MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: RESOURCE MANAGEMENT TECHNIQUES

SUBJECT CODE: SAZ5C

SEMESTER: V

PREPARED BY: PROF.C.DHANA LAKSHMI

Title of the Course/	**Paper -XV**	**RESOURCE MANAGEMENT TECHNIQUES**	
Core	**III Year & Fifth Semester**	Credit: 4	
Objective of the course	This course introduces the concepts of Resource Management Technique		
Course outline	Unit-1: Basics of Operations Research (OR): Characteristics of O.R - Necessity of O.R in Industry -OR and Decision making - Role of computers in O.R. Linear programming: Formulations and Graphical solution (of 2 variables) canonical & standard terms of Linear programming problem. Algebraic solution: Simplex method.		
	Unit-2: Algebraic solution: Charnes method of penalties - two phase simplex method - concept of Duality - properties of duality - Dual simplex method.		
	Unit-3: Transportation model: Definition - formulation and solution of transportation models - the row - minima, column - minima, matrix minima and vogel's approximation methods. Assignment model: Definition of Assignment model - comparison with transportation model - formulation and solution of Assignment model - variations of Assignment problem.		
	Unit-4: Sequencing problem: Processing each of n jobs through m machines - processing n jobs through 2 machines - processing n jobs through 3 machines - processing 2 jobs through m machines - processing n jobs through m machines - travelling salesman problem. Game Theory: Characteristics of games - Maximin, Minimax criteria of optimality - Dominance property - algebraic and graphical method of solution of solving 2 x 2 games.		
	Unit-5 : Pert - CPM: Networks - Fulkerson's Rule - measure of activity - PERT computation - CPM computation - resource scheduling. Simulation: Various methods of obtaining random numbers for use in computer simulation - Additive, multiplicative and mixed types of congruence random number generators - Monte Carlo method of simulation - its advantages and disadvantages.		

1.Recommended Texts
   i. Hamdy A. Taha: ,1996,Operation Research - An Introduction, 5th edition, Prentice Hall of India, Pvt. Ltd., New Delhi .
   ii.. Ackoff R.L. and Sasieni M. W,1968, Fundamentals of Operations Research, John Wiley and sons, New York.
   iii. Charnes A. Cooper W. and Hendersen A.,1953, Introduction to Linear Programming, Wiley and Sons, New York.
   iv. Srinath L.S,1973, PERT and CPM principles and applications, Affiliated East West Press Pvt. Ltd., New York .

UNIT - 1

OPERATIONS RESEARCH

1.5. DEFINITION OF OPERATIONS RESEARCH

Any subject matter when defined to explain what exactly it is, we may find one definition. Always a definition explains what that particular subject matter is. Say for example, if a question is asked what is Boyel's law, we have a single definition to explain the same, irrespective of the language in which it is defined. But if you ask, what Operations research is? The answer depends on individual objective. Say for example a student may say that the Operations research is technique used to obtain first class marks in the examination. If you ask a businessman the same question, he may say that it is the technique used for getting higher profits. Another businessman may say it is the technique used to capture higher market share and so on. Like this each individual may define in his own way depending on his objective. Each and every definition may explain one or another characteristic of Operations Research but none of them explain or give a complete picture of Operations research. But in the academic interest some of the important definitions are discussed below.

(*a*) **Operations Research is the art of winning wars without actually fighting. - Aurther Clarke.**

This definition does not throw any light on the subject matter, but it is oriented towards warfare. It means to say that the directions for fighting are planned and guidance is given from remote area, according to which the war is fought and won. Perhaps you might have read in Mahabharatha or you might have seen some old pictures, where two armies are fighting, for whom the guidance is given by the chief minister and the king with a chessboard in front of them. Accordingly war is fought in the warfront. Actually the chessboard is a model of war field.

(*b*) **Operations Research is the art of giving bad answers to problems where otherwise worse answers are given. - T.L. Satty.**

This definition covers one aspect of decision-making, *i.e.*, choosing the best alternative among the list of available alternatives. It says that if the decisions are made on guesswork, we may face the worse situation. But if the decisions are made on scientific basis, it will help us to make better decisions. Hence this definition deals with one aspect of decision-making and not clearly tells what is operations research.

(*c*) **Operations Research is Research into Operations. - J. Steinhardt.**

This definition does not give anything in clear about the subject of Operations Research and simply says that it is research in to operations. Operations may here be referred as military activities or simply the operations that an executive performs in his organisations while taking decisions. Research in the word means that finding a new approach. That is when an executive is involved in performing his operations for taking decisions he has to go for newer ways so that he can make a better decision for the benefit of his organisation.

(*d*) **Operations Research is defined as Scientific method for providing executive departments a quantitative basis for decisions regarding the operations under their control. - P.M. Morse and G.E. Kimball.**

This definition suggests that the Operations Research provides scientific methods for an executive to make optimal decisions. But does not give any information about various models or methods. But this suggests that executives can use scientific methods for decision-making.

(*e*) **Operations Research is th study of administrative system pursued in the same scientific manner in which system in Physics, Chemistry and Biology are studied in natural sciences.**

This definition is more elaborate than the above given definitions. It compares the subject Operations Research with that of natural science subjects such as Physics, Chemistry and Biology, where while deciding any thing experiments are conducted and results are verified and then the course of action is decided. It clearly directs that Operations Research can also be considered as applied science and before the course of action is decided, the alternatives available are subjected to scientific analysis and optimal alternative is selected. But the difference between the experiments we conduct in natural sciences and operations research is: in natural sciences the research is rigorous and exact in nature, whereas in operations research, because of involvement of human element and uncertainty the approach will be totally different.

(*f*) **Operations Research is the application of scientific methods, techniques and tools to operation of a system with optimum solution to the problem. - Churchman, Ackoff and Arnoff.**

This definition clearly states that the operations research applies scientific methods to find an optimum solution to the problem of a system. A system may be a production system or information system or any system, which involves men, machine and other resources. We can clearly identify that this definition tackles three important aspects of operations research *i.e.* application of scientific methods, study of a system and optimal solution. This definition too does not give any idea about the characteristics of operations research.

(*g*) **Operations Research is the application of the theories of Probability, Statistics, Queuing, Games, Linear Programming etc., to the problems of War, Government and Industry.**

This definition gives a list of various techniques used in Operations Research by various managers to solve the problems under their control. A manager has to study the problem, formulate the problem, identify the variables and formulate a model and select an appropriate technique to get optimal solution. We can say that operations research is a bunch of mathematical techniques to solve problems of a system.

