends upon the arc length of the path. Hence the force is not conservative force.

6.2 Potential Energy and Conservative Force

The potential energy of a particle in a field of force F is defined as the total work done in moving a particle from its existing position to its standard position (zero level of the potential energy) along the curve.
Let O be the origin of an inertia frame of reference fixed in space. Let P_0 be the position (standard) of a particle on a curve C and $P(t)$ be an arbitrary existing position of a particle at any time t. Let

$$
\begin{aligned}
\vec{OP_0} &= \vec{r_0} \\
\vec{OP} &= \vec{r}
\end{aligned}
$$

Analytically we can write, the expression for the potential energy is

$$
\begin{aligned}
U_{(P)} &= \int_r^{r_0} \vec{F}.\vec{dr} \\
&= -\int_{r_0}^{r} \vec{F}.\vec{dr}
\end{aligned}
$$

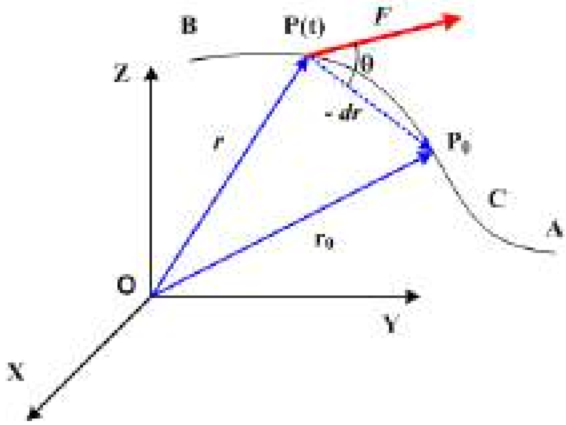


Figure 8: Work done

Theorem 6.1. *A vector field is conservative if and only if it is the gradient of a scalar field.*

$$\vec{F} = -\nabla\vec{U} \qquad (6.1)$$

where $U(r)$ is called the potential field, or the potential energy; the negative sign is a convention whereby the force is directed in the direction of decreasing potential.

Proof Consider Fig. , we can write

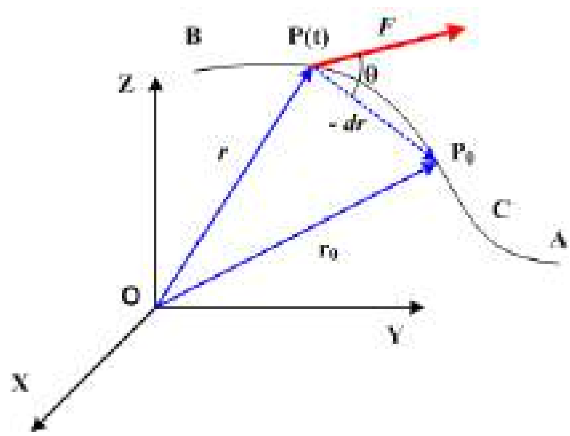


Figure 9: Work done

$$\begin{aligned}
\vec{r} &= \langle x, y, z\rangle \\
\vec{r_0} &= \langle x_0, y_0, z_0\rangle \\
\vec{dr} &= \langle dx, dy, dz\rangle \\
\vec{F} &= \langle F_x, F_y, F_z\rangle \\
U &= U(x, y, z)
\end{aligned}$$

Let P_0 be the zero level of potential energy, then U at P is

$$\begin{aligned}
U_{(P)} &= \int_{(x,y,z)}^{(x_0,y_0,z_0)} \langle F_x, F_y, F_z\rangle.\langle dx, dy, dz\rangle \\
&= -\int_{(x_0,y_0,z_0)}^{(x,y,z)} (F_x dx + F_y dy + F_z dz) \\
&= -\int_{P_0}^{P} \vec{F}.\vec{dr} \qquad (6.2)
\end{aligned}$$

differentiating we have

$$\begin{aligned}
dU_{(P)} &= -d\left[\int_{(x_0,y_0,z_0)}^{(x,y,z)} (F_x dx + F_y dy + F_z dz)\right] \\
&= -(F_x dx + F_y dy + F_z dz) \\
dU(x, y, z) &= (-F_x dx - F_y dy - F_z dz) \\
\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz &= -F_x dx - F_y dy - F_z dz)
\end{aligned}$$

$$\left(\frac{\partial U}{\partial x}+F_x\right)dx+\left(\frac{\partial U}{\partial y}+F_y\right)dy+\left(\frac{\partial U}{\partial z}+F_z\right)dz=0 \qquad (6.3)$$

Since x, y, and z are linearly independent, so dx, dy, and dz are also linearly independent. This implies that the coefficient of dx, dy, and dz must be equal to zero. *i.e.*

$$\begin{aligned}
\frac{\partial U}{\partial x}+F_x &= 0\\
\frac{\partial U}{\partial y}+F_y &= 0\\
\frac{\partial U}{\partial z}+F_z &= 0
\end{aligned}$$

or we have

$$\begin{aligned}
F_x &= -\frac{\partial U}{\partial x}\\
F_y &= -\frac{\partial U}{\partial y}\\
F_z &= -\frac{\partial U}{\partial z}
\end{aligned}$$

Hence $\vec{F}$ can be written as

$$\begin{aligned}
\vec{F} &= \langle F_x, F_y, F_z\rangle = \left\langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right\rangle\\
&= -\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle U(x,y,z)\\
&= -\nabla U
\end{aligned}$$

where

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$$

is an operator.
Conversely suppose that

$$\vec{F} = -\nabla U$$

The work done is

$$
\begin{aligned}
W &= \int_{P}^{P_0} F.dr \\
&= -\int_{P}^{P_0} \nabla U.dr \\
&= -\int_{P}^{P_0} \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle .\langle dx, dy, dz\rangle \\
&= -\int_{P}^{P_0} \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right) \\
&= -\int_{P}^{P_0} dU = -U\Big|_{P}^{P_0} \qquad (\;\; 6.4) \\
&= -U(P_0) + U(P) \\
&= \Delta U
\end{aligned}
$$

Hence the work done along a trajectory r connecting the points P and P_0, is independent of path, so the vector field of force F is conservative.

Theorem 6.2. *A necessary and sufficient condition for a vector field to be conservative is*

$$curl\vec{F} = \vec{O} \tag{6.5}$$

Proof Let $\vec{F}$ is conservative, Then there exist a function $U(x, y, z)$ of class C^2 (second order partial derivatives of U exist and are continuous) and $\vec{F}$ can be expressed as

$$\vec{F} = -\nabla U$$

Apply curl on both sides

$$\begin{aligned} curl\vec{F} &= -curl\ \nabla U \\ &= -\nabla \times \nabla U \end{aligned}$$

This cross product can be written as

$$\begin{aligned} curl\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 U}{\partial y \partial Z} - \frac{\partial^2 U}{\partial z \partial y}\right)\hat{i} - \left(\frac{\partial^2 U}{\partial z \partial x} - \frac{\partial^2 U}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right)\hat{k} \end{aligned}$$

Since U is of class C^2, then $\frac{\partial^2 U}{\partial y \partial Z} = \frac{\partial^2 U}{\partial z \partial y}$ and all other pairs are so. Hence we have

$$\begin{aligned} curl\vec{F} &= \langle 0, 0, 0 \rangle \\ &= \vec{O} \end{aligned}$$

Conversely suppose that

$$curl\vec{F} = \vec{O}$$

Let

$$\vec{F} = \langle P, Q, R \rangle$$

be a force to do the work. Then

$$\begin{aligned} curl\vec{F} &= \vec{O} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} &= \langle 0, 0, 0 \rangle \\ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} &= 0\hat{i} + 0\hat{j} + 0\hat{k} \end{aligned}$$

Since the two vectors are equal, this mean that their corresponding elements are equal. *i.e*

$$\begin{aligned}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) &= 0\\ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) &= 0\\ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) &= 0\end{aligned}$$

or we can write

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial Q}{\partial z}\\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x}\\ \frac{\partial Q}{\partial x} &= \frac{\partial P}{\partial y}\end{aligned}$$

which is possible only if there exist a function U of class C^2 such that

$$\vec{F} = -\vec{\nabla} U$$

Hence $\vec{F}$ is conservative.

Example 6.1. *The force*

$$\vec{F} = \langle x, y, z\rangle$$

is conservative. Also find the corresponding potential function.

Solution For a conservative force we need to show only

$$curl\vec{F} = \vec{O}$$

Next

$$\begin{aligned}curl\vec{F} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}\\ &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{k}\end{aligned}$$

Since $x,\ y,\ z$ are linearly independent, that means

$$\begin{aligned}\frac{\partial z}{\partial y} &= 0 = \frac{\partial y}{\partial z}\\ \frac{\partial x}{\partial z} &= 0 = \frac{\partial z}{\partial x}\\ \frac{\partial y}{\partial x} &= 0 = \frac{\partial x}{\partial y}\end{aligned}$$

Then

$$\begin{aligned}curl\vec{F} &= \langle 0,0,0\rangle\\ &= \vec{O}\end{aligned}$$

Hence the given force $\vec{F}$ is conservative and there exist a function U of class C^2 such that

$$\begin{aligned}\vec{F} &= -\vec{\nabla}U\\ \langle x,y,z\rangle &= \langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\rangle\end{aligned}$$

The two vectors are equal, so there corresponding entries are equal.

$$x = -\frac{\partial U}{\partial x} \quad \text{or} \quad -\frac{\partial U}{\partial x} = x \qquad (6.6)$$

$$y = -\frac{\partial U}{\partial y} \quad \text{or} \quad -\frac{\partial U}{\partial y} = y \qquad (6.7)$$

$$z = -\frac{\partial U}{\partial z} \quad \text{or} \quad -\frac{\partial U}{\partial z} = z \qquad (6.8)$$

Partially integrate (6.6) with respect to x

$$\begin{aligned}-U &= \int x dx + f(y,z)\\ &= \frac{x^2}{2} + f(y,z) \qquad (6.9)\end{aligned}$$

Partially differentiate (6.9) with respect to y

$$-\frac{\partial U}{\partial y} = \frac{\partial f}{\partial y} \qquad (6.10)$$

Using (6.10) in (6.7)

$$\frac{\partial f}{\partial y} = y \qquad (6.11)$$

Partially integrate (6.11) with respect to y

$$f = \frac{x^2}{2} + g(z) \qquad (6.12)$$

Then (6.9) becomes

$$-U = \frac{x^2}{2} + \frac{y^2}{2} + g(z)$$

Partially differentiate (6.13) with respect to z

$$-\frac{\partial U}{\partial z} = \frac{dg}{dz} \qquad (6.13)$$

From (6.8) and (6.13), we can write

$$\frac{dg}{dz} = z \qquad (6.14)$$

Integrating (6.14)

$$g = \frac{z^2}{2} + c \qquad (6.15)$$

Using (6.15) in (6.13) we have

$$-U = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C$$

Ignoring C, the corresponding potential function is

$$U = -\frac{1}{2}\left(x^2 + y^2 + z^2\right) \qquad (6.16)$$

From this potential function, the corresponding conservative force can be calculated as Let the force is

$$\vec{F} = \langle P, Q, R\rangle$$

$$\vec{F} = -\nabla U$$
$$\langle P, Q, R\rangle = \langle -\frac{\partial U}{\partial X}, -\frac{\partial U}{\partial Y}, -\frac{\partial U}{\partial z}\rangle$$

From (6.16), we have

$$-\frac{\partial U}{\partial x} = x$$
$$-\frac{\partial U}{\partial y} = y$$
$$-\frac{\partial U}{\partial z} = z$$

Hence the corresponding conservative force is

$$\vec{F} = \langle x, y, z\rangle$$

Example 6.2. *A particle moves under the action of a force*

$$\vec{F} = \langle 3x^2 + 6xy, 3x^2 - 3y^2, 0 \rangle$$

from $A(1,1,0)$ *to* $B(2,3,0)$. *Then determine*

(a) *Is the force conservative?*

(b) *If yes, find the corresponding potential energy function.*

(c) *The work done from* A *to* B

Solution For a conservative force we need to show only

$$curl\vec{F} = \vec{O}$$

Next

$$\begin{aligned} curl\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 + 6xy & 3x^2 - 3y^2 & 0 \end{vmatrix} \\ &= \left(\frac{\partial(0)}{\partial y} - \frac{\partial\left(3x^2 - 3y^2\right)}{\partial z} \right) \hat{i} + \left(\frac{\partial\left(3x^2 + 6xy\right)}{\partial z} - \frac{\partial(0)}{\partial x} \right) \hat{j} \\ &+ \left(\frac{\partial\left(3x^2 - 3y^2\right)}{\partial x} - \frac{\partial\left(3x^2 + 6xy\right)}{\partial y} \right) \hat{k} \end{aligned}$$

Since $x,\ y,\ z$ are linearly independent, then

$$\begin{aligned} curl\vec{F} &= (0)\,\hat{i} + (0)\,\hat{j} + (6x - 6x)\,\hat{k} \\ &= \langle 0, 0, 0 \rangle \\ &= \vec{O} \end{aligned}$$

Hence the given force $\vec{F}$ is conservative.

(b) Since the given force is conservative, then there exist a function U of class C^2 such that

$$\begin{aligned} \vec{F} &= -\vec{\nabla} U \\ \langle 3x^2 + 6xy, 3x^2 - 3y^2, 0 \rangle &= \langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \rangle \end{aligned}$$

The two vectors are equal, so there corresponding entries are equal.

$$3x^2 + 6xy = -\frac{\partial U}{\partial x} \quad \text{or} \quad -\frac{\partial U}{\partial x} = 3x^2 + 6xy \tag{6.17}$$

$$3x^2 - 3y^2 = -\frac{\partial U}{\partial y} \quad \text{or} \quad -\frac{\partial U}{\partial y} = 3x^2 - 3y^2 \tag{6.18}$$

$$0 = -\frac{\partial U}{\partial z} \quad \text{or} \quad -\frac{\partial U}{\partial z} = 0 \tag{6.19}$$

The given force is a two dimensional force, so we can ignore (6.19) Partially integrate (6.17) with respect to x

$$\begin{aligned} -U &= \int \left(3x^2 + 6xy\right) dx + f(y) \\ &= x^3 + 3x^2 y + f(y) \end{aligned} \tag{6.20}$$

Partially differentiate (6.20) with respect to y

$$-\frac{\partial U}{\partial y} = 3x^2 + \frac{df}{dy} \tag{6.21}$$

Using (6.21) in (6.18)

$$\frac{df}{dy} = -3y^2 \tag{6.22}$$

Partially integrate (6.22) with respect to y

$$f = -y^3 + C \tag{6.23}$$

Then (6.20) becomes

$$-U = x^3 + 3x^2 y - y^3 + C \tag{6.24}$$

Ignoring C, the corresponding potential function is

$$U = -x^3 - 3x^2 y + y^3 \tag{6.25}$$

(c) Work done from A to B can be calculated by using (8.6.4)

$$W = -U\Big|_A^B$$

Using (6.25) the work done is

$$\begin{aligned} W &= \left|x^3 + 3x^2 y - y^3\right|_{(1,1,0)}^{(2,3,0)} \\ &= [(8 + 36 - 27) - (1 + 3 - 1)] = [17 - 3] \\ &= 14\ J \end{aligned}$$

Example 6.3. *Examples of potential energy functions.*

1. For a mass under the influence of earth gravity

$$U(r) = U(z) = mgz$$

2. For a mass suspended on a spring,

$$U(z) = \frac{1}{2}kz^2$$

3. For a planet under the influence of a stars gravity,

$$U(r) = -\frac{GMm}{r^2}$$

7 Law of Conservation of Energy

StatementWithin a closed, isolated system, energy can change form, but the total amount of energy is constant

$$T_{initial} + U_{initial} = T_{final} + U_{final} \quad (7.1)$$

The sum of kinetic energy and potential energy represents the total mechanical energy.
Proof Consider a particle of mass m is moving under the influence of a conservative force field F. If the particle performs a trajectory $r(t)$ connecting the points $r(t_1)$ and $r(t_2)$ then from (8.6.4), we can write

$$\begin{aligned} -U(rt_2) + U(rt_1) &= \int_P^{P_0} \vec{F}.d\vec{r} \\ &= m\int_P^{P_0} \vec{a}.d\vec{r} \end{aligned} \quad (7.2)$$

Using (2.2) and (2.3), (7.2) becomes

$$\begin{aligned} -U(rt_2) + U(rt_1) &= \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \vec{v}dt \\ &= \int_{t_1}^{t_2} \frac{1}{2}m\frac{dv^2}{dt}dt \\ &= \int_{t_1}^{t_2} \frac{d}{dt}Kdt \\ &= T(v(t_2)) - T(v(t_1)) \\ \Delta U &= \Delta T \end{aligned} \quad (7.3)$$

It can also be written as

$$T(v(t_1)) + U(rt_1) \quad = \quad T(v(t_2)) + U(rt_2) \qquad (7.4)$$

7.4 is known as the law of conservation of energy.
Defining the total mechanical energy, or simply the energy,

$$E(r, v) \quad = \quad U(r) + T(v), \qquad (7.5)$$

we conclude that it assumes the same value at time t_1 and t_2, *i.e.*, it is conserved (it is a function of the trajectory whose value remains constant in time).

Theorem 7.1. *In a conservative vector field, the total energy (mechanical) is constant throughout the motion.*

Proof It is another to proof the law of conservation of energy.
It can be proved by showing

$$\frac{d}{dt}E(r, v) \quad = \quad 0$$

Taking time derivative of (8.7.5)

$$\begin{aligned} \frac{d}{dt}E(r, v) &= \frac{d}{dt}\big[U(r) + T(v)\big] \\ &= \frac{d}{dt}\big[U(r(t)) + T(v(t))\big] \qquad (7.6) \end{aligned}$$

Now by chain rule, the first term on right hand side can be written as

$$\begin{aligned} \frac{d}{dt}\big[U(r(t))\big] &= \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \\ &= \vec{\nabla U} \cdot \dot{\vec{r}} \end{aligned}$$

Since $\vec{F}$ is conservative, then by (8.6.1), we can write

$$\frac{d}{dt}\big[U(r(t))\big] \quad = \quad -\vec{F} \cdot \vec{v} \qquad (7.7)$$

Again by chain rule, the second term on right hand side can be written as

$$\begin{aligned} \frac{d}{dt}\big[T(v(t))\big] &= \frac{d}{dt}\big[\frac{1}{2}mv^2\big] \\ &= \frac{1}{2}m\frac{d}{dt}v^2 \\ &= m\frac{d\vec{v}}{dt} \cdot \vec{v} \\ &= \vec{F} \cdot \vec{v} \qquad (7.8) \end{aligned}$$

Using (7.7) and (7.8), (7.6) becomes

$$\frac{d}{dt}E(r,v) = -\vec{F}\cdot\vec{v}+\vec{F}\cdot\vec{v}=0 \qquad (7.9)$$

(7.4) and (7.9) represent the principle of conservation of energy.
Second Method We will show that the sum of kinetic and potential energies is constant. By Newton's second law of motion its equation of motion is

$$F = m\ddot{r} \qquad (7.10)$$

Multiply (7.10) with $\dot{r}=\frac{dr}{dt}$,

$$m\dot{r}\ddot{r} = F\frac{dr}{dt} \qquad (7.11)$$

Integrating (7.11) with respect to t

$$\frac{1}{2}m\dot{r}^2 = \int F\frac{dr}{dt}dt + constant$$

or we can write

$$\frac{1}{2}m\dot{r}^2 - \int F\cdot dr = constant \qquad (7.12)$$

Since the force is conservative, so we have

$$\vec{F} = -\vec{\nabla}U$$

where $U(r)$ is the potential energy and can be written as

$$U = -\int F\cdot dr$$

and the term $\frac{1}{2}m\dot{r}^2$ is the kinetic energy of the system. Using these results, (7.12) becomes

$$T+U = constant \qquad (7.13)$$

Hence the total energy of the system is conserved. The conservation of total mechanical energy when forces are conservative is useful as shows in the following examples.

Example 7.1. *A body is dropped (at rest) from a height of h meters. If the motion is free fall, show that the energy of the system is conserved.*

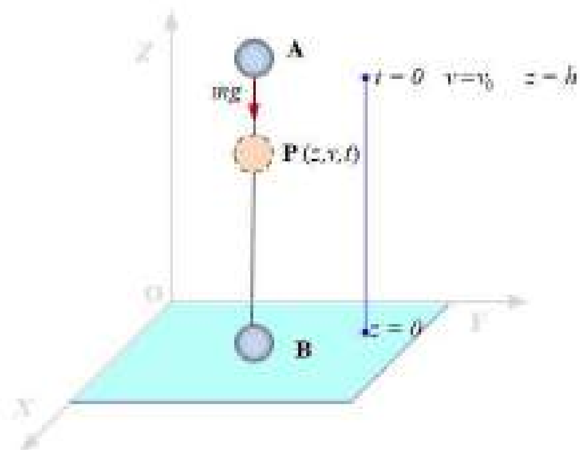


Figure 10: Downward motion

Solution We will show that the sum of kinetic and potential energies is constant. The particle is executing one dimensional motion and its motion is along $z-axis$. Let any time t the particle is at P as shown in the Fig 10. At P the kinetic energy is

$$T = \frac{1}{2}m\dot{z}^2$$

Taking ground (xy plane) as zero level for potential energy, then potential energy is

$$U = mgz$$

By Newton's second law of motion its equation of motion is

$$\begin{aligned} F &= -W \\ m\ddot{z} &= -mg \qquad (7.14) \end{aligned}$$

Multiply (7.14) with $\dot{z}$,

$$m\dot{z}\ddot{z} + mg\dot{z} = 0 \qquad (7.15)$$

(7.15) can be written as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{z}^2 + mgz\right) = 0 \qquad (7.16)$$

The term $\frac{1}{2}m\dot{z}^2$ is the kinetic energy and mgz is the potential energy of the system. Using these results, (7.16) becomes

$$\begin{aligned} \frac{d}{dt}(T+U) &= 0 \\ \frac{dE}{dt} &= 0 \end{aligned} \qquad (7.17)$$

Integrating (7.17), we have

$$E = constant \qquad (7.18)$$

Hence the total energy of the system is conserved.

Example 7.2. *A body is dropped (at rest) from a height of h meters. If the motion is free fall, use energy approach to find speed with which it will hit the ground.*

Solution As the body starts from rest, so the initial data is

$$\begin{aligned} t_0 &= 0 \\ v_0 &= 0 \\ z_0 &= h \end{aligned}$$

One way to solve it is via the equations of motion:
The other way of solving this exercise is with energies. Taking xy plane as zero level for potential energy. At P the potential energy is

$$U(z) = mgz$$

the conservation of energy implies that

$$U(z(0)) + \frac{mv^2(0)}{2} = U(z(t_1)) + \frac{mv^2(t_1)}{2}$$

i.e.,

$$\begin{aligned} mgh + 0 &= 0 + \frac{mv^2(t_1)}{2} \\ v(t_1) &= \sqrt{2hg}. \end{aligned}$$

which gives the exact same answer.

Simple Harmonic Motion

Simple Harmonic Motion (S.H.M) is an interesting special type of motion in nature, having forward and backward oscillation (or) to and fro oscillation about a fixed point. The fixed point is known as the mean position or equilibrium position. When the oscillation is very small we prove the motion is simple harmonic. In this section we study about the resultant of two S.H.M'S of the same period in the same straight line and in two perpendicular lines. Also we find the periodic time of oscillation of a simple pendulum.

Examples

Small oscillation of a cradle, simple pendulum, seconds pendulum, simple equivalent pendulum, transverse vibrations of a plucked violin string etc.

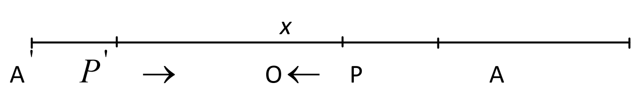
Hooke's law

Tension of an elastic string or spring is directly proportional to its extended length and indirectly proportional to its natural length.

1 Simple Harmonic Motion in a straight line

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.



Let O be a fixed point on the straight line A^1 OA on which a particle is having simple harmonic motion. Take O as the origin and OA as the X axis. Let P be the position of the particle at time t such that OP = x. The magnitude of the acceleration at P is μx where μ is a positive constant. The acceleration at P in the positive direction of the X axis is $-\mu$ x towards O.

Hence the equation of motion of P is $\frac{d^2x}{dt^2} = -\mu x$ ……..(1)

Equation (1) is the fundamental differential equation representing a S.H.M.

If v is the velocity of the particle at time t (1) can be written as

$v\frac{dv}{dx} = -\mu x$ i.e. $\quad vdv = -\mu x\, dx$ …………(2)

Integrating (2), we have $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c$ ………..(3)

Initially let the particle starts from rest at the point A where OA = a

Hence when x=a, $v = 0 = \frac{dx}{dt}$

Putting these in (3), $0 = -\frac{\mu a^2}{2} + c$ or $c = \frac{\mu a^2}{2}$

$\therefore v^2 = -\mu x^2 + \mu a^2 = \mu\left(a^2 - x^2\right)$

$\therefore v = \pm\sqrt{\mu\left(a^2 - x^2\right)}$ ………. (4)

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence we take the negative sign in (4),

$\frac{dx}{dt} = \mathbf{v} = -\sqrt{\mu\left(a^2 - x^2\right)}$ **……..(5)**

$$-\frac{dx}{\sqrt{\left(a^2 - x^2\right)}} = \sqrt{\mu}\, dt$$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t + A$

Initially when t = 0, x = a, $\cos^{-1}1 = 0 + A \Rightarrow$ A=0

$\therefore \cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t$ or $\quad x = a\cos\sqrt{\mu}\, t$ ………. (6)

To get the time from A to A^1, put $x = -a$ in (6)

We have $\cos\sqrt{\mu}\, t = -1 = \cos\pi$, $t = \frac{\pi}{\sqrt{\mu}}$

$\therefore$ The time from A to A' and back = $\dfrac{2\pi}{\sqrt{\mu}}$.

Equation (6) can be written as

x = a cos $\sqrt{\mu}$ t = a cos ($\sqrt{\mu}$ t +2 π) = a cos ($\sqrt{\mu}$ t + 4 π) etc

$= a\cos\sqrt{\mu}\left(t+\dfrac{2\pi}{\sqrt{\mu}}\right) = a\cos\sqrt{\mu}\left(t+\dfrac{4\pi}{\sqrt{\mu}}\right)$ etc.

Differentiating (6),

$\dfrac{dx}{dt} = -a\sqrt{\mu}\ .\sin\sqrt{\mu}\ t$

$= -a\sqrt{\mu}\sin(\sqrt{\mu}\ t+2\pi) = -a\sqrt{\mu}\ \sin(\sqrt{\mu}\ t+4\pi)$ etc.

$= -a\sqrt{\mu}\ \sin\sqrt{\mu}\ (t+\dfrac{2\pi}{\sqrt{\mu}}) = -a\sqrt{\mu}\ \sin\sqrt{\mu}\,(t+\dfrac{4\pi}{\sqrt{\mu}})$ etc.

The values of $\dfrac{dx}{dt}$ are the same if t is increased by $\dfrac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\dfrac{2\pi}{\sqrt{\mu}}$. Hence after a time $\dfrac{2\pi}{\sqrt{\mu}}$ the particle is again at the same point moving with the same velocity in the same direction. Hence the particle has the period $\dfrac{2\pi}{\sqrt{\mu}}$.

T = $\dfrac{2\pi}{\sqrt{\mu}}$; frequency = $\dfrac{1}{T}$ = $\dfrac{2\pi}{\sqrt{\mu}}$

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Amplitude = OA = OA' = a.

The periodic time = $\dfrac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Deductions : 1) Maximum acceleration $= \mu.a = \mu$. (amplitude)

2) Since v = $\sqrt{\mu(a^2-x^2)}$, the greatest value of v is at x = 0 and its

Maximum velocity = a $\sqrt{\mu}$ = $\sqrt{\mu}$. (amplitude) at the centre

General solution of the S.H.M. equation

The S.H.M. equation is $\frac{d^2x}{dt^2} = -\mu x$

i.e. $\frac{d^2x}{dt^2} + \mu x = 0$ ....(1)

(1) is a differential equation of the second order with constant coefficients. Its general solution is of the form

$x = A\cos\sqrt{\mu}\, t + B\sin\sqrt{\mu}\, t$ ....(2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

$x = C\cos(\sqrt{\mu}\, t + \varepsilon)$.... (3) and $x = D\sin(\sqrt{\mu}\, t + \alpha)$ ....(4)

- If the solution of the S.H.M. equation is $x = a\cos(\sqrt{\mu}\, t + \varepsilon)$, the quantity ε is called the **epoch**.

Definition

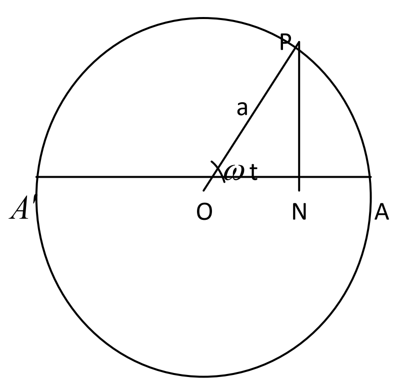
If two simple harmonic motions of the same period can be represented by

$x_1 = a_1\cos(\sqrt{\mu}\, t + \varepsilon_1)$ and $x_2 = a_2\cos(\sqrt{\mu}\, t + \varepsilon_2)$

- The difference in phase = $\frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$
- If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.
- If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phase.

2 Geometrical Representation of S.H.M

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H.M.



Let AA' be the diameter of the circle with centre O and P be the position of the particle at time $t\,secs$. Let N be the foot of the perpendicular drawn from P on the diameter AA'. P moves along the circumference of the circle with uniform speed and describes equal arcs in equal times. Let ω – be the angular velocity. $\therefore \angle AOP = \omega t$

If ON = x, Op = a, then, $x = a \cos(\omega t)$ ……………… (1)

$$\frac{dx}{dt} = -a\omega \sin(\omega t) \quad \text{……………….(2)}$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos(\omega t) = -\omega^2 x \quad \text{……………… (3)}$$

(3) shows that the motion of N is simple harmonic. When P moves along the circumference of the circle starting from A, N oscillates from A to A' and A' to A.

Periodic time of P = Periodic time of N $= \dfrac{2\pi}{\omega}$

(along the circle) (along the diameter)

Problem 1

A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are $x_1, x_2, x_3.$ Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$

Solution:

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that x = a cos $\sqrt{\mu}$ t ….. (1)

Let $x_1, x_2, x_3.$ be the displacements at three consecutive seconds $t_1,\ t_1 + 1,\ t_1 + 2$.

Then x_1 = a cos $\sqrt{\mu}\, t_1$ ...… (2)

x_2 = a cos $\sqrt{\mu}(t_1 + 1)$ = a cos $\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)$ …….(3)

x_3 = a cos $\sqrt{\mu}(t_1 + 2)$ = a cos $\left(\sqrt{\mu}t_1 + 2\sqrt{\mu}\right)$ …….(4)

$\therefore x_1 + x_3 = a\,[\cos\left(\sqrt{\mu}\, t_1 + 2\sqrt{\mu}\right) + \cos\left(\sqrt{\mu}\, t_1\right)]$

$= a.2\cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2}\ .\cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$

$= 2\,a\cos\left(\sqrt{\mu}\; t_1 + \sqrt{\mu}\right).\cos\sqrt{\mu} = 2x_2\,.\cos\sqrt{\mu}$

$\therefore \dfrac{x_1 + x_3}{2x_2} = \cos\sqrt{\mu}\,, \qquad \sqrt{\mu} = \cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)$

Period $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$

Problem 2

If the displacement of a moving point at any time be given by an equation of the form x = a cos ω t + b sin ω t, show that the motion is a simple harmonic motion.

If a = 3, b=4, ω= 2 determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

Given x = a cos ωt + b sin ϖ t ……………… (1)

Differentiating (1) with respect to t,

$\dfrac{dx}{dt} = -a\omega\sin\ \omega t + b\omega\cos\omega t$ ……………………(2)

$\dfrac{d^2x}{dt^2} = -\omega^2\ \cos\ \omega t - b\ \omega^2 \sin\ \omega\ t$

$= -\omega^2(a\cos\ \omega\ t + b\sin\ \omega\ t) = -\omega^2 x$ ….(3)

$\therefore$ The motion is simple harmonic.

The constant μ of the S,H.M. = ω^2.

$\therefore$ Period $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ secs.

Amplitude is the greatest value of x.

When x is maximum, $\frac{dx}{dt} = 0$.

$-a\omega \sin \omega t + b\omega \cos \omega t = 0$ i.e. a sin ωt = b cos ω t or tan ω t = $\frac{b}{a} = \frac{4}{3}$

When tan ω t = $\frac{4}{3}$, sin ω t = $\frac{4}{5}$ and cos ω t = $\frac{3}{5}$

Greatest value of x = a $\times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a + 4b}{5} = \frac{3.3 + 4.4}{5} = 5$

Hence amplitude = 5.

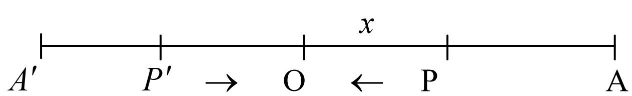
Max. acceleration = μ . Amplitude = 4 x 5 = 20

Max. velocity = $\sqrt{\mu}$. Amplitude = 2 x 5 =10

Problem 3

Show that the energy of a system executing S.H.M. is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration at a distance x from O = μ x.

Force = mass $\times$ acceleration = m μx

If the particle is given displacement dx from P,

work done against the force = m μ x. dx

Total work done in displacing the particle to a distance x

$$= \int_0^x m \;\; \mu x dx = m \;\; \mu \frac{x^2}{2} \qquad \text{.........(1)}$$

Work done = potential energy at P.

If v is the velocity at P. we know that v^2 = $\mu\left(a^2 - x^2\right)$,

$\therefore$ Kinetic energy at P = $\frac{1}{2}$ mv^2 = $\frac{1}{2} m\mu\left(a^2 - x^2\right)$ ……….. (2)

The total energy at P = Potential energy + Kinetic energy

$$= \frac{m\mu x^2}{2} + \frac{m\mu}{2}\left(a^2 - x^2\right) = \frac{m\mu a^2}{2} \ldots\ldots\ldots (3)$$

Total energy at P $\alpha\ a^2$

If n is the frequency, we know that

$$n = \frac{1}{Period} = \frac{1}{\left(\frac{2\pi}{\sqrt{\mu}}\right)} = \frac{\sqrt{\mu}}{2\pi}$$

$\therefore\ \sqrt{\mu} = 2\pi$ n or $\mu = 4\pi^2 n^2$

Total energy = $\frac{1}{2}m.\ 4\pi^2 n^2 a^2 = 2\pi^2 m a^2 n^2\ \ \alpha\ n^2$

Problem 4

A mass of 1 gm. Vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity,

Solution:

Maximum velocity v = $\sqrt{\mu}$. a

Given, frequency $= \frac{1}{T} = 256 = \frac{\sqrt{\mu}}{2\pi}$.

$\therefore\ \sqrt{\mu} = 2 \times 256 \times \pi$.

Given, amplitude = a = 1 millimeter = 1×10^{-1} c.m.

$\therefore$ Maximum velocity, V = $2 \times 256 \times \pi \times \frac{1}{10} = \frac{256\ \pi}{5}$ cm/ sec

Problem 5

In a S.H.M. if f be the acceleration and v the velocity at any time and T is the periodic time. Prove that $f^2T^2 + 4\pi^2v^2$ is constant.

Solution:

Velocity at any time, v $= \sqrt{\mu\left(a^2 - x^2\right)}$

Periodic time $T = \frac{2\pi}{\sqrt{\mu}}, \frac{d^2x}{dt^2} = f.$

For, S.H.M, $\frac{d^2x}{dt^2} = -\mu.x$

$\therefore f = -\mu.x$

$$\therefore f^2T^2 + 4\pi^2v^2 = \mu^2x^2.\frac{4\pi^2}{\mu} + 4\pi\ \mu^2\left(a^2 - x^2\right)$$

$$= 4\pi^2\mu x^2 + 4\pi^2\mu a^2 - 4\pi^2\mu x^2$$

$$= 4\pi^2\mu a^2 \text{ (constant)}$$

Problem 6

A body moving with simple harmonic motion has an amplitude 'a' and period T. Show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$

Solution:

We know, $v^2 = \mu\left(a^2 - x^2\right)$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2}\left(a^2 - x^2\right)$$

$$\therefore v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$$

Problem 7

If the amplitude of a S.H.M. is 'a' and the greatest speed is u, find the period of an oscillation and the acceleration at a given distance from the centre of oscillatin.

Solution:

Given, amplitude = a

Max. velocity = u.

ie) $\sqrt{\mu}\ a = u \Rightarrow \sqrt{\mu} = \frac{u}{a}$

Period of oscillation T = $\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi.a}{u}$ secs.

Acceleration $\frac{d^2x}{dt^2} = \mu\ x = \frac{u^2 x}{a^2}$ units.

Problem 8

A particle, moving in S.H.M. has amplitude 8 cm. If its maximum acceleration is 2cm/sec ^{2}, find (i) its period (ii) maximum velocity and (iii) its velocity when it is 3 cm. from the extreme position

Solution:

Maximum acceleration = 2 cm/ sec ^{2} = $\mu.a. = \mu \times 8$.

$$\therefore \mu = \frac{2}{8} = \frac{1}{4},$$

Period T = $\frac{2\pi}{\sqrt{\mu}} = 2\pi \times \frac{1}{\sqrt{\frac{1}{4}}} = 4\pi$ secs.

Max. velocity = $\sqrt{\mu}.$ a = $\frac{1}{2} \times 8 = 4 cm/$sec.

When the particle is 3 cm from the extreme position, x = 5 cm.

$\therefore$ velocity $^2 = v^2 = \mu\left(a^2 - x^2\right) = \frac{1}{4}(64 - 25) = \frac{39}{4}$.

$\therefore v = \frac{1}{2}\sqrt{39}$ cm / sec.

Problem 9

A particle moves in a straight line. If v be its velocity when at a distance x from a fixed point in the line and $v^2 = \alpha - \beta\ x^2$ where α, β are constants, show that the motion is simple harmonic and determinc its period and amplitude.

Solution:

Given, $v^2 = \alpha - \beta x^2$ ....(1)

Differentiating, $2v.\dfrac{dv}{dt} = -2\beta x\dfrac{dx}{dt}\left[\because v = \dfrac{dx}{dt}\right]$

$$\therefore \frac{dv}{dt} = -\beta x$$

ie) $$\boxed{\frac{d^2x}{dt^2} = -\beta\, x}$$

$\therefore$ The motion is a S.H.M. $\sqrt{\mu} = \sqrt{\beta}$

Period T = $\dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{\beta}}$.

Amplitude is the maximum value of x.

x - is maximum, when $\dfrac{dx}{dt} = 0$

$$\therefore v^2 = \alpha - \beta x^2 = 0, \Rightarrow x = \sqrt{\frac{\alpha}{\beta}}$$

$\therefore$ Amplitude = $\sqrt{\dfrac{\alpha}{\beta}}$

Problem 10

If the distance x of a point moving on a straight line measured from a fixed origin on it and velocity v are connected by the relation $4v^2 = 25 - x^2$, show that the motion is simple harmonic. Find the period and amplitude of the motion.

Solution:

Given, $4v^2 = 25 - x^2$............(1)

Differentiating, $8v.\dfrac{dv}{dt} = -2x.\dfrac{dx}{dt}$

$$\therefore \frac{dv}{dt} = -\frac{1}{4}.x.$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{1}{4}.x.}$$

Hence the motion is a S.H.M. Here $\mu = \dfrac{1}{4}$

$\therefore$ Period $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{4} = 4\pi$ secs.

Amplitude = maximum value of x.

x is maximum when $\dfrac{dx}{dt} = 0$

Ie) $25 - x^2 = 0. \Rightarrow x = \pm 5.$ Maximum value of x= 5.

amplitude = 5

.3 Composition of two simple Harmonic Motions of the same period and in the same straight line

Since the period same, the two separate simple harmonic motions are represented by the same differential equation $\dfrac{d^2x}{dt^2} = -\mu\ x$

Let x_1 and x_2 be the displacements for the separate motions.

$x_1 = a_1 \cos\left(\sqrt{\mu}\ t + \varepsilon_1\right)$, a_1 - amplitude

$x_2 = a_2 \cos\left(\sqrt{\mu}\ t + \varepsilon_2\right)$, a_2 – amplitude

Let x be their resultant displacement, then $x = x_1 + x_2$

ie) $x = a_1 \cos\left(\sqrt{\mu}t + \varepsilon_1\right) + a_2 \cos\left(\sqrt{\mu}t + \varepsilon_2\right)$

$= a_1\left[\cos\sqrt{\mu}t.\cos\varepsilon_1 - \sin\sqrt{\mu}t.\sin\varepsilon_1\right] + a_2\left[\cos\sqrt{\mu}t.\cos\varepsilon_2 - \sin\sqrt{\mu}t.\sin\varepsilon_2\right]$

$= \cos\sqrt{\mu}t\left(a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2\right) - \sin\sqrt{\mu}t\left(a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2\right)$

$= \cos\sqrt{\mu}t.A\cos\varepsilon - \sin\sqrt{\mu}t.A\sin\varepsilon$ ……………….. (1)

where $A\cos\varepsilon = a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2$ ……………..(2)

$A\sin\varepsilon = a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2$ …………… (3)

Squaring (2) and (3) and adding,

$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\left(\varepsilon_1 - \varepsilon_2\right)$ ……….(4)

Dividing (3) by (2), $\tan\ \varepsilon\ = \dfrac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$ ……..(5)

Now (1) becomes $x = A\ .\ \left(\cos\sqrt{\mu}t\cos\varepsilon - \sin\sqrt{\mu}t\sin\varepsilon\right)$

$= A\ .\ \cos\left(\sqrt{\mu}t + \varepsilon\right)$ ……… (6)

The resultant displacement given by (6) also represents a simple harmonic motion of the same period as the individual motions.

4 Composition of two simple Harmonic motions of the same period in two perpendicular directions

If a particle possesses two simple harmonic motions of the same period, in two perpendicular directions, we can prove that its path is an ellipse. Take, two $\perp$r lines as x and y axes. The displacements of the particle can be taken as $x = a_1\ \cos\ \sqrt{\mu}\ t$ …….. (1)

$y = a_2 \cos\left(\sqrt{\mu}\ t\ + \varepsilon\right)$ ……… (2)

Eliminate 't' between (1) and (2)

(2) $\Rightarrow$ $y = a_2 \cos\sqrt{\mu}\ t.\ \cos\ \varepsilon. - a_2 \sin\sqrt{\mu}\ t.\ \sin\ \varepsilon$

$= a_2 \left[\cos\ \varepsilon.\dfrac{x}{a_1} - \sin\ \varepsilon.\sqrt{1 - \dfrac{x^2}{a_1^2}}\right]$ by (1)

$\dfrac{y}{a_2} = \cos\varepsilon.\dfrac{x}{a_1} - \sin\varepsilon.\sqrt{1 - \dfrac{x^2}{a_1{}^2}}$

i.e. $\dfrac{y}{a_2} - \dfrac{x\cos\varepsilon}{a_1} = -Sin\varepsilon.\sqrt{1 - \dfrac{x^2}{a_1^2}}$

Squaring,

$$\frac{y^2}{a_2^2} + \frac{x^2\cos^2\varepsilon}{a_1^2} - \frac{2xy\cos\varepsilon}{a_1 a_2} = \sin^2\varepsilon - \frac{x^2}{a_1^2}\sin^2\varepsilon$$

i.e. $\frac{x^2}{a_1^2} - \frac{2xy}{a_1 a_2}\cos\varepsilon + \frac{y^2}{a_2^2} = \sin^2\varepsilon$ ………. (3)

This is of the form $ax^2 + 2hxy + by^2 = \lambda$ ……. (4)

where $a = \frac{1}{a_1^{\,2}}$, $h = -\frac{\cos\ \varepsilon}{a_1 a_2}$, $b = \frac{1}{a_2^{\,2}}$

(4) represents a conic with centre at the origin.

Also, $ab - h^2 = \frac{1}{a_1^{\,2} a_2^{\,2}} - \frac{\cos^2\ \varepsilon}{a_1^{\,2} a_2^{\,2}} = \frac{\sin^2\varepsilon}{a_1^{\,2} a_2^{\,2}} = +ve$

Hence (3) represents an ellipse.

If $\varepsilon = 0$, equation (3)$\Rightarrow \frac{x}{a_1} - \frac{y}{a_2} = 0$ (straight line).

If $\varepsilon = \pi$, (3) $\Rightarrow \frac{x}{a_1} + \frac{y}{a_2} = 0$ (straight line).

If $\varepsilon = \frac{\pi}{2}$, (3) $\Rightarrow \frac{x^2}{a_1^{\,2}} + \frac{y^2}{a_2^{\,2}} = 1$ (ellipse).

If $\varepsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path is the circle $x^2 + y^2 = a_1^{\,2}$

Problem 11

Show that the resultant of two simple harmonic motions in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a simple harmonic motion of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\frac{\tan^{-1} 2}{2\pi}$ of a period.

Solution:

Let the two displacements be

$x_1 = a_1 \cos\left(\sqrt{\mu}\ t + \varepsilon_1\right)$…………(1) [$\because$ they have equal periodic time,

μ is same]

$x_2 = a_2 \cos\left(\sqrt{\mu}\ t + \varepsilon_2\right)$ ………… (2)

Where $a_2 = 2a_1$ and $\frac{\varepsilon_2 - \varepsilon_1}{\sqrt{\mu}}$ = phase difference (given) = $\frac{1}{4} \times \frac{2\pi}{\sqrt{\mu}}$

$\therefore \varepsilon_2 - \varepsilon_1 = \frac{\pi}{2}$ or $\varepsilon_2 = \frac{\pi}{2} + \varepsilon_1$

We know that the resultant displacement is x = A cos $\left(\sqrt{\mu}\ t + \varepsilon\right)$….. (3)

where $A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)$

$$= a_1^2 + 4a_1^2 + 4a_1^2 \cos(-90^0) = 5a_1^2$$

$\therefore$ amplitude of the resultant motion = A = $a_1\sqrt{5}$

Also $\tan \varepsilon = \frac{a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2}{a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2}$

$$[\because A \sin \varepsilon = a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2,\ \ A \cos \varepsilon = a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2]$$

$$= \frac{a_1 \sin \varepsilon_1 + 2a_1 \sin(90^0 + \varepsilon_1)}{a_1 \cos \varepsilon_1 + 2a_1 \cos(90^0 + \varepsilon_1)}$$

i.e. $\frac{\sin \varepsilon}{\cos \varepsilon} = \frac{\sin \varepsilon_1 + 2\cos \varepsilon_1}{\cos \varepsilon_1 - 2 \sin \varepsilon_1}$

$$\sin\ \varepsilon \cos \varepsilon_1 - 2 \sin \varepsilon \sin \varepsilon_1 = \sin \varepsilon_1 \cos \varepsilon + 2 \cos \varepsilon_1\ \cos \varepsilon$$

$\therefore$ sin $(\varepsilon - \varepsilon_1) = 2\cos(\varepsilon - \varepsilon_1)$ i.e. tan $(\varepsilon - \varepsilon_1) = 2$ $\therefore \varepsilon - \varepsilon_1 = \tan^{-1} 2$

$$\therefore \frac{\varepsilon - \varepsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{2\pi} \left(\frac{2\pi}{\sqrt{\mu}}\right)$$

$$= \frac{\tan^{-1} 2}{2\pi} \text{ of a period}$$

Problem 12

Two simple harmonic motions in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6, find the amplitude and phase of the resulting motion

Solution:

Let the two S.H.M. in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ be,

$x_1 = a_1.\cos\sqrt{\mu}$ t …………(2)

$x_2 = a_2\left(\cos\sqrt{\mu}\ t+\varepsilon\right)$……..(2)

given, A cos $\varepsilon = 4 = a_1$, A sin ε $= 6 = a_2$

$\therefore$ Amplitude of the resultant motion A $= \sqrt{(A\cos\varepsilon)^2 + (ASin\varepsilon)^2}$

$= \sqrt{16+36} \quad = \sqrt{52}$

$$\boxed{A = 2\sqrt{13}}$$

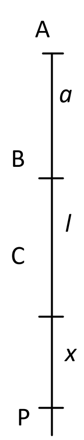
$$\tan\ \varepsilon = \frac{A\ Sin\varepsilon}{A\ Cos\varepsilon} = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \quad \boxed{\varepsilon = \tan^{-1}\left(\frac{3}{2}\right)}$$

which is the phase of the resulting motion.

5 Motion of a particle suspended by a spiral spring

A particle is suspended from a fixed point by a spiral spring of natural length a and modulus λ. If it is displaced slightly in the vertical direction, discuss the subsequent motions



Let AB = a, natural length of the spring which is fixed at A. Let m be the mass of the particle connected at B, which pulls the spring and comes to rest at C such that the increased length BC = *l*. At C, the mass 'm' is in equilibrium. Hence the downward force mg and the upward force T must be equal at C. ie) T = mg

But, by Hooke's law, T = $\frac{\lambda l}{a}$

$$\therefore \frac{\lambda l}{a} = mg \text{ .... (1)}$$

Let the particle be slightly displaced vertically downwards through a distance and then released. It will begin to move upwards. Let P be the subsequent position of the particle so that CP= x

The forces acting at P are the weight and the upward tension.

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = \text{Resultant downward force} = mg - \text{Tension at P.}$$

$$= mg - \frac{\lambda}{a}(AP\text{-}AB)$$

$$= mg - \frac{\lambda}{a}(BP) = mg - \frac{\lambda}{a}(l + x)$$

$$= -\frac{\lambda x}{a} \quad [\because mg = \frac{\lambda l}{a}] \quad \text{by (1)}$$

i.e. $\frac{d^2x}{dt^2} = -\frac{\lambda}{a\ m}x$ ..... (2)

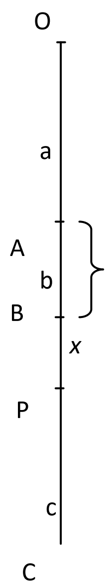
Equation (2) represents a S.H.M.

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{\lambda}{am}}} = 2\pi\sqrt{\frac{am}{\lambda}}$$

Problem 13

Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off. Show that the distance of M from the upper end of the string at time t is a+b+ c cos $\sqrt{\frac{g}{b}}$ t, where a is the unstretched length of the string, and b and c are the distances by which it would be stretched when supporting M and M', respectively.

Solution



Let OA = a be the natural length of the elastic string, which is fixed at O. When the string supports M,

Mg = upward Tension.

By Hooke's law,

upward Tension at B = $\frac{\lambda b}{a}$

$\therefore Mg = \frac{\lambda b}{a}$ ……………….. (1)

When the string supports M^1,

M^1g = upward Tension at C = $\frac{\lambda c}{a}$

ie) $M^1g = \frac{\lambda c}{a}$ …………………….. (2)

(1) + (2) $\Rightarrow M + M' = \frac{\lambda}{a}(b+c)$

ie) At C, M + M' is in equilibrium.

When M' falls off, M will move towards B.

Let P be the position of M at time t seconds such that BP = x

Forces acting at P are,

(i) Weight Mg ii) Upward tension

$\therefore$ At P, equation of motion of M is $M.\frac{d^2x}{dt^2}$ = resultant downward force.

$$= Mg - \frac{\lambda}{a}(OP - OA)$$

$$= Mg - \frac{\lambda}{a}(AP)$$

$$= Mg - \frac{\lambda}{a}(b + x)$$

$$= Mg - \frac{\lambda b}{a} - \frac{\lambda}{a}x$$

$$= -\frac{\lambda}{a}x \qquad \text{by (1)}$$

$$\boxed{\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{aM}.\ x}$$

$\therefore$ The motion of M at P is simple harmonic

Amplitude = BC = c

$\therefore$ Displacement = $x = c.\cos\sqrt{\frac{\lambda}{aM}}\ t$

$$= c.\cos\sqrt{\frac{g}{b}}.\ t \qquad \text{by (1)}$$

$\therefore$ Distance of M from O at time t = OP = OA + AB + BP

= a + b + x

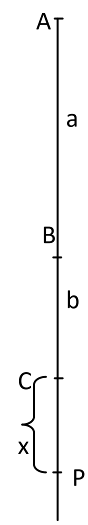
= a + b + c. cos $\sqrt{\frac{g}{b}}.\ t$

Problem 14

A Particle of mass m is tied to one end of an elastic string which is suspended from the other end. The extension caused in its length is b. If the particle is pulled down and let go, show that it executes simple harmonic motion and that the period is $2\pi\sqrt{\dfrac{b}{g}}$

Solution:

Let AB be the natural length of the elastic string. When m is tied at the other end, extended length is b. and the mass is in equilibrium at C.



$\therefore$ At C, mg = T = $\dfrac{\lambda b}{a}$ ____________(1)

When the mass is pulled down and released let P be the subsequent position such that CP = x

At P, equation of motion is

$$m.\frac{d^2x}{dt^2} = \text{ resultant downward force}$$

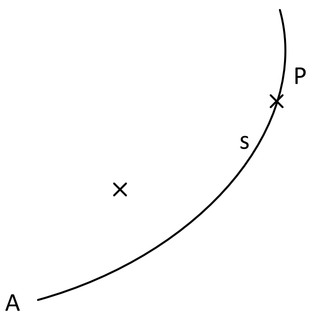
$$= mg - \frac{\lambda(b+x)}{a} = -\frac{\lambda x}{a} \quad \text{[by (1)]}$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{am}.x. \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

(2) shows that the motion is simple harmonic

$$\therefore \text{Period } T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\dfrac{\lambda}{am}}} = 2\pi\sqrt{\frac{am}{\lambda}} = 2\pi\sqrt{\frac{b}{g}} \quad \text{by (1)}$$

6 Simple Harmonic Motion On a Curve



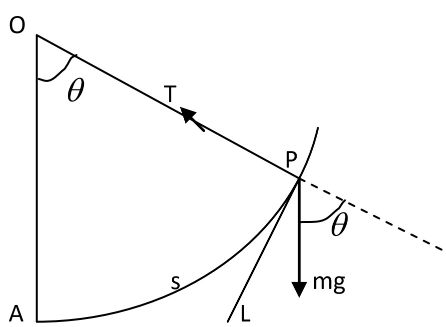
If P is the position of a particle on a curve at time t and if the tangential acceleration at P varies as the arcual distance of P measured from a fixed point A on the curve and is directed towards A, then the motion of P is said to be simple harmonic.

The differential equation for the S.H.M. on a curve will be of the form $\frac{d^2s}{dt^2} = -\mu s$, s is the arc distance AP.

7 Simple pendulum

A simple pendulum consists of a small heavy particle or bob suspended from a fixed point by means of a light inextensible string and oscillating in a vertical plane.

Period of oscillation of a simple pendulum



Let OA = l be the length of the pendulum where O is the point of suspension. Let 'm' be the mass of the bob and P be the position of the bob in time t secs and arc AP = s, $A\hat{O}P = \theta$

The two forces acting are i) mg$(\downarrow)$ ii) Tension T along PO.

mg is resolved into two components i) mg $\cos\theta$ along OP.

ii) mg $\sin\theta$ along PL.

UNIT III
PROJECTILES

3.1 Projectiles.

Definitions:

i. A particle projected into the air in any direction with any velocity is called a **projectile**.

ii. **The angle of projection** is the angle made by the initial velocity with the horizontal plane through the point of projection.

iii. **The velocity of projection** is the velocity with which the particle is projected.

iv. **The trajectory** is the path described by the projectile.

v. The **range on a plane** through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.

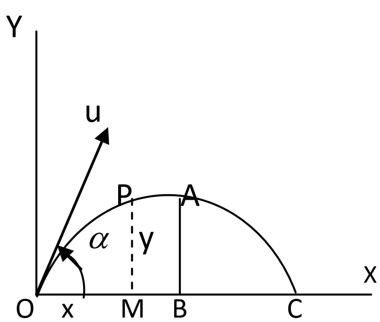
vi. **The time of flight** is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

Two fundamental principles

i. The horizontal velocity remains constant throughout the motion.

ii. The vertical component of the velocity will be subjected to retardation *g*.

3.2 Equation of the path of the projectile



Let a particle be projected from O, with initial velocity u and α be the angle of projection. Take OX and OY as x and y axes respectively. Let P (x,y) be the position of the particle in time t secs. Now u can be divided into two components as u cosα in the horizontal direction and $u\sin\alpha$ in the vertical direction.

Now, horizontal velocity $u\cos\alpha$ is constant throughout the motion.

$\therefore x = (u\cos\alpha)\ t$ ………………… (1)

Vertical velocity is subjected to retardation 'g'

$\therefore y = (u\sin\alpha)\ t - \frac{1}{2}gt^2$ ………………. (2)

Eliminate 't' using (1) and (2)

$$(1) \Rightarrow t = \frac{x}{u\cos\alpha}$$

$$(2) \Rightarrow y = u\sin\alpha\frac{x}{u\cos\alpha} - \frac{1}{2}g.\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \text{………………. (3)}$$

$$= \frac{x\tan\alpha.2u^2\cos^2\alpha - gx^2}{2u^2\cos^2\alpha}$$

$$2u^2\cos^2\alpha.y = x.2u^2\sin\alpha\cos\alpha - gx^2$$

$$\therefore gx^2 - 2u^2\sin\alpha\cos\alpha.x = -2u^2\cos^2\alpha.y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = \frac{-2u^2\cos^2\alpha}{g}y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} - \frac{2u^2\cos^2\alpha}{g}.y$$

ie) $\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$……… (4)

Shifting the origin to $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$

$$X^2 = -\frac{2u^2\cos^2\alpha}{g}.Y \qquad \text{……………….. (5)}$$

(5) is the equation of a parabola of the form $X^2 = -4aY$,

whose latus-rectum is $\frac{2u^2 \cos^2 \alpha}{g} = \frac{2}{g}(u \cos \alpha)^2$

$= \frac{2}{g}(horizontal\ velocity)^2$

Vertex is $\left(\frac{u^2 \sin \alpha . \cos \alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g}\right)$

3.3 Characteristics of the motion of the projectile

1. Greatest height attained by a projectile.

2. Time taken to reach the greatest height.

3. Time of flight.

4. The range on the horizontal plane through the point of projection.

Derive formula for the characteristics

3.3.1 Greatest height h

When the particle reaches the highest point at A, its direction is horizontal.

$\therefore$ At A, vertical velocity = 0

Let AB = h.

Consider the vertical motion and using the formula "$v^2 = u^2 + 2aS$"

$O = (u \sin \alpha)^2 - 2g.h \qquad \therefore h = \frac{u^2 \sin^2 \alpha}{2g}$

- Highest point of the path is the vertex of the parabola.

3.3.2 Time taken to reach the greatest height T

Let T be the time taken to travel from O to reach the greatest height at A.

At A final vertical velocity is zero

At O initial vertical velocity is $u \sin \alpha$

Using the formula "$v = u + at$"

$O = u \sin \alpha - gT \quad \therefore \quad \boxed{T = \frac{u \sin \alpha}{g}}$

3.3.3 Time of flight t

Let t be the time taken to travel from O to C along its path. At C, vertical distance traveled is zero. Consider the vertical motion and by the formula $S = ut + \frac{1}{2}at^2$

$$O = u\sin\alpha \; .t - \frac{1}{2}gt^2$$

ie) $t\left(u\sin\alpha - \frac{1}{2}gt\right) = 0$

$\therefore t = 0$ or $u\sin\alpha - \frac{1}{2}gt = 0$

ie) $t = 0$ or $t = \frac{2u\sin\alpha}{g} = 2\left(\frac{u\sin\alpha}{g}\right) = 2T$

t = 0 gives the time of projection.

$$\therefore \text{Time of flight } t = \frac{2u\sin\alpha}{g}$$

- ❖ Time of flight = 2 x time taken to reach the greatest height.

3.3.4 The range on the horizontal plane through the point of projection R

Range R = OC = horizontal distance traveled during the time of flight.

= horizontal velocity x time of flight

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

- ❖ Horizontal range R = $\frac{2(u\cos\alpha)(u\sin\alpha)}{g} = \frac{2UV}{g}$

Where U – initial horizontal velocity, V – initial vertical velocity.

Problem 1

A body is projected with a velocity of 98 metres per sec. in a direction making an angle $\tan^{-1} 3$ with the horizon; show that it rises to a vertical height of 441 metres and that its time of flight is about 19 sec. Find also horizontal range through the point of projection (g=9.8 metres / sec^2)

Solution:

Given u = 98; $\alpha = \tan^{-1}3$ i.e $\tan\alpha = 3$

$$\therefore \sin\alpha = \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{3}{\sqrt{10}}$$

$$\cos\alpha = \frac{\sin\alpha}{\tan\alpha} = \frac{1}{\sqrt{10}}$$

$$\text{Greatest height} = \frac{u^2\sin^2\alpha}{2g} = \frac{98\times98\times9}{10\times2\times9.8} = 441 \text{ metres}$$

$$\text{Time of flight} = \frac{2u\sin\alpha}{g} = \frac{2\times98\times3}{\sqrt{10}\times9.8} = 6\sqrt{10}$$

$$= 6\times3.162 = 18.972 = 19 \text{ secs. nearly}$$

$$\text{Horizontal range} = \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{2\times98\times98}{9.8}\times\frac{3}{\sqrt{10}}\times\frac{1}{\sqrt{10}} = 588 \text{ metres}$$

Problem 2

If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection

Solution

Let u be the initial velocity and α the angle of projection

$$\text{Greatest height} = \frac{u^2\sin^2\alpha}{2g}$$

Horizontal range = $\dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

Given $\dfrac{u^2 \sin^2\alpha}{2g} = \dfrac{1}{4} \times \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

i.e $\dfrac{u^2 \sin^2\alpha}{2g} = \dfrac{u^2 \sin\alpha \cos\alpha}{2g}$

i.e $\sin\alpha = \cos\alpha \Rightarrow \tan\alpha = 1 \quad \therefore \alpha = 45^0$

Problem 3

A particle is projected so as to graze the tops of two parallel walls, the first of height 'a' at a distance b from the point of projection and the second of height b at a distant 'a' from the point of projection. If the path of particle lies in a plane perpendicular to both the walls, find the range on the horizontal plane and show that the angle of projection exceeds $\tan^{-1}3$.

Solution:

Let u be the initial velocity, α be the angle of projection.

Equation to the path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

i.e $y = xt - \dfrac{gx^2}{2u^2}\left(1+t^2\right)$ where $t = \tan\alpha$ ……. (1)

The tops of the two walls are (b, a) and (a, b) lie on (1)

$\therefore a = bt - \dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ……… (2)

b = $at - \dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (3)

From (2), $a - bt = -\dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ………. (4)

From (3), $b - at = -\dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (5)

Dividing (4) by (5), $\frac{a-bt}{b-at}=\frac{b^2}{a^2}$

i.e $b^3 - ab^2 t = a^3 - a^2bt \quad \Rightarrow \quad t(a^2b - ab^2) = a^3 - b^3$

$$\therefore t = \frac{a^3-b^3}{a^2b-ab^2} = \frac{(a-b)\left(a^2+ab+b^2\right)}{ab(a-b)} = \frac{a^2+ab+b^2}{ab}$$

$$\therefore \tan\alpha = \frac{a^2+ab+b^2}{ab} = \frac{\left(a^2-2ab+b^2\right)+3ab}{ab} = \frac{(a-b)^2}{ab}+3 \text{.....(6)}$$

$$(6) \Rightarrow \tan\alpha > 3 \quad or \ \alpha > \tan^{-1} 3$$

From (4), $\frac{g\left(1+t^2\right)}{2u^2} = \frac{a-bt}{-b^2} = \frac{bt-a}{b^2}$

$$= \frac{\frac{b\left(a^2+ab+b^2\right)}{ab} - a}{b^2} = \frac{a^2+ab+b^2-a^2}{ab^2}$$

$$= \frac{b(a+b)}{ab^2} = \frac{a+b}{ab} \qquad \text{........... (7)}$$

Horizontal range = $\frac{u^2 \sin 2\alpha}{g} = \frac{2u^2 t}{g\left(1+t^2\right)} \quad \because \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

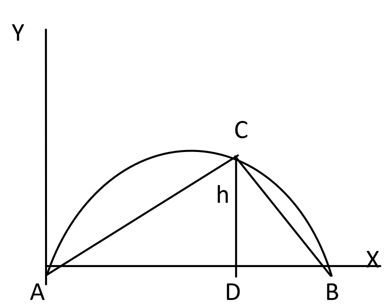
$$= t.\frac{ab}{a+b} \text{ from (7)}$$

$$= \frac{\left(a^2+ab+b^2\right)}{ab}\cdot\frac{ab}{a+b} = \frac{a^2+ab+b^2}{a+b}$$

Problem 4

A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, show that $\tan\alpha$ = tan A + tan B

Solution:



Let u be the velocity of projection and α the angle of projection and let t secs be the time taken from A to C. Draw CD $\perp$ AB and let CD = h.

Consider the vertical motion, h = vertical distance described in time t

$$= u \sin \alpha \cdot t - \frac{1}{2} g t^2$$

AD = horizontal distance described in time t = u cos $\alpha \cdot$ t

$$\text{From } \Delta\text{CAD}, \tan A = \frac{CD}{AD} = \frac{h}{AD} = \frac{u \sin \alpha \cdot t - \frac{1}{2} g t^2}{u \cos \alpha \cdot t}$$

$$= \tan \alpha - \frac{gt}{2u \cos \alpha} \qquad \ldots\ldots (1)$$

$$\text{AB = horizontal range} = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$\therefore \text{DB = AB} - \text{AD} = \frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t$$

$$\text{From } \Delta\text{CDB}, \ \tan B = \frac{CD}{DB} = \frac{h}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t\right)}$$

$$= \frac{u \sin \alpha . t - \frac{1}{2} g t^2}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha . t\right)}$$

$$= \frac{gt\left(u \sin \alpha - \frac{1}{2} gt\right)}{u \cos \alpha (2u \sin \alpha - gt)}$$

$$= \frac{gt(2u \sin \alpha - gt)}{2u \cos \alpha (2u \sin \alpha - gt)} = \frac{gt}{2u \cos \alpha} \ldots\ldots\ldots\ldots (2)$$

(1) + (2) $\Rightarrow$ tanA + tanB = tanα

Problem 5

Show that the greatest height which a particle with initial velocity v can reach on a vertical wall at a distance 'a' from the point of projection is $\frac{v^2}{2g} - \frac{ga^2}{2v^2}$ Prove also that the greatest height above the point of projection attained by the particle in its fight is $v^6/2g(v^4 + g^2a^2)$

Solution:

Equation to the path is $y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$ …….. (1)

Put x = a in (1), $y = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$

$y = at - \frac{ga^2}{2v^2}(1+t^2)$ where t = tan α …….. (2)

y is a function of t. $\therefore$ y is maximum when $\frac{dy}{dt} = 0$ and $\frac{d^2y}{dt^2}$ is negative.

Differentiating (2) with respect to t,

$$\frac{dy}{dt} = a - \frac{ga^2}{2v^2}\cdot 2t = a - \frac{ga^2t}{v^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{v^2} = \text{negative}$$

So y is maximum when $a - \frac{ga^2t}{v^2} = 0$ or $t = \frac{v^2}{ga}$ ……(3)

Put $t = \frac{v^2}{ga}$ in (2)

Max value of $y = a\cdot\frac{v^2}{ga} - \frac{ga^2}{2v^2}\left(1 + \frac{v^4}{g^2a^2}\right)$

$$= \frac{v^2}{g} - \frac{ga^2}{2v^2} - \frac{v^2}{2g} = \frac{v^2}{2g} - \frac{ga^2}{2v^2}$$

Greatest height during the flight

$$= \frac{v^2\sin^2\alpha}{2g} = \frac{v^2}{2g}\cdot\frac{1}{\cos ec^2\alpha} = \frac{v^2}{2g(1+\cot^2\alpha)}$$

$$= \frac{v^2}{2g\left(1 + \frac{g^2a^2}{v^4}\right)} \text{ from (3)}$$

$$= \frac{v^6}{2g\left(v^4 + g^2a^2\right)}$$

Problem 6

a. A projectile is thrown with a velocity of 20 m/sec. at an elevation 30^0. Find the greatest height attained and the horizontal range.

b. A particle is projected with a velocity of 9.6 metres at an angle of 30^0. Find

i. The time of flight

ii. the greatest height of the particle.

Solution:

Given $u = 20\text{m/sec}; \ \alpha = 30^0$

Greatest height = $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{20^2\left(\sin 30^0\right)^2}{2 \times 9.8} = 5.1m$

Horizontal range = $\dfrac{u^2 \sin 2\alpha}{g} = \dfrac{20^2 \cdot \sin 60^0}{9.8} = 35.35m$

Problem 7

(a) A particle is projected under gravity in a vertical plane with a velocity u at an angle α to the horizontal. If the range on the horizontal be R and the greatest height attained by h, show that $\dfrac{u^2}{2g} = h + \dfrac{R^2}{16h}$ and $\tan\alpha = \dfrac{4h}{R}$.

(b) A particle is projected so that on its upward path, it passes through a point x feet horizontally and y feet vertically from the point of projection. Show that, if R be the horizontal range, the angle of projection is $\tan^{-1}\left(\dfrac{y}{x} \cdot \dfrac{r}{R - x}\right)$.

Solution:

a) $h+\dfrac{R^2}{16h}=\dfrac{u^2\sin^2\alpha}{2g}+\dfrac{\left(\dfrac{2u^2\sin\alpha\cos\alpha}{g}\right)^2}{16\left(\dfrac{u^2\sin^2\alpha}{2g}\right)}$

$$=\frac{u^2\sin^2\alpha}{2g}+\frac{u^2\cos^2\alpha}{2g}=\frac{u^2}{2g}$$

b) Equation of the path is, $y=x\tan\alpha-\dfrac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore x\tan\alpha=y+\frac{gx^2}{2u^2\cos^2\alpha}$$

$$\therefore \tan\alpha=\frac{y}{x}+\frac{gx}{2u^2\cos^2\alpha} \qquad \text{........... (1)}$$

We have $R=\dfrac{2u^2\sin\alpha\cdot\cos\alpha}{g}\Rightarrow g=\dfrac{2u^2\sin\alpha\cos\alpha}{R}$

$\therefore (1)\Rightarrow \tan\alpha=\dfrac{y}{x}+\dfrac{x}{2u^2\cos^2\alpha}\times\dfrac{2u^2\cdot\sin\alpha\cos\alpha}{R}=\dfrac{y}{x}+\dfrac{x\tan\alpha}{R}$

$\therefore \tan\alpha\left(1-\dfrac{x}{R}\right)=\dfrac{y}{x}$

ie $\tan\alpha\left(\dfrac{R-x}{R}\right)=\dfrac{y}{x}$ or $\tan\alpha=\dfrac{y}{x}.\dfrac{R}{R-x}$

$\therefore \alpha=\tan^{-1}\left(\dfrac{y}{x}\cdot\dfrac{R}{R-x}\right)$

Problem 8

If the time of flight of a shot is T seconds over a range of x metres, show that the elevation is $\tan^{-1}\left(\dfrac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.

Solution:

Given, horizontal range R = x metres

Time of flight $T = \dfrac{2u\sin\alpha}{g}$ ………. (1)

where α -is the angle of projection

$$\therefore x = \frac{2u^2 \sin\alpha\cos\alpha}{g}$$

$\therefore$ (1) $\Rightarrow$ gT = 2u sinα . $\Rightarrow$ $\boxed{u = \dfrac{gT}{2\sin\alpha}}$

$$\therefore x = \frac{2\cdot g^2T^2\cdot\sin\alpha\cos\alpha}{4\sin^2\alpha\cdot g} = \frac{1}{2}gT^2\cdot\cot\alpha$$

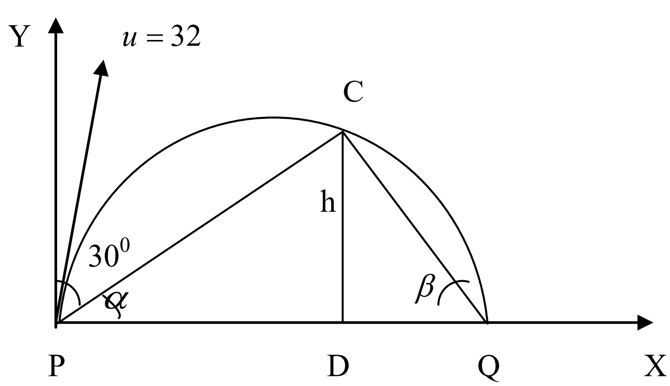
$$\therefore \tan\alpha = \frac{gT^2}{2x} \Rightarrow \boxed{\alpha = \tan^{-1}\left(\frac{gT^2}{2x}\right)}$$

Maximum height = $\dfrac{u^2\sin^2\alpha}{2g} = \dfrac{g^2T^2}{4\sin^2\alpha}\cdot\dfrac{\sin^2\alpha}{2g} = \dfrac{gT^2}{8}$

Problem 9

A particle is projected from a point P with a velocity of 32m per second at an angle of 30^0 with the horizontal. If PQ be its horizontal range and if the angles of elevation from P and Q at any instant of its flight be α and β respectively, show that $\tan\alpha + \tan\beta = \dfrac{1}{\sqrt{3}}$

Solution:



Given, initial velocity u = 32 m/sec, 30^0 is the angle of projection. P-be the point of projection. 't' – be the time taken from P to C.

Let CD = h = $u \sin \alpha . t - \frac{1}{2} g t^2$

$$h = (32 . \sin 30^0)\ t - \frac{1}{2} g t^2 = \text{ vertical distance described in t secs}$$

$$= 16t - \frac{1}{2} g t^2$$

PD = horizontal distance described in t secs = $u \cos \alpha . t$

$$= \left(32 \cos 30^0\right) t \qquad = 32 \cdot \frac{\sqrt{3}}{2} t \qquad = 16\sqrt{3} t.$$

From Δ PCD , $\tan \alpha = \frac{h}{PD} = \frac{h}{16\sqrt{3}t}$ …….. (1)

From Δ QCD, $\tan \beta = \frac{h}{DQ} = \frac{h}{PQ - PD}$, PQ = range

ie $$\tan \beta = \frac{h}{\left(\frac{2(32)^2 \sin 30^0 \cdot \cos 30^0}{g}\right) - 16\sqrt{3}t}$$

$$= \frac{hg}{512\sqrt{3} - 16\sqrt{3}gt} \qquad \text{………(2)}$$

$$\therefore (1) + (2) \Rightarrow \tan \alpha + \tan \beta = \frac{h}{16\sqrt{3}} \left[\frac{1}{t} + \frac{g}{32 - gt}\right]$$

$$= \frac{\left(16t - \frac{1}{2} g t^2\right)}{16\sqrt{3}} \left[\frac{32 - gt + gt}{t(32 - gt)}\right]$$

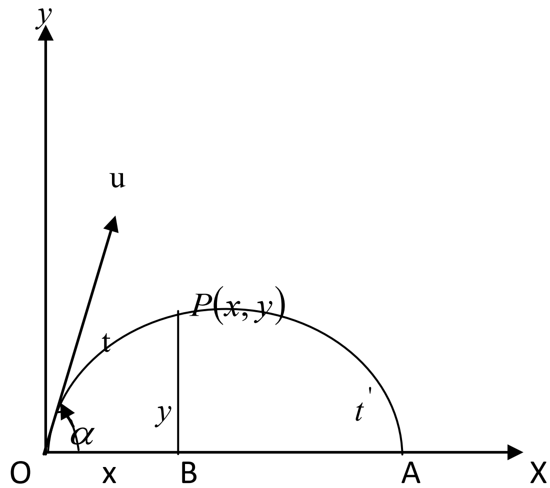
$$= \frac{t\ \left(32 - gt\right)}{32\sqrt{3}} \times \frac{32}{t(32 - gt)} = \frac{1}{\sqrt{3}}$$

$\therefore$ $$\boxed{\tan \alpha + \tan \beta = \frac{1}{\sqrt{3}}}$$

Problem 10

A particle is projected and after time t reaches a point P. If t is the lime it takes to move from P to the horizontal plane through the point of projection, prove that the height of P above the plane is $\frac{1}{2}gt\ t'$

Solution:



Let u be the velocity of projection, α be the angle of projection, P be the position of the particle after t secs. Let t' be the time taken to travel from P to A

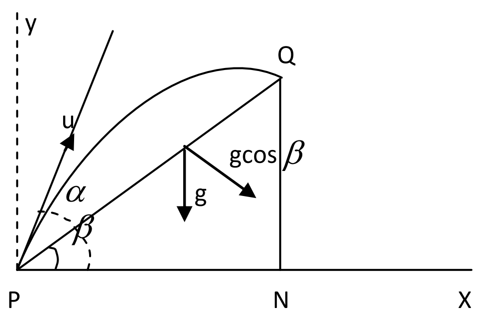
$\therefore$ We have $t + t'$ = time of flight = $\frac{2u\sin\alpha}{g}$ $\therefore u\sin\alpha = \frac{g(t+t')}{2}$

Now, y = vertical distance described in t secs $= (u\sin\alpha)\ t - \frac{1}{2}gt^2$

$$= \frac{g(t+t')\ t}{2} - \frac{1}{2}gt^2 \quad = \frac{gtt'}{2}$$

$\therefore$ Height of P above the plane = $= \frac{gtt'}{2}$

3.4 Range on an inclined Plane:



Let P be the point of projection on a plane of inclination β, u be the velocity of projection at an angle α with the horizontal. The particle strikes the inclined plane at Q. Then PQ = r is the range on the inclined plane. Take PX and PY as x and y axes.

Draw $QN \perp PX$.

From $\Delta PQN,\ PN = r\cos\beta,\ QN = r\sin\beta$

$Q(r\cos\beta, r\sin\beta)$ lies on the path. $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore r\sin\beta = r\cos\beta.\tan\alpha - \frac{g(r\cos\beta)^2}{2u^2\cos^2\alpha}$$

Dividing by r we get $\dfrac{gr\cos^2\beta}{2u^2\cos^2\alpha} = \cos\beta.\dfrac{\sin\alpha}{\cos\alpha} - \sin\beta$

$$\therefore r = \frac{2u^2\cos^2\alpha}{g\cos^2\beta}\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}\right]$$

$$\boxed{r = \frac{2u^2\cos\alpha}{g\cos^2\beta}\sin(\alpha - \beta)}$$

3.5 Maximum range on the inclined plane, given u the velocity of projection and β the inclination of the plane:

Range r on the inclined plane is

$$r = \frac{2u^2 \cos\alpha \sin(\alpha - \beta)}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}\left[\sin(2\alpha - \beta) - \sin\beta\right] \quad \text{..... (1)}$$

Now u and β are given, g constant.

So r is maximum when $\left[\sin(2\alpha - \beta) - \sin\beta\right]$ is maximum.

i.e. when sin $(2\alpha - \beta)$ is maximum.

i.e.when. $2\alpha - \beta = \frac{\pi}{2}$

$$\therefore \quad \boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}} \quad \text{for maximum range.}$$

From (1), maximum range on the inclined plane

$$= \frac{u^2}{g\cos^2\beta}(1 - \sin\beta) = \frac{u^2}{g(1 + \sin\beta)}$$

3.5.1 Time of flight T (up an inclined plane):

From the figure in 6.11, the time taken to travel from P to Q is the time of flight. Consider the motion perpendicular to the inclined plane. At the end of time T, the distance travelled perpendicular to the inclined plane S = 0, component of g perpendicular to the inclined plane is $g\cos\beta$, initial velocity perpendicular to the inclined plane is $u\sin(\alpha - \beta)$.

$$0 = u\sin(\alpha - \beta)T - \frac{1}{2}g\cos\beta . T^2 \qquad \text{using } "S = ut + \frac{1}{2}at^2"$$

$$\boxed{\therefore T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}}$$

3.5.2 Greatest distance S of the projectile from the inclined plane and show that it is attained in half the total time of flight:

Consider the motion perpendicular to the inclined plane. The initial velocity perpendicular to the plane is u sin (α - β) and this is subjected to an acceleration gcos β in the same direction but acting downwards. Let S be the greatest distance travelled by the particle perpendicular to the inclined plane. At the greatest distance the velocity becomes parallel to the inclined plane and hence the velocity perpendicular to the plane is zero.