(*h*) **Operations Research is the use of Scientific Methods to provide criteria or decisions regarding man-machine systems involving repetitive operations.**

This definition talks about man- machine system and use of scientific methods and decision-making. It is more general and comprehensive and exhaustive than other definitions. Wherever a study of system involving man and machine, the person in charge of the system and involved in decision-making will use scientific methods to make optimal decisions.

(*i*) **Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with problems that confront the executive, when he tries to achieve a thorough going rationally in dealing with his decision problem. - D.W. Miller and M.K. Starr.**

This definition also explains that operations research uses scientific methods or logical means for getting solutions to the executive problems. It too does not give the characteristics of Operations Research.

(*j*) **Operations Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, materials and money in industry, business, Government and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcome of alternative decisions, strategies or controls. The purpose is to help management to determine its policy and actions scientifically. - Operations Society of Great Britain.**

The above definition is more elaborate and says that operations research applies scientific methods to deal with the problems of a system where men, material and other resources are involved and the system under study may be industry, defense or business etc, gives this definition. It also say that the manager has to build a scientific model to study the system which must be provided with facility to measure the outcomes of various alternatives under various degrees of risk, which helps the managers to take optimal decisions.

In addition to the above there are hundreds of definitions available to explain what Operations Research is? But many of them are not satisfactory because of the following reasons.

 (*i*) Operations Research is not a well-defined science like Physics, Chemistry etc. All these sciences are having well defined theory about the subject matter, where as operations research do not claim to know or have theories about operations. Moreover, Operations Research is not a scientific research into the control of operations. It is only the application of mathematical models or logical analysis to the problem solving. Hence none of the definitions given above defines operations research precisely.

(*ii*) The objective of operations research says that the decisions are made by brain storming of people from various walks of life. This indicates that operations research approach is inter- disciplinary approach, which is an important character of operations research. This aspect is not included in any of the definitions hence they are not satisfactory.

(*iii*) The above-discussed definitions are given by various people at different times and stages of development of operations research as such they have considered the field in which they are involved hence each definition is concentrating on one or two aspects. No definition is having universal approach.

But salient features of above said definitions are:
  * Operations Research uses Scientific Methods for making decisions.
  * It is interdisciplinary approach for solving problems and it uses the knowledge and experience of experts in various fields.

* While analyzing the problems all aspects are considered and examined and analyzed scientifically for finding the optimal solution for the problem on hand.
* As operations research has scientific approach, it improves the quality of answers to the problems.
* Operations research provides scientific base for decision-making and provide scientific substitute for judgement and intuition.

1.6. CHARACTERISTICS OF OPERATIONS RESEARCH

After considering the objective and definitions of Operations Research, now let us try to understand what are the characteristics of Operations Research.

(*a*) **Operations Research is an interdisciplinary team approach.**

The problems an operations research analyst face is heterogeneous in nature, involving the number of variables and constraints, which are beyond the analytical ability of one person. Hence people from various disciplines are required to understand the operations research problem, who applies their special knowledge acquired through experience to get a better view of cause and effects of the events in the problem and to get a better solution to the problem on hand. This type of team approach will reduce the risk of making wrong decisions.

(*b*) **Operations Research increases the creative ability of the decision maker.**

Operations Research provides manager mathematical tools, techniques and various models to analyse the problem on hand and to evaluate the outcomes of various alternatives and make an optimal choice. This will definitely helps him in making better and quick decisions. A manager, without the knowledge of these techniques has to make decisions by thumb rules or by guess work, which may click some times and many a time put him in trouble. Hence, a manager who uses Operations Research techniques will have a better creative ability than a manager who does not use the techniques.

(*c*) **Operations Research is a systems approach.**

A business or a Government organization or a defense organization may be considered as a system having various sub-systems. The decision made by any sub-system will have its effect on other sub-systems. Say for example, a decision taken by marketing department will have its effect on production department. When dealing with Operations Research problems, one has to consider the entire system, and characteristics or sub- systems, the inter-relationship between sub-systems and then analyse the problem, search for a suitable model and get the solution for the problem. Hence we say Operations Research is a Systems Approach.

1.7. SCOPE OF OPERATIONS RESEARCH

The scope aspect of any subject indicates, the limit of application of the subject matter/techniques of the subject to the various fields to solve the variety of the problems. But we have studied in the objective, that the subject Operations Research will give scientific base for the executives to take decisions or to solve the problems of the systems under their control. The system may be business, industry, government or defense. Not only this, but the definitions discussed also gives different versions. This indicates that the techniques of Operations Research may be used to solve any type of problems. The problems may pertain to an individual, group of individuals, business, agriculture, government or

defense. Hence, we can say that there is no limit for the application of Operations Research methods and techniques; they may be applied to any type of problems. Let us now discuss some of the fields where Operations Research techniques can be applied to understand how the techniques are useful to solve the problems. In general we can state that whenever there is a problem, simple or complicated, we can use operations research techniques to get best solution.

(*i*) **In Defense Operations**

In fact, the subject Operations research is the baby of World War II. To solve war problems, they have applied team approach, and come out with various models such as resource allocation model, transportation model etc.In any war field two or more parties are involved, each having different resources (manpower, ammunition, etc.), different courses of actions (strategies) for application. Every opponent has to guess the resources with the enemy, and his courses of action and accordingly he has to attack the enemy. For this he needs scientific, logical analysis of the problem to get fruitful results. Here one can apply the techniques like *Linear Programming, Game theory, and inventory models etc.* to win the game. In fact in war filed every situation is a competitive situation. More over each party may have different bases, such as Air force, Navy and Army. The decision taken by one will have its effect on the other. Hence proper co-ordination of the three bases and smooth flow of information is necessary. Here operations research techniques will help the departmental heads to take appropriate decisions.

(*ii*) **In Industry**

After the II World War, the, Industrial world faced a depression and to solve the various industrial problems, industrialist tried the models, which were successful in solving their problems. Industrialist learnt that the techniques of operations research can conveniently applied to solve industrial problems. Then onwards, various models have been developed to solve industrial problems. Today the managers have on their hand numerous techniques to solve different types of industrial problems. In fact *decision trees, inventory model, Linear Programming model, Transportation model, Sequencing model, Assignment model and replacement models* are helpful to the managers to solve various problems, they face in their day to day work. These models are used to minimize the cost of production, increase the productivity and use the available resources carefully and for healthy industrial growth. An industrial manager, with these various models on his hand and a computer to workout the solutions (today various packages are available to solve different industrial problems) quickly and preciously.