Using the formula "$v^2 = u^2 + 2as$"

$$0 = [u\sin(\alpha - \beta)]^2 - 2g\cos\beta.S$$

$$\boxed{S = \frac{u^2.\sin^2(\alpha - \beta)}{2.g\cos\beta}}$$

3.5.3 Time taken to reach the greatest distance t :

When the particle is at the greatest distance from the inclined plane, its velocity becomes parallel to the inclined plane and the velocity perpendicular to the inclined plane is zero. So, if t is the time taken to reach the greatest distance, using the formula

"$v = u + at$"

$$\therefore 0 = [u\sin(\alpha - \beta)] - g\cos\beta \cdot t$$

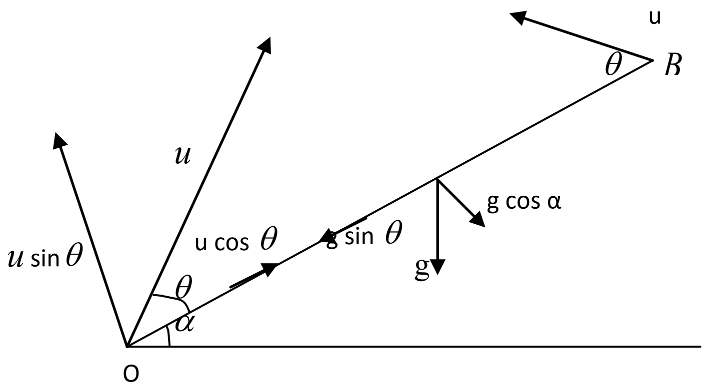
i.e. $\boxed{t = \frac{u\sin(\alpha - \beta)}{g\cos\beta}}$

Note : Time of flight T = $\frac{2u\sin(\alpha - \beta)}{g\cos\beta}$ = 2.t = 2 × time taken to reach the greatest distance.

Problem 11

Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$

Solution



Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane. u has two components u$\cos\theta$ along the upward inclined plane and u$\sin\theta$ perpendicular to the inclined plane. g has two components, g sin α along the downward inclined plane and g$\cos\alpha$ perpendicular to the inclined plane and downwards.

Consider the motion perpendicular to the inclined plane. Let T be the time of flight. Distance travelled perpendicular to the inclined plane in time T = 0

$$\therefore 0 = u\sin\theta \cdot T - \frac{1}{2} g\cos\alpha \cdot T^2 \qquad \left(\because S = ut + \frac{1}{2}at^2\right)$$

i.e. $T = \dfrac{2u\sin\theta}{g\cos\alpha}$

Range up the plane = R_1

R_1= distance travelled along the plane in time T

$$= u\cos\theta \cdot T - \frac{1}{2} g\sin\alpha \cdot T^2$$

$$= u\cos\theta \cdot \frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2} g\sin\alpha \cdot \frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha) = \frac{u^2}{g\cos^2\alpha}\cdot 2\cos(\theta+\alpha)\sin\theta$$

$$= \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha)-\sin\alpha\right]$$

R_1 is maximum, when $\sin(2\theta+\alpha)=1$

$\therefore$ Maximum range up the plane

$$= \frac{u^2}{g\cos^2\alpha}(1-\sin\alpha) = \frac{u^2}{g(1+\sin\alpha)} \quad \ldots\ldots\ldots\ldots (1)$$

When the particle is projected down the plane from B at the same angle θ to the plane, the time of flight T has the same value $\frac{2u\sin\theta}{g\cos\alpha}$. The component of the initial velocity along the inclined plane is u $\cos\theta$ downwards and the component of acceleration g sin α is also downwards.

Range down the plane = R_2

R_2 = distance travelled along the plane in time T

$$= u\cos\theta\cdot T + \frac{1}{2}g\sin\alpha\cdot T^2$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha) = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha)+\sin\alpha\right]$$

R_2 is maximum, when sin $(2\theta - \alpha) = 1$.

Maximum range down the plane

$$= \frac{u^2}{g\cos^2\alpha}(1+\sin\alpha) = \frac{u^2}{g(1-\sin\alpha)} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \frac{Max\cdot range\ down\ the\ plane}{Max\cdot range\ up\ the\ plane} = \frac{u^2}{g(1-\sin\alpha)}\cdot\frac{g(1+\sin\alpha)}{u^2} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

Problem 12

A particle is projected at an angle α with a velocity u and it strikes up an inclined plane of inclination β at right angles to the plane. Prove that (i) cotβ = 2tan(α - β) (ii) cotβ = tan $\alpha - 2\tan\beta$. If the plane is struck horizontally, show that $\tan\alpha = 2\tan\beta$.

Solution:

The initial velocity and acceleration are split into components along the plane and perpendicular to the plane.

The time of flight is $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ ….. (1)

Since the particle strikes the inclined plane normally, its velocity parallel to the inclined plane at the end of time T is = 0.

i.e. 0 = u cos (α - β) – g sin $\beta \cdot T$

$$T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{….. (2)}$$

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \text{ from (1) and (2)}$$

i.e. cot β = 2 tan ($\alpha - \beta$) …… (i)

i.e. $\cot\beta = \dfrac{2(\tan\alpha - \tan\beta)}{1+\tan\alpha\tan\beta}$, Simplifying we get

cot β + tan α = 2 tan $\alpha - 2$ tan β

cot β = tan $\alpha - 2$ tan β ….. (ii)

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T = 0. Initial vertical velocity = u sinα, and acceleration in this direction = g (downwards).

Vertical velocity in time T = u sin $\alpha -$ gT

$\therefore$ u sin $\alpha - gT = 0$ or $T = \dfrac{u\sin\alpha}{g}$ ….. (3)

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\sin\alpha}{g}$$ from (1) and (3)

Simplifying we get

$2\sin(\alpha - \beta) = \sin\alpha\cos\beta$

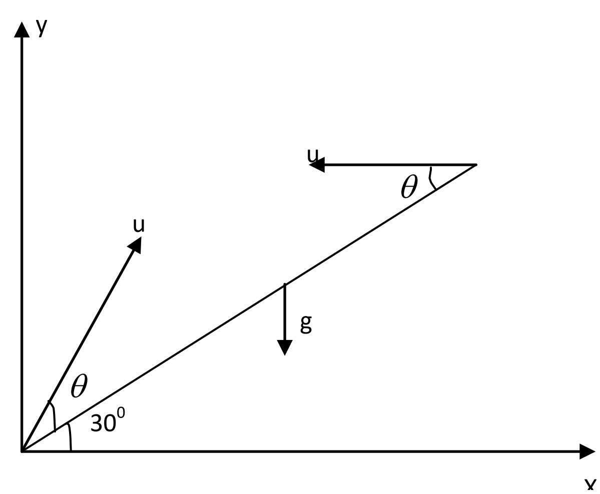
$2(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin\alpha\cos\beta$.

$\sin\alpha\cos\beta = 2\cos\alpha\sin\beta$ or $\tan\alpha = 2\tan\beta$

Problem 13

The greatest range with a given velocity of projection on a horizontal plane is 3000 metres. Find the greatest ranges up and down a plane inclined at 30^0 to the horizon.

Solution:



Let u be the velocity of projection, θ be the inclination of direction of projection with the plane. Given $\dfrac{u^2}{g} = 3000\,m \Rightarrow u^2 = 3000 \times g$

At the end of time t, distance travelled perpendicular to the inclined plane is zero.

$$\therefore 0 = u\sin\theta \cdot T - \frac{1}{2} g\cos 30^0 \cdot T^2$$

$$0 = u\sin\theta \cdot T - \frac{1}{2} g \cdot \frac{\sqrt{3}}{2} \cdot T^2$$

$$\boxed{\therefore T = \frac{4u\sin\theta}{g\sqrt{3}}}$$

Range up the inclined plane, S = $u\cos\theta \cdot T - \frac{1}{2} g \cdot \sin 30^0 \cdot T^2$

$$= u\cos\theta \cdot \frac{4u\sin\theta}{g\sqrt{3}} - \frac{1}{4} \cdot g \cdot \times \frac{16u^2\sin^2\theta}{3g^2}$$

$$= \frac{4u^2\sin\theta\cos\theta}{g\sqrt{3}} - \frac{4u^2\sin^2\theta}{3g}$$

S = $\frac{4u^2\sin\theta}{3g}\left[\sqrt{3}\cos\theta - \sin\theta\right]$

Max. range is got when $\sin(2\theta + 30^0) = 1$

i.e. $2\theta + 30^0 = 90^0 \therefore \theta = 30^0$

Max. range up the inclined plane

$$= S_{\max} = \frac{4u^2\sin 30^0}{3g}\left[\sqrt{3}\cos 30^0 - \sin 30^0\right]$$

$$= \frac{4u^2 \times \frac{1}{2}}{3g}\left[\sqrt{3} \times \frac{\sqrt{3}}{2} - \frac{1}{2}\right] = \frac{2}{3} \times 3000 \quad S_{\max} = 2000m$$

$\therefore$ Range down the inclined plane = $\frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta - \alpha) + \sin\alpha\right]$

Max. range down the inclined plane

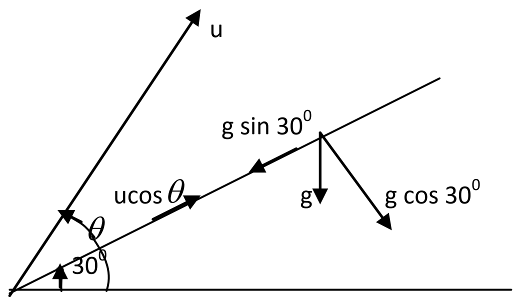
$$= \frac{u^2}{g \cdot \cos^2 30^0}\left[1 + \sin 30^0\right] = \frac{4u^2}{3g}\left[1 + \frac{1}{2}\right]$$

$$= \frac{2u^2}{g} = 2 \times 3000 = 6000m$$

Problem 14

An inclined plane is inclined at an angle of 30^0 to the horizon. Show that, for a given velocity of projection, the maximum range up the plane is 1/3 of the maximum range down the plane.

Solution:



Max. range up the plane $= \dfrac{u^2}{g \cdot \cos^2 30^0}\left[1-\sin 30^0\right] = \dfrac{2u^2}{3g}$

Max. range down the plane $= \dfrac{u^2}{g \cdot \cos^2 30^0}\left[1+\sin 30^0\right]$

$$= \frac{4u^2}{3g} \times \frac{3}{2} = \frac{2u^2}{g}$$

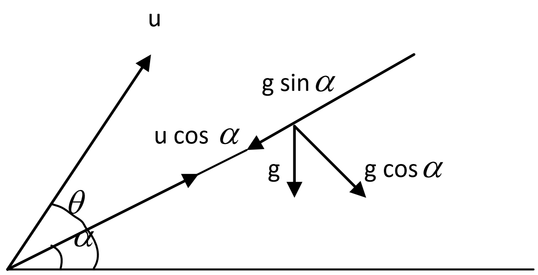
Max. range up the plane $= \dfrac{1}{3} \times \dfrac{2u^2}{g}$

$$= \frac{1}{3} \times \max \cdot \textit{range down the plane}$$

Problem 15

If the greatest range down an inclined plane is three times its greatest range up the plane then show that the plane is inclined at 30^0 to the horizon..

Solution



Greatest range down the inclined plane R_1

$$R_1 = \frac{u^2}{g\cos^2\alpha}[1+\sin\alpha]$$

Greatest range down the inclined plane R_2

$$R_2 = \frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

Given, $R_1 = 3R_2$

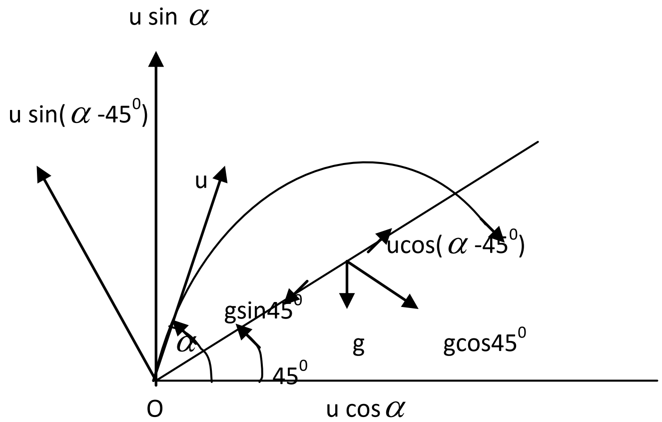
i.e. $$\frac{u^2}{g\cos^2\alpha}[1+\sin\alpha] = 3\cdot\frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

$$\sin\alpha = \frac{1}{2} \quad \therefore \alpha = 30^0$$

Problem 16

A particle is projected in a vertical plane at an angle α to the horizontal from the foot of a plane whose inclination to the horizon is 45^0. Show that the particle will strike the plane at right angles if tan α =3.

Solution:



When the particle strikes the plane at right angles, velocity parallel to the plane is zero.

$$\therefore O = u\cos(\alpha - 45^0) - g\cdot\sin 45^0 \cdot T$$

$$\therefore T = \frac{u\cos(\alpha-45^0)}{g\sin 45^0} = \frac{u\cos(\alpha-45^0)}{g\cdot\frac{1}{\sqrt{2}}} \qquad \ldots\ldots (1)$$

Also, time of flight, $T = \dfrac{2u\cdot\sin\left(\alpha - 45^0\right)}{g\cdot\cos 45^0}$ ……. (2)

$$(1)\,\&\,(2) \Rightarrow \frac{u\cos\left(\alpha-45^0\right)}{g\cdot\frac{1}{\sqrt{2}}} = \frac{2u\cdot\sin\left(\alpha-45^0\right)}{g.\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \cos\left(\alpha - 45^0\right) = 2\cdot\sin(\alpha - 45^0) \;\Rightarrow 2\cdot\tan\left(\alpha-45^0\right) = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - \tan 45^0}{1+\tan\alpha\cdot\tan 45^0}\right] = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - 1}{1+\tan\alpha}\right] = 1$$

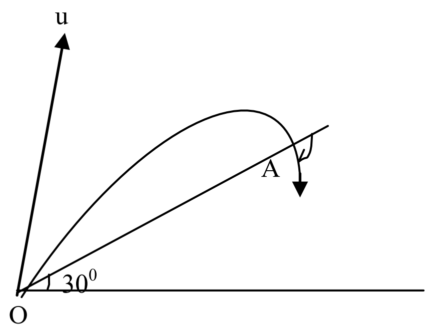
i.e. $2(\tan\alpha - 1) = 1 + \tan\alpha$

$\therefore \tan\alpha = 3$

Problem 17

A particle is projected with speed u so as to strike at right angles a plane through the point of projection inclined at 30^0 to the horizon. Show that the range on this inclined plane is $\frac{4u^2}{7g}$

Solution:



Since u is the velocity of projection, $\beta = 30^0$ is the inclination of the inclined plane, we have proved, Range on the inclined plane = OA

$$= \frac{2u^2 \cdot \sin\beta}{g\left(1 + 3\sin^2\beta\right)}$$

$$= \frac{2u^2 \cdot \sin 30^0}{g\left(1 + 3\sin^2 30^0\right)}$$

$$= \frac{2u^2 \times \frac{1}{2}}{g\left(1 + \frac{3}{4}\right)} \qquad = \frac{4u^2}{7g}$$

3. 6 Impulsive Forces

3.6.1 Impulse:

The term impulse of force is defined as follows:

(1) The impulse of a constant force F during a time interval T is defined as the product FT.

Let f be the constant acceleration produced on a particle of mass m on which F acts and u, v be respectively the velocity at the beginning and end of the period T.

Then v-u = fT and F = mf.

Hence the impulse I = FT = mfT = m(v-u)

=change of momentum produced.

(2) The impulse of a variable force F during a time interval T is defined to be the time integral of the force for that interval.

i.e. Impulse I = $\int_o^T$ Fdt. This is got as follows. During a short interval of time Δt, the force F can be taken to be constant and hence elementary impulse in this interval = F. Δt. Hence the impulse during the whole time T for which the force F acts is the sum of such impulses and

$$= \underset{\Delta t \to 0}{\text{Lt}} \sum_{t-0}^{T} F.\Delta t = \int_{o}^{T} F\,dt.$$

Since F is variable, F = m . $\frac{dv}{dt}$

So impulse = $\int_o^T m \frac{dv}{dt} dt$ = mv – mu, where u and v are the velocities at the beginning and end of the interval and hence this is also equal to the change of momentum produced.

Thus whether a force is a variable or constant,

its impulse = change of momentum produced.

3.6.2 Impulsive Force:

The change of momentum produced by a variable force P acting on a body of mass m from time t = t_1 to t = t_2 is $\int_{t1}^{t2}$ P dt. Suppose P is very large but the time interval t_2 - t_1 during which it acts is very small. It is quite possible that the above definite integral tends to a finite limit. Such a force is called an impulsive force.

Thus an impulsive force is one of large magnitude which acts for a very short period of time and yet produces a finite change of momentum.

Theoretically an impulsive force should be infinitely great and the time during which it acts must be very small. This, of course, is never realized in practice, but approximate examples are (1) the force produced by a hammer-blow (2) the impact of a bullet on a target. In such cases the measurement of the magnitude of the actual force is impracticable but the change in momentum produced may be easily measured. Thus an impulsive force is measured by its impulse i.e. the change of momentum it produces.

Since an impulsive force acts only for a short time on a particle, during this time the distance travelled by a particle having a finite velocity is negligible. Also suppose a body is acted upon by impulsive forces is very short, during this time, the effect of the ordinary finite forces can be neglected.

3.7. Collision of Elastic Bodies

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yields slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the force of restitution. Also, the properly which causes a solid body to recover its shape is called elasticity. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be inelastic. When a body completely regains its shape after a collision, it is said to be perfectly elastic. If it does not come to its original shape, it is said to be perfectly inelastic.

Definitions:

Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.

Two bodies are said to impinge obliquely, if the direction of motion of either body or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the line of impact. Thus, in the cause of two spheres, the line of impact is the line joining their centres.

3.8. Fundamental Laws of Impact:

1. Newton's Experimental Law (NEL):

When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the *coefficient (or modulus) of elasticity (or restitution or resilience).*

This law can be put symbolically as follows: If u_1, u_2 are the components of the velocities of two impinging bodies along their common normal before impact and v_1, v_2 their component velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution, then

$$\frac{v_2 - v_1}{u_2 - u_1} = -e.$$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which $e = 0$ are said to be inelastic. For perfectly elastic bodies, $e=1$. Probably, there are no bodies in nature coming strictly under wither of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

2. Motion of two smooth bodies perpendicular to the line of Impact:

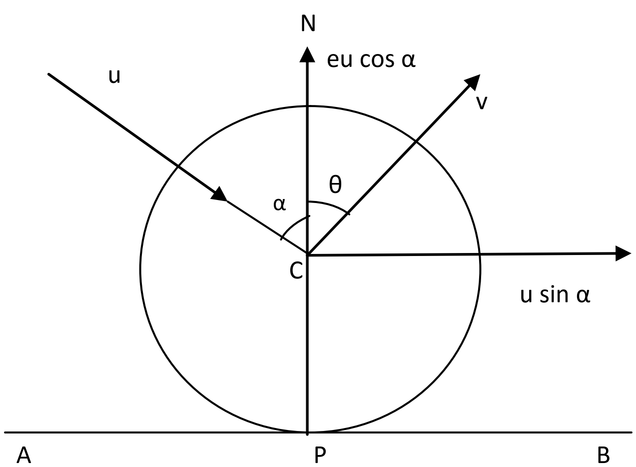
When two smooth bodies impinge, the only force between them at the time of impact is the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence the velocity of either body resolved in a direction perpendicular to the line of impact is not altered by impact.

3. Principle of Conservation of Momentum (PCM) :

We can apply the law of conservation of momentum in the case of two impinging bodies. The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum of their moments before impact, all momenta being measured along the common normal.

3.9. Impact of a smooth sphere on a fixed smooth plane:

A smooth sphere, or particle whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.



Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure.

Since the plane and the sphere are smooth, the only force acting during impact is the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere, resolved in a direction parallel to the plane is unaltered by the impact.

Hence $v \sin\theta = u \sin\alpha$ … (1)

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is (-e) time its relative velocity along the common normal before impact. Hence

$$v \cos \theta - 0 = -e(-u \cos \alpha - 0)$$

i.e. $v \cos \theta = eu \cos \alpha$ …(2)

Squaring (1) and (2), and adding, we have

$v^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha)$

i.e. $v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$ ... (3)

Dividing (2) by (1), we have $\cot \theta = e \cot \alpha$ ... (4)

Hence the (3) and (4) give the velocity and direction of motion after impact.

Corollary 1: If e = 1, we find that from (3) v = u and from (4) $\theta = \alpha$. Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Cor. 2: If e = 0, then from (2), $v \cos \theta = 0$ and from (3), $v = u \sin \alpha$. Hence $\cos \theta = 0$ i.e. $\theta = 90^{\circ}$. Hence the inelastic sphere slides along the plane with velocity $u \sin \alpha$

Cor. 3: If the impact is direct we have $\alpha = 0$. Then $\theta = 0$ and from (3) v=cu. Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.

Cor. 4: The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$I = mv \cos \theta - (-mu \cos \alpha)$

$= m(v \cos \theta + u \cos \alpha)$

$= m(cu \cos \alpha + u \cos \alpha)$

$= mu \cos \alpha (1 + e)$

Cor. 5: Loss of kinetic energy due to the impact

$$= \frac{1}{2} mu^2 - \frac{1}{2} mv^2 = \frac{1}{2} mu^2 - \frac{1}{2} mu^2 \ (\sin^2 \alpha \ + e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} mu^2 (1 - \sin^2 \alpha \ + e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} mu^2 (\cos^2 \alpha \ - e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} (1 - e^2)\, mu^2 \cos^2 \alpha$$

If the sphere is perfectly elastic, e = 1 and the loss of kinetic energy is zero.

Problem 18

A particle falls from a height h upon a fixed horizontal plane: if e be the coefficient of restitution, show that the whole distance described before the particle has finished rebounding is $h\left(\frac{1+e^2}{1-e^2}\right)$. Show also that the whole time taken is $\frac{1+e}{1-e}.\sqrt{\frac{2h}{g}}$.

Solution:

Let u the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$. After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity eu. After the second impact, it rebounds with a velocity c^2u and the process is repeated a number of times. The velocities after the third, fourth etc. impacts are e^3u e^4u etc.

The height ascended after the first impact with velocity eu is $\frac{(\text{velocity})^2}{2g}$

$$= \frac{e^2u^2}{2g}$$

The height ascended after the second impact with velocity e^2u is $e^4u^2/2g$ and so on.

∴ Total distance travelled before the particle stops rebounding

$$= h + 2\left(\frac{e^2u^2}{2g} + \frac{e^4u^2}{2g} + \frac{e^6u^2}{2g} + \dots\dots\dots\right)$$

$$= h + \frac{2 \,.\, e^2u^2}{2g}\ (1 + e^2 + e^4 + \dots\dots\dots \text{ to } \infty)$$

$$= h + \frac{e^2u^2}{g} \cdot \frac{1}{1-e^2}$$

$$= h + \frac{e^2.2gh}{g} \cdot \frac{1}{1-e^2}$$

$$= h\,(1 + \frac{2e^2}{1-e^2})$$

$$= h. \frac{(1+e^2)}{(1-e^2)}$$

Considering the motion before the first impact, we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken, u = 0 + gt.

$$\therefore t = \frac{u}{g} = \frac{\text{velocity}}{g}$$

Time interval between the first and second impacts is

= 2 x time taken for gravity to reduce the velocitiy to 0.

= 2. velocity / g

= 2 eu / g.

Similarly time interval between the second and third impacts

= 2 e^2 u/g and so on.

So total time taken

$$= \frac{u}{g} + 2(\frac{eu}{g} + \frac{e^2u}{g} + \frac{e^3u}{g} + \dots \dots \infty)$$

$$= \frac{u}{g} + \frac{2\,e\,u}{g}\ (1 + e + e^2 + \dots \dots \dots \text{to } \infty)$$

$$= \frac{u}{g} + \frac{2\,e\,u}{g} \cdot \frac{1}{1-e} = \frac{u}{g}[1 + \frac{2e}{1-e}]$$

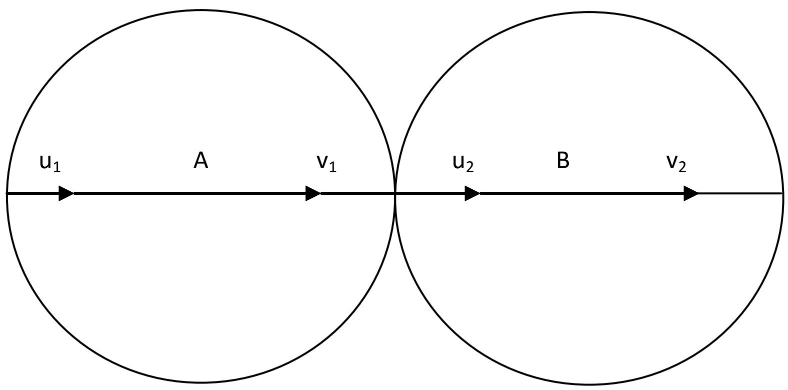
$$= \frac{u}{g} + (\frac{1+e}{1-e})$$

$$= \frac{\sqrt{2gh}}{g}(\frac{1+e}{1-e}) = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}.$$

3.10 Direct impact of two smooth spheres:

A smooth sphere of mass m_1 impinges directly with velocity u_1 on another smooth sphere of mass m_2, moving in the same direction with velocity u_2. If the coefficient of restitution is e, to find their velocities after the impact:

Solution:



AB is the line of impact, i.e. the common normal. Due to the impact there is no tangential force and hence, for either sphere the velocity along the tangent is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere tangential velocity after impact = its tangent velocity before impact = 0. So, after impact, the spheres will move only in the direction AB. Let their velocities be v_1 and v_2.

By Newton's experimental law, the relative velocity of m_2 with respect to m_1 after impact is (-e) times the corresponding relative velocity before impact.

$\therefore v_2 - v_1 = -e\,(u_2 - u_1)$ …….(1)

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$\therefore m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$ …….(2)

(2) – (1) x m_2 gives

$v_1 (m_1 + m_2) = m_1 u_1 + m_2 u_2 + em_2 (u_2 - u_1)$

$= m_2 u_2 (1 + e) + (m_1 - em_2) u_1$

$$\therefore v_1 = \frac{m_2 u_2 (1+e) + (m_1 - em_2) u_1}{m_1 + m_2} \quad \text{... (3)}$$

(1) x m_1 + (2) gives

$$v_2 (m_1 + m_2) = - em_1 (u_2 - u_1) + m_1u_1 + m_2u_2$$

$$= m_1u_1 (1 + e) + (m_2 - em_1) u_2$$

$$\therefore v_2 = \frac{m_1 u_1 (1 + e) + (m_2 - em_1) u_2}{m_1 + m_2} \quad \text{... (4)}$$

Equations (3) and (4) give the velocities of the spheres after impact.

Note: If one sphere say m_2 is moving originally in a direction opposite to that of m_1, the sign of u_2 will be negative. Also it is most important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in this direction. If either of them is actually in the opposite direction, the value obtained for it will turn to be negative.

In writing equation (1) corresponding to Newton's law, the velocities must be subtracted in the same order on both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and $m_1 = m_2$. Then, from equations (3) and (4), we have

$$v_1 = \frac{m_1 u_2 . 2 + 0}{2m_1} = u_2 \text{ and } v_2 = \frac{m_1 u_1 . 2 + 0}{2m_1} = u_1.$$

i.e. *If two equal perfectly elastic spheres impinge directly, they interchange their velocities.*

Cor: 2. The impulse of the blow on the sphere A of mass m_1 = change of momentum of A = $m_1 (v_1 - u_1)$.

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 - em_2)u_1}{m_1 + m_2} - u_1\right]$$

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 u_1 - em_2 u_1 - m_1 u_1 - m_2 u_1}{m_1 + m_2}\right]$$

$$= \frac{m_1 [m_2 u_2 (1+e) - m_2 u_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 \ m_2 \,(1+e)\,(u_2 - u_1)}{m_1 + m_2}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinge directly; to show that there is a loss of kinetic energy and to find the amount:

Let m_1 m_2 be the masses of the spheres, u_1 and u_2, v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By Newton's law, $v_2 - v_1 = -e\,(u_2 - u_1)$ … (1)

By the principle of conservation of momentum,

$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ ….(2)

Total kinetic energy before impact

$$= \frac{1}{2} m_1 u_1^{\ 2} + \frac{1}{2} m_2u_2^2$$

and total kinetic energy after impact

$$= \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2$$

Change in K.E. = initial K.E. – final K.E.

$$= \frac{1}{2} m_1u_1^2 + \frac{1}{2} m_2u_2^2 - \frac{1}{2} m_1v_1^2 - \frac{1}{2} m_2v_2^2$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_2 (u_2 - v_2)(u_2 + v_2)$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_1 (v_1 - u_1)(u_2 + v_2)$$

$$[\because \ m_2 (u_2 - v_2) = m_1 (v_1 - u_1) \text{ from (2)}]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)\,[u_1 - u_2 - (v_2 - v_1)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)\,[u_1 - u_2 + e\,(u_2 - u_1)] \text{ using (1)}$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 - u_2)(1 - e) \quad …..(3)$$

Now, from (2), $m_1 (u_1 - v_1) = m_2 (v_2 + u_2)$

$\therefore \frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1}$ and each $= \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$

i.e. each $= \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2}$

$$= \frac{(u_1 - u_2) - e(u_2 - u_1)}{m_1 + m_2} \quad \text{using (1)}$$

$$= \frac{(u_1 - u_2)\ (1 + e)}{m_1 + m_2}$$

$$\therefore u_1 - v_1 = \frac{m_2 (u_1 - u_2)\ (1 + e)}{m_1 + m_2} \text{ and substituting this in (3),}$$

$$\text{Change in K.E.} = \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)(1 + e)(u_1 - u_2)\ (1 - e)}{m_1 + m_2}$$

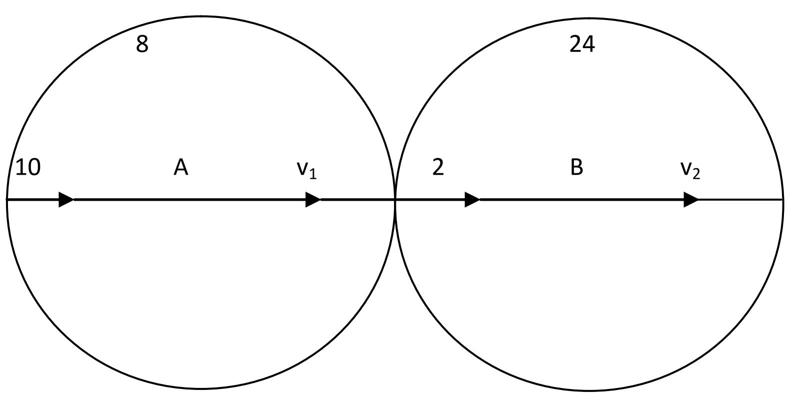
$$= \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)^2 (1 - e^2)}{m_1 + m_2} \qquad \ldots(4)$$

As $e < 1$, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the case, when $e=1$, i.e. only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total K.E. is unchanged by impact.

Problem 19

A ball of mass 8 gm. moving with a velocity of 10 cm. per sec. impinges directly on another of mass 24 gm., moving at 2cm per sec. in the same direction. If e = ½, find the velocities after impact. Also calculate the loss in kinetic energy.

Solution:



Let v_1 and v_2 cm. per sec. be the velocities of the masses 8gm and 24 gm respectively after impact.

By Newton's Law, $v_2 - v_1 = -\frac{1}{2}(2-10) = 4$ …… (1)

By the principle of momentum,

$24v_2 + 8v_1 = 24 \times 2 + 8 \times 10 = 128$

i.e. $3v_2 + v_1 = 16$

Solving (1) and (2), $v_1 = 1$ cm. / sec., $v_2 = 5$ cm./ sec.

The K.E. before impact $= \frac{1}{2} . 8.10^2 + \frac{1}{2} . 24.2^2$

$= 448$ dunes

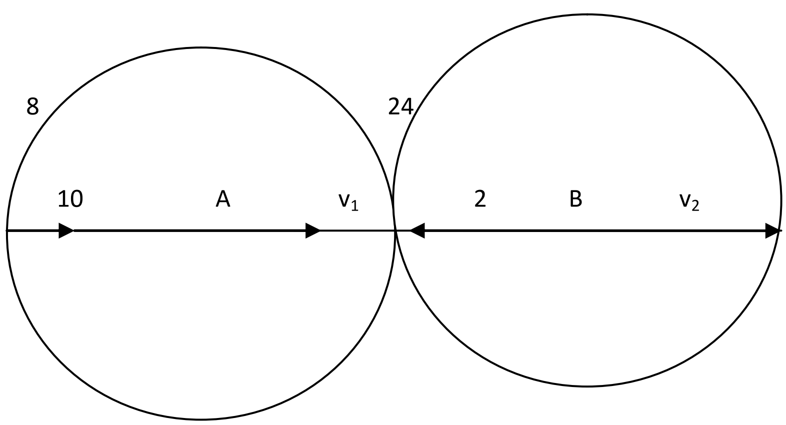
The K.E. after impact $= \frac{1}{2} . 8.1^2 + \frac{1}{2} . 24.5^2 = 304$ dines

$\therefore$ Loss in K.E. = 144 dynes

Problem 20

If the 24 gm.mass in the previous question be moving in a direction opposite to that of the 8 gm. mass, find the velocities after impact.

Solution:



Let v_1 and v_2 cm/sec. be the velocities of the 8gms and 24 gms mass respectively after impact.

By Newton's law,

$V_2 - v_1 = -\frac{1}{2}(-2-10) = 6$ ……… (1)

By conservation of momentum,

$24v_2 + 8v_1 = 24 \times (-2) + 8 \times 10 = 32$ i.e. $3v_2 + v_1 = 4$ ………….. (2)

Solving (1) and (2), $v_1 = -\frac{7}{2}$ cm/sec $v_2 = \frac{1}{2}$ cm / sec.

The negative sign of v_1 shows that the direction of motion of the 8 gm. Mass is reversed, as we had taken the direction left to right as positive and assumed v1 to be in this direction. Since v2 is positive, the 24gm. ball moves from left to right after impact, so that its direction of motion is also reversed.

Problem 21

A ball overtakes another ball of m times its mass, which is moving with $\frac{1}{n}$ th of its velocity in the same direction. If the impact reduces the first ball to rest, prove that the coefficient of elasticity is $\frac{m+n}{m\ (n-1)}$

Deduce that $m > \frac{n}{n-2}$

Taking AB as positive direction (as shown in the previous diagram), let the mass of the first ball be k and u its velocity along AB before impact. Then, for the second ball, the mass is mk and $\frac{u}{n}$ is the velocity before impact. After impact, the first ball is reduced to rest and let v be the velocity of the second ball.

By Newton's law of impact, we have

$v - 0 = -e. \left(\frac{u}{n} - u \right)$ i.e. $v = \frac{eu(n-1)}{n}$ .. (1)

By principle of conservation of momentum along AB,

$K \times 0 + mk. V = ku + mk. \frac{1}{n} u$

i.e. $mv = u + \frac{m}{u} u = \frac{u\ (m+n)}{n}$ ..(2)

Substituting value of v from (1) in (2), 12 have

$\frac{meu\ (n-1)}{n} = \frac{u(m+n)}{n}$ or $e = \frac{(m+n)}{m\ (n-1)}$

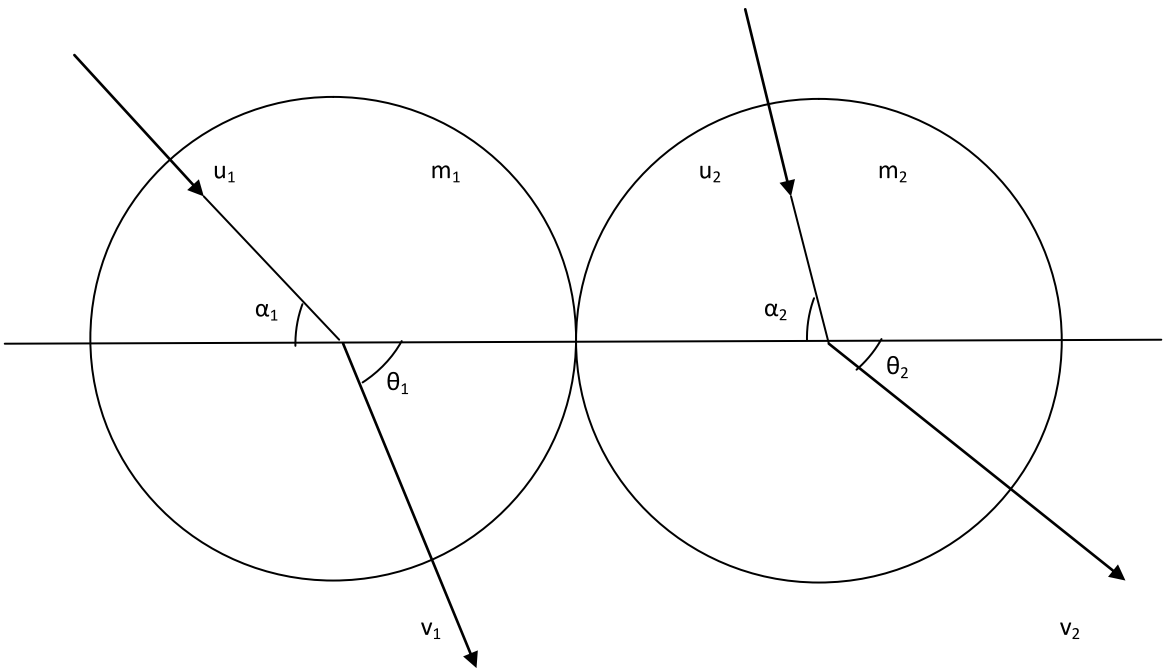
Now e is positive and less than 1.

$\therefore$ m (n – 1) > m + n i.e. mn – 2m > n

$\therefore$ m (n – 2) > n or $m > \frac{n}{n-2}$

3.11 Oblique impact of two smooth spheres:

A smooth sphere of mass m_1 impinges obliquely with velocity u_1 on another smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact make angles α_1 and α_2 respectively with line joining the centres of the spheres and if the coefficient of restitution be e, to find the velocities and directions of motion after impact.



Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line of centres and therefore, for each sphere the velocities in the tangential direction are not affected by impact.

$\therefore$ v1 sin θ1 = u1 sin α1 … (1) and

$v_2 \sin \theta_2 = u_2 \sin \alpha_2$ … (2)

By Newton's law concerning velocities along the common normal AB,

$v_2 \cos \theta_2 - v_1 \cos \theta_1 = -e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$ ... (3)

By the principle of conservation of momentum along AB,

$m_2 . v_2 \cos \theta_2 + m_1 v_1 \cos \theta_1 = m_2 u_2 \cos \alpha_2 + m_1 u_1 \cos \alpha_1$ ….(4)

(4) – (3) x m2 gives

$$v_1 \cos \theta_1 . (m_1 + m_2) = m_2 u_2 \cos \alpha_2 + m_1 u_1 \cos \alpha_1 + em_2 (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$$

i.e. $v_1 \cos \theta_1 = \dfrac{u_1 \cos \alpha_1 (m_1 - em_2) + m_2 u_2 \cos \alpha_2 (1+e)}{m_1 + m_2}$ … (5)

(4) + (3) x m_1 gives

$V_2 \cos \theta_2 = \dfrac{u_2 \cos \alpha_2 (m_2 - em_1) + m_1 u_1 \cos \alpha_1 (1+e)}{m_1 + m_2}$ … (6)

From (1) and (5), by squaring and adding, we obtain v_1^2 and by division, we have $\tan \theta_1$. Similarly from (2) and (6) we get v_2^2 and $\tan \theta_2$. Hence the motion after impact is completely determined.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and m_1=m_2.

Then from equations (5) and (6) we have

$V_1 \cos \theta_1 = \dfrac{0 + m_1 u_2 \cos \alpha_2 .2}{2 m_1} = u_2 \cos \alpha_2$

And $V_2 \cos \theta_2 = \dfrac{0 + m_1 u_1 \cos \alpha_2 .2}{2 m_1} = u_1 \cos \alpha_1$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centres.

Corollary 2. Usually, in most problems on oblique impact, one of the spheres is at rest. Suppose m_2 is at rest i.e. $u_2 = 0$.

From equation (2), $v_2 \sin \theta_2 = 0$ i.e. $\theta_2 = 0$. Hence m2 moves along AB after impact. This is seen independently, since the only force on m_2 impact is along the line of centres.

Corollary 3:

The impulse of the blow on the sphere A of mass m_1

= change of momentum of A along the common normal

$= m_1 \ (v_1 \ \cos\theta_1 \ - \ u_1 \ \cos\alpha_1)$

$$= m_1 \left[\frac{u_1 \cos \alpha_1 (m_1 - em_2) + m_2 u_2 \cos \alpha_2 \ (1+e)}{m_1 + m_2} - u_1 \cos \alpha_1\right]$$

$$= m_1 \left[\frac{m_1 u_1 \cos \alpha_1 - em_2 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 + em_2 u_2 \cos \alpha_2 - m_1 u_1 \cos \alpha_1 - m_2 u_1 \cos \alpha_1}{m_1 + m_2}\right]$$

$$= \frac{m_1 \ [m_2 u_2 \cos \alpha_2 (1+e) - m_2 u_1 \cos \alpha_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 \ m_2 (1+e)}{m_1 + m_2} (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

Loss of kinetic energy due to oblique impact of two smooth spheres:

Two spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centres, come into collision. To find an expression for the loss of kinetic energy:

The velocities perpendicular to the line of centres are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact if we replace in the expression (4) on page 236, the quantities u_1 and u_2 by $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ respectively.