(*iii*) **In Planning For Economic Growth**

In India we have five year planning for steady economic growth. Every state government has to prepare plans for balanced growth of the state. Various secretaries belonging to different departments has to co-ordinate and plan for steady economic growth. For this all departments can use Operations research techniques for planning purpose. The question like how many engineers, doctors, software people etc. are required in future and what should be their quality to face the then problems etc. can be easily solved.

(*iv*) **In Agriculture**

The demand for food products is increasing day by day due to population explosion. But the land available for agriculture is limited. We must find newer ways of increasing agriculture yield. So the selection of land area for agriculture and the seed of food grains for sowing

must be meticulously done so that the farmer will not get loss at the same time the users will get what they desire at the desired time and desired cost.

(*v*) **In Traffic control**

Due to population explosion, the increase in the number and verities of vehicles, road density is continuously increasing. Especially in peak hours, it will be a headache to control the traffic. Hence proper timing of traffic signaling is necessary. Depending on the flow of commuters, proper signaling time is to be worked out. This can be easily done by the application of *queuing theory*.

(*vi*) **In Hospitals**

Many a time we see very lengthy queues of patient near hospitals and few of them get treatment and rest of them have to go without treatment because of time factor. Some times we have problems non-availability of essential drugs, shortage of ambulances, shortage of beds etc. These problems can be conveniently solved by the application of operations research techniques.

The above-discussed problems are few among many problems that can be solved by the application of operation research techniques. This shows that Operations Research has no limit on its scope of application.

1.8. PHASES IN SOLVING OPERATIONS RESEARCH PROBLEMS OR STEPS IN SOLVING OPERATIONS RESEARCH PROBLEMS

Any Operations Research analyst has to follow certain sequential steps to solve the problem on hand. The steps he has to follow are discussed below:

First he has to study the situation and collect all information and formulate the statement of the problem. Hence the first step is the *Formulation of the problem*. The figure 1.3 shows the various steps to be followed.

1.8.1. Formulation of the Problem

The Operations Research analyst or team of experts first have to examine the situation and clearly define what exactly happening there and identify the variables and constraints. Similarly identify what is the objective and put them all in the form of statement. The statement must include a) a precise description goals or objectives of the study, b) identification of controllable and uncontrollable variables and c) restrictions of the problem. The team should consult the personals at the spot and collect information, if something is beyond their reach, they have to consult duty engineers available and understand the facts and formulate the problem. Let us consider the following statement:

Statement: A company manufactures **two products *X* and *Y***, by using the **three machines *A*, *B*, and *C*.** Each unit of *X* takes **1 hour** on machine A, **3 hours** on machine B and **10 hours** on machine *C*. Similarly, product *Y* takes **one hour, 8 hours and 7 hours** on Machine *A*, *B*, and *C* respectively. In the coming planning period, **40 hours of machine *A*, 240 hours of machine *B* and 350 hours of machine *C*** is available for production. Each unit of *X* **brings a profit of Rs 5/-** and *Y* **brings Rs. 7 per unit. How much of *X* and *Y* are to be manufactured** by the company **for maximizing the profit**?

The team of specialists prepares this statement after studying the system. As per requirement this must include the variables, constraints, and objective function.
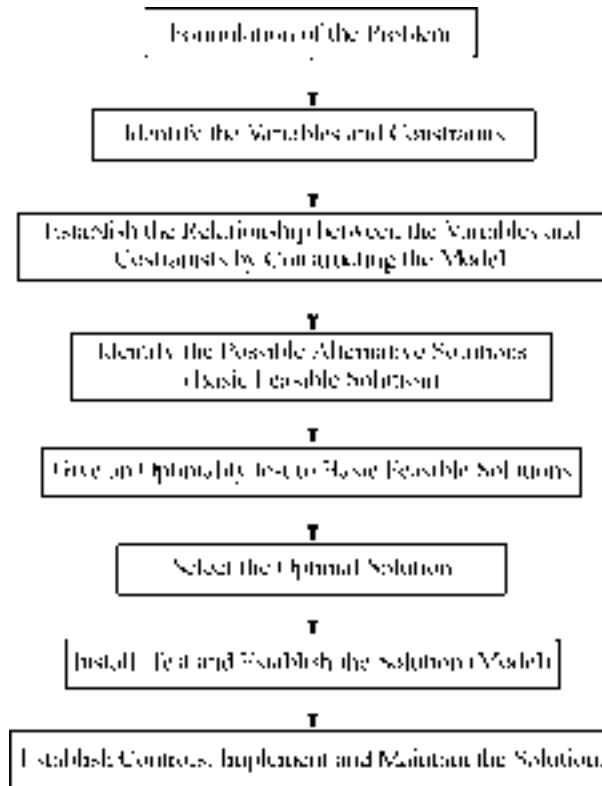
Figure 1.3. Phases of Solving Operations Research Problems.

1.8.2. Variables

The Company is manufacturing two products *X* and *Y*. These are the two variables in the problem. When they are in the problem statement they are written in *capital letters.* Once they are entered in the model small letters (lower case) letters are used (*i.e,. x* and *y*). We have to find out how much of *X* and how much of *Y* are to be manufactured. Hence they are variables. In linear programming language, these are known as **competing candidates**. Because they compete to use or consume available resources.

1.8.3. Resources and Constraints

There are three machines *A*, *B*, and C on which the products are manufactured. These are known as resources. The capacity of machines in terms of machine hours available is the available resources. The competing candidates have to use these available resources, which are limited in nature. Now in the above statement, machine *A* has got available 40 hours and machine B has available a capacity of 240 hours and that of machine *C* is 350 hours. The products have to use these machine hours in required proportion. That is one unit of product *X* consumes one hour of machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly, one unit of *Y* consumes one hour of machine *B*, 8 hours of machine *B* and 7 hours of machine *C*. These machine hours given are the available resources and they are limited in nature and hence they are **constraints** given in the statement.

1.8.4. Objective of the Problem

To maximise the profit how much of *X* and *Y* are to be manufactured? That is **maximization of the profit or maximization of the returns** is the objective of the problem. For this in the statement it is given that the profit contribution of X is Rs 5/- per unit and that of product Y is Rs. 7/- per unit.