Therefore the loss is $= \frac{1}{2} \frac{m_1 \ m_2}{m_1 + m_2} (1 - e^2) \ \ (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$

We shall now derive this independently.

Let v_1 and v_2 be the velocities of the spheres after impact, in directions inclined at angles θ1 and θ2 respectively to the line of centres. As explained in § 8.7 the tangential velocity of each sphere is not altered by impact.

$\therefore v_1 \sin \theta_1 = u_1 \sin \alpha_1$ … (1) and $v_2 \sin \theta_2 = u_2 \sin \alpha_2$ … (2)

By Newton's of rule

$v_2 \cos\theta_2 - v_1 \cos\theta_1 = - e (u_2 \cos\alpha_2 - u_1 \cos\alpha_1) \ldots (3)$

By conservation of momenta,

$m_2 v_2 \cos\theta_2 + m_1 v_1 \cos\theta_1 = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$

i.e. $m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) = m_2 (v_2 \cos\theta_2 - u_2 \cos\alpha_2) \ldots (4)$

Change in K.E.

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_1^2 - \frac{1}{2} m_1 v_1^{\ 2} - \frac{1}{2} m_2 v_2^{\ 2}$$

$$= \frac{1}{2} m_1 u_1^2 (\cos^2\alpha_1 + \sin^2\alpha_1) + \frac{1}{2} m_2 u_2^{\ 2} (\cos^2\alpha_2 + \sin^2\alpha_2)$$

$$- \frac{1}{2} m_1 v_1^2 (\cos^2\theta_1 + \sin^2\theta_1) - \frac{1}{2} m_2 v_2^2 (\cos^2\theta_2 + \sin^2\theta_2)$$

$$= \frac{1}{2} m_1 u_1^2 \cos^2\alpha_1 + \frac{1}{2} m_2 u_2^2 \cos^2\alpha_2 - \frac{1}{2} m_1 v_1^2 \cos^2\theta_1$$

$$- \frac{1}{2} m_2 v_2^2 \cos^2\theta_2 \text{ using (1) and (2)}$$

$$= \frac{1}{2} m_1 (u_1^2 \cos\alpha_1 - v_1^2 \cos^2\theta_1) + \frac{1}{2} m_2 (u_2^2 \cos^2\alpha_2 - v_2^2 \cos^2\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$+ \frac{1}{2} m_2 (u_2 \cos\alpha_2 + v_2 \cos\theta_2)(u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$- \frac{1}{2} (u_2 \cos\alpha_2 + v_2 \cos\theta_2).\ m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) \text{ using (4)}$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)(u_1 \cos\alpha_1 + v_1 \cos\theta_1 - u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)[u_1 \cos\alpha_1 + u_2 \cos\alpha_2$$

$$+ e (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)] \text{ Using (3)}$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)(u_1 \cos\alpha_1 - u_2 \cos\alpha_2)(1 - e) \quad \ldots..(5)$$

Now from (4),

$$\frac{u_1 \cos\alpha_1 - v_1 \cos\theta_1}{m_2} = \frac{v_2 \cos\theta_2 - u_2 \cos\alpha_2}{m_1}$$

and each $= \dfrac{u_1 \cos\alpha_1 - v_1 \cos\theta_1 + v_2 \cos\theta_2 - u_2 \cos\alpha_2}{m_1 + m_2}$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2) + (v_2 \cos \theta_2 - v_1 \cos \theta_1)}{m_1 + m_2}$$

$$= \frac{u_1 \cos \alpha_1 - u_2 \cos \alpha_2 - e \ (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)}{m_1 + m_2} \text{ using (3)}$$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1 + e)}{m_1 + m_2}$$

$$\therefore u_1 \cos \alpha_1 - v_1 \cos \theta_1 = \frac{m_2 (1+e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

Substituting in (5),

Change in K.E. $= \frac{1}{2} \frac{m_1 m_2 (1+e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$
$\quad x \ (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1 + e)$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1 - e^2)(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$$

If the spheres are perfectly elastic, e = 1 and the loss of kinetic energy is zero.

Problem 22

A ball of mass 8 gms. moving with velocity 4 cms. Per sec. impinges on a ball os mass 4 gms. Moving with velocity 2 cm. per sec. If their velocities before impact be inclined at angle 30° and 60° to the joining their centres at the moment of impact, find their velocities after impact when e = $\frac{1}{2}$

Solution:

In the diagram in the oblique impact of two smooth spheres, let $m_1 = 8$ $u_1 = 4$ $\alpha_1 = 30^0$, $m_2 = 4$, $u_2 = 2$, $\alpha_2 = 60^0$

Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angles θ_1 and θ_2 respectively to the line of centres.

The tangential velocity of each sphere is not affected by impact

$\therefore v_1 \sin \theta_1 = 4 \sin 30^o = 2$ …(1)

and $v_2 \sin \theta_2 = 2 \sin 60^o = \sqrt{3}$ …(2)

By Newton's Law,

$v_2 \cos \theta_2 - v_1 \cos \theta_1 = - e (2 \cos 60^o - 4 \cos 30^o)$

$= - \frac{1}{2} (2. \frac{1}{2} - 4. \frac{\sqrt{3}}{2})$

$\frac{1}{2} (2 \sqrt{3} - 1)$ …(3)

By conservation of momenta along AB,

$4v_2 \cos \theta_2 + 8v_1 \cos \theta_1 = 4 \text{ x } 2 \cos 60^o + 8 \text{ x } 4 \cos 30^o = 4 + 16 \sqrt{3}$

i.e. $v_2 \cos \theta_2 + 2v_1 \cos \theta_1 = 1 + 4\sqrt{3}$ …(4)

$\therefore 3v_1 \cos \theta_1 = 1 + 4\sqrt{3} - \frac{1}{2}(2\sqrt{3} - 1) = \frac{3 + 6\sqrt{3}}{2}$

i.e. $v_1 \cos \theta^1 = \frac{1+ 2\sqrt{3}}{2}$ …(5)

From (4), $v_2 \cos \theta_2 = 1 + 4\sqrt{3} - 1 - 2\sqrt{3} = 2\sqrt{3}$ …(6)

From (1) and (5), $v_1^2 = 2^2 + (\frac{1+ 2\sqrt{3}}{2})^2$

$= 4 + \frac{1+ 4\sqrt{3} + 12}{4} = \frac{29 + 4\sqrt{3}}{4}$

$\therefore v_1 = \frac{29- 4\sqrt{3}}{2}$ cm. per sec.

Dividing (1) by (5), $\tan \theta 1 = \frac{4}{1+2\sqrt{3}}$

From (2) and (6)

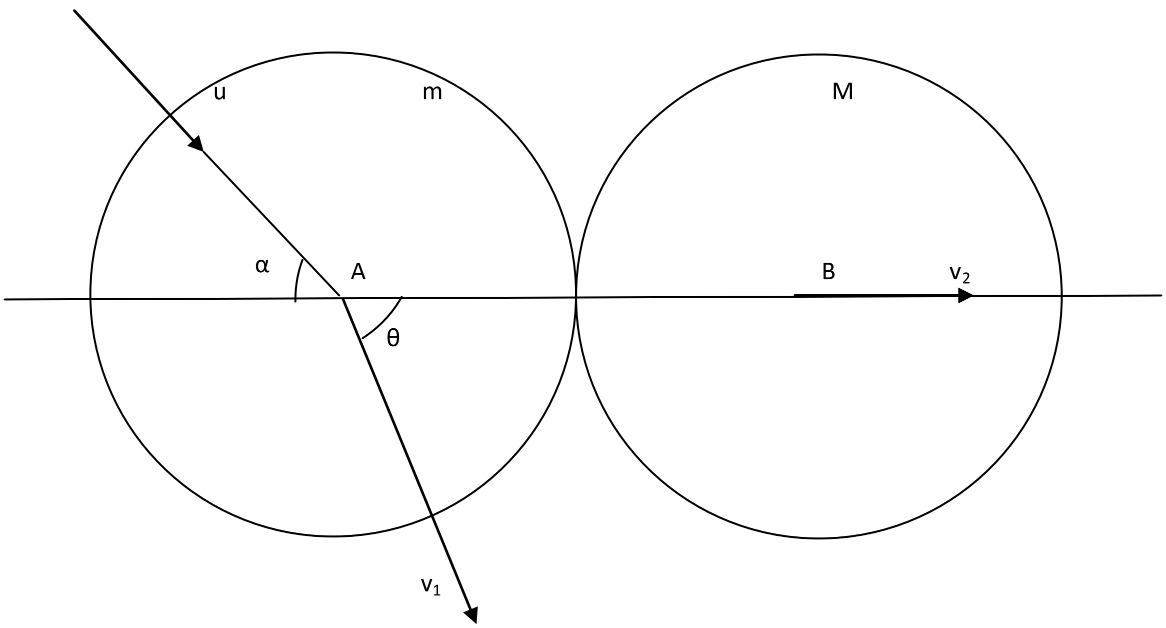
$v_2{}^2 = 3 + 12 = 15$ and $\therefore v_2 = \sqrt{15}$ cm / sec

Dividing (2) by (6), $\tan \theta_2 = \frac{1}{2}$

Problem 23

A smooth sphere of mass m impinges obliquely on a smooth sphere of mass M which is at rest. Show that if m = eM, the directions of motion after impact are at right angles. (e is the coefficient of restitution)

Solution:



Considering the sphere M, its tangential velocity before impact is zero and hence after impact also, its tangential velocity is zero.

(∵ During impact, there is no force acting along the common tangent).

Hence, after impact, M will move along AB. Let its velocity be v_2. Let the velocity of m be v_1 at an angle θ to AB, after impact.

By Newton's rule $v_2 - v_1 \cos\theta = - e(0 - u\cos\alpha)$

i.e. $v_2 - v_1 \cos\theta = eu\cos\alpha$ ...(1)

By conservation of momenta along AB,

$M. v_2 + m v_1 \cos\theta = M. 0 + m. u\cos\alpha$ …(2)

Multiplying (1) by M and subtracting from (2),

$mv_1 \cos\theta + M v_1 \cos\theta = mu \cos\alpha - M eu \cos\alpha$

i.e. $v_1 \cos\theta = \frac{u \cos\alpha\,(m - eM)}{m + M} = \frac{u \cos\alpha .0}{m + M}$ ($\because$ m = e M)

$= 0$

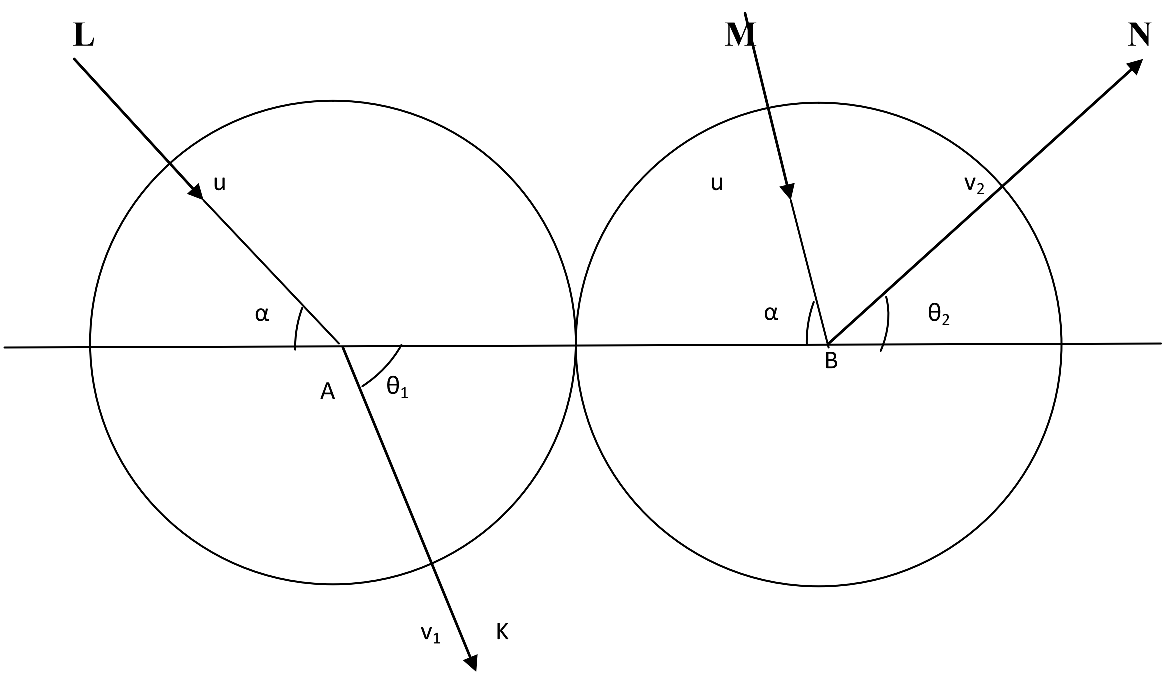
$\therefore \cos\theta = 0$ or $\theta = 90^o$

i.e. The direction of motion of m is perpendicular to AB.

Problem 24

Two equal elastic balls moving in opposite parallel direction with equal speeds impinge on one another. If the inclination of their direction of motion to the line of centres be $\tan^{-1}(\sqrt{e})$ where e is the coefficient of restitution, show that their direction of motion will be turned through a right angle.

Solution:



Let m be the mass of either sphere: AB is the line of impact. Before impact, the directions of motion are LA and BM making the same acute angle α with AB as shown in the figure. Let u be their velocity.

After impact, let the sphere A proceed in the direction AK with velocity v_1 at an angle θ_1 to AB and the sphere B proceed in the direction BN with velocity v_2 at an angle θ_2 to AB.

The tangential velocity of either sphere is not affected by impact.

$\therefore v_1 \sin \theta_1 = u \sin \alpha$ … (1) and

$v_2 \sin \theta_2 = u \sin \alpha$ …(2)

By Newton's Law, (resolving all velocities along AB),

$v_2 \cos \theta_2 - v_1 \cos \theta_1 = - e (- u \cos \alpha - u \cos \alpha)$

i.e. $v_2 \cos \theta_2 + v_1 \cos \theta_1 = 2 eu \cos \alpha$ … (3)

By conservation of momenta along AB,

$m (v_2 \cos \theta_2) + m. v_1 \cos \theta_1 = m (-u \cos \alpha) + mu \cos \alpha$

i.e. $v_2 \cos \theta_2 + v_1 \cos \theta_1 = 0$ …(4)

(4) – (3) gives $v_1 \cos \theta_1 = - 2 eu \cos \alpha$

$\therefore v_1 \cos \theta_1 = - eu \cos \alpha$ …(5)

From (4), $v_2 \cos \theta_2 = - v_1 \cos \theta_1 = eu \cos \alpha$ …(6)

Dividing (1) by (5),

$\tan \theta_1 = - \frac{1}{e} \tan \alpha = - \frac{1}{e} \sqrt{e}$ ($\because \alpha = \tan^{-1} \sqrt{e}$ given)

$= - \frac{1}{\sqrt{e}} = - \frac{1}{\tan \alpha} = - \cot \alpha = \tan (90^o + \alpha)$

$\therefore \theta_1 = - 90^o + \alpha$

Dividing (2) by (6), $\tan \theta_2 = \frac{1}{e} \tan \alpha = \cot \alpha = \tan (90^o - \alpha)$

$\therefore \theta_2 = 90^o - \alpha$.

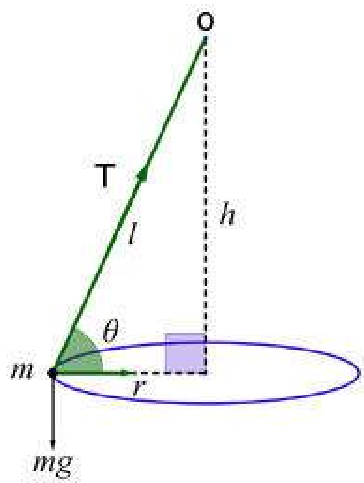
Hence their directions of motion are turned through a right angle.

UNIT IV
CIRCULAR MOTION

Conical pendulum

Problems concerning the conical pendulum assume no air resistance and that the string has no mass and cannot be stretched.

Solution of problems involves resolving forces on the mass vertically and horizontally. In this way the speed of the mass, the tension in the string and the period of revolution can be ascertained.



resolving forces vertically on the mass

$$mg = T\sin\theta$$

$$T = \frac{mg}{\sin\theta} \qquad \text{(i}$$

resolving forces horizontally on the mass

$$T\cos\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{r\cos\theta} \qquad \text{(ii}$$

eliminating T by combining (i and (ii

$$\frac{mg}{\sin\theta} = \frac{mv^2}{r\cos\theta}$$

$$gr = v^2\tan\theta$$

$$v = \sqrt{\frac{gr}{\tan\theta}}$$

but $\tan\theta = \frac{h}{r}$

$\Rightarrow$ $$v = \sqrt{gr\left(\frac{r}{h}\right)}$$

$\Rightarrow$ $$v = \sqrt{\left(\frac{gr^2}{h}\right)}$$

$\Rightarrow$ $$\underline{v = r\sqrt{\left(\frac{g}{h}\right)}}$$

$$\text{the period of revolution} = \frac{\text{circumference}}{\text{speed}}$$

$$= \frac{2\pi r}{v}$$

substituting for v using $v = r\sqrt{\frac{g}{h}}$

$$\text{the period of revolution} = \frac{2\pi r}{r\sqrt{\frac{g}{h}}} = \frac{2\pi r}{r}\sqrt{\frac{h}{g}}$$

$$\underline{\text{the period of revolution} = 2\pi\sqrt{\frac{h}{g}}}$$

Example

A 20g mass moves as a conical pendulum with string length 8x and speed v.
if the radius of the circular motion is 5x find:
i) the string tension(assume g =10 ms^{-2} , ans. to 2 d.p.)
ii) v in terms of x, g

i)

$l = 8x \quad r = 5x \quad \cos^{-1}\theta = \frac{5}{8} \quad \theta = 51.3°$

$m = 20\text{g} \equiv 0.02\text{kg} \quad g = 10\text{ms}^{-2}$

resolving vertically

$$T\sin\theta = mg$$

$$T = \frac{mg}{\sin\theta}$$

$$= \frac{0.02 \times 10}{0.7804} = 0.2563$$

$\underline{\text{Ans. the string tension } T \text{ is } 0.26\text{N (2.d.p.)}}$

ii)

resolving horizontally

$$T\cos\theta = \frac{mv^2}{r}$$

substituting for T, from $T = \dfrac{mg}{\sin\theta}$ above

$$mg\frac{\cos\theta}{\sin\theta} = \frac{mv^2}{r}$$

$$\frac{gr}{\tan\theta} = v^2$$

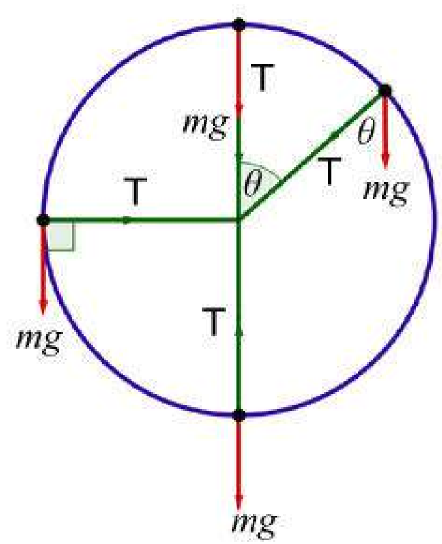
$$v = \sqrt{\frac{gr}{\tan\theta}}$$

substituting for $r = 5x$, $\theta = 51.3^\circ$

$$v = \sqrt{\frac{5gx}{\tan(51.3^\circ)}} = \sqrt{\frac{5}{\tan(51.3^\circ)}}\sqrt{gx} = 2\sqrt{gx}$$

Ans. velocity v in terms of g, x is $2\sqrt{gx}$

Mass performing vertical circular motion under gravity



Consider a mass m performing circular motion under gravity, the circle with radius r .
The centripetal force on the mass varies at different positions on the circle.

top $\quad mg + T = \frac{mv^2}{r}$

middle $\quad T = \frac{mv^2}{r}$

bottom $\quad T - mg = \frac{mv^2}{r}$

string at an angle θ to the vertical

$$mg\cos\theta + T = \frac{mv^2}{r}$$

For many problems concerning vertical circular motion, energy considerations(KE & PE) of particles at different positions are used to form a solution.

Example #1

A 50g mass suspended at the end of a light inextensible string performs vertical motion of radius 2m.
if the mass has a speed of 5 ms^{-1} when the string makes an angle of 30° with the vertical, what is the tension?
(assume g =10 ms^{-2} , answer to 1 d.p.)

$m = 50\text{g} \equiv 0.05\text{kg} \quad v = 5\text{ms}^{-1} \quad \theta = 30^\circ \quad r = 2\text{m}$

$g = 10\text{ms}^{-2}$

the centripetal force is the sum of the tension in the string and the component of the weight along the string

$$\Rightarrow \quad mg\cos\theta + T = \frac{mv^2}{r}$$

$$\Rightarrow \quad T = \frac{mv^2}{r} - mg\cos\theta$$

$$= \frac{(0.05)(5)^2}{2} - (0.05)(10)\cos 30^\circ$$

$$= 0.625 - 0.433 = 0.192$$

Ans. tension in string is 0.2N

Example #2

A 5kg mass performs circular motion at the end of a light inextensible string of length 3m. If the speed of the mass is 2 ms^{-1} when the string is horizontal, what is its speed at the bottom of the circle?
(assume g =10 ms^{-2})

$v_H = 2ms^{-1}$ $\quad r = 3m \quad$ $g = 10ms^{-1}$
v_B speed at bottom of circle

PE is measured relative to the bottom of the circle

KE + PE string horizontal = KE + PE at bottom

$$\tfrac{1}{2}mv_H^2 + mgr = \tfrac{1}{2}mv_B^2 + 0$$

$$v_H^2 + 2gr = v_B^2$$

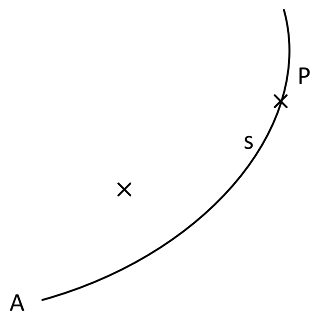
$$v_B = \sqrt{v_H^2 + 2gr}$$

$$= \sqrt{(2)^2 + 2 \times (10) \times (3)}$$

$$= \sqrt{4 + 60} = \sqrt{64} = 8$$

Ans. speed at bottom of circle is 8 ms^{-1}

Simple Harmonic Motion On a Curve



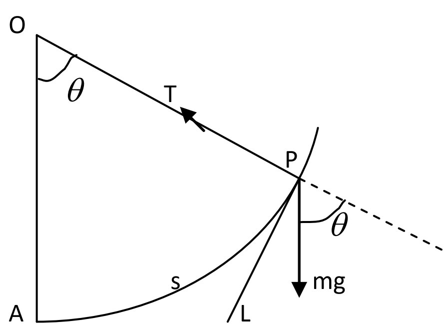
If P is the position of a particle on a curve at time t and if the tangential acceleration at P varies as the arcual distance of P measured from a fixed point A on the curve and is directed towards A, then the motion of P is said to be simple harmonic.

The differential equation for the S.H.M. on a curve will be of the form $\frac{d^2s}{dt^2} = -\mu s$, s is the arc distance AP.

Simple pendulum

A simple pendulum consists of a small heavy particle or bob suspended from a fixed point by means of a light inextensible string and oscillating in a vertical plane.

Period of oscillation of a simple pendulum



Let OA = l be the length of the pendulum where O is the point of suspension. Let 'm' be the mass of the bob and P be the position of the bob in time t secs and arc AP = s, $A\hat{O}P = \theta$

The two forces acting are i) mg$(\downarrow)$ ii) Tension T along PO.

mg is resolved into two components i) mg $\cos\theta$ along OP.

ii) mg $\sin\theta$ along PL.

mg $\cos\theta$ and T balances each other.

The equation of motion at P is $m.\frac{d^2s}{dt^2} = -mg.\sin\theta$ .... (1)

[Negative sign shows that mg $\sin\theta$ is towards A.]

When θ is small, sin $\theta \cong \theta$

$$\therefore \frac{d^2s}{dt^2} = -g.\theta \qquad \text{.......................... (2)}$$

But s = $l\theta$, $\theta = \frac{s}{l}$, $\therefore \frac{d^2s}{dt^2} = -\frac{g}{l}.s$ .... (3)

(3) shows that the motion of the bob at P is **simple harmonic when θ is small.**

Hence $\mu = \frac{g}{l}$

Period T = $\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$

Simple equivalent pendulum

A simple pendulum which oscillates in the same time as the given pendulum is called the Simple Equivalent Pendulum.

Consider two motions represented by the equations.

$\frac{d^2x}{dt^2} = -\mu x$ ...... (1)

$\frac{d^2s}{dt^2} = -\frac{g}{l}s$ ....(2)

We know that (1) and (2) are S.H. motions and (2) is the equation of motion of a simple pendulum.

They represent equivalent motions, if $\mu = \frac{g}{l}$ i.e. $\boxed{l = \frac{g}{\mu}}$

The length of the simple equivalent pendulum is $\frac{g}{\mu}$.

The Seconds Pendulum

A seconds pendulum is one whose period of oscillation is 2 seconds.

Hence if l is its length, we have $2 = 2\pi\sqrt{\frac{l}{g}}$ $\therefore$ $l = \frac{g}{\pi^2}$

The length of the seconds pendulum is $\frac{g}{\pi^2}$

Note : Since the time of oscillation of a seconds pendulum is 2 secs, it makes 43200 oscillation per day. If it gains n seconds a day, it makes $43200 + \frac{n}{2}$ oscillations in 86,400 secs.

Hence its period = $\dfrac{86400}{43200 + \frac{n}{2}}$ ……………………. (1)

If it loses n seconds a day, it makes $43200 - \frac{n}{2}$ oscillation in 864000 secs.

So its period = $\dfrac{86400}{43200 - \frac{n}{2}}$ …………………………(2)

Problem 1

Find the length of a simple pendulum which oscillates 56 times in 55 seconds

Solution:

Given, T = $\frac{55}{56}$ secs.

But $T = 2\pi\sqrt{\frac{l}{g}}$ l- length of the pendulum

$$\therefore\ 2\pi\sqrt{\frac{l}{g}} = \frac{55}{56}$$

$$\therefore\ \sqrt{\frac{l}{g}} = \frac{55}{56\times 2\pi} = \frac{55\times 7}{56\times 2\times 22} = \frac{5}{32}$$

$$\therefore\ \frac{l}{g} = \left(\frac{5}{32}\right)^2 = \frac{25}{1024}$$

$$\therefore l = \frac{25}{1024}\times 9.8 = 0.239\,m.$$

Problem 2

Show that an incorrect seconds pendulum of a clock which loses x seconds a day must be shortened by $\frac{x}{432}$ percent of its length in order to keep correct time.

Solution:

Let l, l^1 be the correct and incorrect lengths of the seconds pendulum of a clock

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{l}{g}} = \frac{86400}{43200} = 2 \text{ secs} \quad \text{\_\_\_\_\_\_\_\_\_(1)}$$

When it loses x seconds a day,

$$2\pi\sqrt{\frac{l^1}{g}} = \frac{86400}{43200 - \frac{x}{2}} \quad \text{\_\_\_\_\_\_\_\_(2)}$$

$$\frac{(2)}{(1)} \Rightarrow \sqrt{\frac{l^1}{l}} = \frac{43200}{43200 - \frac{x}{2}} = \frac{1}{1 - \frac{x}{86400}}$$

$$\therefore \frac{l^1}{l} = \frac{1}{\left(1 - \frac{x}{86400}\right)^2} = \left(1 - \frac{x}{86400}\right)^{-2} = 1 + \frac{2x}{86400} \text{ (approximately)}$$

ie) $\frac{l^1}{l} = 1 + \frac{x}{43200}$

$$\therefore l^1 = l + \frac{x}{43200} l$$

ie) $l^1 = l + \frac{x}{432}$ Percent of l

$\therefore$ Length should be shortened by $\frac{x}{432}$ percent of its length in order to keep correct time.

Problem 3

A pendulum whose length is l makes m oscillations in 24 hours. When its length is slightly altered, it makes m+n oscillations in 24 hours. Show that the diminution of the length is $\frac{2nl}{m}$ nearly.

Solution:

Given , when the length of the pendulum is l, it makes 'm' oscillations in 24 hrs.

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g}} = \frac{24}{m} \qquad \_\_\_\_\_\_\_\_\_\_(1)$$

When its length is altered, let $l - l^1$ be its length and it makes m+n oscillations per day.

$$\therefore\ \text{Periodic time } T = 2\pi\sqrt{\frac{l-l^1}{g}} = \frac{24}{m+n} \qquad \_\_\_\_\_\_\_\_\_\_\_(2)$$

$$\therefore\ \frac{(1)}{(2)} \Rightarrow \frac{m+n}{m} = \sqrt{\frac{l}{l-l^1}}$$

ie) $\sqrt{\dfrac{1}{1-\dfrac{l^1}{l}}} = 1 + \dfrac{n}{m}$

ie) $\left(1 - \dfrac{l^1}{l}\right)^{-1/2} = 1 + \dfrac{n}{m}$

ie) $1 + \dfrac{l^1}{2l} = 1 + \dfrac{n}{m}$ (nearly)

$\therefore l' = \dfrac{2nl}{m}$ nearly

Problem 4

A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another. Compare the acceleration due to gravity at the two places.

Solution:

Let g_1 , g_2 be the acceleration due to gravity at the two places where the pendulum gains 10 secs per day and loses 10 secs per day respectively.

When it gains, Periodic time $= 2\pi\sqrt{\frac{l}{g_1}} = \frac{24 \times 60 \times 60}{43200 + 5}$ ______(1)

When it loses, Periodic time $= 2\pi\sqrt{\frac{l}{g_2}} = \frac{24 \times 60 \times 60}{43200 - 5}$ _______(2)

where l is the length of the pendulum

$\therefore \frac{(1)}{(2)} \Rightarrow \sqrt{\frac{g_2}{g_1}} = \frac{43195}{43205} \quad \therefore \frac{g_1}{g_2} = \frac{(43205)^2}{(43195)^2}$

Problem 5

If l_1 is the length of an imperfectly adjusted seconds pendulum which gains n seconds in one hour and l_2 the length of one which loses n seconds in one hour at the same place, show that the true length of the seconds pendulum is $\frac{4\ l_1\ l_2}{l_1 + l_2 + 2\sqrt{l_1 l_2}}$

Solution:

Let l be the true length of the seconds pendulum. For the same place g is constant,

$\therefore \text{T} = 2\pi\sqrt{\frac{l}{g}} = 2 \text{ secs}$ _______ (1)

Let l_1 be the length of the pendulum, when it gains n seconds in one hour.

$\therefore \text{Period} = 2\pi\sqrt{\frac{l_1}{g}} = \frac{3600}{1800 + \frac{n}{2}}$ ____________(2)

Let l_2 - be the length of the pendulum, when it loses n seconds in one hour.

$\therefore \text{Period } 2\pi\sqrt{\frac{l_2}{g}} = \frac{3600}{1800 - \frac{n}{2}}$ _________(3)

$$\frac{(1)}{(2)} \Rightarrow \sqrt{\frac{l}{l_1}} = \frac{1800 + \frac{n}{2}}{1800} = 1 + \frac{n}{3600} \quad \text{\_\_\_\_\_\_\_\_\_\_(4)}$$

$$\frac{(1)}{(3)} \Rightarrow \sqrt{\frac{l}{l_2}} = \frac{1800 - \frac{n}{2}}{1800} = 1 - \frac{n}{3600} \quad \text{\_\_\_\_\_\_\_\_(5)}$$

(4) +(5) $\Rightarrow \sqrt{\frac{l}{l_1}} + \sqrt{\frac{l}{l_2}} = 2$

Squaring, $\frac{l}{l_1} + \frac{l}{l_2} + \frac{2l}{\sqrt{l_1 l_2}} = 4$

$$\therefore\ l \left(\frac{l_2 + l_1}{l_1 l_2} + \frac{2}{\sqrt{l_1 l_2}} \right) = 4$$

i.e.) $l \left(\frac{l_1 + l_2 + 2\sqrt{l_1 l_2}}{l_1 l_2} \right) = 4$

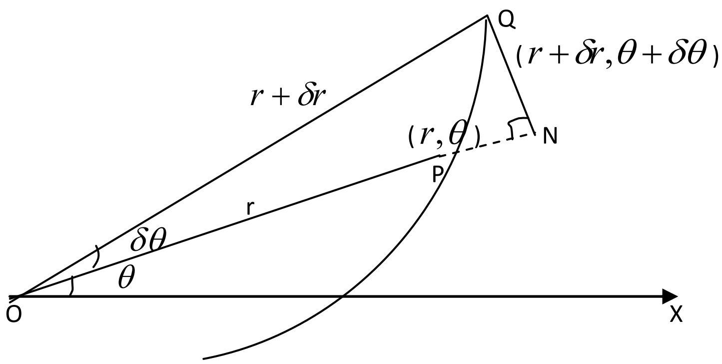
$$\boxed{\therefore\ l = \frac{4 l_1 l_2}{l_1 + l_2 + 2\sqrt{l_1 l_2}}}$$

MOTION UNDER THE ACTION OF CENTRAL FORCES

In this unit we study components of velocities and accelerations in two mutually perpendicular directions. We deal with the motion under the action of a force always directed towards a fixed point and derive formulae for various velocities and accelerations together with polar form and pedal form of central orbits.

1 Velocity and acceleration in polar co- ordinates

Radial and Transverse velocities



Consider a particle moves in a plane curve. Let P (r, θ) be its position in time t and $Q(r+\delta r, \theta+\delta\theta)$ be its position in time t+δt. Take O – as the pole and OX- as initial line. Velocity along the radius vector OP in the direction of r increasing is called the radial velocity and the velocity in the direction $\perp$ r to OP in the direction of θ increasing is called the transverse velocity.

$$\text{Radial velocity at P} = \lim_{\delta t \to 0} \left[\frac{displacement\ along\ OP\ in\ time\ \delta t}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \frac{PN}{\delta t} = \lim_{\delta t \to 0} \frac{ON - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[1 + \frac{(\delta\theta)^2}{2!} + \ldots\ldots\right] - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta t)(1) - r}{\delta t}$$, neglecting higher powers of $\delta\theta$

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}$$

$\therefore$ Radial velocity $= \dot{r}$

Transverse velocity at P = $\lim_{\delta t \to 0} \frac{QN}{\delta t} = \lim_{\delta t \to 0} \frac{(r+\delta r).\sin\delta\theta}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \ldots\ldots\ldots\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\delta\theta}{\delta t}$$, neglecting higher powers of $\delta\theta$

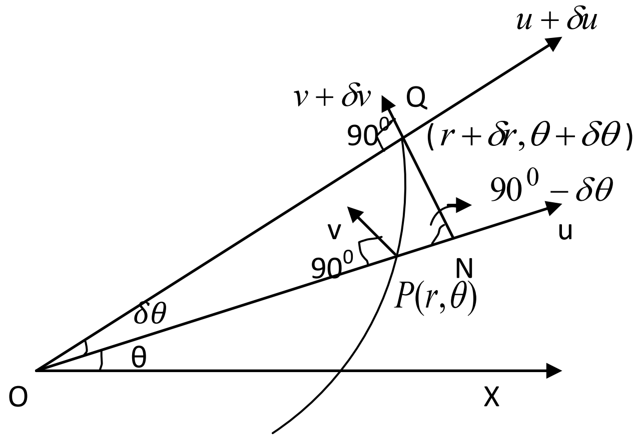
$$= \lim_{\delta t \to 0} \frac{r\ \delta\theta}{\delta t} = \left(r.\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}\right)$$

$$= r\frac{d\theta}{dt} = r\ \dot{\theta}$$

Transverse velocity = r $\dot{\theta}$

Radial and Transverse Accelerations

Let u, v be the radial and transverse velocities at (r,θ) and $(u+\delta u)$ and $(v+\delta v)$be the radial and transverse velocities at Q $(r+\delta r, \theta+\delta\theta)$



$$\text{Radial acceleration} = \lim_{\delta t\to 0}\left[\frac{Change\ \ of\ \ velocity\ along\ \ OP\ in\ \ time\ \ \delta t}{\delta t}\right]$$

$$= \lim_{\delta t\to 0}\left[\frac{(u+\delta u)\cos\delta\theta-(v+\delta v)\cos(90^\circ-\delta\theta)}{\delta t}\right]-u$$

$$= \lim_{\delta t\to 0}\ \frac{[(u+\delta u)[1]-(v+\delta v)(\delta\theta)-u]}{\delta t}$$

$$= \lim_{\delta t\to 0}\ \frac{\delta u - v\delta\theta}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\delta u}{\delta t} - v\lim_{\delta t\to 0}\frac{\delta\theta}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt},\ \text{where u} = \frac{dr}{dt}, v = r\frac{d\theta}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\frac{d\theta}{dt}.\frac{d\theta}{dt}$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$$

$$\therefore Radial\ \ acceleration = \ddot{r} - r\dot{\theta}^2$$

Transverse acceleration = $\lim_{\delta t \to 0} \dfrac{\left[\text{Change in velocity perpendicular to } OP \text{ in time } \delta t\right]}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{\left[(u+\delta u)\sin\delta\theta + (v+\delta v)\sin(90^\circ - \delta\theta)\right] - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\left[(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta\right] - v}{\delta t}$$

when $\delta\theta$ is small, $\sin\delta\theta \approx \delta\theta$

$$= \lim_{\delta t \to 0} \frac{\left[(u+\delta u)(\delta\theta) + (v+\delta v)(1) - v\right]}{\delta t} \quad \text{and} \quad \cos\delta\theta \approx 1$$

$$= \lim_{\delta t \to 0}\left[\frac{u\delta\theta + \delta v}{\delta t}\right] = u\frac{d\theta}{dt} + \frac{dv}{dt} \quad \text{Where } u = \frac{dr}{dt}, v = r\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}.\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$= \frac{dr}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}.\frac{dr}{dt}$$

$$= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}.\frac{d\theta}{dt}.$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$\therefore \textit{Transverse acceleration} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

		Magnitude
1	Radial Component of velocity	$\dot{r}$
2	Transverse Component of velocity	$r\dot{\theta}$
3	Radial component of acceleration	$\ddot{r} - r\dot{\theta}^2$
4	Transverse component of acceleration	$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$

Corollary

(1) Suppose the particle P is describing a circle of radius 'a'. Then r = a throughout the motion

Hence $\ddot{r} = 0$ and the **radial acceleration =** $\ddot{r} - r\dot{\theta}^2$

$$= 0 - a\dot{\theta}^2 = -a\dot{\theta}^2$$

Transverse acceleration $= \frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{a}a^2\ddot{\theta} = a\ddot{\theta}$

(2) The magnitude of the resultant velocity of P

$$= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2)}$$

and the magnitude of the resultant acceleration

$$= \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + [\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta})]^2}$$

Problem 1

The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + C = \frac{\mu}{2r^2}$ where C is a constant. Show also that the accelerations along and perpendicular to the radius vector are $2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ *and* $\mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$

Solution:

Radial velocity = $\frac{dr}{dt} = \lambda r^2$ …. (1)

Transverse velocity = $r\frac{d\theta}{dt} = \mu\theta^2$ ….. (2)

Dividing (2) by (1), we have

$r\frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2}$ $i.e.\lambda\frac{d\theta}{\theta^2} = \frac{\mu}{r^3}dr$

Integrating, $-\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + C$

i.e. $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + C$ …. (3)

(3) is the equation of the path,

Differentiating (1) $\frac{d^2r}{dt^2} = \lambda \cdot 2r\frac{dr}{dt} = 2\lambda^2 r^3$ using (1)

Radial acceleration = $\ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$

$$= 2\lambda^2 r^3 - r(\frac{\mu\theta^2}{r})^2 = 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \text{ using (2)}$$

Transverse acceleration $= \frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\cdot\frac{d}{dt}(r^2\frac{\mu\theta^2}{r})$

$$= \frac{1}{r}\cdot\frac{d}{dt}(\mu\ r\theta^2) = \frac{\mu}{r}(r^2\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt})$$

$$= \frac{\mu}{r}(2r\cdot\theta\frac{\mu\theta^2}{r} + \theta^2\cdot\lambda r^2) = \mu[\frac{2\mu\theta^3}{r} + \lambda r\theta^2]$$

Problem 2

The velocities of a particle along and perpendicular to the radius from a fixed origin are λ r and $\mu\ \theta$; find the path and show that the acceleration along and perpendicular to the radius vector are $\lambda^2 r - \frac{\mu^2\theta^2}{r}$ and $\mu\ \theta\left(\lambda + \frac{\mu}{r}\right)$

Solution:

Given, radial velocity = $\dot{r} = \frac{dr}{dt} = \lambda r$ _________(1)

Transverse velocity = r$\dot{\theta} = \mu\theta$ _________(2)

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= \lambda\ \dot{r} - r\left(\frac{\mu\theta}{r}\right)^2 \qquad \text{[by (1) \& (2)]}$$

$$= \lambda(\lambda\ r) - \frac{\mu^2\theta^2}{r} = \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

Transverse acceleration $= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2.\frac{\mu\theta}{r}\right)$

$$= \frac{1}{r}\frac{d}{dt}\left(\mu\ r\ \theta\right) = \frac{1}{r}.\mu\left[r\dot{\theta} + \theta\ \dot{r}\right]$$

$$= \frac{\mu}{r}\left[r.\frac{\mu\theta}{r} + \theta.\lambda\ r\right]$$

$$\boxed{\text{Transverse ace.} = \mu\theta\left[\frac{\mu}{r} + \lambda\right]}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\mu\theta}{\lambda\ r} = \frac{\mu}{\lambda}.\frac{\theta}{r}.$$

i.e. $\frac{r\,d\theta}{dr} = \frac{\mu}{\lambda}.\frac{\theta}{r} \quad \therefore \frac{d\theta}{\theta} = \frac{\mu}{\lambda}.\frac{dr}{r^2}$

Integrating, $\log\theta = \frac{\mu}{\lambda}\left[\frac{r^{-1}}{-1}\right] + C;$ C – constant

$$= -\frac{\mu}{\lambda\ r} + C$$

i.e. $\boxed{\log \quad \theta = c - \frac{\mu}{\lambda r}}$

which is the equation of the path

Problem 3

The velocities of a particle along and perpendicular to the radius vector from a fixed origin area and b. Find the path and the acceleration along and perpendicular to the radius vector.