1.8.5. To establish relationship between variables and constraints and build up a model

Let us say that company manufactures x units of *X* and y units of *Y*. Then as one unit of *x* consumes one hour on machine A and one unit of y consumes one hour on machine A, the total consumption by manufacturing *x* units of *X* and y units of *Y* is, $1x + 1y$ and this should not exceed available capacity of 40 hours. Hence the **mathematical relationship in the form of mathematical model is $1x + 1y \leq$ 40**. This is for resource machine *A*. Similarly for machine *B* and machine *C* we can formulate the mathematical models. They appear as shown below:

$3x + 8y \leq$ 240 for machine *B* **and $10x + 7y \leq$ 350** for machine *C*. Therefore, the mathematical model for these resources are:

$$1x + 1y \leq 40$$
$$3x + 8y \leq 240 \text{ and}$$
$$10x + 7y \leq 350.$$

Similarly for objective function as the company manufacturing x units of *X* and y units of *Y* and the profit contribution of *X* and Y are Rs.5/- and Rs 7/- per unit of *X* and *Y* respectively, the total profit earned by the company by manufacturing *x* and *y* units is **$5x + 7y$.** This we have to maximise. Therefore **objective function is Maximise $5x + 7y$.** At the same time, we have to remember one thing that the company can manufacture any number of units or it may not manufacture a particular product, for example say $x = 0$. But it cannot manufacture negative units of *x* and *y*. Hence one more constraint is to be introduced in the model *i.e.* **a non - negativity constraint.** Hence the mathematical representation of the contents of the statement is as given below:

Maximise Z = $5x + 7y$ Subject to a condition (written as s.t.) ⟶ **OBJECTIVE FUNCTION.**

$1x + 1y \leq$ **40**

$3x + 8y \leq$ **240** ⟶ **STRUCTURAL CONSTRAINTS.**

$10x + 7y \leq$ **350 and**

Both *x* and *y* are $\geq$ 0 ⟶ **NON-NEGATIVITY CONSTRAINT.**

1.8.6. Identify the possible alternative solutions (or known as Basic Feasible Solutions or simply BFS)

There are various methods of getting solutions. These methods will be discussed later. For example we go on giving various values (positive numbers only), and find various values of objective function. All these are various Basic Feasible Solutions. For example $x = 0,1,2,3$, etc. and $y = 0,1,2,3$ etc are all feasible values as far as the given condition is concerned. Once we have feasible solutions on hand go on asking is it maximum? Once we get maximum value, those values of x and y are optimal values. And the value of objective function is **optimal value of the objective function**. These two steps we shall discuss in detail in the next chapter.

1.8.7. Install and Maintain the Solution

Once we get the optimal values of x and y and objective function instructions are given to the concerned personal to manufacture the products as per the optimal solution, and maintain the same until further instructions.

1.9. MEANING AND NECESSITY OF OPERATIONS RESEARCH MODELS

Management deals with reality that is at once complex, dynamic, and multifacet. It is neither possible nor desirable, to consider each and every element of reality before deciding the courses of action. It is impossible because of time available to decide the courses of action and the resources, which are limited in nature. More over in many cases, it will be impossible for a manager to conduct experiment in real environment. For example, if an engineer wants to measure the inflow of water in to a reservoir through a canal, he cannot sit on the banks of canal and conduct experiment to measure flow. He constructs a similar model in laboratory and studies the problem and decides the inflow of water. Hence for many practical problems, a model is necessary. We can **define an operations research model as some sort of mathematical or theoretical description of various variables of a system representing some aspects of a problem on some subject of interest or inquiry. The model enables to conduct a number of experiment involving theoretical subjective manipulations to find some optimum solution to the problem on hand.**

Let us take a very simple example. Say you have a small child in your house. You want to explain to it what is an elephant. You can say a story about the elephant saying that it has a trunk, large ears, small eyes etc. The child cannot understand or remember anything. But if you draw small drawing of elephant on a paper and show the trunk, ears, eyes and it will grasp very easily the details of elephant. When a circus company comes to your city and take elephants in procession, then the child if observe the procession, it will immediately recognize the elephant. This is the exact use of a model. In your classrooms your teacher will explain various aspects of the subject by drawing neat sketches on the black board. You will understand very easily and when you come across real world system, you can apply what all you learnt in your classroom. Hence through a model, we can explain the aspect of the subject / problem / system. The inequalities given in section 1.8.5 is a mathematical model, which explains clearly the manufacturing system, given in section 1.8.1. (***Here we can say a model is a relationship among specified variables and parameters of the system***)**.**

1.9.1. Classification of Models

The models we use in operations research may broadly classified as:

(*i*) Mathematical and Descriptive models, and (*ii*) Static and Dynamic Models.

Mathematical and Descriptive Models

(*i*) **Descriptive Model**

A descriptive model explains or gives a description of the system giving various variables, constraints and objective of the system or problem. In article 1.8.1 gives the statement of the problem, which is exactly a descriptive model. The drawback of this model is as we go on reading and proceed; it is very difficult to remember about the variables and constraints, in case the problem or description of the system is lengthy one. It is practically impossible to keep on reading, as the manager has to decide the course of action to be taken timely.

Hence these models, though necessary to understand the system, have limited use as far as operations research is concerned.

(**ii**) **Mathematical Model**

In article, 1.8.2 we have identified the variables and constraints and objective in the problem statement and given them mathematical symbols x and y and a model is built in the form of an inequality of $\leq$ type. Objective function is also given. This is exactly a mathematical model, which explains the entire system in mathematical language, and enables the operations research person to proceed towards solution.

1.9.2. Types of Models

Models are also categorized depending on the *structure, purpose, nature of environment, behaviour, by method of solution and by use of digital computers.*

(**a**) **Classification by Structure**

(*i*) *Iconic Models:* These models are scaled version of the actual object. For example a toy of a car is an iconic model of a real car. In your laboratory, you might have seen Internal Combustion Engine models, Boiler models etc. All these are iconic models of actual engine and boiler etc. They explain all the features of the actual object. In fact a globe is an iconic model of the earth. These models may be of enlarged version or reduced version. Big objects are scaled down (reduced version) and small objects, when we want to show the features, it is scaled up to a bigger version. In fact it is a descriptive model giving the description of various aspects of real object. As far as operations research is concerned, is of less use. The advantages of these models: are It is easy to work with an iconic model in some cases, these are easy to construct and these are useful in describing static or dynamic phenomenon at some definite time. The limitations are, we cannot study the changes in the operation of the system. For some type of systems, the model building is very costly. It will be sometimes very difficult to carry out experimental analysis on these models.

(*ii*) *Analogue Model:* In this model one set of properties are used to represent another set of properties. Say for example, blue colour generally represents water. Whenever we want to show water source on a map it is represented by blue colour. Contour lines on the map is also analog model. Many a time we represent various aspects on graph by defferent colours or different lines all these are analog models. These are also not much used in operations research. The best examples are warehousing problems and layout problems.