Solution:

Radial velocity = $\dot{r} = \frac{dr}{dt} = a$ _______(1)

Transverse velocity = $r\dot{\theta} = r\frac{d\theta}{dt} = b$ _____(2)

Radial acceleration = $\ddot{r} = r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

Now, $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{da}{dt} = 0$

$\therefore$ Radial acceleration $= -r\left(\frac{b}{r}\right)^2 = -\frac{b^2}{r}$

Transverse acceleration $= \frac{1}{r}.\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}.\frac{d}{dt}\left(r^2.\frac{b}{r}\right)$

$= \frac{1}{r}.\frac{d}{dt}(br) = \frac{b}{r}.\frac{dr}{dt} = \frac{ab}{r}$

To find the path

$\frac{(2)}{(1)} \Rightarrow \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{b}{a}$ i.e. $r\frac{d\theta}{dr} = \frac{b}{a} \Rightarrow \frac{dr}{r} = \frac{a}{b}d\theta$

Integrating, log r = $\frac{a}{b}\theta + c,$ where C – is constant

$\boxed{\therefore r = A.e^{\frac{a\theta}{b}}}$ is the equation of the path.

Problem 4

A point moves so that its radial and transverse velocities are always 2 λ a θ and λr. Show that its accelerations in these two directions are λ^2(2a-r) and that its path is the curve r = a $\theta^2 + C$.

Solution:

Given, radial velocity $\dot{r} = \dfrac{dr}{dt} = 2\lambda\, a\theta$ ____________ (1)

Transverse velocity r $\dot{\theta}$ = r$\dfrac{d\theta}{dt} = \lambda r$ __________ (2) $\Rightarrow \dot{\theta} = \lambda$

Radial acceleration (R.A) = $\ddot{r} - r\dot{\theta}^2 = 2\,\lambda a\dfrac{d\theta}{dt} - r\left[\lambda^2\right]$

$$= 2\ \lambda a \cdot \lambda - r\lambda^2$$

$$\boxed{\mathrm{R}\cdot A = \lambda^2(2a - r)}$$

Transverse acceleration (T.A) = $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\cdot\lambda\right) = \frac{\lambda}{r}\cdot 2r\ \dot{r}$$

$$= \frac{\lambda}{r}.\ 2r\ .\ 2\,\lambda a\theta$$

$$\boxed{\mathrm{T.A} = 4\ \lambda^2 a\theta}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\lambda\ r}{2\lambda a\theta} = \frac{r}{2a\theta}\ \text{ i.e. r } \frac{d\theta}{dr} = \frac{r}{2a\theta}$$

$$\therefore 2a\ \theta d\theta = dr$$

Integrating, 2a $\dfrac{\theta^2}{2} + C = r, C$ - constant

$\boxed{\mathrm{r} = \mathrm{C} + \mathrm{a}\theta^2}$ is the equation of the path.

Problem 5

If a point moves so that its radial velocity is k times its transverse velocity then show that its path is an equiangular spiral.

Solution:

Given, radial velocity = $k\times$ transverse velocity

i.e. $\dot{r}=k.r\dot{\theta}$

i.e. $\frac{dr}{dt}=k.r.\frac{d\theta}{dt}$

$$\therefore\frac{dr}{r}=k.d\theta$$

Integrating, log r = k θ + log A, A – constant

i.e. $\log\left(\frac{r}{A}\right)=k\theta \quad \therefore\frac{r}{A}=e^{k\theta}$

$$\boxed{r = A\ e^{k\theta}}$$

which is an equiangular spiral.

Problem 6

If the radial and transverse velocities of a particle are always proportional to each other, show that the equation of the path is of the form r = A. $e^{k\theta}$, where A and k are constants.

Solution:

Given radial velocity α transverse velocity

i.e $\dot{r}\ \alpha\ r\dot{\theta}\Rightarrow\dot{r}=k.r\dot{\theta}$, k – constant

$$\Rightarrow\frac{dr}{r}=k.d\theta$$

Integrating, log r = Kθ+ log A

log r – log A = k θ

ie) log $\left(\frac{r}{A}\right)=k\theta$

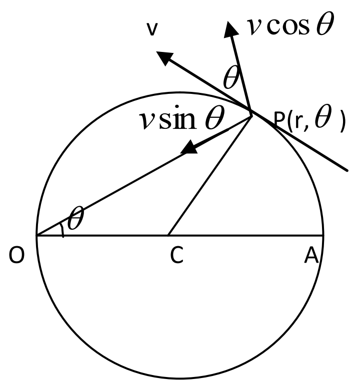
i.e log $\left(\frac{r}{A}\right) = k..\theta$

$\Rightarrow \frac{r}{A} = e^{k\theta} \Rightarrow$ r = A.e$^{k\theta}$

Problem 7

A point moves in a circular path of radius 'a' so that its angular velocity about a fixed point in the circumference of the circle is constant, equal to ω.Show that the resultant acceleration of the point at every point of the path is of constant magnitude 4 aω^2.

Solution:



Let O – be the fixed point (pole), OC – initial line. Polar equation of the circle is r = 2 a cos θ. Let P (r, θ) be the position at time't' Angular velocity about O is $\dot{\theta} = w$ (constant)

Radial velocity $= \dot{r} = -(2a\sin\theta)\dot{\theta} = -2a\omega\sin\theta$

$$\ddot{r} = -(2a\omega\cos\theta)\dot{\theta} = -2a\omega^2\cos\theta$$

$$= -\omega^2(2a\cos\theta)$$

$$= -\omega^2.r$$

Radial acceleration at P $= \ddot{r} - r\dot{\theta}^2$

$= -\omega^2 r - r.\omega^2$

$= -2\omega^2 r = -2\omega^2(2a\cos\theta)$

$= -4a\omega^2\cos\theta$

Transverse acceleration at P $= \frac{1}{r}.\frac{d(r^2\dot{\theta})}{dt} = \frac{1}{r}.w.2r\,\dot{r}$

$$= 2\ \omega(-2a\omega\sin\theta) = -4a\omega^2\sin\theta$$

$\therefore$ Resultant acceleration $= \sqrt{(-4a\omega^2\cos\theta)^2 + (-4a\omega^2\sin\theta)^2}$

$$= 4a\ \omega^2$$

Problem 8

A point moves with uniform speed v along a cardioid r = a (1+ cos θ). Show that (i) its angular velocity ω about the pole is $v\frac{\sec\theta/2}{2a}$ (ii) the radial component of the acceleration is constant equal to $\frac{3v^2}{4a}$ (iii) the magnitude of the resultant acceleration is $\frac{3v\omega}{2}$.

Solution:

Given, path is r = a $(1+\cos\theta)$ ………………(1)

Uniform speed v = resultant velocity = $\sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

(1) $\Rightarrow \dot{r} = a(-\sin\theta)\dot{\theta}$

$$\ddot{r} = -a[\sin\theta.\ddot{\theta} + \dot{\theta}\cos\theta.\dot{\theta}]$$

$$= -(a\cos\theta)\dot{\theta}^2 - a\ddot{\theta}\sin\theta.$$

$$\therefore v = \sqrt{a^2\dot{\theta}^2\sin^2\theta + [a(1+\cos\theta)\dot{\theta}]^2}$$

$$= \sqrt{a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2(1 + 2\cos\theta + \cos^2\theta)}$$

$$= \sqrt{a^2\dot{\theta}^2 + a^2\dot{\theta}^2(1 + 2\cos\theta)}$$

$$= \sqrt{2a^2\dot{\theta}^2 + 2\cos\theta.a^2\dot{\theta}^2}$$

$$= a\ \dot{\theta}\sqrt{2}\ \sqrt{1+\cos\theta}$$

$$= \sqrt{2}\ a\ \dot{\theta}\sqrt{2\cos^2\theta/2}$$

$$\boxed{v = 2\ a\dot{\theta}.\cos\theta/2}$$

$$\therefore \dot{\theta} = \omega = \frac{v}{2a.\cos\theta/2} = \left(\frac{v}{2a}\right).\sec\theta/2$$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= -(a\cos\theta)\dot{\theta}^2 - (a\sin\theta)\ddot{\theta} \quad - a(1+\cos\theta)\dot{\theta}^2$$

$$= -a(1+2\cos\theta)\dot{\theta}^2 - \dot{\theta}(a\sin\theta)\,{}^1\!/_2\left[\frac{v}{2a}\sec{}^{\theta}\!/_2 . \tan{}^{\theta}\!/_2\right]$$

$$= -a(1+2\cos\theta)\dot{\theta}^2 - \frac{a}{2}\sin\theta . \tan{}^{\theta}\!/_2\left[\frac{v}{2a}.\sec{}^{\theta}\!/_2\right]\dot{\theta}$$

$$= -a(1+2\cos\theta)\left[\frac{v}{2a}.\sec{}^{\theta}\!/_2\right]^2 - \frac{a}{2}\sin\theta\tan{}^{\theta}\!/_2\left[\frac{v}{2a}\sec.{}^{\theta}\!/_2\right]^2$$

$$= -a\left(\frac{v^2}{4a^2}\right)\left(\sec^2{}^{\theta}\!/_2\right)\left[(1+2\cos\theta) + \frac{1}{2}\tan{}^{\theta}\!/_2 . \sin\theta\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_2\right)\left[(1+2\cos\theta) + \sin^2\frac{\theta}{2}\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_2\right)\left[(1+2\cos\theta) + \frac{1}{2}(1-\cos\theta)\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_2\right)\left[\frac{3}{2}(1+\cos\theta)\right]$$

$$= -\frac{3}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_2\right)[(1+\cos\theta)]$$

$$= -\frac{3}{4}\left(\frac{v^2}{a}\right)\sec^2\frac{\theta}{2} . \cos^2\frac{\theta}{2} = -\frac{3}{4}\left(\frac{v^2}{a}\right)$$

R.A = constant

Transverse acceleration $= \frac{1}{r}.\frac{d}{dt}\left(r^2\dot{\theta}\right)$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left[a^2(1+\cos\theta)^2\cdot\frac{v}{2a}\cdot\sec{}^{\theta}\!/_2\right]$$

$$= \frac{1}{r}\frac{d}{dt}\left[a^2.\left(2\cos^2{}^{\theta}\!/_2\right)^2\cdot\frac{v}{2a}\sec{}^{\theta}\!/_2\right]$$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left[2a.\cos^3 \theta\!/\!_2 .v\right]$$

$$= \frac{1}{r}\times 2a.v.\left[3\cos^2 \theta\!/\!_2\cdot\left(-\sin \theta\!/\!_2\right)\dot{\theta}\!/\!_2\right]$$

$$= -\frac{3av}{r}\cdot\cos^2 \theta\!/\!_2\cdot\left[\frac{v}{2a}\sec\theta\!/\!_2\right] . \sin \frac{\theta}{2}$$

$$= -\frac{3v^2}{2r}\left[\cos\theta\!/\!_2\cdot\sin \theta\!/\!_2\right]$$

$$= -\frac{3v^2}{2\left[a(1+\cos\theta)\right]}\times\left[\cos \theta\!/\!_2\cdot\sin \theta\!/\!_2\right]$$

$$= -\frac{3v^2}{2a.2\cos^2 \theta\!/\!_2}\cdot\cos\theta\!/\!_2\cdot\sin\theta\!/\!_2$$

$$\boxed{\text{T.A} = -\frac{3v^2}{4a}\cdot\tan\theta\!/\!_2}$$

$\therefore$ Resultant acceleration $= \pm\sqrt{(R.A)^2+(T.A)^2}$

$$= \pm\sqrt{\left(-\frac{3}{4}\frac{v^2}{a}\right)^2+\left(\frac{-3v^2}{4a}\tan\theta\!/\!_2\right)^2}$$

$$= \pm\sqrt{\frac{9v^4}{16a^2}\left(1+\tan^2 \theta\!/\!_2\right)}$$

$$= \pm\sqrt{\frac{9v^4}{16a^2}\cdot\sec^2 \theta\!/\!_2}$$

$$= \pm\frac{3v^2}{4a}\sec\theta\!/\!_2$$

$$= \pm\left[\frac{3v\omega}{2}\right]$$

2 Differential Equation of central orbits

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Obtain the differential equation of its path.

Take O as the pole and a fixed line through O as the initial line. Let P (r, θ) be the polar coordinates of the particle at time t and m be its mass. Also let P be the magnitude of the central acceleration along PO.

The equations of motion of the particle are

$$m\left(\ddot{r}-r\dot{\theta}^{2}\right) = -mP$$

$$\text{i.e. } \ddot{r}-r\dot{\theta}^{2} = -P \qquad \ldots\ldots (1)$$

$$\text{and } \frac{m}{r}\cdot\frac{d}{dt}\left(r^{2}\dot{\theta}\right) = 0$$

$$\text{i.e. } \frac{1}{r}\cdot\frac{d}{dt}\left(r^{2}\dot{\theta}\right) = 0 \qquad \ldots\ldots (2)$$

Equation (2) shows that the transverse component of the acceleration is zero throughout the motion.

From (2), $r^{2}\dot{\theta}$ = constant = h .... (3)

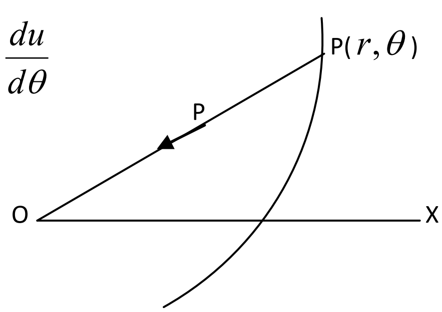
To get the polar equation of the path, we have to eliminate t between (1) and (3).

put $u = \frac{1}{r}$

From (3), $\dot{\theta} = \frac{h}{r^{2}} = h\,u^{2}$

$$\text{Also } \dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^{2}}\frac{du}{dt} = -\frac{1}{u^{2}}\frac{du}{dt} = -\frac{1}{u^{2}}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{u^{2}}\frac{du}{d\theta}\cdot h\,u^{2} = -h\frac{du}{d\theta}$$



$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\cdot\frac{d\theta}{dt}$$

$$= -h\frac{d^{2}u}{d\theta^{2}}\cdot hu^{2} = -h^{2}u^{2}\frac{d^{2}u}{d\theta^{2}}$$

Substitute r and $\dot{\theta}$ in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta} - \frac{1}{u}h^2u^4 = -\text{P ie } h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \text{P}$$

ie) $u + \dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2}$ .....(4)

(4) is the differential equation of a central orbit, in polar coordinates.

Perpendicular from the pole on the tangent - Formulae in polar coordinates

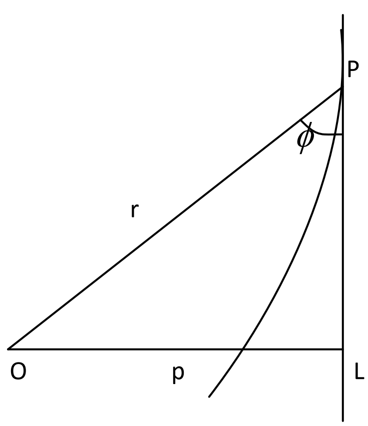
Let φ be the angle made by the tangent at P with the radius vector OP.

We know that $\tan\varphi = r\dfrac{d\theta}{dr}$ .... (1)

From O draw OL perpendicular to the tangent at P and let OL= p.

Then $\sin\varphi = \dfrac{OL}{OP} = \dfrac{p}{r}$

$\therefore$ p = r sin φ .... (2)



Now eliminate φ between (1) and (2).

From (2), $\dfrac{1}{p^2} = \dfrac{1}{r^2\sin^2\varphi} = \dfrac{1}{r^2}\operatorname{cosec}^2\varphi$

$$= \frac{1}{r^2}\left(1 + \cot^2\varphi\right)$$

$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right], \text{(by (1))}$$

i.e. $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ..... (3)

Using r = $\frac{1}{u}, \frac{dr}{d\theta} = \frac{dr}{du} \cdot \frac{du}{d\theta} = -\frac{1}{u^2} \cdot \frac{du}{d\theta}$

Hence (3) becomes

$$\frac{1}{P^2} = u^2 + u^4 . \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2$$

i. e) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ ……………… (4)

3 Pedal equation (or) (p, r) equation of the central orbit

We have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ ……… (1)

Differentiating both sides of (1) with respect to θ,

$$-\frac{2}{p^3} \cdot \frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right) \quad \text{…….. (2)}$$

But the differential equation is u + $\frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

Hence (2) becomes $-\frac{1}{p^3} \cdot \frac{dp}{d\theta} = \frac{P}{h^2u^2} \cdot \frac{du}{d\theta}$

i.e. $-\frac{1}{p^3} dp = \frac{P}{h^2u^2} du = \frac{P}{h^2} r^2 d\left(\frac{1}{r}\right)$

$$= \frac{P\, r^2}{h^2} \times -\frac{1}{r^2} dr = -\frac{P}{h^2} dr$$

$$\frac{h^2}{p^3} \cdot \frac{dp}{dr} = P \quad \text{………(3)}$$

is the (p, r) equation or the pedal equation to the central orbit.

Problem 9

Find the law of force towards the pole under which the curve

$r^n = a^n . \cos n\,\theta$ can be described.

Solution:

Given $r^n = a^n \cos n\theta$

Put $r = \frac{1}{u}$, the equation is $u^n a^n \cos n\,\theta = 1$ …… (1)

Taking logarithms,

$n \log u + n \log a + \log \cos n\,\theta = 0$ ……. (2)

Differentiating (2) with respect to θ

$$n\,\frac{1}{u}\frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

ie) $\frac{du}{d\theta} = u \tan n\theta$ …………….. (3)

Differentiating (3) with respect to θ,

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta . \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \text{ using (3)}$$

$$u + \frac{d^2u}{d\theta} = u + nu \sec^2 n\,\theta + u \tan^2 n\,\theta$$

$$= nu \sec^2 n\,\theta + u\left(1 + \tan^2 n\theta\right)$$

$$= nu \sec^2 n\theta + u \sec^2 n\theta = (n+1)u \sec^2 n\theta$$

$$= (n+1)u.u^{2n}a^{2n} \text{ using (1)}$$

$$= (n+1)a^{2n}u^{2n+1}$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2.(n+1)a^{2n}u^{2n+1}$$

$$= (n+1)a^{2n}.h^2.u^{2n+3}$$

$$= (n+1)a^{2n}.h^2.\frac{1}{r^{2n+3}} \qquad \text{……….. (4)}$$

$$\therefore \text{ P } \alpha \frac{1}{r^{2n+3}}$$

Important notes

(i) When n=1, the equation is r = a cos θ. The curve is a circle and P α $1 / r^5$.

(ii) When n = 2, the equation is $r^2 = a^2 \cos 2\theta$. This is the Lemniscate of Bernowli and P α $\frac{1}{r^7}$.

(iii) When n = $\frac{1}{2}$, the equation is $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos\frac{\theta}{2}$

i.e. $r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta)$

This is a cardioid and P α $\frac{1}{r^4}$

(iv) When n = $-\frac{1}{2}$, the equation is $r^{\frac{-1}{2}} = a^{\frac{-1}{2}}.\cos\frac{\theta}{2}$

i.e. $a^{\frac{1}{2}} = r^{\frac{1}{2}}\cos\frac{\theta}{2}$

So $r = \frac{a}{\cos^2\frac{\theta}{2}} = \frac{2a}{1+\cos\theta}$ i.e $\frac{2a}{r} = 1+\cos\theta$

This is a parabola and P α $\frac{1}{r^2}$

(v) When n = - 2 , the equation is $r^{-2} = a^{-2}.\cos 2\theta$

i.e. $r^2\cos 2\theta = a^2$ (rectangular hyperbola)

Problem 10

A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

Solution:

The polar equation to the ellipse, with pole at focus is

$$\frac{l}{r} = 1 + \text{e} \cos\, \theta \qquad \text{........ (1)}$$

where e is the eccentricity and *l* is the semi latus-rectum.

From (1), u = $\frac{1}{r} = \frac{1+e \cos\theta}{l}$

Hence $\frac{du}{d\theta} = -\frac{e \sin\theta}{l}$ and $\frac{d^2u}{d\theta^2} = -\frac{e \cos\theta}{l}$

u+ $\frac{d^2u}{d\theta^2} = \frac{1+e \cos\theta}{l} - \frac{e \cos\theta}{l} = \frac{1}{l}$

We know that $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{l}$

Hence P = $\frac{h^2u^2}{l} = \frac{\mu}{r^2}$, **where** $\mu = \frac{h^2}{l}$

i.e. ***The force varies inversely as the square of the distance from the pole.***

Now, $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

$$= \left(\frac{1+e \cos\theta}{l}\right)^2 + \left(\frac{e \sin\theta}{l}\right)^2 = \frac{1+2e \cos\theta + e^2}{l^2}$$

Also h = pv where v is the linear velocity

$\therefore$ v$^2 = \frac{h^2}{p^2} = \frac{h^2\left(1+2e \cos\theta + e^2\right)}{l^2}$

$$= \frac{\mu l}{l^2}\left[1 + e^2 + 2\left(\frac{l}{r} - 1\right)\right] \text{ from (1)}$$

$$= \frac{\mu}{l}\left(e^2 + \frac{2l}{r} - 1\right) = \frac{\mu}{l}\left[\frac{2l}{r} - \left(1 - e^2\right)\right]$$

$$= \mu\left[\frac{2}{r} - \frac{\left(1-e^2\right)}{l}\right] \qquad \text{........ (2)}$$

Now a and b are the semi major and minor - axes of the ellipse.

$$\therefore\ l = \frac{b^2}{a} = \frac{a^2\left(1-e^2\right)}{a} = a\left(1-e^2\right)$$

Put $l = a\left(1-e^2\right)$ in (2)

$$\mathbf{v}^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right],\ \therefore V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]}$$

Areal velocity = $\frac{h}{2}$

Area of the ellipse = πab

$$\textbf{Periodic Time T} = \frac{\pi\ ab}{\left(\frac{h}{2}\right)} = \frac{2\pi\ ab}{h}$$

$$= \frac{2\pi\ ab}{\sqrt{\mu l}} = \frac{2\pi\ ab}{\sqrt{\mu}.b}.\sqrt{a}$$

$$= \frac{2\pi}{\sqrt{\mu}}.a^{\frac{3}{2}}$$

Problem 11

Find the law of force towards the pole under which the curves can be described.

i) $r^2 = a^2 \cos 2\theta$

[Hint : Put n = 2 in problem 9, (i.e., $r^n = a^n.\cos n\theta$)]

ii) $r^{1/2} = a^{1/2} \cos \theta/2$

[Hint : Put n = $1/2$ in problem 9,(i.e.,$r^n = a^n \cos n\theta$)]

iii.) $r^n \cos n\theta = a^n$

Solution:

$a^n u^n = \cos n\theta \left[\because r = \frac{1}{u}\right]$ ________(1)

Take log both sides, and differentiate n log a + n log u = log cosn θ

$$\frac{n}{u}\cdot\frac{du}{d\theta} = \frac{1}{\cos n\theta}(-\sin n\theta)n$$

$$\therefore \frac{du}{d\theta} = -u\tan n\theta \quad \text{________(2)}$$

$$\therefore \frac{d^2u}{d\theta^2} = -\left[u.\sec^2 n\theta.n + (\tan n\theta)\frac{du}{d\theta}\right]$$

$$= -\left[un\sec^2 n\theta - u.\tan^2 n\theta\right]$$

$$= u\tan^2 n\theta - un.\sec^2 n\theta$$

We have, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

i.e., $u + u\tan^2 n\theta - un.\sec^2 n\theta = \frac{P}{h^2u^2}$

$u\sec^2 n\theta(1-n) = \frac{p}{h^2u^2}$

i.e $P = h^2(1-n).u^3.\frac{1}{a^{2n}.u^{2n}} = \frac{h^2(1-n)}{a^{2n}}.\frac{1}{u^{2n-3}}$

$$= \frac{h^2(1-n)}{a^{2n}}.r^{2n-3}$$

$$\boxed{\therefore P \propto r^{2n-3}}$$

iv) $\mathbf{r}^n = A\cos n\theta + B.\sin n\theta$

Solution:

This equation can be taken as

$r^n = \lambda.\cos(n\theta+\alpha), \lambda, \alpha$ are constants.

$$\therefore 1 = \lambda . u^n . \cos(n\theta + \alpha), \because r = \frac{1}{u}$$

Take log both sides and differentiate,

$$0 = \log \lambda + n \log u + \log \cos (n\theta + \alpha)$$

$$\therefore n . \frac{1}{u} . \frac{du}{d\theta} + \frac{1}{\cos(n\theta + \alpha)} [-\sin(n\theta + \alpha)] \; n = 0$$

$$\therefore \frac{du}{d\theta} = u . \tan(n\theta + \alpha) \quad \_\_\_\_\_\_\_(1)$$

$$\therefore \frac{d^2u}{d\theta^2} = u . \sec^2(n\theta + \alpha) . n + \tan(n\theta + \alpha) . \frac{du}{d\theta}$$

$$= nu. \sec^2(n\theta + \alpha) + u . \tan^2(n\theta + \alpha)$$

$$= n.u. \sec^2(n\theta + \alpha) + u . [\sec^2(n\theta + \alpha) - 1]$$

$$\therefore u + \frac{d^2u}{d\theta^2} = (n+1) u \sec^2(n\theta + \alpha) = \frac{P}{h^2 u^2}$$

$$\therefore P = h^2 . (n+1) . u^3 . \sec^2(n\theta + \alpha)$$

$$= h^2 (n+1) . u^3 (\lambda u^n)^2$$

$$= h^2 \lambda^2 (n+1) . u^{2n+3} = \frac{\lambda^2 h^2 (n+1)}{r^{2n+3}}$$

$\therefore$

$$\boxed{P \quad \alpha \frac{1}{r^{2n+3}}}$$

v) a = r sin nθ

Solution:

Take log and differentiate au = $\sin n\theta$

$$\log (au) = \log \sin n \theta \qquad \left[\because r = \frac{1}{u}\right]$$

i.e. log a + log u = log (sin nθ)

$$\therefore \frac{1}{u}.\frac{du}{d\theta} = \frac{1}{\sin n\theta}.\cos n\theta.n$$

$$\boxed{\frac{du}{d\theta} = nu.\cot n\theta}$$

$$\therefore \frac{d^2u}{d\theta^2} = n\left[u.\left(-\operatorname{cosec}^2 n\theta\right)n + \cot n\theta.\frac{du}{d\theta}\right]$$

$$= \text{n} \left[-nu.\operatorname{cosec}^2 n\theta + nu.\cot^2 n\theta\right]$$

$$= \text{n}^2.\text{u}\left[\cot^2 n\theta - \operatorname{cosec}^2 n\theta\right] = -n^2u.$$

$$\therefore u + \frac{d^2u}{d\theta^2} = u - n^2u = u\left(1-n^2\right)$$

But, $\text{u} + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = u\left(1-n^2\right)$

$$\therefore P = h^2u^3\left(1-n^2\right)$$

$$= \text{h}^2\left(1-n^2\right)u^3 = \frac{h^2\left(1-n^2\right)}{r^3}$$

$\therefore$

$$\boxed{\text{P} \;\alpha \frac{1}{r^3}}$$

vi) r = a sin nθ

Solution:

1 = au. sin nθ. $\left[\because r = \frac{1}{u}\right]$

Take log and differentiate,

0 = log a + log u + log sin n θ

$$\frac{1}{u}.\frac{du}{d\theta} + \frac{1}{\sin n\theta}\left(\cos n\theta\right).n = 0$$

i.e $\frac{1}{u}.\frac{du}{d\theta}+n.\cot n\theta=0$

$\therefore$ $$\boxed{\frac{du}{d\theta}=-nu.\cot n\theta}$$

$$\therefore\frac{d^2u}{d\theta^2}=-n\left[u.\left(-\cos ec^2 n\theta.n\right)+\cot n\theta.\frac{du}{d\theta}\right]$$

$$=-n\left[-nu.\cos ec^2 n\theta-nu\cot^2 n\theta\right]$$

$$=n^2u\left[\cos ec^2 n\theta+\cot^2 n\theta\right]$$

$$\therefore u+\frac{d^2u}{d\theta^2}=u+n^2u\cos ec^2 n\theta+n^2u\cot^2 n\theta$$

$$=\text{u}+n^2u.\frac{a^2}{r^2}+n^2u\left(\cos ec^2 n\theta-1\right)$$

$$=u+\frac{n^2a^2}{r^3}+n^2u.\frac{a^2}{r^2}-\frac{n^2}{r}$$

$$=\frac{1}{r}+2\frac{n^2a^2}{r^3}-\frac{n^2}{r}=\frac{2n^2a^2}{r^3}-\frac{\left(n^2-1\right)}{r}$$

But $u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2}$

$$\therefore\frac{2n^2a^2}{r^3}-\frac{\left(n^2-1\right)}{r}=\frac{P}{h^2u^2}$$

$$\therefore P=h^2\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^2}\right]$$

$$\therefore P\alpha\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^3}\right]$$

vii) $\frac{a}{r} = e^{n\theta}$

Solution:

Given $\frac{a}{r} = e^{n\theta}$

$\therefore au = e^{n\theta}$ __________(1) $\left[\because r = \frac{1}{u}\right]$

Differentiating, a.$\frac{du}{d\theta} = e^{n\theta} n$

$$\therefore \frac{du}{d\theta} = \frac{n}{a}.e^{n\theta}$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{n}{a}.e^{n\theta}.n$$

$$= \frac{n^2}{a}.e^{n\theta}$$

$$\therefore u + \frac{d^2u}{d\theta} = u + \frac{n^2}{a}.e^{n\theta} = \frac{e^{n\theta}}{a} + \frac{n^2}{a} e^{n\theta}$$

$$= \frac{e^{n\theta}}{a}\left(1+n^2\right) = u\left(1+n^2\right) \text{ by (1)}$$

But, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = u\left(1+n^2\right)$

$$\therefore P = h^2u^3\left(1+n^2\right) = h^2\left(1+n^2\right)u^3$$

$$= \frac{h^2\left(1+n^2\right)}{r^3}$$

$\therefore$ $\boxed{P \;\alpha\; \frac{1}{r^3}}$

viii) r = a. e $^{\theta \cot \alpha}$

Solution:

Given r = a. e $^{\theta \cot \alpha}$

$1 = au.\ e^{\theta \cot \alpha}$ ____(1) $\left[\because u = \frac{1}{r}\right]$

Differentiating w.r.to θ,

$$0 = a\left[u.e^{\theta \cot \alpha}.\cot \alpha + e^{\theta \cot \alpha}.\frac{du}{d\theta}\right]$$

$$\therefore \frac{du}{d\theta} = -\frac{u.e^{\theta \cot \alpha}.\cot \alpha}{e^{\theta \cot \alpha}} = -u \cot \alpha$$

$$\therefore \frac{d^2u}{d\theta^2} = -\cot \alpha \quad .\frac{du}{d\theta} = \text{u } \cot^2 \alpha$$

$$\therefore u + \frac{d^2u}{d\theta} = u + u\cot^2 \alpha = u\left(1 + \cot^2 \alpha\right) = u.\cos ec^2 \alpha$$

But u+ $\frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = u.\cos ec^2 \alpha$

$$\therefore P = h^2u^3.\cos ec^2 \alpha$$

$$= \frac{h^2.\cos ec^2 \alpha}{r^3}$$

$$\boxed{\therefore P \alpha \frac{1}{r^3}}$$

ix) r = a cosh n θ

Solution:

$1 = au.\ \cosh n\ \theta$ _________(1) $\left[\because r = \frac{1}{u}\right]$

Differentiating w.r.to θ, a$\left[u.n.\sinh n\theta + \cosh n\theta.\dfrac{du}{d\theta}\right] = 0$

$$\therefore \frac{du}{d\theta} = -nu \tanh n\theta \quad \_\_\_\_\_\_\_\_(2)$$

$$\therefore \frac{d^2u}{d\theta^2} = -n\left[un\operatorname{sec}h^2 n\theta + \tanh n\theta.\frac{du}{d\theta}\right]$$

$$= -n\left[nu.\operatorname{sec}h^2 n\theta - nu\tanh^2 n\theta\right]$$

$$= -n^2u\left[\operatorname{sec}h^2 n\theta - \tanh^2 n\theta\right]$$

$$\therefore \frac{d^2u}{d\theta^2} = -n^2u\left[\operatorname{sec}h^2 n\theta + \operatorname{sec}h^2 n\theta - 1\right] \left[\because \operatorname{sec}h^2\theta + \tanh^2\theta = 1\right]$$

$$= -n^2u\left[2\operatorname{sec}h^2 n\theta - 1\right]$$

$$= -n^2u\left[2a^2u^2 - 1\right]$$

$$\therefore u + \frac{d^2u}{d\theta^2} = -2n^2a^2u^3 + n^2u + u \quad = -2n^2a^2u^3 + \left(n^2+1\right) u.$$

But, u+$\dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2}$

$$\therefore \frac{P}{h^2u^2} = -2n^2a^2u^3 + \left(n^2+1\right) u.$$

$$\therefore P = -2n^2a^2h^2.u^5 + h^2\left(n^2+1\right) u^3$$

$$= -\frac{2n^2a^2h^2}{r^5} + \frac{\left(n^2+1\right)h^2}{r^3}$$

$$\boxed{P\,\alpha - \frac{2n^2a^2}{r^5} + \frac{\left(n^2+1\right)}{r^3}}$$

x) r cosh n θ = a

Solution:

Given r cosh n θ = a

$\therefore$ au = cosh nθ .... (1) $\left[\because \frac{1}{r} = u\right]$

Differentiating w.r.to θ,

$$a.\frac{du}{d\theta} = n.\sinh n\theta$$

$$\therefore a.\frac{d^2u}{d\theta^2} = n^2.\cosh n\theta$$

$$\frac{d^2u}{d\theta^2} = \frac{n^2}{a}\cosh n\theta$$

But, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

$$\therefore u + \frac{n^2}{a}.\cosh n\theta = \frac{P}{h^2u^2}$$

$$u + \frac{n^2}{a}.au = \frac{P}{h^2u^2} \qquad \text{[from (1)]}$$

i.e. $u + n^2u = \frac{P}{h^2u^2}$

$$\therefore P = h^2u^3\left(1+n^2\right) = \frac{h^2\left(n^2+1\right)}{r^3}$$

$$\boxed{\therefore P\alpha\frac{1}{r^3}}$$

Problem 12

Find the central acceleration under which the conic $\frac{l}{r}$ = 1 + e cos θ, can be described.

Solution:

Given equation is, $lu = 1 + e\cos\theta$ $\qquad \because \frac{1}{r} = u$

$$\therefore u = \frac{1 + e\cos\theta}{l} = \frac{1}{l} + \frac{e}{l}.\cos\theta$$

$$\therefore \frac{du}{d\theta} = -\frac{e}{l}.\sin\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta = \frac{1}{l}$$

$$\therefore P = \frac{h^2u^2}{l} = \frac{h^2}{l}.\frac{1}{r^2} = \frac{\mu}{r^2} \quad \left[\because \frac{h^2}{l} = \mu\right]$$

$$\boxed{\therefore P \alpha \frac{1}{r^2}}$$

4 Apses and apsidal distances

Definition

If there is a point A on a central orbit at which the velocity of the particle is perpendicular to the radius OA, then the point A is called an apse and the length OA is the apsidal distance.

Note : At an apse, the particle is moving at right angles to the radius vector.

We know that $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ where $u = \frac{1}{r}$

At an apse, $p = r = \frac{1}{u}$. $\therefore$ At an apse, $\frac{du}{d\theta} = 0$

Given the law of force to the pole, find the orbit

Given the central acceleration P, we find the path. We use the equation.

$$u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2} \qquad \text{..... (1)}$$

To solve equation (1), we multiply both sides by $2\frac{du}{d\theta}$, we have

$$2u.\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}=2\frac{P}{h^2u^2}\cdot\frac{du}{d\theta}$$

i.e. $\frac{d}{d\theta}(u)^2+\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2=\frac{2P}{h^2u^2}\cdot\frac{du}{d\theta}$

Integrating with respect to θ,

$$u^2+\left(\frac{du}{d\theta}\right)^2=\int\frac{2P}{h^2u^2}du+\text{constant} \qquad \text{...... (2)}$$

Problem 13

A particle moves with an acceleration $\mu\left[3au^4-2(a^2-b^2)u^5\right]$ and is projected from an apse at a distance (a + b) with a velocity $\frac{\sqrt{\mu}}{a+b}$. Prove that the equation to its orbit is r = $a+b\cos\theta$.

Solution:

Given $P=\mu\left[3au^4-2\left(a^2-b^2\right)u^5\right]$

The differential equation to the path is

$$u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2}=\frac{\mu}{h^2}\left[3au^2-2\left(a^2-b^2\right)u^3\right] \qquad \text{.... (1)}$$

Multiplying (1) by $2\frac{du}{d\theta}$ and integrating with respect to θ we get

$$u^2+\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{h^2}\int\left[3au^2-2\left(a^2-b^2\right)u^3\right]du+C$$

$$= \frac{2\mu}{h^2}\left[au^3 - 2\left(a^2 - b^2\right)\frac{u^4}{2}\right] + C \quad \text{................ (2)}$$

Now h = pv = constant = $p_o v_o$ where p_o and v_o are the initial values of p and v respectively.

Given $v_o = \frac{\sqrt{\mu}}{a+b}$ and p_o = a + b as the particle is projected from an apse

Hence $h = (a+b)\frac{\sqrt{\mu}}{a+b} = \sqrt{\mu}$ i.e. $\boxed{h^2 = \mu}$

So (2) becomes $u^2 + \left(\frac{du}{d\theta}\right)^2 = 2[au^3 - (a^2 - b^2)\frac{u^4}{2}] + c$ ….. (3)

Initially at the apse $\frac{du}{d\theta} = 0$ and $u = \frac{1}{a+b}$

Hence substituting these in (3), we have

$$\frac{1}{(a+b)^2} = 2\left[\frac{a}{(a+b)^3} - \frac{(a^2 - b^2)}{2(a+b)^4}\right] + C$$

$$= \frac{2a}{(a+b)^3} - \frac{(a-b)}{(a+b)^3} + C = \frac{1}{(a+b)^2} + C$$

$$\Rightarrow C = 0$$

$$(3) \Rightarrow \left(\frac{du}{d\theta}\right)^2 = 2au^3 - \left(a^2 - b^2\right) u^4 - u^2$$

$$\frac{du}{d\theta} = \sqrt{2au^3 - (a^2 - b^2)u^4 - u^2} = u\sqrt{2au - (a^2 - b^2)u^2 - 1} \quad \text{… (4)}$$

i.e $\frac{du}{u\sqrt{2au - (a^2 - b^2)u^2 - 1}} = d\theta$

Put $u = \frac{1}{r}$ $\quad\therefore du = -\frac{1}{r^2} dr$

$$-\frac{1}{r^2} \cdot r \frac{dr}{\sqrt{\frac{2a}{r} - \frac{\left(a^2 - b^2\right)}{r^2} - 1}} = d\theta$$

$$-\frac{dr}{\sqrt{2ar - (a^2 - b^2) - r^2}} = d\theta$$

i.e $\frac{-dr}{\sqrt{b^2 - (r-a)^2}} = d\theta$

Integrating, $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta + \alpha$ …… (5) where α is constant.