(*iii*) *Symbolic Models or Mathematical Models*: In these models the variables of a problem is represented by mathematical symbols, letters etc. To show the relationships between variables and constraints we use mathematical symbols. Hence these are known as symbolic models or mathematical models. These models are used very much in operations research. Examples of such models are Resource allocation model, Newspaper boy problem, transportation model etc.

(**b**) **Classification by utility**

Depending on the use of the model or purpose of the model, the models are classified as D*escriptive, Predictive and Prescriptive models.*

(*i*) *Descriptive model*: The descriptive model simply explains certain aspects of the problem or situation or a system so that the user can make use for his analysis. It will not give full details and clear picture of the problem for the sake of scientific analysis.

(*ii*) *Predictive model*: These models basing on the data collected, can predict the approximate results of the situation under question. For example, basing on your performance in the examination and the discussions you have with your friends after the examination and by verification of answers of numerical examples, you can predict your score or results. This is one type of predictive model.

(*iii*) *Prescriptive models*: We have seen that predictive models predict the approximate results. But if the predictions of these models are successful, then it can be used conveniently to prescribe the courses of action to be taken. In such case we call it as Prescriptive model. Prescriptive models prescribe the courses of action to be taken by the manager to achieve the desired goal.

(*c*) **Classification by nature of environment**

Depending on the environment in which the problem exists and the decisions are made, and depending on the conditions of variables, the models may be categorized as *Deterministic models* and *Probabilistic models.*

(*i*) *Deterministic Models*: In this model the operations research analyst assumes complete certainty about the values of the variables and the available resources and expects that they do not change during the planning horizon. All these are deterministic models and do not contain the element of uncertainty or probability. The problems we see in Linear Programming, assumes certainty regarding the values of variables and constraints hence the Linear Programming model is a Deterministic model.

(*ii*) *Probabilistic or Stochastic Models*: In these models, the values of variables, the pay offs of a certain course of action cannot be predicted accurately because of element of probability. It takes into consideration element of risk into consideration. The degree of certainty varies from situation to situation. A good example of this is the sale of insurance policies by Life Insurance Companies to its customers. Here the failure of life is highly probabilistic in nature. The models in which the pattern of events has been compiled in the form of probability distributions are known as Probabilistic or Stochastic Models.

(*d*) **Classification depending on the behaviour of the problem variables**

Depending on the behaviour of the variables and constraints of the problem they may be classified as *Static Models* or *Dynamic models*.

(*i*) *Static Models*: These models assumes that no changes in the values of variables given in the problem for the given planning horizon due to any change in the environment or conditions of the system. All the values given are independent of the time. Mostly, in static models, one decision is desirable for the given planning period.

(*ii*) *Dynamic Models*: In these models the values of given variables goes on changing with time or change in environment or change in the conditions of the given system. Generally, the dynamic models then exist a series of interdependent decisions during the planning period.

(*e*) **Classification depending on the method of getting the solution**

We may use different methods for getting the solution for a given model. Depending on these methods, the models are classified as *Analytical Models* and *Simulation Models*.

(*i*) *Analytical Models:* The given model will have a well-defined mathematical structure and can be solved by the application of mathematical techniques. We see in our discussion that the Resource allocation model, Transportation model, Assignment model, Sequencing model etc. have well defined mathematical structure and can be solved by different mathematical techniques. For example, Resource allocation model can be solved by Graphical method or by Simplex method depending on the number of variables involved in the problem. All models having mathematical structure and can be solved by mathematical methods are known as Analytical Models.

(*ii*) *Simulation Models:* The meaning of simulation is *imitation*. These models have mathematical structure but cannot be solved by using mathematical techniques. It needs certain experimental analysis. To study the behaviour of the system, we use random numbers. More complex systems can be studied by simulation. Studying the behaviour of laboratory model, we can evaluate the required values in the system. Only disadvantage of this method is that it does not have general solution method.

1.9.3. Some of the Points to be Remembered while Building a Model

* When we can solve the situation with a simple model, do not try to build a complicated model.
* Build a model that can be easily fit in the techniques available. Do not try to search for a technique, which suit your model.
* In order to avoid complications while solving the problem, the fabrication stage of modeling must be conducted rigorously.
* Before implementing the model, it should be validated / tested properly.
* Use the model for which it is deduced. Do not use the model for the purpose for which it is not meant.
* Without having a clear idea for which the model is built do not use it. It is better before using the model; you consult an operations research analyst and take his guidance.
* Models cannot replace decision makers. It can guide them but it cannot make decisions. Do not be under the impression, that a model solves every type of problem.
* The model should be as accurate as possible.
* A model should be as simple as possible.
* Benefits of model are always associated with the process by which it is developed.

1.9.4. Advantages of a Good Model

(*i*) A model provides logical and systematic approach to the problem.
(*ii*) It provides the analyst a base for understanding the problem and think of methods of solving.
(*iii*) The model will avoid the duplication work in solving the problem.
(*iv*) Models fix the limitation and scope of an activity.
(*v*) Models help the analyst to find newer ways of solving the problem.
(*vi*) Models saves resources like money, time etc.
(*vii*) Model helps analyst to make complexities of a real environment simple.

(*viii*) Risk of tampering the real object is reduced, when a model of the real system is subjected to experimental analysis.

(*ix*) Models provide distilled economic descriptions and explanations of the operation of the system they represent.

1.9.5. Limitations of a Model

(*i*) Models are constructed only to understand the problem and attempt to solve the problem; they are not to be considered as real problem or system.

(*ii*) The validity of any model can be verified by conducting the experimental analysis and with relevant data characteristics.

1.9.6. Characteristics of a Good Model

(*i*) The number of parameters considered in a model should be less to understand the problem easily.

(*ii*) A good model should be flexible to accommodate any necessary information during the stages of building the model.

(*iii*) A model must take less time to construct.

(*iv*) A model may be accompanied by lower and upper bounds of parametric values.

1.9.7. Steps in Constructing a Model

(*i*) *Problem environment analysis and formulation*: One has to study the system in all aspects, if necessary make relevant assumptions, have the decision for which he is constructing the model in mind and formulate the model.

(*ii*) *Model construction and assumptions*: Identify the main variables and constraints and relate them logically to arrive at a model.

(*iii*) *Testing the model*: After the formulation, before using check the model for its validity.

1.10. Methods of Solving Operations Research Problems

There are three methods of solving an operations research problem. They are:

(*i*) Analytical method, (*ii*) Iterative Method, (*iii*) The Monte-Carlo Technique.

(*i*) *Analytical Method*: When we use mathematical techniques such as differential calculus, probability theory etc. to find the solution of a given operations research model, the method of solving is known as analytical method and the solution is known as analytical solution. Examples are problems of inventory models. This method evaluates alternative policies efficiently.

(*ii*) *Iterative Method (Numerical Methods)*: This is trial and error method. When we have large number of variables, and we cannot use classical methods successfully, we use iterative process. First, we set a trial solution and then go on changing the solution under a given set of conditions, until no more modification is possible. The characteristics of this method is that the trial and error method used is laborious, tedious, time consuming and costly. The solution we get may not be accurate one and is approximate one. Many a time we find that

after certain number of iterations, the solution cannot be improved and we have to accept it as the expected optimal solution.

(*iii*) *Monte-Carlo Method*: This method is based on random sampling of variable's values from a distribution of the variable. This uses sampling technique. A table of random numbers must be available to solve the problems. In fact it is a simulation process.

1.11. SOME IMPORTANT MODELS (PROBLEMS) WE COME ACROSS IN THE STUDY OF OPERATIONS RESEARCH

1. Linear Programming Model

This model is used for resource allocation when the resources are limited and there are number of competing candidates for the use of resources. The model may be used to maximise the returns or minimise the costs. Consider the following two situations:

(*a*) A company which is manufacturing variety of products by using available resources, want to use resources optimally and manufacture different quantities of each type of product, which yield different returns, so as to maximise the returns.

(*b*) A company manufactures different types of alloys by purchasing the three basic materials and it want to maintain a definite percentage of basic materials in each alloy. The basic materials are to be purchased from the sellers and mix them to produce the desired alloy. This is to be done at minimum cost.

Both of them are resource allocation models, the case (*a*) is maximisation problem and the case (*b*) is minimisation problem.

(*c*) Number of factories are manufacturing the same commodities in different capacities and the commodity is sent to various markets for meeting the demands of the consumers, when the cost of transportation is known, the linear programming helps us to formulate a programme to distribute the commodity from factories to markets at minimum cost. The model used is transportation model.

(*d*) When a company has number of orders on its schedule, which are to be processed on same machines and the processing time, is known, then we have to allocate the jobs or orders to the machines, so as to complete all the jobs in minimum time. This we can solve by using Assignment model.

All the above-discussed models are Linear Programming Models. They can be solved by application of appropriate models, which are linear programming models.

2. Sequencing Model

When a manufacturing firm has some job orders, which can be processed on two or three machines and the processing times of each job on each machine is known, then the problem of processing in a sequence to minimise the cost or time is known as Sequencing model.

3. Waiting Line Model or Queuing Model

A model used for solving a problem where certain service facilities have to provide service to its customers, so as to avoid lengthy waiting line or queue, so that customers will get satisfaction from effective service and idle time of service facilities are minimised is waiting line model or queuing model.

4. Replacement Model

Any capital item, which is continuously used for providing service or for producing the product is subjected to wear and tear due to usage, and its efficiency goes on reducing. This reduction in efficiency can be predicted by the increasing number of breakdowns or reduced productivity. The worn out parts or components are to be replaced to bring the machine back to work. This action is known as maintenance. A time is reached when the maintenance cost becomes very high and the manager feels to replace the old machine by new one. This type of problems known as replacement problems and can be solved by replacement models.

5. Inventory Models

Any manufacturing firm has to maintain stock of materials for its use. This stock of materials, which are maintained in stores, is known as inventory. Inventory is one form of capital or money. The company has to maintain inventory at optimal cost. There are different types of inventory problems, depending the availability and demand pattern of the materials. These can be solved by the application of inventory models.

In fact depending on the number of variables, characteristics of variables, and the nature of constraints different models are available. These models, we study when we go chapter wise.

QUESTIONS

1. Trace the history of Operations Research.
2. Give a brief account of history of Operations Research.
3. Discuss the objective of Operations Research.
4. "Operations Research is a bunch of mathematical techniques to break industrial problems". Critically comment.
5. What is a Operations Research model? Discuss the advantages of limitation of good Operations Research model.
6. Discuss three Operations Research models.
7. What is a decision and what are its characteristics.
8. Briefly explain the characteristics of Operations Research.
9. Discuss the various steps used in solving Operations Research problems.
10. Discuss the scope of Operations Research.

Linear Programming Models
(Resource Allocation Models)

2.1. INTRODUCTION

A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is *linear programming*.

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is **LINEAR PROGRAMMING**. As a decision making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

2.2. PROPERTIES OF LINEAR PROGRAMMING MODEL

Any linear programming model (problem) must have the following properties:
   (*a*) **The relationship between variables and constraints must be linear.**
   (*b*) **The model must have an objective function.**
   (*c*) **The model must have structural constraints.**
   (*d*) **The model must have non-negativity constraint.**
Let us consider a product mix problem and see the applicability of the above properties.

Example 2.1. A company manufactures two products X and Y, which require, the following resources. The resources are the capacities machine M_1, M_2, and M_3. The available capacities are 50,25,and 15 hours respectively in the planning period. Product X requires 1 hour of machine M_2 and 1 hour of machine M_3. Product Y requires 2 hours of machine M_1, 2 hours of machine M_2 and 1 hour of machine M_3. The profit contribution of products X and Y are Rs.5/- and Rs.4/- respectively.

The contents of the statement of the problem can be summarized as follows:

Machines	Products		Availability in hours
	X	Y	
M_1	0	2	50
M_2	1	2	25
M_3	1	1	15
Profit in Rs. Per unit	5	4	

In the above problem, Products *X* and *Y* are competing candidates or variables.

Machine capacities are available resources. Profit contribution of products *X* and *Y* are given. Now let us formulate the model.

Let the company manufactures *x* units of *X* and y units of *Y*. As the profit contributions of *X* and *Y* are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit *Z*, hence objective function is:

Maximize $Z = 5x + 4y$ ⎯⎯⟶ OBJECTIVE FUNCTION.

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

For Machine M_1 $0x + 2y \leq 50$
For Machine M_2 $1x + 2y \leq 25$ and ⎯⎯⟶ **LINEAR STRUCTURAL CONSTRAINTS.**
For machine M_3 $1x + 1y \leq 15$

But the company can stop production of *x* and *y* or can manufacture any amount of *x* and *y*. It cannot manufacture negative quantities of *x* and *y*. Hence we have write,

Both *x* and *y* are ≥ 0 . ⎯⎯⟶ NON -NEGATIVITY CONSTRAINT.