If θ is measured from the apse line, r = a + b and θ = 0.

$$\cos^{-1}(\frac{a+b-a}{b}) = 0 + \alpha$$

i.e $\cos^{-1} 1 = \alpha \quad \therefore \alpha = 0$

Hence (5) becomes $\cos^{-1}(\frac{r-a}{b}) = \theta$

i.e $\frac{r-a}{b} = \cos\theta$

r = a + b $\cos\theta$

Problem 14

A particle moves with a central acceleration equal to $\mu \div$ (distance) and is projected from an apse at a distance 'a' with a velocity equal to n times that which would be acquired in falling from infinity. Show that the other apsidal distance is $\frac{a}{\sqrt{n^2-1}}$

Solution:

"Velocity from infinity" means the velocity that acquired by the particle in falling with the given acceleration from infinity to the particular point given.

If x is the distance at time t from the centre in this motion, the equation is $\ddot{x} = -\frac{\mu}{x^5}$

Multiply by $2\dot{x}$ and integrate

$$\dot{x}^2 = -2\mu\int \frac{1}{x^5}dx + A = \frac{\mu}{2x^4} + A$$

Where $x = \infty$, $\dot{x} = 0$. Hence A = 0 and $\dot{x}^2 = \frac{\mu}{2x^4}$

When x = a, $\dot{x}^2 = \frac{\mu}{2x^4}$ and $\dot{x} = \sqrt{\frac{\mu}{2a^4}}$

Hence v_o = initial velocity of projection $= n\sqrt{\frac{\mu}{2a^4}} = \frac{n}{a^2}\sqrt{\frac{\mu}{2}}$

For the central orbit, P = $\frac{\mu}{r^5} = \mu u^5$

The differential equation of the path is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = \frac{\mu}{h^2}u^3$$

Multiplying (1) by $2\frac{du}{d\theta}$ and integrate with respect to θ,

$$u^2 + (\frac{du}{d\theta})^2 = \frac{2\mu}{h^2}\int u^3 du + C = \frac{\mu}{2h^2}u^4 + C \qquad \text{...... (2)}$$

Initial values are $p_o = a$, $v_o = \frac{n}{a^2}\sqrt{\frac{\mu}{2}}$

Hence h = $p_o v_o = \frac{n}{a}\sqrt{\frac{\mu}{2}} \quad or \quad h^2 = \frac{n^2\mu}{2a^2}$ i.e. $\frac{\mu}{2h^2} = \frac{a^2}{n^2}$

$$\therefore\ u^2 + (\frac{du}{d\theta})^2 = \frac{a^2u^4}{n^2} + C \qquad \text{..... (3)}$$

Initially at an apse, $\frac{du}{d\theta} = 0$ and $u = \frac{1}{a}$

So from (3), $\frac{1}{a^2} = \frac{1}{n^2a^2} + C \quad \therefore$ $\boxed{C = \frac{1}{a^2} - \frac{1}{n^2a^2}}$

$$\therefore u^2 + (\frac{du}{d\theta})^2 = \frac{a^2u^4}{n^2} + \frac{1}{a^2} - \frac{1}{n^2a^2} \qquad \text{.... (4)}$$

To get the apsidal distance put $\frac{du}{d\theta} = 0\ \ in\ \ (4)$

Hence $\frac{a^2u^4}{n^2} + \frac{1}{a^2} - \frac{1}{a^2n^2} - u^2 = 0$

i.e $a^4 u^4 + n^2 - 1 - a^2 n^2 u^2 = 0$

or $a^4 u^4 - n^2 a^2 u^2 + (n^2 - 1) = 0$

i.e. $(a^2 u^2 - 1)\,[a^2 u^2 - (n^2 - 1)] = 0$

i.e. $a^2 u^2 = 1$ or $a^2 u^2 = n^2 - 1$

i.e $\quad au = 1$ or $au = \sqrt{n^2 - 1}$

$u = \dfrac{1}{a}$ gives the point of projection

$\therefore$ apsidal distance is $u = \dfrac{\sqrt{n^2 - 1}}{a}$ i.e. $r = \dfrac{a}{\sqrt{n^2 - 1}}$

Problem 15

A particle is moving with central acceleration $\mu(r^5 - c^4 r)$ being projected from an apse at a distance C with velocity $C^3\sqrt{\dfrac{2\mu}{3}}$, Show that its path is the curve $x^4 + y^4 = c^4$

Solution:

Differential equation of the path is

$$\frac{p}{h^2 r^2} = u + \frac{d^2 u}{d\theta^2} \qquad \text{............ (1)}$$

Given, $P = \mu(r^5 - c^4 r) = \mu\left(\dfrac{1}{u^5} - \dfrac{c^4}{u}\right)$

$$\therefore \frac{\mu}{h^2}\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right) = u + \frac{d^2 u}{d\theta^2}$$

$$\therefore h^2\left(\frac{d^2 u}{d\theta^2} + u\right) = \mu\left(u^{-7} - c^4 \cdot u^{-3}\right)$$

Multiply by $2\frac{du}{d\theta}$ and integrate,

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right] + c_1 \quad \ldots\ldots (2)$$

Initially, r = c, ie. $u = \frac{1}{c},\ v = c^3\sqrt{\frac{2\mu}{3}},\ \frac{du}{d\theta} = 0$

$$\therefore c^6\left(\frac{2\mu}{3}\right) = h^2\left[0 + \frac{1}{c^2}\right] = 2\mu\left[-\frac{1}{6}c^6 + \frac{c^6}{2}\right] + c_1$$

$$\boxed{\therefore h^2 = \frac{2}{3}\mu c^8} \quad , \quad \boxed{c_1 = 0}$$

$$(2) \Rightarrow \frac{2}{3}\mu c^8\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right]$$

$$\therefore \frac{c^8}{3}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{-1}{6u^6} + \frac{c^4}{2u^2}$$

$$\therefore c^8\left(\frac{du}{d\theta}\right)^2 = 3\left[\frac{-1}{6u^6} + \frac{c^4}{2u^2}\right] - c^8u^2$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]$$

$$\therefore c^4 \cdot \frac{du}{d\theta} = \pm \frac{1}{4u^3}\left[\sqrt{1-\left(4c^4u^4-3\right)^2}\right]$$

$$\therefore 4u^3c^4 du = -\sqrt{1-\left(4c^4u^4-3\right)^2}\, d\theta$$

ie) $-\dfrac{4c^4u^3du}{\sqrt{1-\left(4c^4u^4-3\ \right)^2}} = d\theta$

$$\therefore \int -\frac{16c^4u^3du}{\sqrt{1-\left(4c^4u^4-3\right)^2}} = \int 4d\theta\,.$$

$$\therefore \cos^{-1}\left(4c^4u^4-3\right) = 4\theta + c_2 \qquad \ldots\ldots (3)$$

Initially, $u = \dfrac{1}{c},\ \theta = 0\ \Rightarrow c_2 = 0$

$$\therefore \cos^{-1}\left(4c^4u^4-3\right) = 4\theta$$

$$\therefore 4c^4u^4 - 3 = \cos 4\theta$$

$$\therefore 4c^4 = r^4(3+\cos 4\theta) = r^4(3+2\cos^2 2\theta - 1)$$

$$= r^4(2+2\cos^2 2\theta) = r^4[2+2(2\cos^2\theta-1)^2]$$

$$= r^4[2+2(4\cos^4\theta - 4\cos^2\theta+1)]$$

$$= r^4\left[4+4\left(2\cos^4\theta - 2\cos^2\theta\right)\right]$$

$$= 4r^4[1+2\cos^4\theta - 2\cos^2\theta]$$

$$= 4r^4[\cos^4\theta + (\cos^4\theta - 2\cos^2\theta + 1)]$$

$$= 4r^4[\cos^4\theta + (1-\cos^2\theta)^2]$$

$$4c^4 = 4r^4[\cos^4\theta + \sin^4\theta]$$

$$= 4[(r\cos\theta)^4 + (r\sin\theta)^4] = 4[x^4 + y^4]$$

$$\boxed{\therefore c^4 = x^4 + y^4}$$ where $x = r\cos\theta, y = r\sin\theta$

Problem 16

In a central orbit the force is $\mu\, u^3(3 + 2a^2u^2)$; if the particle be projected at a distance 'a' with a velocity $\sqrt{5\mu/a^2}$ in a direction making an angle $\tan^{-1}(1/2)$ with the radius, show that the equation to the path is r = a $\tan\theta$.

Solution:

The differential eqn. of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{p}{h^2u^2} = \frac{\mu\, u^3(3 + 2a^2u^2)}{h^2u^3}$$

$$\therefore h^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu(3u + 2a^2u^3)$$

Multiply by 2 $\frac{du}{d\theta}$ and integrating,

$$v^2 = a^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu[3u^2 + a^2u^4] + C \qquad \_\_\_\_\_\_\_\_(1)$$

Also, p = r sin ϕ

$\therefore$ Initially, P_o = a sin ϕ_0

Now, $\phi_0 = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \tan\phi_0 = \frac{1}{2}$

$\therefore \sin\phi_0 = \frac{1}{\sqrt{5}}$

$\therefore po = a\sin\phi_0 = \frac{a}{\sqrt{5}}$ __________(2)

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Initially, $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{po^2} = \frac{5}{a^2}$

Also, initially, $v = \sqrt{\frac{5\mu}{a^2}}$ given.

$$\therefore (1) \Rightarrow \mu\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\left[3u^2 + a^2u^4\right] + \frac{\mu}{a^2}$$

i.e. $\left(\frac{du}{d\theta}\right)^2 = 2u^2 + a^2u^4 + \frac{1}{a^2} = \frac{2a^2u^2 + a^4u^4 + 1}{a^2}$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{a^4u^4 + 2a^2u^2 + 1}{a^2} = \left(\frac{a^2u^2 + 1}{a}\right)^2$$

$$\therefore \frac{du}{d\theta} = \pm\left(\frac{a^2u^2 + 1}{a}\right)$$

i.e. $\int -\frac{a\,du}{a^2u^2 + 1} = \int d\theta$

$\therefore \cot^{-1}(au) = \theta + c_1$.

Initially, $u = \frac{1}{a}, \theta = \frac{\pi}{4}$ $\therefore c_1 = 0$

$\therefore \cot^{-1}(au) = \theta$. $\therefore au = \cot\theta$

$\therefore \frac{a}{r} = \frac{1}{\tan\theta}$ $\therefore$ $\boxed{r = a\tan\theta}$

Problem 17

A particle is projected from an apse at a distance 'a' with a velocity from infinity, the acceleration being μu^7 show that the equation to its path is $r^2 = a^2 \cos 2\theta$

Solution:

Eqn. of motion is, force = - ma

$$\therefore \mu u^7 = -\frac{d^2x}{dt^2} = -\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\frac{d\left(\frac{dx}{dt}\right)}{dx}.\frac{dx}{dt}$$

We know $v = \frac{dx}{dt} \Rightarrow$ $\boxed{\frac{\mu}{x^7} = -v\frac{dv}{dx}}$

$$\therefore \int_0^v 2v\,dv = \int_{x=\infty}^{a} -2\mu x^{-7} dx$$

$$\therefore v^2 = -2\mu\left[\frac{x^{-6}}{-6}\right]_0^a = 2\frac{\mu a^{-6}}{6} = \frac{\mu}{3a^6}$$

Now, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

$$\therefore h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu u^7}{u^2} = \mu u^5$$

Multiply by $2\frac{du}{d\theta}$; and integrating,

$$\therefore h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}.\frac{d^2u}{d\theta^2}\right] = 2\mu u^5 \frac{du}{d\theta}$$

$$\therefore h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\frac{\mu.u^6}{6} + C.$$

i.e. $\frac{h^2}{p^2}=\frac{\mu\ u^6}{3}+C.$

Initially, $v=V, u=\frac{1}{a}$ Also at an apse $\frac{du}{d\theta}=0$

$$\therefore V^2=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{3}.\frac{1}{a^6}+C \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_}(2)$$

i.e. $\frac{\mu}{3a^6}=\frac{\mu}{3a^6}+C\Rightarrow 0=C$

(2) $\Rightarrow$ Also, $\boxed{h^2=\frac{\mu}{3}\frac{a^2}{a^6}=\frac{\mu}{3a^4}}$

$$\therefore \frac{\mu}{3a^4}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu\ u^6}{3}$$

$$\therefore \left(\frac{du}{d\theta}\right)^2+u^2=a^4u^6$$

$$\therefore \left(\frac{du}{d\theta}\right)^2=a^4u^6-u^2$$

Also, u = $\frac{1}{r}$, $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

$$\therefore \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^4}{r^6}-\frac{1}{r^2}=\frac{a^4-r^4}{r^6}, \frac{dr}{d\theta}=\pm\frac{\sqrt{a^4-r^4}}{r}$$

$$\therefore \frac{-r\ dr}{\sqrt{a^4-r^4}}=d\theta$$

Put $z=r^2$ $\therefore dz=2rdr$

$$\therefore \frac{-dz}{\sqrt{a^4-z^2}}=2d\theta,$$

$$\therefore \int -\frac{dz}{\sqrt{(a^2)^2 - z^2}} = 2\int d\theta = 2\theta$$

i.e. $\cos^{-1}\left(\frac{z}{a^2}\right) = 2\theta + C_1$

Initially, r=a, i.e. z=r$^2 = a^2; \theta = 0 \Rightarrow C_1 = 0$

$$\therefore \cos^{-1}\left(\frac{z}{a^2}\right) = 2\theta \Rightarrow \frac{z}{a^2} = \cos 2\theta$$

$$\text{i.e.) } \frac{r^2}{a^2} = \cos 2\theta$$

$$\therefore \boxed{r^2 = a^2 \cos 2\theta}$$

5 Inverse Square Law

Newton's Law of Attraction

The mutual attraction between two particles of masses m_1 and m_2 placed at a distance 'r' apart is a force of magnitude $\gamma \frac{m_1 m_2}{r^2}$ where γ is a constant, known as the constant of gravitation.

Problem 18

A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to $\frac{\mu}{(distance)^2}$; Show that its path is a conic section and distinguish between the three cases that arise .

Solution:

Given $P = \frac{\mu}{r^2}$.

The (p, r) equation to the path is $\frac{h^2}{p^3} \cdot \frac{dp}{dr} = P = \frac{\mu}{r^2}$ .... (1)

i. e. $h^2 \frac{dp}{p^3} = \mu \frac{dr}{r^2}$

Integrate, $\frac{h^2}{-2p^2} = -\frac{\mu}{r} +$ constant

$$\frac{h^2}{p2} = \frac{2\mu}{r} + C \qquad \text{....... (2)}$$

We know (p, r) equation of a parabola is $p^2 = ar$

(p, r) equation of an ellipse is $\frac{b^2}{p^2} = \frac{2a}{r} - 1$

(p, r) equation of a hyperbola is $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

Comparing these equations with equation (2)

We get (2) is a parabola if C = 0

(2) is an ellipse if C is negative

(2) is a hyperbola if C is positive

Hence (2) always represents a conic section

Since h = pv where v is the velocity in the orbit at any point P distant r from the pole, equation (2) becomes

$$v^2 = \frac{2\mu}{r} + C$$

$$v^2 - \frac{2\mu}{r} = C \qquad \text{........ (4)}$$

Now, C is zero, negative or positive according as v^2 is equal to, less than or greater than $\frac{2\mu}{r}$.

Hence the path is a parabola, an ellipse or a hyperbola according as $v^2 =, < or > \frac{2\mu}{r}$.

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UNIT V

Moment of Inertia

5.0 Introduction

In this unit, a new concept known as Moment of inertia is introduced. Two important theorems on moment of inertia are explained. The moments of inertia of many standard bodies are derived. Some problems are also worked out. Later, the motion of a rigid body rotating about a fixed horizontal axis is discussed in detail. The kinetic energy, angular momentum and the moment of the effective forces about the axis of rotation are derived. Many examples are given. Compound pendulum and its period of oscillation are explained. The lengths of the simple equivalent pendulum are calculated for many rigid bodies. The motion of a circular disc rolling down an inclined plane is also discussed.

5.1 OBJECTIVES

After studying this unit, you will know about

* The moment of inertia
* The parallel axes theorem.
* The perpendicular axes theorem.
* The moments of inertia of many standard bodies.
* The motion of a rigid body rotating about a fixed horizontal axes.
* The compound pendulum.
* The period of oscillation of a compound pendulum.
* The length of a simple equivalent pendulum.
* The motion of a circular disc rolling down a rough inclined plane.

5.2 Definition

The moment, of inertia of a particle of mass m, about a line l is mr^2 where r is the perpendicular distance of the particle from the line.

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The moment of inertia of a rigid body about a line l is $\sum mr^2$ or $\int r^2 dm$ Where dm is the elementary mass of the rigid body.

5.3 Theorem of Parallel axes:

If I be the moment of inertia of a rigid body of mass M about a line l and if I_G be the moment of inertia about a parallel line through its centre of gravity G, then $I = I_G + Md^2$ where d is the distance between the parallel axes.

5.4 Theorem of Perpendicular axes

If I_x and I_y be the moments of inertia of a rigid body about two perpendicular lines ox and oy in the plane of a lamina and if I_z be the moment of inertia about a line which is perpendicular to both ox and oy, then

$$I_z = I_x + I_y$$

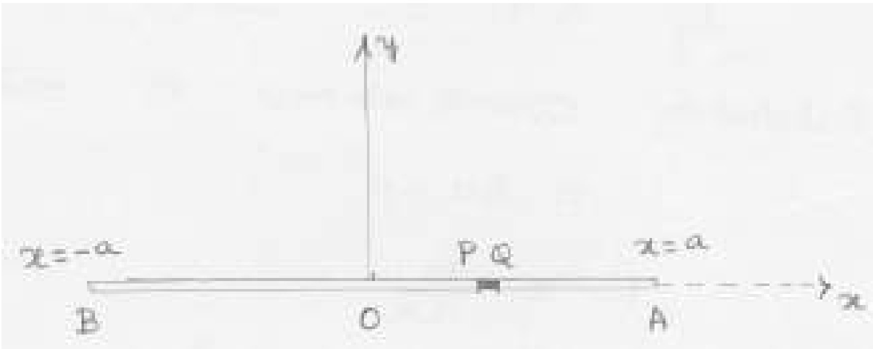
Note:

The parallel axes theorem is applicable to both two dimensional and three dimensional bodies.

The perpendicular axes theorem is applicable to only two dimensional bodies.

Moment of Inertia of some standard bodies.

1. To find the moment of inertia of a rod.



Let BOA be a uniform rod of length 2α.

Let o be the midpoint of the rod.

Let oy $\perp^r$ BOA.

We will find the M.I about OY

We will take o as origin and OA as x axis.

Let PQ be an elementary mass of the rod.

It will be a particle of thickness 8x at a distance x from o.

Elementary mass of PQ is $\delta m = \delta x.\rho$

Where ρ is the density of the rod.

M.I of the elementary mass about $oy = (\delta x \rho).x^2$

$$I_y = \rho\left(\frac{x^3}{3}\right)_{-a}^{a}$$

$$I_y = \rho\left[\frac{a^3}{3} - \left(-\frac{a^3}{3}\right)\right]$$

$$I_y = \rho.\frac{2a^3}{3} \qquad \rightarrow (1)$$

Let M be the mass of the rod

$$\therefore M = 2a.\rho$$

$$\therefore \rho = \frac{M}{2a}$$

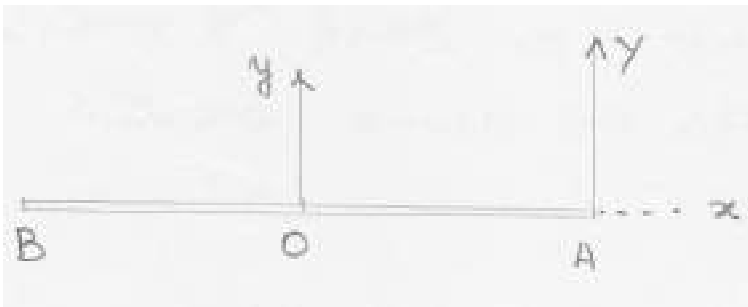
$$(1) \Rightarrow \begin{array}{l} I_y = \frac{M}{2a}.\frac{2a^3}{3} \\ I_y = M.\frac{a^2}{3} \end{array}$$

Note: If AY is a line 11^{p} to oy, then by the parallel axes theorem, M.I about AY is

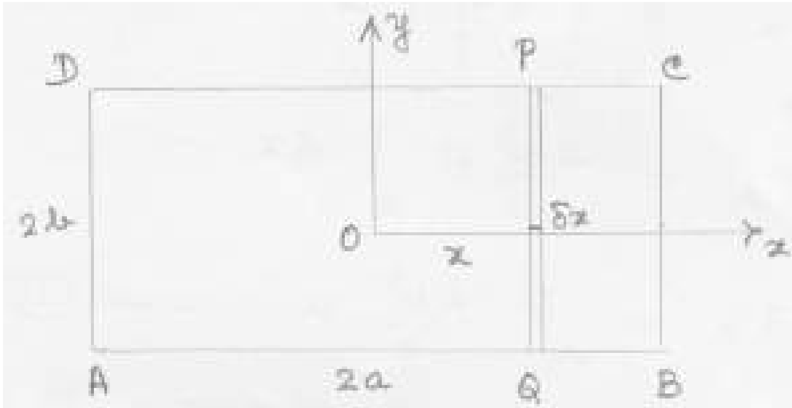
I= $I_G + Md^2$

I= $M.\frac{a^2}{3} + M.a^2$

$$I_y = M.\frac{4a^2}{3}$$



2. To find the M.I of a rectangular lamina



Let ABCD be a rectangular lamina of sides 2a and 2b.

Let o be the centre of the lamina.

Let ox and oy be two $\perp^r$ lines parallel to the sides as shown in the figure.

We will find the M.I about ox

Let PQ be an elementary mass of the lamina. It will be a thin rod of length wb and thickness δx at a distance x from o.

Elementary mass of PQ is $\delta m = 2b.\delta x.\rho$

Where ρ is the density of the lamina.

M.I of the elementary mass about ox = $(2b\delta x\rho).\ \frac{b^2}{3}$

$\therefore$ M.I of the lamina about ox $= I_x = \int_{x=-a}^{a} 2\frac{b^3}{3}\rho dx$

$$I_x = 2\int_{x=0}^{a} 2\frac{b^3}{3}\rho dx$$

$$I_x = 4\frac{b^3}{3}\rho\int_{x=0}^{a} dx$$

$$I_x = \rho.4\frac{b^3}{3}.a. \qquad \rightarrow (1)$$

Let M be the mass of the lamina.

$$\therefore M = 2a.2b.\rho$$

Then, $\therefore \rho = \frac{M}{4ab}$

$(1) \Rightarrow \qquad I_x = \frac{M}{4ab}.\frac{4b^3}{3}.a$

$$I_x = M.\frac{b^3}{3}$$

$\therefore$ M.I about ox is $I_x = M.\frac{b^3}{3}$

111^{ly} M.I about oy is $I_y = M.\frac{a^2}{3}$

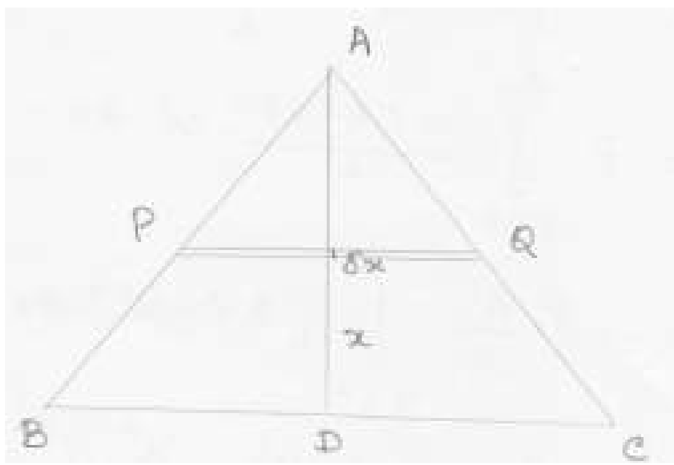
If oz is an axis $\perp^r$ to both ox and oy,

$$I_z = I_x + I_y$$

$$I_z = M.\frac{b^2}{3} + M.\frac{a^2}{3}$$

$$I_z = M\frac{\left(a^2 + b^2\right)}{3}$$

5.5.3 To find the M.I of a triangular lamina.



Let ABC be a triangular lamina of base a and height h.

We will the M.I about the base BC

Let PQ be an elementary mass of the lamina.It will be a thin rod of length PQ and thickness δx at a distance x from BC.

Elementary mass of PQ is δm =PQ.δx.ρ

Where ρ is the density.

M.I of the elementary mass about BC = (PQ.δx.ρ).x^2

Therefore, M.I of the triangular lamina about BC = $\int_{x=0}^{h} PQ.\rho.x^2 dx$

$\Delta ABC\,|||^{r}\ \Delta APQ.$

$$\therefore \frac{BC}{PQ}=\frac{h}{h-x}$$

$$PQ=\frac{BC(h-x)}{h}$$

$$PQ=\frac{a(h-x)}{h}.$$

$\therefore (1)\Rightarrow$

$$I=\rho\int_{x=0}^{h}\frac{a(h-x)}{h}.x^{2}.dx.$$

$$I=\rho\frac{a}{h}\int_{x=0}^{h}\left[h.\frac{x^{3}}{3}-\frac{x^{4}}{4}\right]dx$$

$$I=\rho\frac{a}{h}\left[h.\frac{x^{3}}{3}-\frac{x^{4}}{4}\right]_{0}^{h}$$

$$I=\rho\frac{a}{h}\left[h.\frac{h^{3}}{3}-\frac{h^{4}}{4}\right]$$

$$I=\rho.\frac{a}{h}.\frac{h^{4}}{12}$$

$$I=\rho.\frac{a}{12}h^{3}\qquad\rightarrow(2)$$

Let M be the mass of the triangular lamina.

$$\therefore M=\frac{1}{2}a.h.\rho$$

,

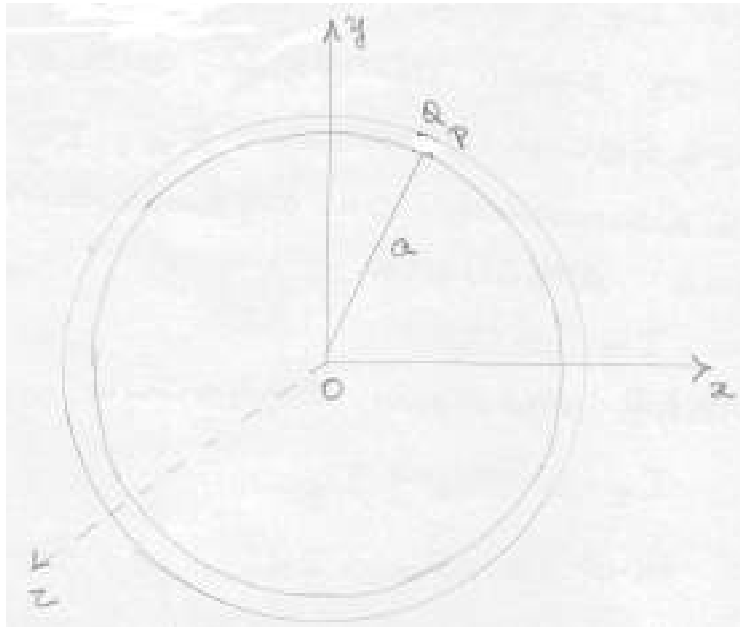
$$\therefore \rho=\frac{2M}{ah}$$

$(2)\Rightarrow$

$$I=\frac{2M}{ah}.\frac{a}{12}.h^{3}$$

$$I=M.\frac{h^{2}}{6}$$

4. To find the M.I of a circular ring



Let O be the centre and a be the radius of a circular ring.

Let OX and OY be two perpendicular lines in the plane of the ring.

Let OZ be perpendicular to both ox and oy.

We will find the M.I about oz

Let PQ be an elementary mass of the ring.

Elementary mass of PQ= δ m.

M.I of the circular ring about $OZ = \sum \delta m.a^2$

$$I_z = a^2 \left(\Sigma \delta m \right)$$

$$I_z = a^2.M$$

Where M is the mass of the ring.

$$\therefore I_z = M.a^2 \qquad \rightarrow (1)$$

I_x, I_y be the M.I of the circular ring about ox and oy. Since the ring is symmetrical about ox and oy, we get

$$I_x = I_y.$$

By the perpendicular axes theorem,

$$I_z = I_x + I_y$$
$$M.a^2 = I_x + I_x$$
$$\therefore 2I_x = M.a^2$$
$$III^{by} I_y = M.\frac{a^2}{2}.$$

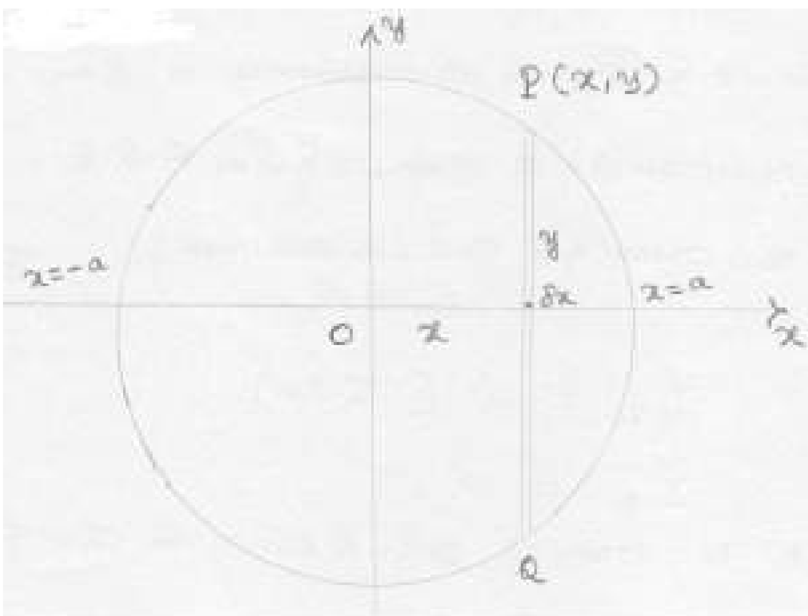
$\therefore$ M.I of the circular ring about a radius $= M.\frac{a^2}{2}.$

M.I of the circular ring about a line through the centre perpendicular to the plane of the ring $= M.a^2$

5.5.5 To find the M.I of a circular lamina

Let O be the centre and 'a' be the radius of the circular lamina.

Let ox and oy be two perpendicular lines in the plane of the lamina.



We will find the M.I about ox

Let PQ be an elementary mass of the circular lamina.

It will be a thin rod of length 2y and thickness δx at a distance x from O.

Elementary mass of PQ is $\delta m = 2y.\delta x.\rho$

Where ρ is the density

M.I of the elementary mass about ox $= (2y.\delta x.\rho).\frac{y^2}{3}$

M.I of the circular lamina about ox $=\int_{x=-a}^{a}\frac{2}{3}.y^3\rho.dx.$

$$I_x=\frac{2}{3}\rho\int_{x=-a}^{a}y^2dx \qquad \rightarrow(1)$$

On a circle,

$$x^2+y^2=a^2$$
$$x=a\cos\theta$$
$$y=a\sin\theta$$
$$dx=-a\sin\theta d\theta$$

$x=-a,\ -a=a\cos\theta$

$-1=\cos\theta\ \therefore\theta=\pi$

x = a; a= a cos θ $\therefore\theta=0$

$$\therefore(1)\Rightarrow I_x=\frac{2}{3}\rho\int_{\theta=\pi}^{0}(a\sin\theta)^3(-a\sin\theta)d\theta$$

$$\theta=\pi$$

$$I_x=\frac{2}{3}\rho a^4\int_{\theta=0}^{\pi}\sin^4\theta d\theta$$

$$I_x=\frac{2}{3}\rho a^4.2\int_{\theta=0}^{\pi/2}\sin^4\theta.d\theta$$

$$I_x=\rho.\frac{4}{3}.a^4.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$I_x=\rho.\frac{\pi a^4}{4} \qquad \rightarrow(2)$$

Let M be the mass of the lamina.

Then, $M=\pi a^2.\rho$

$$\therefore\rho=\frac{M}{\pi a^2}$$

$$\therefore(2)\Rightarrow I_x=\frac{M}{\pi a^2}.\frac{\pi a^4}{4}$$

$$I_x=M.\frac{a^2}{4}$$

$\therefore$ M.I of a circular lamina about a radius $=M.\frac{a^2}{4}$.

Let oz be a line $\perp^r$ to both ox and oy

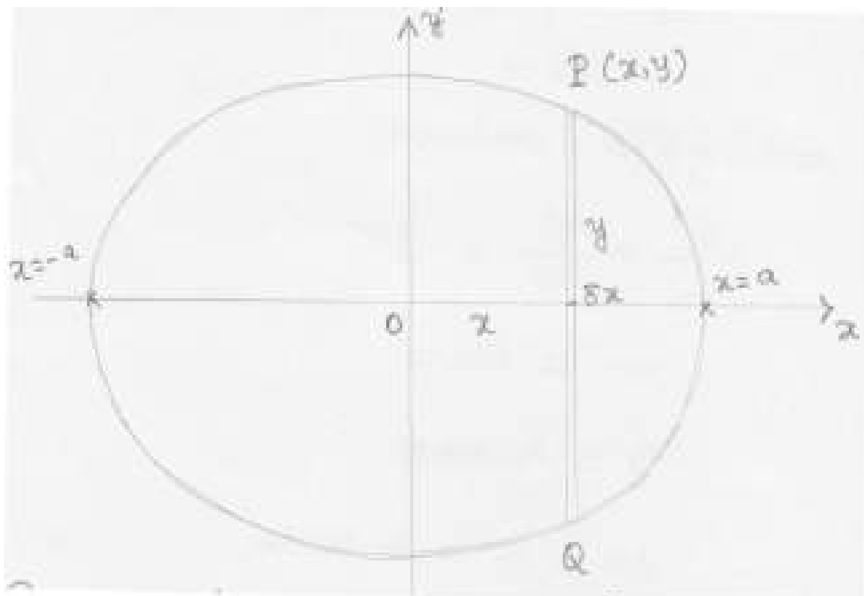
Then M.I about oz is

$$I_z = I_x + I_y$$

$$I_z = M.\frac{a^2}{4} + M.\frac{a^2}{4}$$

$$I_z = M.\frac{a^2}{2}$$

6. To find the M.I of an elliptical lamina.



Let o be the centre of the elliptical lamina.

Let 2a, 2b be the lengths of the major and minor axes.

Let ox and oy be two perpendicular lines in the plane of the lamina as shown in the figure.

We will find the M.I about ox.

Let PQ be an elementary mass of the lamina. It will be a thin rod of length 2y and thickness δx, at a distance x from O.

Elementary mass of PQ is $\delta m = 2y\delta x.\rho.$

Where ρ is the density.

M.I of the elementary mass about ox $= (2y\delta x\rho).\frac{y^2}{3}$

M.I of the elliptical lamina about ox $= \int_{x=-a}^{a} \frac{2y^3}{3}\rho dx$

$$I_x = \frac{2}{3}\rho\int_{x=-a}^{a} y^2 dx \qquad \rightarrow (1)$$

On the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$x = a\cos\theta$$

$$y = b\sin\theta$$

$$dx = -a\sin\theta d\theta$$

$x - a$ gives $\quad -a = a\cos\theta$

$$-1 = \cos\theta$$

$$\theta = \pi$$

$x = a$ gives $\quad a = a\cos\theta$

$$1 = \cos\theta$$

$$\theta = 0$$

$(1) \Rightarrow$

$$I_x = \rho.\frac{2}{3}\int_{\theta=\pi}^{0} (b\sin\theta)^3 (-a\sin\theta) d\theta$$

$$I_x = \rho.\frac{2}{3}ab^3\int_0^{\pi} \sin^4\theta d\theta$$

$$I_x = \rho.\frac{2}{3}ab^3.2\int_0^{\pi/2} \sin^4\theta.d\theta$$

$$I_x = \rho.\frac{4}{3}ab^3.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$I_2 = \rho.\pi.\frac{a}{4}b^3 \qquad \rightarrow (2)$$

Let M be the mass of the elliptical lamina,

$$M = \pi ab.\rho$$

$$\therefore \rho = \frac{M}{\pi ab}$$

$(2) \Rightarrow$

$$I_x = \frac{M}{\pi ab}.\pi.\frac{a}{4}.b^3$$

$$\therefore I_x = M.\frac{b^2}{4}.$$

III^{ly}, M.I about oy is $I_y = M.\frac{a^2}{4}$.

Let oz be a line perpendicular to both ox and oy. Then, M.I about oz is

$$I_z = I_x + I_y$$

$$I_z = M.\frac{b^2}{4} + M.\frac{a^2}{4}$$

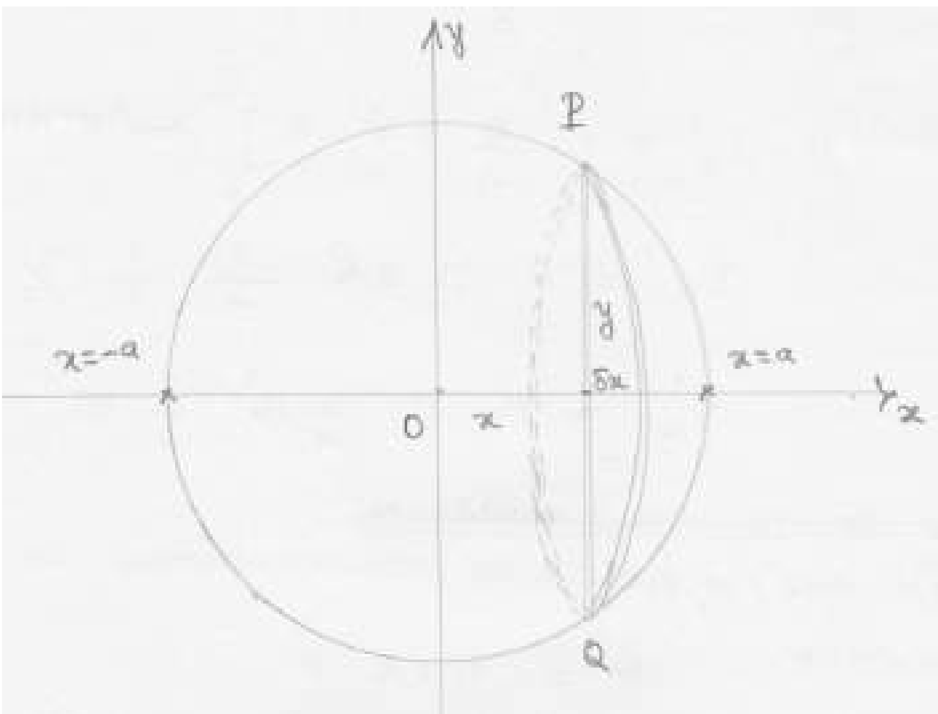
$$I_z = \left(\frac{a^2 + b^2}{4}\right)$$

5.5.7 To find the M.I of a solid sphere

Let O be the centre and 'a' be the radius of a solid sphere.

Let ox and oy be two perpendicular lines through o.

We will find the M.I about ox.



Let PQ be an elementary mass of the solid sphere.

It will be a circular lamina of radius y and thickness δx at a distance x from o.

Elementary mass of PQ is $\delta m = \pi y^2.\delta x.\rho$.

Where ρ is the density

M.I of the elementary mass about ox $= \left(\pi y^2 \delta x \rho\right).\frac{y^2}{2}$

M.I of the solid sphere about ox $= \int_{x=-a}^{a} \pi \frac{y^4}{2} \rho dx$

$$I_x = \frac{\pi}{2} \rho \int_{x=-a}^{a} y^4 dx \qquad \rightarrow (1)$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$dx = -a\sin\theta d\theta$$

$x = -a$ gives $\quad -a = a\cos\theta$

$$-1 = \cos\theta$$

$$\theta = \pi$$

$x = a$ gives $\quad a = a\cos\theta$

$$1 = \cos\theta$$

$$\theta = 0$$

$\therefore (1) \Rightarrow$

$$I_x = \frac{\pi\rho}{2} \int_{\theta=\pi}^{0} (a\sin\theta)^4 . (-a\sin\theta d\theta)$$

$$I_x = \frac{\pi\rho}{2} . a^5 \int_{\theta=0}^{\pi} \sin^5\theta d\theta$$

$$I_x = \frac{\pi\rho}{2} . a^5 . 2 \int_{0}^{\pi/2} \sin^5\theta d\theta$$

$$I_x = \pi\rho a^5 . \frac{4}{5} . \frac{2}{3} . 1$$

$$I_x = \frac{8}{15} \rho . \pi a^5 \qquad \rightarrow (2)$$

Let M be the mass of the solid sphere . Then,

$$M = \frac{4}{3} \pi a^3 \rho$$

$$\therefore \rho = \frac{3M}{4\pi a^3}$$

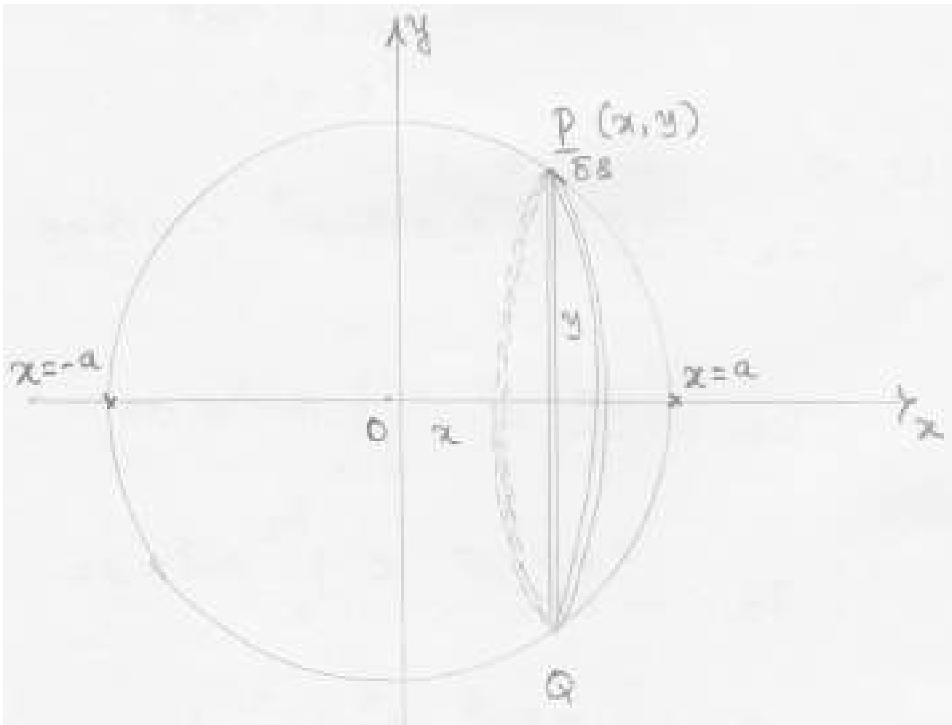
$$(2) \Rightarrow I_x = \frac{3M}{4\pi a^3} . \frac{8}{15} . \pi a^5$$

$$I_x = M . \frac{2}{5} . a^2$$

$\therefore$ M.I of a solid sphere about a radius $= M . \frac{2}{5} . a^2$

To find the M.I of a hollow sphere

Let o be the centre and 'a' be the radius of a hollow sphere.



s

Let ox and oy be two perpendicular lines through o.

We will find the M.I about ox

Let PQ be an elementary mass of the hollow sphere.

It will be a circular ring of radius y and thickness δs at a distance x from o.

Elementary mass of PQ is $= 2\pi y\delta s.\rho$

Where ρ is the density.

M.I of the elementary mass about ox $= (2\pi y\delta s\rho).y^2$

M.I of the hollow sphere about ox $= \int_{x=-a}^{a} 2\pi y^3\rho ds$

$$I_x = 2\pi\rho\int_{x=-a}^{a} y^3 ds \qquad \rightarrow (1)$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$s = a\theta$$

$$ds = ad\theta$$

$x = -a$ gives $\quad -a = a\cos\theta$

$-1 = \cos\theta$

$\theta = \pi$

$x = a$ gives $\quad a = a\cos\theta$

$1 = \cos\theta$

$\theta = 0$

$$\therefore (1) \Rightarrow I_x = 2\pi\rho\int_{\theta=\pi}^{0} (a\sin\theta)^3 .ad\theta$$

$$I_x = 2\pi\rho.a^4\int_{\theta=0}^{\pi} \sin^3\theta d\theta$$

$$I_x = 2\pi\rho.a^4.2\int_{0}^{\pi/2} \sin^2\theta d\theta$$

$$I_x = 2\pi\rho.a^4.2.\frac{2}{3}.1$$

$$I_x = \frac{8}{3}\pi a^4\rho \qquad \rightarrow (2)$$

Let M be the mass of the hollow sphere Then, .

$$M = 4\pi a^2.\rho$$

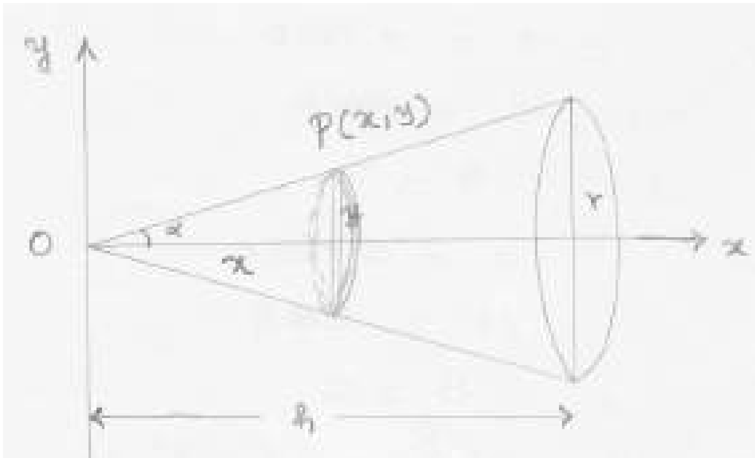
$$\therefore \rho = \frac{M}{4\pi a^2}$$

$$(2) \Rightarrow I_x = \frac{8}{3}\pi a^4.\frac{M}{4\pi a^2}$$

$$I_x = M.\frac{2}{3}a^2$$

$\therefore$ M.I of the hollow sphere about a radius $= M.\frac{2}{3}a^2$

To find the M.I of a solid cone



Let O be the vertex

Ox be the axis

α be the semi-vertical angle

r be the radius

and h be the height of the solid cone

We will find the M.I about ox

Let PQ be an elementary mass of the cone. It will be circular disc of radius y and thickness δx at a distance x from o.

Elementary mass of PQ is $\delta m = \pi y^2 . \delta x . \rho$

Where ρ is the density

M.I of the elementary mass about ox $= \left(\pi y^2 \delta x \rho\right) . \frac{y^2}{2}$

M.I of the solid cone about ox $= \int_{x=0}^{h} \pi \frac{y^4}{2} \rho dx$

$$I_x = \frac{\pi\rho}{2} \int_{x=0}^{h} y^4 dx \qquad \left[\tan\alpha = \frac{y}{x} \; y = x\tan\alpha\right]$$

$$I_x = \frac{\pi\rho}{2}\int_{x=0}^{h}(x\tan\alpha)^4\,dx$$
$$I_x = \frac{\pi\rho}{2}\tan^4\alpha\int_{x=0}^{h}x^4dx$$
$$I_x = \frac{\pi\rho}{2}\tan^4\alpha\left(\frac{x^5}{5}\right)_0^h$$
$$I_x = \frac{\pi\rho}{2}\tan^4\alpha.\frac{h^5}{5}$$
$$I_x = \rho.\frac{\pi}{10}.h^5\tan^4\alpha \qquad \rightarrow(1)$$

Let M be the mass of the cone

$$M = \frac{1}{3}\pi r^2 h.\rho$$
$$\rho = \frac{3M}{\pi r^2 h}$$

$(1)\Rightarrow$

$$I_x = \frac{3M}{\pi r^2 h}.\frac{\pi}{10}h^5\tan^4\alpha$$
$$I_x = M.\frac{3}{10.r^2}h^4\tan^4\alpha \qquad \left[\tan\alpha = \frac{r}{h} \;\; h\tan\alpha = r\right]$$
$$I_x = M.\frac{3}{10.r^2}(h\tan\alpha)^4$$
$$I_x = M.\frac{3}{10r^2}r^4$$

Next, we will find the M.I about oy

Elementary mass of PQ is $\delta m = \pi y^2\delta x.\rho$

M.I of the elementary mass about oy $= \pi y^2\delta x.\rho.\frac{y^2}{4} + \pi y^2\delta x\rho.x^2$

$\therefore$ M.I of the cone about oy $I_y = \int_{x=0}^{h}\left[\pi\frac{y^4}{4}\rho + \pi x^2y^2\rho\right]dx$

$$I_y = \int_{x=0}^{h} \frac{\left(\pi \tan^4 \alpha\rho + 4\pi \tan^2 \alpha\rho\right)}{4} x^4 dx$$

$$I_y = \frac{1}{4}.\pi \tan^2 \alpha\rho\left(\tan^2 \alpha + 4\right).\left(\frac{x^5}{5}\right)_0^h$$

$$I_y = \frac{\pi}{4}.\tan^2 \alpha.\frac{3M}{\pi r^2 h}\left(4 + \tan^2 \alpha\right).\frac{h^5}{5}$$

$$I_y = M\frac{3}{20.r^2}\left(h^2 \tan^2 \alpha\right).\left(4 + \tan^2 \alpha\right).h^2$$

$$I_y = M.\frac{3}{20.r^2} r^2.h^2\left(4 + \tan^2 \alpha\right)$$

$$I_y = M.\frac{3}{20} h^2\left(4 + \tan^2 \alpha\right)$$

5.5.10 Self assessment problems I

1. If a and b be external and internal radii of a hollow sphere, prove that its moment of inertia about a diameter is $M.\frac{2}{5}\frac{\left(a^5 - b^5\right)}{\left(a^3 - b^3\right)}$.

2. Find the moment of inertia of a solid cube about an edge

Until now , we have seen the motion of a particle, under the action of different kinds of forces.

Next we will discuss the motion of a rigid body

5.6 Motion of a rigid body

A rigid body rotates a fixed horizontal axis passing through the body. To find

i) the kinetic energy of the rigid body

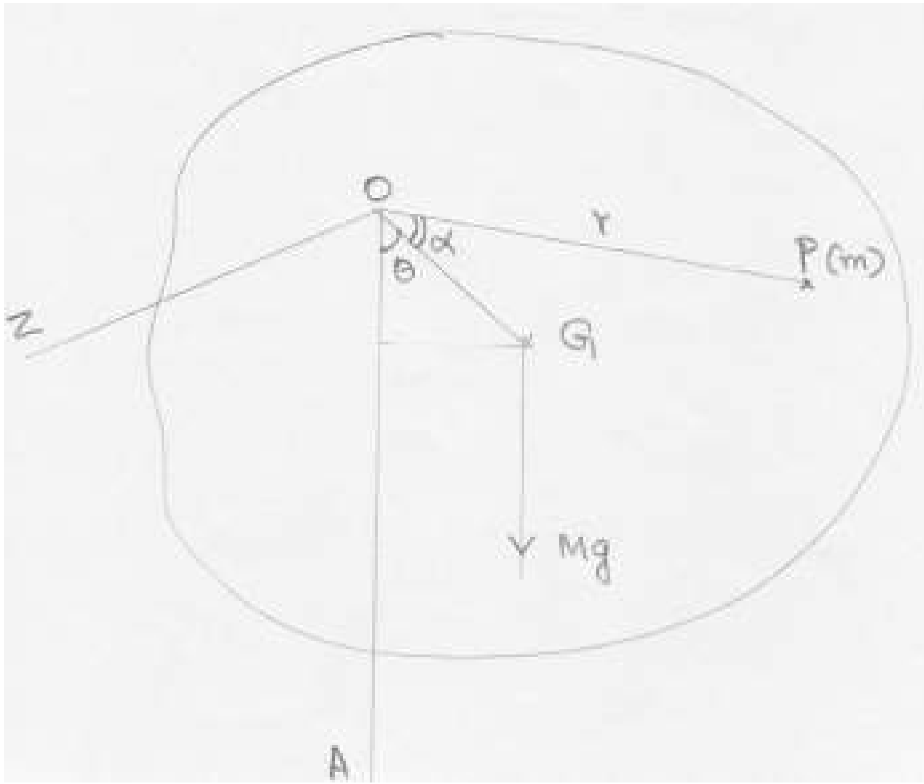
ii) the angular momentum and

iii) the moment of the effective forces about the axis of rotation

Let a rigid body of mass M rotate about a fixed horizontal axis oz.

Let OA be the fixed vertical line through O

Let G be the centre of gravity of the rigid body.



Let m be an elementary mass of the rigid body at P where OP = r.

In a rigid body, the distance between any two fixed points remains constant throughout the motion.

Therefore for all the positions of the rigid body, $G\hat{O}P = \alpha$ is constant.

When the rigid body rotates about the axis oz at any time t, let $A\hat{O}G = \theta$

OZA is a fixed plane in space and OZG is a fixed plane in the rigid body.

$\therefore$ $A\hat{O}G = \theta$ is the angle between these two fixed planes..

As the rigid body rotates about oz, at any time t,

$$\text{the angular velocity of P} = \frac{d}{dt}(\theta + \alpha)$$

$$= \frac{d\theta}{dt} + \frac{d\alpha}{dt}$$

$$\therefore \text{The angular velocity of P } = \frac{d\theta}{dt} + 0 = \frac{d\theta}{dt}$$

[Since α is constant, $\frac{d\alpha}{dt} = 0$]

The angular acceleration of P $= \frac{d^2}{dt^2}(\theta + \alpha)$

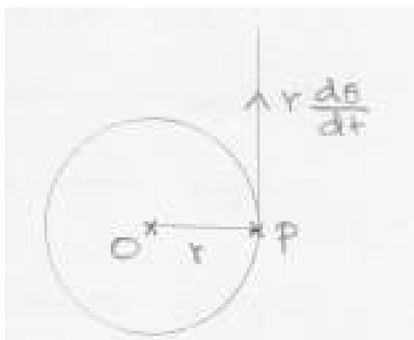
$$= \frac{d^2\theta}{dt^2} + \frac{d^2\alpha}{dt^2}$$

$$= \frac{d^2\theta}{dt^2} + 0$$

$$= \frac{d^2\theta}{dt^2}$$

$\therefore$ All the particles of the rigid body rotate with the same angular velocity $\frac{d\theta}{dt}$ and same angular acceleration $\frac{d^2\theta}{dt^2}$

i) To find the Kinetic energy



When the rigid body rotates about the axis oz,P will move along a circle with centre o and radius r.

$\therefore$ It will have a velocity $r = \frac{d\theta}{dt}$, along the tangent at ρ.

$\therefore$ Kinetic energy of the elementary mass at ρ}= $\frac{1}{2}m(r\frac{d\theta}{dt})^2$

[If a particle of mass m moves with a velocity v, then its K.E= $\frac{1}{2}mv^2$

$$\therefore \text{Kinetic energy of the rigid body}\} = \sum \frac{1}{2} mr^2 \left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2}\theta^2\left(\sum mr^2\right)$$

$$= \frac{1}{2}\dot{\theta}^2 MK^2$$

$$\therefore \text{K.E of the rigid body} = \frac{1}{2} MK^2 \dot{\theta}^2$$

Where MK^2 is the moment of inertia of the rigid body about the axis of rotation oz.

Here K is called the radius of gyration.

ii) To find the angular momentum.

When the rigid body rotates about the axis oz, the elementary mass at ρ moves along a circle with centre o and radius r.

$\therefore$ It has a velocity $r\frac{d\theta}{dt}$ along the tangent at P.

$\therefore$ Momentum of the elementary mass at P = $mr\frac{d\theta}{dt}$

[If a particle of mass m moves with a velocity v, then its momentum is mv].

Taking moments about o,

the moment of momentum of the elementary mass about oz} =”Force $\times$ distance”

$$= r.mr\frac{d\theta}{dt}$$

$$\therefore \text{Moment of momentum of the rigid body about oz} = \sum mr^2 \frac{d\theta}{dt}$$

$$= \sum mr^2 \theta^2$$

$$= \dot{\theta}(\sum mr^2)$$

$$= \dot{\theta} MK^2$$

$$\therefore \text{Angular momentum of the rigid body} = MK^2\dot{\theta}$$

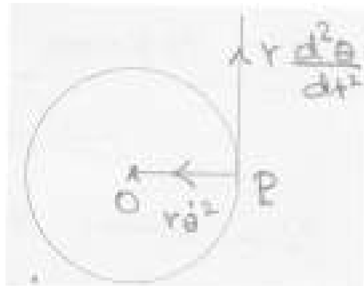
iii) To find the moment of the effective forces about the axis of rotation.

When the rigid body rotates about the fixed horizontal axis oz, ρ moves along a circle with centre o and radius r.

$\therefore$ It has a tangential acceleration $r\frac{d^2\theta}{dt^2}$ and normal acceleration r ($\frac{(\frac{d\theta}{dt^2})}{}$

We will find the moment about oz.

Since the normal acceleration $r\theta^{.2}$ passes through o, its moment about oz=0.



$\therefore$ Moment of the effectice forces acting on the elementary mass about oz $= r.\left(mr\frac{d^2\theta}{dt^2}\right)$

$\therefore$ Moment of the effective forces acting on the rigid body about oz

$$= \sum mr^2 \frac{d^2\theta}{dt^2}$$

$$= \frac{d^2\theta}{dt^2}\left(\sum mr^2\right)$$

$$= \ddot{\theta} Mk^2$$

$$= Mk^2\ddot{\theta}.$$

$\therefore$ Moment of the effective forces about oz $= Mk^2\ddot{\theta}.$

$\therefore$ i) Kinetic Energy $= \frac{1}{2}Mk^2\dot{\theta}$

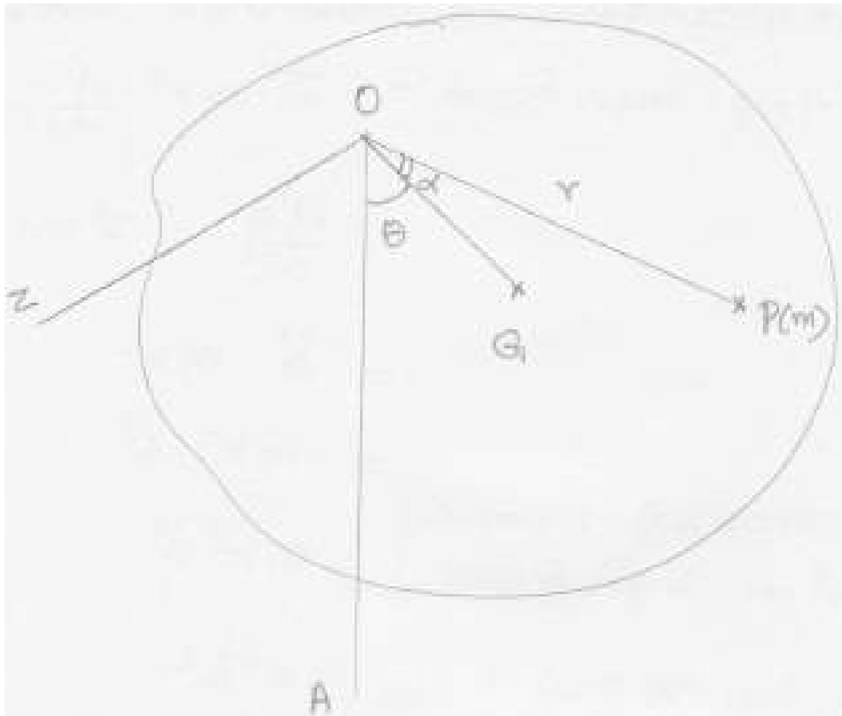
ii) Angular momentum $= Mk^2\dot{\theta}$

iii) Moment of the effective forces about oz $= Mk^2\ddot{\theta}$

5.7 Conservation of Angular Momentum.

If a rigid body rotates about a fixed horizontal axis, and is the sum of the moments of the external forces about its axis of rotation is zero, then its angular momentum about its axis of rotation is constant.

Proof:



Let a rigid body of mass M rotate about a fixed horizontal axis oz passing through the body.

Then, the moment of the effective forces acting on the rigid body about its axis of rotation is $MK^2\ddot{\theta}$.

Here MK^2 is the moment of inertia of the rigid body about oz.

$\therefore$ The equation of motion of the rigid body is

$$MK^2\ddot{\theta} = L \text{..........(1)}$$

Where L is the sum of the moments of the external forces about oz.

$$(1) \Rightarrow \frac{d}{dt}(MK^2\dot{\theta}) = L \text{........(2)}$$

Here $MK^2\dot{\theta}$ is the angular momentum of the rigid body about oz.

i.e$_1$ When a rigid body rotates about a fixed horizontal axis, its rate of change of angular moments of the external forces about the axis of rotation oz.

If L=o,then

$$(2) \Rightarrow \frac{d}{dt}(MK^2\dot{\theta}) = o$$

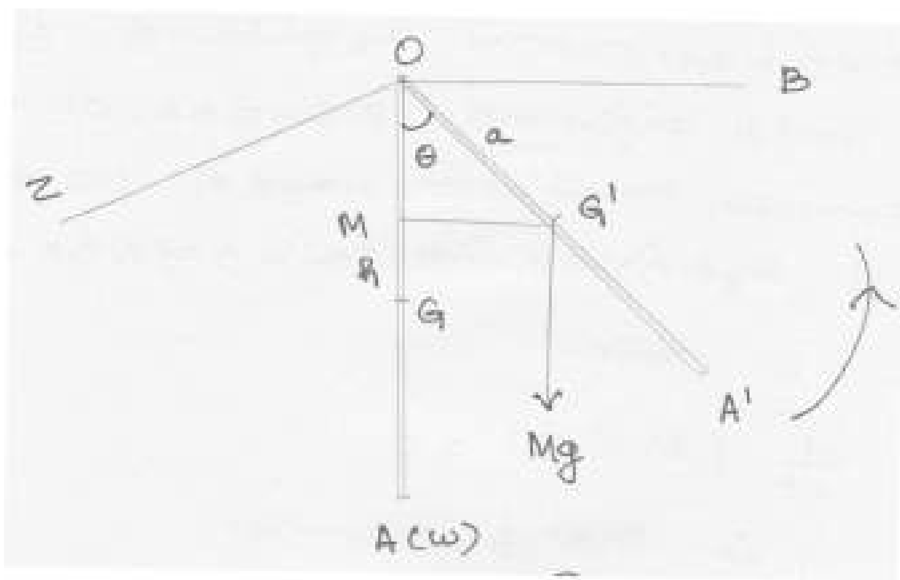
$\therefore MK^2\dot{\theta} =$Constant.

$\therefore$ When a rigid body rotates about a fixed horizontal axis oz, if the sum of the moments of the external forces acting on it about its axis of rotation is zero, then its angular momentum about its axis of rotation remains constant.

This is known as the principle of Conservation of angular momentum.

5.7.1 Problems

1. A uniform rod of length 2a is free to turn about its one end, which is fixed. When it is hanging vertically, it is projected with an angular velocity w. In the horizontal position, if it comes to rest, prove that $w=\sqrt{\frac{3g}{2a}}$



Let OA= 2a, the length of the rod

G- centre of gravity.

M- mass of the rod.

Let the rod rotate about a fixed horizontal axis oz.

When the rod is hanging vertically, in the position OA, it is projected with an angular velocity w.

At any time t, let OA' be the position of the rod. Let G' be the position of the centre of gravity at that instant.

Let $G'M \perp^r OA$.

Let $GM = h$ and $A\hat{O}G' = \theta$.

Now,

Loss in kinetic energy= Work done.

Initial K.E-Final K.E=Force $\times$ distance moved.

$$\frac{1}{2}MK^2\omega^2 - \frac{1}{2}MK^2\dot{\theta}^2 = Mg \times h.....(1)$$

Where MK^2 is the moment of inertia of the rod about oz.

$$\therefore Mk^2 = M.\frac{4a^2}{3}$$

$\therefore (1) \Rightarrow$

$$\frac{1}{2}MK^2(\omega^2 - \dot{\theta}^2) = Mgh$$

[In $\Delta OMG', \cos\theta = \frac{OM}{a}$
$OM = a\cos\theta$]

$$\frac{1}{2}.M.\frac{4a^2}{3}(\omega^2 - \dot{\theta}^2) = Mgh$$

$$\frac{2a^2}{3}(\omega^2 - \dot{\theta}^2) = g(OG - OM)$$

$$\frac{2a^2}{3}(\omega^2 - \dot{\theta}^2) = g(a - a\cos\theta) \_\_\_(1)$$

When the rod reaches the horizontal position, its angular velocity becomes zero.

i.e, then $\theta = 90^\circ, \dot{\theta} = 0$

$\therefore (1) \Rightarrow$

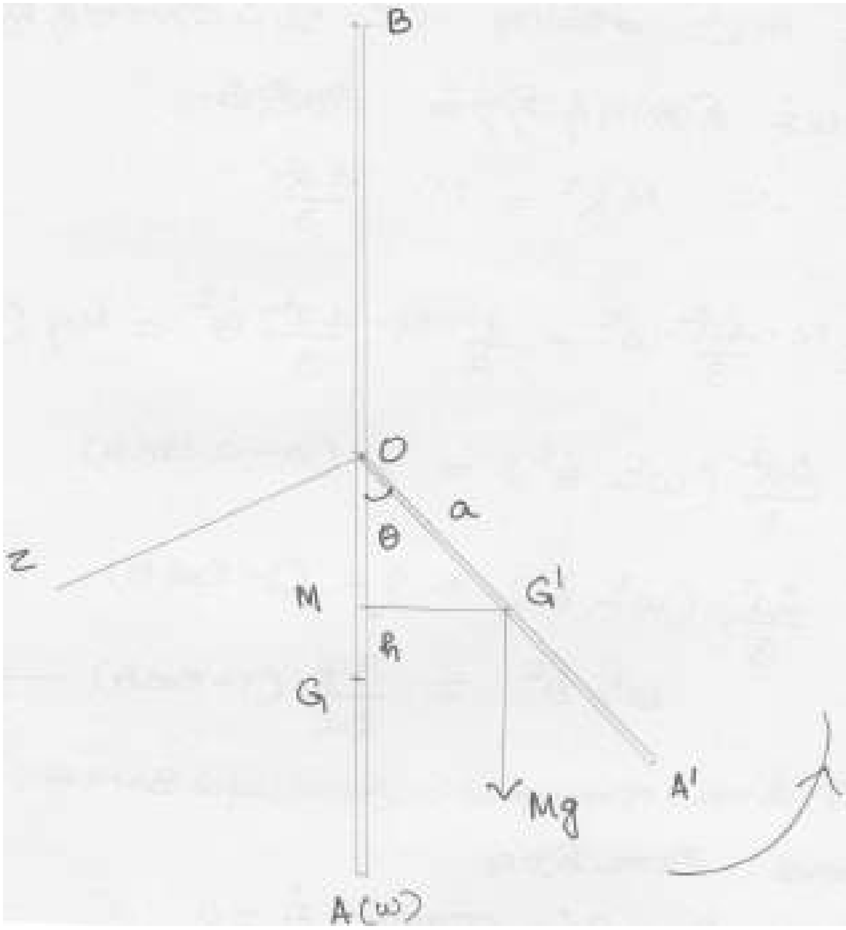
$$\frac{2a^2}{3}(\omega^2 - o) = g(a - a\cos 90^\circ)$$

$$\frac{2a^2}{3.}\omega^2 = g.a \qquad (\cos 90^\circ = 0)$$

$$\therefore \omega^2 = \frac{3g}{2a}$$

$$\therefore \omega = \sqrt{\frac{3g}{2a}}$$

2. A uniform rod of length 2a is at rest hanging from one end. An angular velocity w about a horizontal axis through the fixed and is communicated to it. If it just makes complete revolutions, prove that $\omega = \frac{\sqrt{3g}}{a}$



Let OA= 2a, length of the rod.

G- centre of gravity.

M- mass of the rod.

Let the rod rotate about a fixed horizontal axis oz.

When the rod hangs vertically in the position OA, it is projected with an angular velocity ω.

At any time t, let OA' be the position of the rod. Let G' be the position of the centre of gravity at that instant.

Let $G'M \perp^r OA$.

Let GM= h and $A\hat{O}G = \theta$.

Now, loss in K.E= work done.

Initial K.E-Final K.E=Force × distance moved

$\frac{1}{2}MK^2\omega^2 - \frac{1}{2}MK^2\dot{\theta}^2 = Mg \times h$......(1)

Where MK^2 is the moment of inertia of the rod about oz.

$$\therefore MK^2 = M.\frac{4a^2}{3}$$

$$(1) \Rightarrow$$

$$\frac{1}{2}M.\frac{4a^2}{3}w^2 - \frac{1}{2}M.\frac{4a^2}{3}\dot{\theta}^2 = Mg.h$$

$$\frac{2a^2}{3}(w^2 - \dot{\theta}^2) = g.GM$$

$$\frac{2a^2}{3}(w^2 - \dot{\theta}^2) = g(OG - OM)$$

$$\frac{2a^2}{3}(w^2 - \dot{\theta}^2) = g(a - a\cos\theta)$$

$$\frac{2a^2}{3}(w^2 - \dot{\theta}^2) = ga(1 - \cos\theta)$$

$$w^2 - \dot{\theta}^2 = \frac{3g}{2a}(1 - \cos\theta) \_\_\_(2)$$

If the rod has to make a complete revolution, when the rod reaches the upward vertical position , $\dot{\theta} = 0$

ie in the position OB, $\theta = \pi, \dot{\theta} = 0.$

$\therefore (2) \Rightarrow$

$$\omega^2 - 0 = \frac{3g}{2a}[1 - \cos\pi]$$

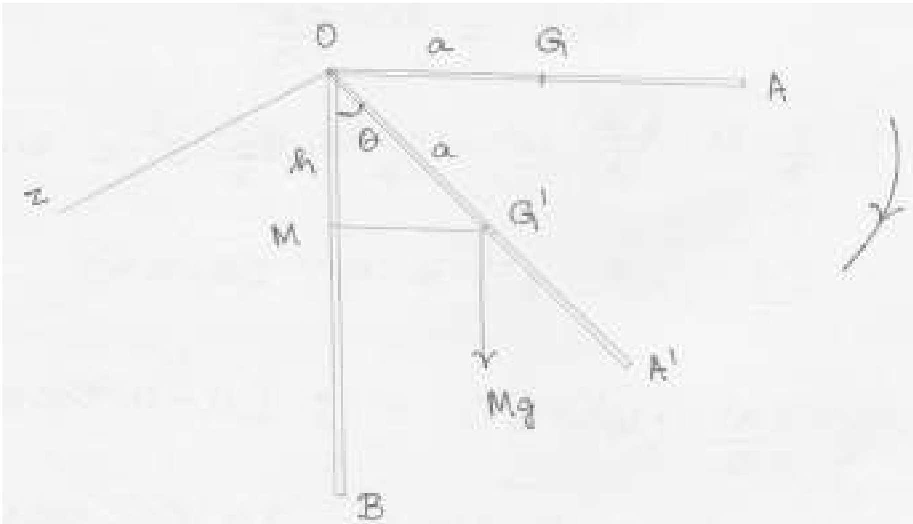
$$\omega^2 = \frac{3g}{2a}[1 - (-1)]$$

$$\omega^2 = \frac{3g}{2a}.2$$

$$\omega^2 = \frac{3g}{a}$$

$$\therefore \omega = \sqrt{\frac{3g}{a}}$$

3. A uniform rod of length 2a can turn freely about one end. If it is released from the horizontal position, prove that its angular velocity when it is first vertical is $\sqrt{\frac{3g}{2a}}$



Let $OA = 2a$, be the length of the rod

$G -$ centre of gravity

$M -$ Mass of the rod

Let the rod rotate about a fixed horizontal axis oz.

When the rod is in the horizontal position OA, it is held at rest and then released.

At any time t, let OA' be the position of the rod.

Let G' be the position of the centre of gravity at that instant.

Let OB be the downward vertical position of the rod.

Let $G'M \perp OB$.

Let $B\hat{O}A = \theta$.

Let $OM = h$.

Now,

Loss in K.E = Work done

Initial K.E – Final K.E = Force $\times$ distance moved

$$0 - \frac{1}{2}MK^2\dot{\theta}^2 = Mg(-h)$$

$$\frac{1}{2}MK^2\dot{\theta}^2 = Mgh \qquad \rightarrow (1)$$

where MK^2 is the moment of inertia of the rod about oz.

$$\therefore MK^2 = M.\frac{4a^2}{3}$$

$$\therefore (1) \Rightarrow \qquad \frac{1}{2}.M.\frac{4a^2}{3}\dot{\theta}^2 = Mg.a\cos\theta$$

$$\dot{\theta}^2 = \frac{3g}{2a}\cos\theta$$

When the rod becomes vertical,

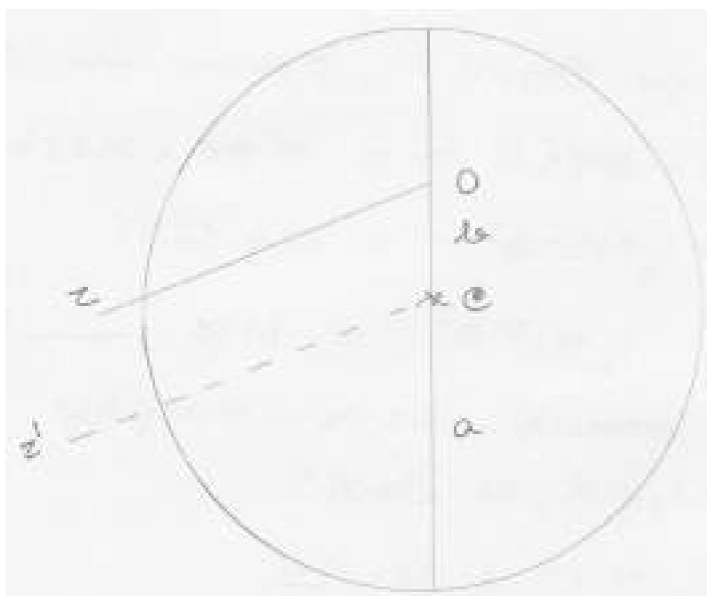
$$\theta = 0, \text{let } \dot{\theta} = \omega$$

$$\therefore \omega^2 = \frac{3g}{2a}.\cos 0^\circ$$

$$\omega^2 = \frac{3g}{2a}.1$$

$$\therefore \omega = \sqrt{\frac{3g}{2a}}$$

4. A uniform circular disc of radius ‘a’ and mass m can revolve about a fixed horizontal axis perpendicular to its plane at a distance ‘b’ from the centre. Find its Kinetic energy.



Let C be the centre and a be the radius of the circular disc.

Let the disc rotate about a fixed horizontal axis oz.

Let oc = b.

When the rod rotates about the fixed horizontal axis oz, Kinetic energy of the disc $= \frac{1}{2} MK^2 \dot{\theta}^2 \qquad \rightarrow (1)$

Where MK^2 is the moment of inertia of the disc about oz.

Let Cz' be an axis parallel to oz.

We know that M.I of the disc about $Cz' = M.\frac{a^2}{2}$.

By parallel axis theorem,

M.I of the disc about OZ= M.I about Cz' + $M.b^2$

$$MK^2 = M.\frac{a^2}{2} + M.b^2$$

$$MK^2 = M\frac{\left(a^2 + 2b^2\right)}{2}$$

$(1) \Rightarrow$

Kinectic energy of the disc $= M\frac{\left(a^2 + 2b^2\right)}{2}\dot{\theta}^2$

5.7.2 Self assessment problems II

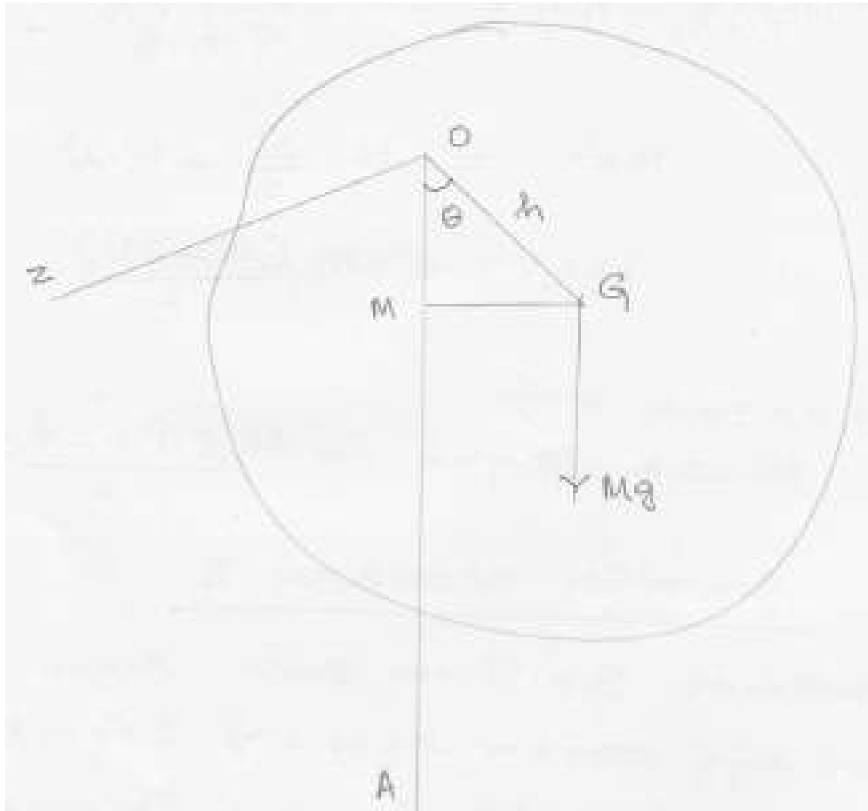
1. A uniform rod of length 2a can turn freely about one end. When it is in the upward vertical position, it is released. When it comes to the horizontal position, find its angular velocity.

2. A square lamina of side 2a can rotate about a fixed horizontal axis passing through one vertex perpendicular to the plane of the lamina . Find the kinectic energy of the lamina.

5.8 COMPOUND PENDULUM

A rigid body oscillating freely under gravity about a fixed horizontal axis is called a compound pendulum.

B.W To find the period of oscillation of a compound pendulum



Let a rigid body of mass M oscillate about a fixed horizontal axis oz.

Let G be the centre of gravity.

Let OA be the fixed vertical line through o.

Let OQ=h and $GM \perp OA$.

When the rigid body oscillates about oz, at any time t, let $A\hat{O}G = \theta$.

The only external force acting on the rigid body is its weight $Mg \downarrow atG$.

Taking moments about oz,

Where MK^2 is the moment of inerti of the rigid body about oz.

$$MK^2\ddot{\theta} = -Mg.GM \quad \rightarrow (1)$$

Where MK^2 is the moment of inertia of the rigid body about oz.

In ΔOGM,

$$\sin\theta = \frac{GM}{OG}$$

$$\sin\theta = \frac{GM}{h}$$

$$\therefore GM = h\sin\theta$$

$$\therefore (1) \Rightarrow \quad MK^2\ddot{\theta} = -Mgh\sin\theta \quad \to (2)$$

For small values of θ in radiands,

$\sin\theta = \theta$ nearly

$$\therefore (2) \Rightarrow \quad MK^2\ddot{\theta} = -Mgh\theta$$

$$\ddot{\theta} = -\frac{gh}{K^2}\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{K^2}\theta \quad \to (3)$$

This equation is of the form

$$\frac{d^2x}{dt^2} = -n^2x$$

$\therefore$ This is a simple harmonic motion.

$$\therefore \text{Its period } T = \frac{2\pi}{n}$$

$$T = \frac{2\pi}{\sqrt{gh/K^2}}$$

$$\text{period } T = 2\pi\sqrt{\frac{K^2}{gh}}$$

5.8.2 Simple equivalent Pendulum

A simple pendulum whose period of oscillation is equal to that of a compound pendulum is called a simple equivalent pendulum.