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

2.2.1. Basic Assumptions

The following are some important assumptions made in formulating a linear programming model:

1. It is assumed that the decision maker here is *completely certain* (*i.e.*, deterministic conditions) regarding all aspects of the situation, *i.e.*, availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.
2. It is assumed that the relationship between variables in the problem and the resources available *i.e.*, constraints of the problem exhibits *linearity*. Here the term linearity implies proportionality and additivity. This assumption is very useful as it simplifies modeling of the problem.
3. We assume here *fixed technology*. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.
4. It is assumed that the *profit contribution of a product remains constant*, irrespective of level of production and sales.

5. It is assumed that the decision variables are *continuous*. It means that the companies manufacture products in fractional units. For example, company manufacture 2.5 vehicles, 3.2 barrels of oil etc. This is referred too as the assumption of *divisibility*.

6. It is assumed that *only one decision* is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a *single stage decision problem*. (Note: Dynamic Programming problem is a multistage decision problem).

7. All variables are restricted to *nonnegative values* (*i.e.*, their numerical value will be ≥ 0).

2.2.2. Terms Used in Linear Programming Problem

Linear programming is a method of obtaining an optimal solution or programme (say, product mix in a production problem), when we have limited resources and a good number of *competing candidates to consume* the limited resources in *certain proportion*. The term linear implies the condition of proportionality and additivity. The *programme* is referred as a course of action covering a specified period of time, say planning period. The manager has to find out the best course of action in the interest of the organization. This best course of action is termed as *optimal course of action or optimal solution* to the problem. A programme is optimal, when it *maximizes or minimizes* some measure or criterion of effectiveness, such as profit, sales or costs.

The term *programming* refers to a systematic procedure by which a particular program or plan of action is designed. Programming consists of a series of instructions and computational rules for solving a problem that can be worked out manually or can fed into the computer. In solving linear programming problem, we use a systematic method known as *simplex method* developed by American mathematician George B. Dantzig in the year 1947.

The candidates or activity here refers to number of products or any such items, which need the utilization of available resources in a certain required proportion. The available resources may be of any nature, such as money, area of land, machine hours, and man-hours or materials. But they are *limited in availability* and which are desired by the activities / products for consumption.

2.2.3. General Linear Programming Problem

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below:

$Z = c_1x_1 + c_2 x_2 + \dots c_n x_n$ subjects to the conditions, ⟶ OBJECTIVE FUNCTION

$a_{11}x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1j} x_j + \dots + ..a_{1n} x_n \ (\geq, =, \leq) \ b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots\dots\dots + a_{2j} x_j + \dots\dots + a_{2n}x_n \ (\geq, =, \leq) \ b_2$

Structural Constraints

....

....

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mj} x_j \dots + a_{mn} x_n \ (\geq, =, \leq) \ b_m$

and all x_j are = 0 ⟶ NON NEGETIVITY CONSTRINT.

Where j = 1, 2, 3, ... n

Where all c_j s, b_i s and a_{ij} s are constants and x_j s are decision variables.

To show the relationship between left hand side and right hand side the symbols ≤ , = , ≥ are used. Any one of the signs may appear in real problems. Generally ≤ sign is used for maximization

problems and $\geq$ sign is used for minimization problems and in some problems, which are known as mixed problems we may have all the three signs. The word optimize in the above model indicates either maximise or minimize. The linear function, which is to be optimized, is the objective function. The inequality conditions shown are constraints of the problem. Finally all x_i s should be positive, hence the non-negativity function.

The steps for formulating the linear programming are:

1. *Identify the unknown decision variables to be determined and assign symbols to them.*
2. *Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.*
3. *Identify the objective or aim and represent it also as a linear function of decision variables.*

Construct linear programming model for the following *problems*:

2.3. MAXIMIZATION MODELS

Example 2.2. A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs. 2/- per unit and type B a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Solution: Here shirts A and B are problem variables. Let the store stock 'a' units of A and 'b' units of B. As the profit contribution of A and B are Rs.2/- and Rs.5/- respectively, objective function is:

Maximize $Z = 2a + 5b$ subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt A and 300 units of shirt B and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity.

And non-negativity constraint is both a and b are ≥ 0. Hence the model is:

Maximize $Z = 2a + 5b$ s.t.

$$1a + 0b \leq 400$$
$$0a + 1b \leq 300$$
$$1a + 1b \leq 600 \text{ and}$$
Both a and b are ≥ 0.

Problem 2.3. A ship has three cargo holds, forward, aft and center. The capacity limits are:

Forward 2000 tons, 100,000 cubic meters
Center 3000 tons, 135,000 cubic meters
Aft   1500 tons, 30,000 cubic meters.

The following cargoes are offered, the ship owners may accept all or any part of each commodity:

Commodity	Amount in tons.	Volume/ton in cubic meters	Profit per ton in Rs.
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

Solution: Problem variables are commodities, *A*, *B*, and *C*. Let the shipping company ship '*a*' units of *A* and '*b*' units of *B* and '*c*' units of *C*. Then Objective function is:

Maximize $Z = 60a + 80b + 50c$ s.t.

Constraints are:

Weight constraint: $6000a + 4000b + 2000c \leq 6,500$ (= 2000+3000+1500)

The tonnage of commodity is 6000 and each ton occupies 60 cubic meters, hence there are 100 cubic meters capacity is available.

Similarly, availability of commodities *B* and *C*, which are having 80 cubic meter capacities each. Hence capacity inequality will be:

$100a + 80b + 80c \leq 2,65,000$ (= 100,000+135,000+30,000). Hence the l.p.p. Model is:

Maximise $Z = 60a + 80b + 50c$ s.t. $100a = 6000/60 = 100$

$6000a + 4000b + 2000c \leq 6,500$ $80b = 4000/50 = 80$

$100a + 80b + 80c \leq 2,65,000$ and $80c = 2000/25 = 80$ etc.

a,b,c all ≥ 0

2.4. MINIMIZATION MODELS

Problem 2.4. A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin *A* and vitamin *D*. Doctor advises him to consume vitamin *A* and *D* regularly for a period of time so that he can regain his health. Doctor prescribes tonic *X* and tonic *Y*, which are having vitamin *A*, and *D* in certain proportion. Also advises the patient to consume **at least** 40 units of vitamin *A* and 50 units of vitamin Daily. The cost of tonics *X* and *Y* and the proportion of vitamin *A* and *D* that present in *X* and *Y* are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	Tonics		Daily requirement in units.
	X	Y	
A	2	4	40
D	3	2	50
Cost in Rs. per unit.	5	3	

Solution: Let patient purchase *x* units of *X* and y units of *Y*.