For a simple pendulum of length l,

$$\text{Period } T = 2\pi\sqrt{\frac{l}{g}} \quad \to (1)$$

$$\text{For a compound pendulum } T = 2\pi\sqrt{\frac{K^2}{gh}} \quad \to (2)$$

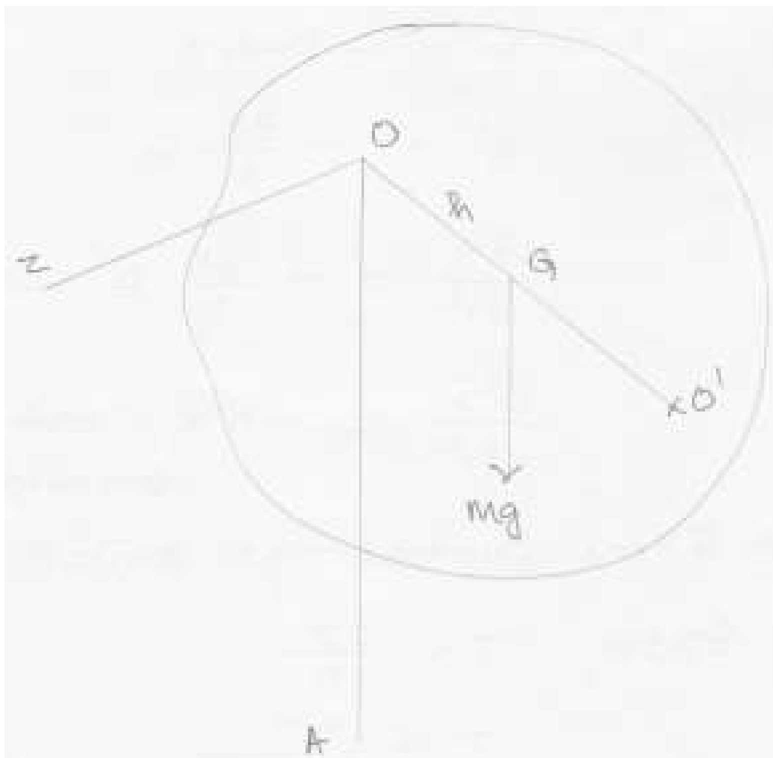
comparing$(1),(2)$

we get $l=\frac{K^2}{h}$

$\therefore$ Length of the simple equivalent pendulum $l=\frac{K^2}{h}$.

5.9 Centre of suspension and centre of oscillation

Let a rigid body oscillate about a fixed horizontal axis oz.



Let G be the centre of gravity.

Let OG=h

Let MK^2 be the moment of inertia of the rigid body about oz.

Let O' be a point on OG, such that

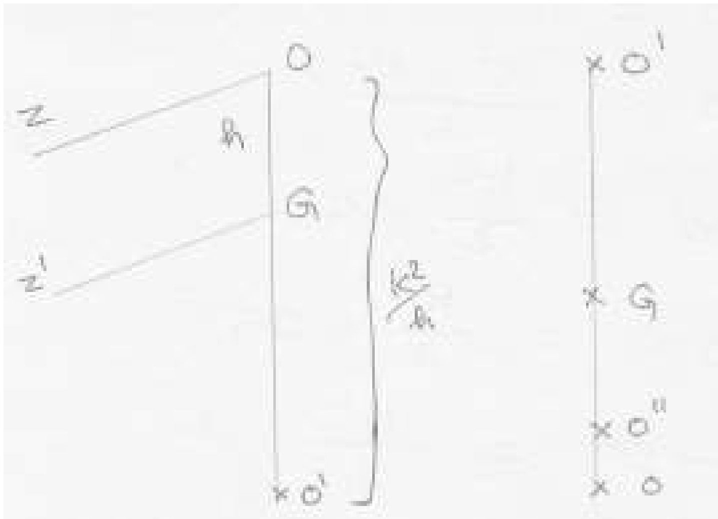
i.e., $OO'=\frac{K^2}{h}$.

Then, o is the centre of suspension and

O' is the centre of oscillation.

5.9.1 To show that the centre of suspension and centre of oscillation are interchangeable.

Let a rigid body oscillate about a fixed horizontal axis oz..



Let o be the centre of Suspension,

G be the centre of gravity

And O' be the centre of oscillation.

Let OG=h

$$OO' = \frac{K^2}{h} \qquad \rightarrow (1)$$

Where Mk^2 is the M.I of the rigid body about oz.

Let MK^2 br yhr M.I of the rigid body about a parallel axis GZ'

Then, by the parallel axes theorem,

$$"I = I_G + Md^2"$$

$$MK^2 = MK^2 + Mh^2$$

$$\therefore K^2 = K^2 + h^2 \qquad \rightarrow (2)$$

$$\therefore (1) \Rightarrow \qquad OO' = \frac{K^2 + h^2}{h}$$

$$OO' = \frac{K^2}{h} + h$$

$$OO' = \frac{K^2}{h} + OG$$

$$OO' - OG = \frac{K^2}{h}$$

$$O'G = \frac{K^2}{OG}$$

$$\therefore OG.O'G = K^2 \qquad \rightarrow (3)$$

Next, when O' becomes the centre of suspension, let O'' be the centre of oscillation.

Then from (2), we get

$$O'G.O''G = K^2 \qquad \rightarrow (2)$$

$(3) \& (4) \Rightarrow$

$$\frac{OG.O'G}{O'G.O''G} = \frac{K^2}{K^2}$$

$$\frac{OG}{O''G} = 1$$

$$\therefore OG = O''G$$

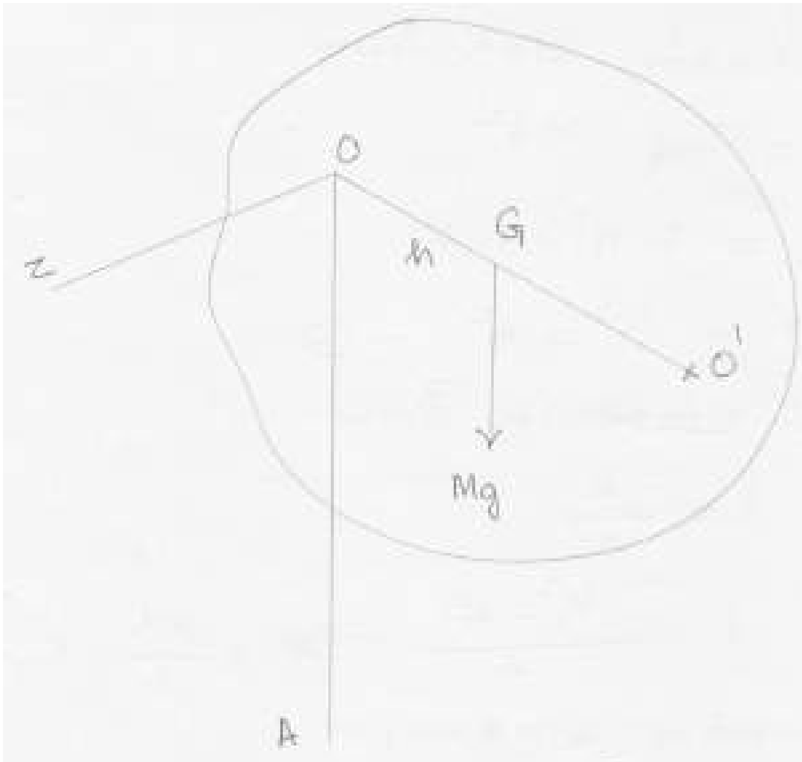
$\therefore O$ and O'' coincide.

i.e, When $\therefore O'$ becomes the centre of suspension,

O becomes the centre of oscillation.

$\therefore$ The centre of oscillation and the centre of suspension are interchangeable.

5.9.2 To find the minimum period of oscillation of a compound pendulum.



Let a rigid body oscillate about a fixed horizontal axis oz.

Period of oscillation $T = 2\pi\sqrt{\frac{K^2}{gh}}$

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \rightarrow (1)$$

Where MK^2 is the M.I of the rigid body about oz.

Let OG=h.

Let O'' be the centre of oscillation.

Then $\therefore OO' = \frac{K^2}{h}$

Let $l = \frac{K^2}{h}$ __(2)

Let MK^2 be the M.I of the rigid body about a paralled axis GZ'.

Then, by paralled axes theorem,

$$I = I_G + Md^2$$
$$MK^2 = MK^2 + Mh^2$$
$$K^2 = K^2 + h^2 \qquad \rightarrow (2)$$

$$\therefore (2) \Rightarrow \qquad l = \frac{K^2 + h^2}{h}$$

$$l = \frac{K^2}{h} + h$$

Differentiating w.r.t.h

$$\frac{dl}{dh} = -\frac{K^2}{h^2} + 1$$

Again differentiating w.r.t. h

$$\frac{d^2l}{dh^2} = \frac{2K^2}{h^3}$$

For maximum or minimum value of l

$$\frac{dl}{dh} = 0$$
$$-\frac{K^2}{h^2} + 1 = 0$$
$$\frac{K^2}{h^2} = 0$$
$$K^2 = h^2$$
$$K = h$$
$$\therefore h = K$$

when $h = K$, $\frac{d^2l}{dh^2} = \frac{2K^2}{K^3}$

$$= \frac{2}{K} > 0$$

$\therefore$ l is minimum when h=k

Minimum value of $l = \frac{K^2 + h^2}{h}$

$$= \frac{K^2 + K^2}{K}$$

$$= 2K.$$

$\therefore$ Minimum period of oscillation

$$T_1 = 2\Pi\sqrt{\frac{K^2}{gh}}$$

$$T_1 = 2\Pi\sqrt{\frac{l}{g}}$$

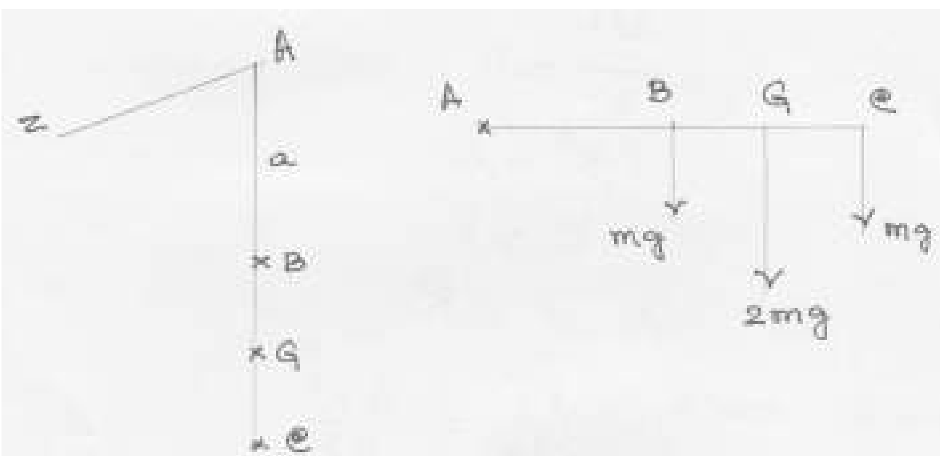
$$T_1 = 2\pi\sqrt{\frac{2K}{g}}$$

5.9.3 PROBLEMS

1. A weightless red ABC of length 2a is capable to rotate about a horizontal axis through A. A particle of mass m is attached at B and another of mass m is attached at C. Find the length of the simple equivalent pendulum.

ABC=2a, length of the rod.

Let the rod oscillate about the fixed horizontal axis AZ.



A particle of mass m is attached at B and another particle of mass m is attached at C.

Let G be the common centre of gravity.

Let AB=h.

Taking moments about A,

$$2mg.AG = mg.AB + mg.AC$$
$$2mg.h = mg.a + mg.2a$$
$$2h = 3a$$
$$h = \frac{3a}{2} \qquad \rightarrow (1)$$

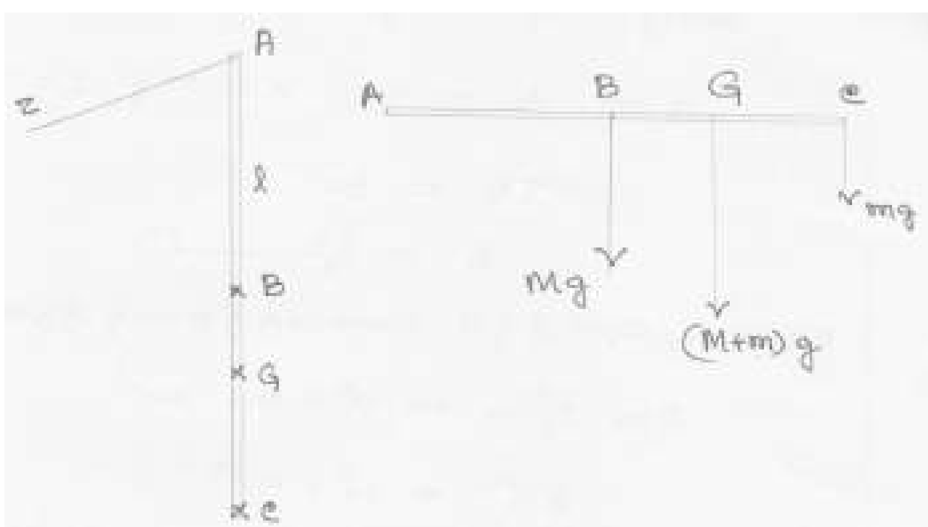
Considering the moment of inertia about AZ,

$$2m.K^2 = m.a^2 + m.(2a)^2$$
$$2K^2 = a^2 + 4a^2$$
$$K^2 = \frac{5a^2}{2} \qquad \rightarrow (2)$$

$\therefore$ The length of the simple equivalent pendulum is

$$l = \frac{K^2}{h}$$
$$l = \frac{\left(\frac{5a^2}{2}\right)}{\left(\frac{3a}{2}\right)}$$
$$l = \frac{5a}{3}$$

2. A heavy uniform rod of length 2l and mass M has a mass m attached to its one end. The whole system oscillates about a fixed horizontal axis through the other end. Show that the period of oscillation is $4\pi\sqrt{\frac{(M+3m)l}{3(M+2m)g}}$ vd ãWÎf.



Let $AC = 2l,$ be the length of the rod.

The weight of the rod $Mg\downarrow$ acts at the mid point B.

Also a weight $mg\downarrow$ acts at C.

Let AG=h.

Taking moments about A,

$$(M+m)g.OZ = Mg.l + mg.2l$$

$$(M+m).g.h = (M+2m)gl$$

$$\therefore h = \frac{(M+2m)l}{(M+m)g} \qquad \rightarrow (1)$$

Let the rod oscillate about the fixed horizontal axis AZ.

Considering the moments about AZ,

$$(M+m).K^2 = M.\frac{4l^2}{3} + m.(2l)^2$$

$$(M+m)K^2 = \frac{4l^2}{3}(M+3m)$$

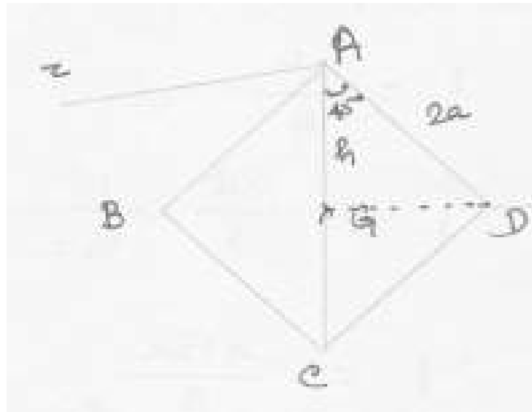
$$\therefore K^2 = \frac{4l^2(M+3m)}{3(M+m)} \qquad \rightarrow (2)$$

Period of oscillation $T = 2\pi\sqrt{\frac{K^2}{gh}}$

$$T = 2\pi\sqrt{\frac{4l^2(M+3m)(M+m)}{3(M+m).g(M+2m).l}}$$

$$T = 4\pi\sqrt{\frac{(M+3m)l}{3(M+2m)g}}$$

3. A square lamine of side 2a oscillates about a horizontal axis through one of its corners which is perpendicular to the plane of the lamina. Find the length of the simple equivalent pendulum.



Let ABCD be a square lamina of side 2a.

Let it oscillate about fixed horizontal axis AZ.

Let G be the centre of gravity.

Let AG=h.

Then, $\frac{AG}{AD} = \cos 45^\circ$

$$\frac{h}{2a} = \frac{1}{\sqrt{2}}$$

$$h = \frac{2a}{\sqrt{2}}$$

$$h = \sqrt{2}a \qquad \rightarrow (1)$$

Let the Moment of inertia of the lamina about AZ. be MK^2.

By the theorem of parallel axes,

$$I = I_G + Md^2$$

$$MK^2 = M.\frac{2a^2}{3} + M.\left(\sqrt{2}a\right)^2$$

$$K^2 = \frac{2a^2 + 6a^2}{3}$$

$$K^2 = \frac{8a^2}{3} \qquad \rightarrow (2)$$

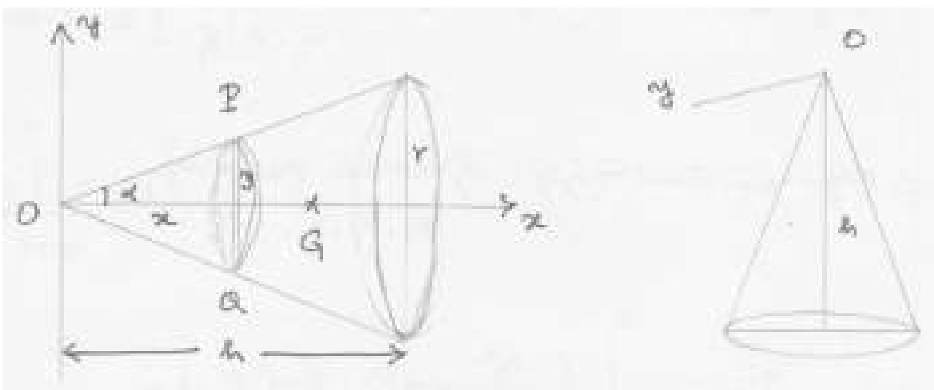
The length of the simple equivalent pendulum is

$$l = \frac{K^2}{h}$$

$$l = \frac{8a^2}{3} \times \frac{1}{\sqrt{2}a}$$

$$l = \frac{4\sqrt{2}a}{3}$$

4. A solid homogeneous right circular cone of height h and semivertical angle α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{h}{5}\left(4 + \tan^2 \alpha\right)$.



Let o be the vertex

H be the height

R be the radius

And α be the semivertical angle of the cone.

Let G be the centre of gravity of the cone.

$$OG = \frac{3}{4}h \qquad \rightarrow (1)$$

The cone oscillates about the fixed horizontal axis oy.

We will find the M.I of the cone about oy.

Let PQ be an elementary mass of the cone.

It will be a circular disc of radius y and thickness δx at a distance x from o.

Elementary mass of PQ is $\delta m = \pi y^2 \delta x . p$

Where ρ is the density.

By the theorem of parallel axes,

$$I = I_G + Md^2$$

M.I of the elementary mass about oy}$= I_y = \int_{x=0}^{h}\left(\frac{y^2}{4}+x^2\right)dm$

$$I_y = \int_{x=0}^{h}\left(\frac{y^2}{4}+x^2\right)\pi y^2 \rho dx$$

$$I_y = \int_{x=0}^{h}\left[\frac{x^2\tan^2\alpha}{4}+x^2\right].\pi.(x\tan\alpha)^2.\rho.dx. \qquad \left[\tan\alpha = \frac{y}{x}\ y = x\tan\alpha\right]$$

$$I_y = \int_{x=0}^{h} x^2\left[\frac{\tan^2\alpha+4}{4}\right].\pi.x^2\tan^2\alpha.\rho.dx.$$

$$I_y = \frac{(4+\tan^2\alpha)}{4}.\tan^2\alpha.\pi.\rho\int_{x=0}^{h} x^4 dx$$

$$I_y = \rho.\frac{\pi}{4}(4+\tan^2 x).\tan^2\alpha.\left(\frac{x^5}{5}\right)_0^h$$

$$I_y = \rho.\frac{\pi}{4}(4+\tan^2 x).\tan^2\alpha.\frac{h^5}{5} \qquad \rightarrow (2)$$

Next, let M be the mass of the cone.

$$\therefore M = \frac{1}{3}\pi r^2 h.\rho$$

$$M = \frac{1}{3}\pi.h^2\tan^2\alpha.h.\rho \qquad \left[\tan\alpha = \frac{h}{r}\ h = r\tan\alpha\right]$$

$$\therefore \rho = \frac{3M}{\pi h^3\tan^2\alpha} \qquad \rightarrow (3)$$

$\therefore (2) \Rightarrow$

$$I_y = \frac{3M}{\pi h^3\tan^2\alpha}.\frac{\pi}{20}h^5\tan^2\alpha(4+\tan^2\alpha)$$

$$\text{i.e., } MK^2 = M.\frac{3}{20}h^2(4+\tan^2\alpha)$$

$$\therefore K^2 = \frac{3}{20}h^2(4+\tan^2\alpha) \qquad \rightarrow (4)$$

$\therefore$ Length of the S.E.P. $= \frac{K^2}{h}$

$$l = \frac{K^2}{h}$$

$$l = \frac{3h^2\left(4 + \tan^2\alpha\right)}{20} \times \frac{4}{3h}$$

$$l = \frac{h}{5}\left(4 + \tan^2\alpha\right).$$

5.9.4 Self Assessment Problems III

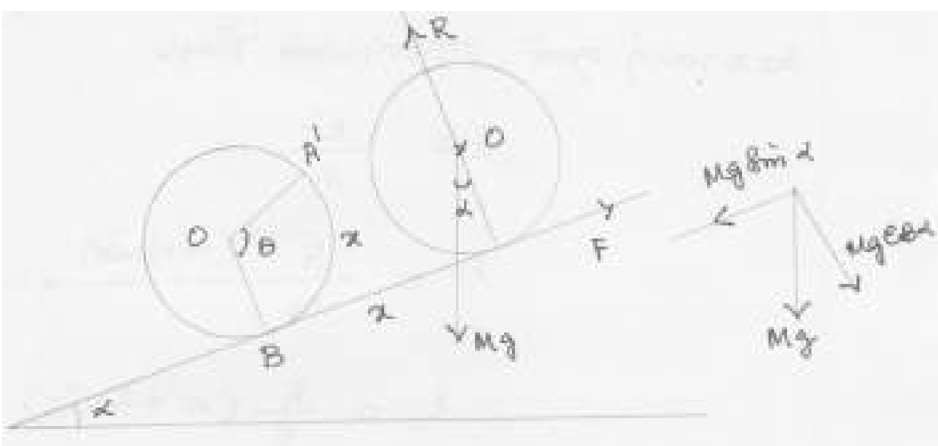
1. An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation. Show that $e = \frac{1}{2}$.

2. A rod of mass m has a mass m attached to its one end. If the rod can oscillate about an axis passing through the other end, find the period of oscillation.

5.10 Motion of a uniform circular disc rolling down an inclined plane.

A uniform circular disc of radius a rolls down a rough inclined plane.

If it starts from rest, find its acceleration and the distance traveled in t seconds.



$a -$ radius

$o -$ centre

$M -$ mass of the circular disc

$\alpha -$ angle of inclination of the inclined plane

$A -$ Initial point of contact of the disc with the inclined plane x- distance moved in time t seconds

θ- angle of rotation

R – normal reaction

F – frictional force

B-point of contact after t seconds

A' – position of the point A after t seconds

$x = A'B = a\theta.$

The forces are acting as shown in the figure.

Resolving the forces along and perpendicular to the inclined plane, we ge t

$$"F = ma"$$

$$M\ddot{x} = Mg\sin\alpha - F \qquad \rightarrow (1)$$

$$O = R - Mg\cos\alpha \qquad \rightarrow (2)$$

The moment of the effective forces about the axis of rotation is

$$MK^2\ddot{\theta} = F.a \qquad \rightarrow (3)$$

Where MK^2 is the M.I of the circular disc about the axis of rotation.

$$\therefore MK^2 = M.\frac{a^2}{2}$$

$(3) \Rightarrow$

$$M\frac{a^2}{2}\ddot{\theta} = Fa$$

$$\therefore M\frac{a}{2}\ddot{\theta} = F \qquad \rightarrow (4)$$

Next,

$x = a\theta$

$\dot{x} = a\dot{\theta}$

$\ddot{x} = a\ddot{\theta}$

$\therefore \ddot{\theta} = \frac{\ddot{x}}{a}$

$(4) \Rightarrow$

$$M\frac{a}{2}\frac{\ddot{x}}{a} = F$$

$$\therefore F = M\frac{\ddot{x}}{2} \qquad \rightarrow (6)$$

$(1) \Rightarrow$

$$M\ddot{x} = Mg\sin\alpha - M\frac{\ddot{x}}{2}$$

$$\therefore \ddot{x} + \frac{\ddot{x}}{2} = g\sin\alpha$$

$$\frac{3}{2}\ddot{x} = g\sin\alpha$$

$$\therefore \ddot{x} = \frac{2g}{3}\sin\alpha \qquad \rightarrow (7)$$

$$"s = ut + \frac{1}{2}at^2"$$

$$x = 0.t + \frac{1}{2}.\frac{2g}{3}\sin\alpha.t^2$$

$$x = \frac{g}{3}t^2\sin\alpha.$$

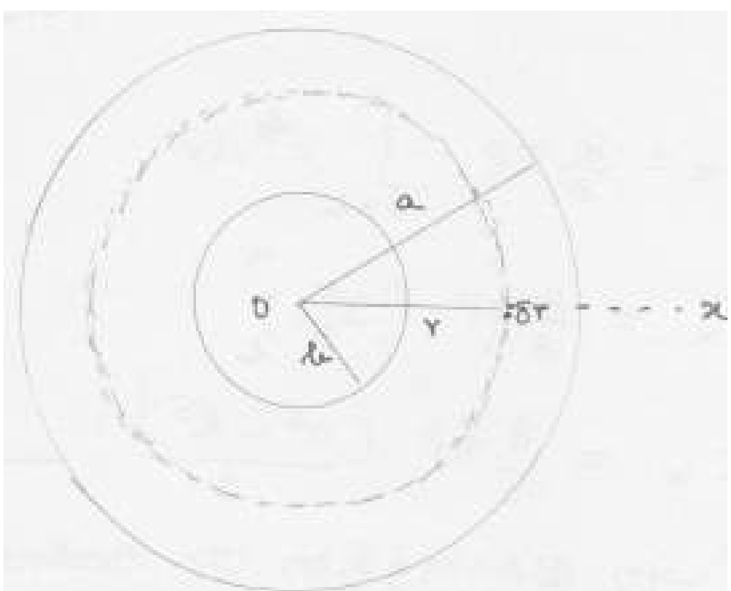
$\therefore$ Acceleration $\ddot{x} = \frac{2g}{3}\sin\alpha.$

Distance moved in t seconds $x = \frac{g}{3}t^2\sin\alpha.$

Answers

Self – assessment problems I

1.



Let o be the centre and a,b be the outer and inner radii of the hollow sphere.

We will find the M.I about a radius ox

Let us consider an elementary mass of the hollow sphere.

It will be a hollow sphere of centre o, radius r and thickness δr.

As r varies from b to a we will get the hollow sphere

Elementary mass $\delta m = 4\pi r^2 \delta r.\rho.$

M.I of the elementary mass about ox $= \left(4\pi r^2 \delta r.\rho\right).\frac{2}{3} r^2$

M.I of the whole body about ox= $\int_{r=b}^{a} \frac{8}{3} \pi r^4 \rho dr$

$$I_x = \frac{8}{3}\pi\rho \int_{r=b}^{a} r^4 dr$$

$$I_x = \frac{8}{3}\pi\rho \left(\frac{r^5}{5}\right)_b^a$$

$$I_x = \frac{8}{3}\pi\rho \frac{\left(a^5 - b^5\right)}{5} \qquad \rightarrow (1)$$

Let M be the mass of the hollow sphere . Then

$$\therefore M = \left(\frac{4}{3}\pi a^3 - \frac{4}{3}\pi b^3\right).\rho$$

$$M = \frac{4\pi}{3}\left(a^3 - b^3\right)\rho$$

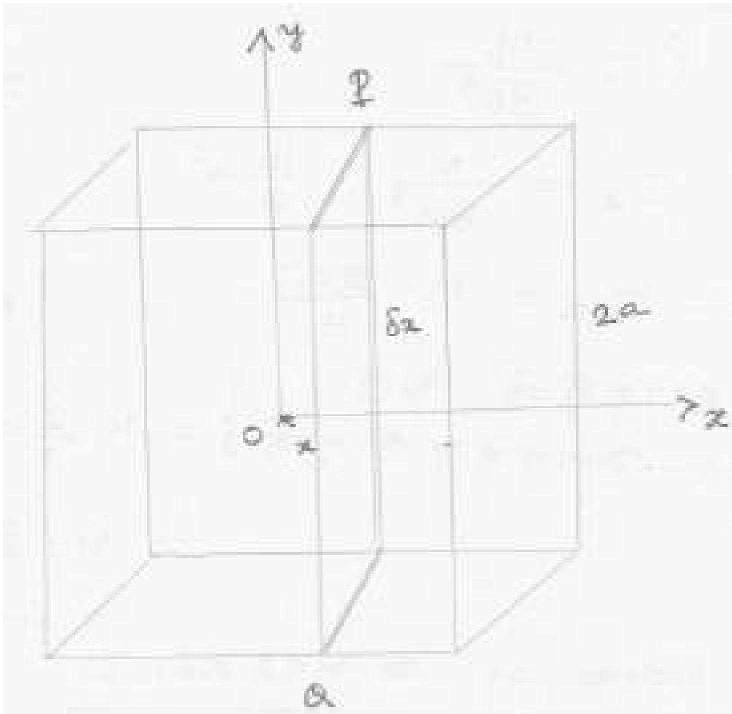
$$\therefore \rho = \frac{3M}{4\pi\left(a^3 - b^3\right)} \qquad \rightarrow (2)$$

$(1) \Rightarrow$

$$I_x = \frac{8}{3}\pi \frac{\left(a^5 - b^5\right)}{5} \times \frac{3M}{4\pi\left(a^3 - b^3\right)}$$

$$I_x = M.\frac{2}{5}\frac{\left(a^5 - b^5\right)}{\left(a^3 - b^3\right)}$$

2. Let 2a be the side of the cube. Let ox and oy be two perpendicular axes through the centre o.



First we will find the M.I about ox.

Let PQ be an elementary mass of the cube. It will be a square lamina of side 2a.

Elementary mass of PQ is $\delta m = 2a.2a.\delta x.\rho$.

Where ρ is the density

M.I of the elementary mass about ox $= \left(4a^2\rho\delta x\right).\frac{2a^2}{3}$

M.I of the cube about ox. $I_x = \int_{x=-a}^{a} \frac{8}{3} a^4 \rho dx$

$$I_x = \frac{8}{3} a^4 \rho \left(x\right)_{-a}^{a}$$

$$I_x = \rho.\frac{8}{3} a^4.2a \qquad \rightarrow (1)$$

Let M be the mass of the cube.

Then, $M = 2a.2a.2a.\rho$

$\therefore \rho = \frac{M}{8a^3}$

$(1) \Rightarrow$

$$I_x = \frac{M}{8a^3} \cdot \frac{16}{3} a^5$$

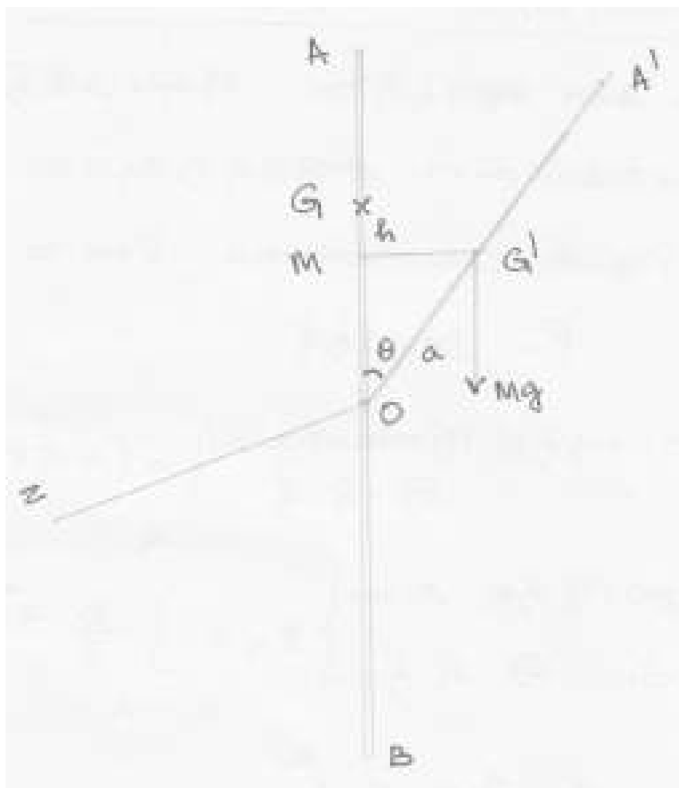
$$\therefore I_x = M.\frac{2a^2}{3}$$

$$\therefore \text{M.I of the cube about the edge AB } = M.\frac{2}{3}a^2 + M.\left(\frac{2a}{\sqrt{2}}\right)^2$$

$$= M.\frac{8a^2}{3}$$

Self assessment problems II

1.



$OA = 2a$, length of the rod

$G -$ Centre of gravity

$M -$ mass of the rod

Let the rod rotate about a fixed horizontal axis oz.

Initiallly the rod is held at rest in the upward vertical position and then released. At any time t, let OA' be the position of the rod.

Let the rod rotate an angle θ in t secs.

Let G' be the position of the C.G at time t secs.

Let $G'M \perp OA$

Let $GM = h$

Let $A\hat{O}A = \theta$

Now ,

Loss in kinetic energy = Work done

Initial kinetic energy – Final kinetic energy = Work done

$$0 - \frac{1}{2}MK^2\dot{\theta}^2 = -Mg.h \qquad \rightarrow (1)$$

Where Mk^2 is the M.I of the rod about oz

$$\therefore MK^2 = M.\frac{4a^2}{3}$$

$\therefore (1) \Rightarrow$

$$\frac{1}{2}.M.\frac{4a^2}{3}\dot{\theta}^2 = Mg.(a - a\cos\theta)$$

$$\dot{\theta}^2 = \frac{3g}{2a}(1 - \cos\theta)$$

When the rod reaches the downward vertical position,

$$\theta = \pi,\ \dot{\theta} = w$$

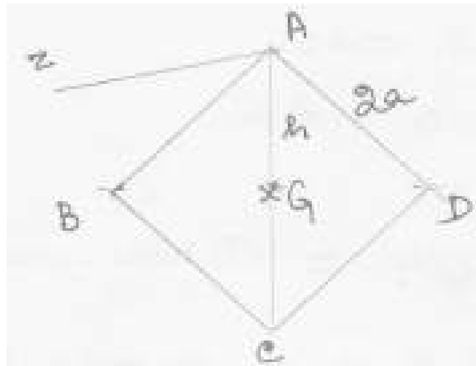
$$\therefore w^2 = \frac{3g}{2a}[1 - \cos\pi]$$

$$w^2 = \frac{3g}{2a}[1 - (-1)]$$

$$w^2 = \frac{3g}{2a}.2$$

$$\therefore w = \sqrt{\frac{3g}{a}}$$

2.



Let ABCD be a square lamina of side 2a. Let the lamina rotate about a fixed horizontal axis AZ.

When the lamina rotates about the axis AZ, at any time t, let θ be the angle of rotation.

Kinetic energy of the lamina $= \frac{1}{2} MK^2 \dot{\theta}^2 \qquad \rightarrow (1)$

Where MK^2 is the M.I of the lamina about AZ.

By the parallel axes theorem,

$$I = I_G + Md^2$$

$$MK^2 = M.\frac{2a^2}{3} + \left(\sqrt{2}a\right)^2$$

$$MK^2 = M.\frac{8a^2}{3}$$

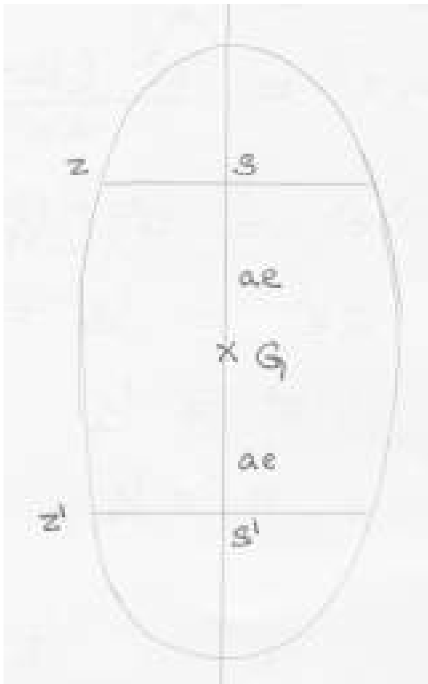
$\therefore (1) \Rightarrow$

K.E of the lamina $= \frac{1}{2} M.\frac{8a^2}{3} \dot{\theta}^2$

$$= M.\frac{4a^2}{3} \dot{\theta}^2$$

Self assessment problems III

1.



Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let the lamina oscillate about the latus rectum sz as a fixed horizontal axis.

Let G be the ceentre of gravity. Then

$$SG=h=ae \qquad \rightarrow(1)$$

Let Mk^2 be the M.I of the lamina about sz.

By parallel axis theorem,

$$I=I_G+Md^2$$

$$Mk^2=M.\frac{a^2}{4}+M(ae)^2$$

$$k^2=a^2\left(\frac{1}{4}+e^2\right) \qquad \rightarrow(2)$$

Given that the other focus

$$ss^1 = \frac{k^2}{h}$$

$$2ae = \frac{a^2\left(\frac{1}{4}+e^2\right)}{ae}$$

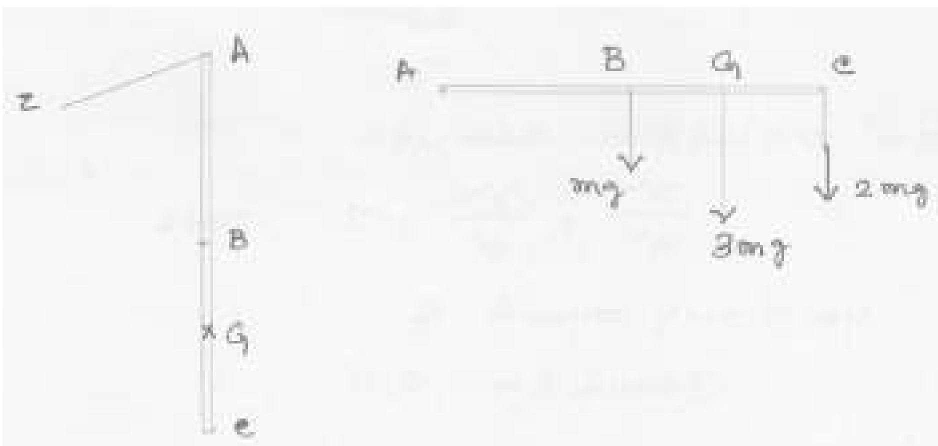
$$2a^2e^2 = a^2\left(\frac{1}{4}+e^2\right)$$

$$2e^2 = \frac{1}{4}+e^2$$

$$e^2 = \frac{1}{4}$$

$$e = \frac{1}{2}$$

2.



AC=2a, length of the rod

B be the midpoint of the rod.

The weight of the rod $mg\downarrow$ acts at B.

Another weight $2mg\downarrow$ is attached to C

Let Gbe the common centre of gravity.

AG=h

Taking moments about A,

3mg.h=mg.a+2mg.2a

3h=a+4a

$$h = \frac{5a}{3} \qquad \rightarrow(1)$$

Let the rod oscillate about the fixed horizontal axis AZ.

Considering M.I about AZ,

$$3m.k^2 = m.\frac{4a^2}{3} + 2m(2a)^2$$

$$3mk^2 = m.\frac{28a^2}{g}$$

$$k^2 = \frac{28a^2}{g} \qquad \rightarrow (2)$$

Period of oscillation $T = 2\pi\sqrt{\frac{k^2}{gh}}$

$$T = 2\pi\sqrt{\frac{28a^2}{gg} \times \frac{3}{5a}}$$

$$T = 4\pi\sqrt{\frac{7a}{15g}}$$

5.12 Books for reference

1. Mechanics – By P. Duraipandian
Emarald Publishers, 135, Anna Salai,
Chennai – 600 002.

2. Dynamics – By S. Narayanan
S. Chand & co
Chennai.