Objective function: Minimize $Z = 5x + 3y$

Inequality for vitamin *A* is $2x + 4y \geq 40$ (Here **at least** word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).

Similarly the inequality for vitamin D is $3x + 2y \geq 50$.

For non–negativity constraint the patient cannot consume negative units. Hence both *x* and *y* must be ≥ 0.

Now the l.p.p. model for the problem is:

Minimize $Z = 5x + 3y$ s.t.

$2x + 4y \geq 40$

$3x + 2y \geq 50$ and

Both x and y are ≥ 0.

Problem 2.5. A machine tool company conducts a job-training programme at a ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of 10 trainees hired, only seven complete the programme successfully. (The unsuccessful trainees are released). Trained machinists are also needed for machining. The company's requirement for the next three months is as follows:

January: 100 machinists, February: 150 machinists and March: 200 machinists.

In addition, the company requires 250 trained machinists by April. There are 130 trained machinists available at the beginning of the year. Pay roll cost per month is:

Each trainee Rs. 400/- per month.

Each trained machinist (machining or teaching): Rs. 700/- p.m.

Each trained machinist who is idle: Rs.500/- p.m.

(Labour union forbids ousting trained machinists). Build a l.p.p. for produce the minimum cost hiring and training schedule and meet the company's requirement.

Solution: There are three options for trained machinists as per the data given. (*i*) A trained machinist can work on machine, (*ii*) he can teach or (*iii*) he can remain idle. It is given that the number of trained machinists available for machining is fixed. Hence the unknown decision variables are the number of machinists goes for teaching and those who remain idle for each month. Let,

'*a*' be the trained machinists teaching in the month of January.

'*b*' be the trained machinists idle in the month of January.

'*c*' be the trained machinists for teaching in the month of February.

'*d*' be the trained machinists remain idle in February.

'*e*' be the trained machinists for teaching in March.

'*f*' be the trained machinists remain idle in the month of March.

The constraints can be formulated by the rule that the number of machinists used for (machining + teaching + idle) = Number of trained machinists available at the beginning of the month.

For January $100 + 1a + 1b \geq 130$

For February, $150 + 1c + 1d = 130 + 7a$ (Here $7a$ indicates that the number of machinist trained is $10 \times a = 10a$. But only 7 of them are successfully completed the training *i.e.* $7a$).

For the month of March, $200 + 1e + 1f \geq 130 + 7a + 7c$

The requirement of trained machinists in the month of April is 250, the constraints for this will be

$130 + 7a + 7c + 7e \geq 250$ and the objective function is

Minimize $Z = 400 (10a + 10c + 10e) + 700 (1a + 1c + 1e) + 400 (1b + 1d + 1f)$ and the non-negativity constraint is a, b, c, d, e, f all ≥ 0. The required model is:

Minimize $Z = 400 (10a + 10c + 10e) + 700 (1a + 1c + 1e) + 400 (1b + 1d + 1f)$ s.t.

$100 + 1a + 1b \geq 130$

$150 + 1c + 1d \geq 130 + 7a$

$200 + 1e + 1f \geq 130 + 7a + 7c$

$130 + 7a + 7c + 7e \geq 250$ and

a, b, c, d, e, f all ≥ 0.

2.5. METHODS FOR THE SOLUTION OF A LINEAR PROGRAMMING PROBLEM

Linear Programming, is a method of solving the type of problem in which two or more **candidates** or **activities** are competing to utilize the available limited resources, with a view to **optimize** the **objective function** of the problem. The objective may be to maximize the **returns** or to minimize the **costs**. The various methods available to solve the problem are:

1. The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.

2. The Systematic Trial and Error method, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence this method is not discussed here.

3. The Vector method. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.

4. The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.

2.5.1. Graphical Method

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (*X*- axis and *Y*-axis). More over as we have non-negativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Some times the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant. The procedure of the method will be explained in detail while solving a numerical problem. The characteristics of Graphical method are:

(*i*) Generally the method is used to solve the problem, when it involves two decision variables.

(*ii*) For three or more decision variables, the graph deals with planes and requires high imagination to identify the solution area.

(*iii*) Always, the solution to the problem lies in first quadrant.

(*iv*) This method provides a basis for understanding the other methods of solution.

Problem 2.6. A company manufactures two products, *X* and *Y* by using three machines *A*, *B*, and *C*. Machine *A* has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines *B* and *C* during the coming week is 24 hours and 35 hours respectively. One unit of

product *X* requires one hour of Machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly one unit of product *Y* requires 1 hour, 8 hour and 7 hours of machine *A*, *B* and *C* respectively. When one unit of *X* is sold in the market, it yields a profit of Rs. 5/- per product and that of *Y* is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Solution: The details given in the problem is given in the table below:

Machines	*Products* (*Time required in hours*).		*Available capacity in hours.*
	X	*Y*	
A	1	1	4
B	3	8	24
C	10	7	35
Profit per unit in Rs.	5	7	

Let the company manufactures x units of *X* and y units of *Y*, and then the L.P. model is:

Maximise $Z = 5x + 7y$ s.t.

$1x + 1y \leq 4$

$3x + 8y \leq 24$

$10x + 7y \leq 35$ and

Both x and y are ≥ 0.

As we cannot draw graph for inequalities, let us consider them as equations.

Maximise $Z = 5x + 7y$ s.t.

$1x + 1y = 4$

$3x + 8y = 24$

$10x + 7y = 35$ and both x and y are ≥ 0

Let us take machine *A*. and find the boundary conditions. If $x = 0$, machine *A* can manufacture $4/1 = 4$ units of y.
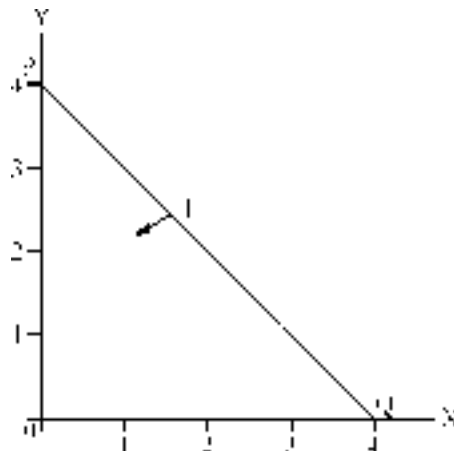
Figure 2.1 Graph for machine A

Similarly, if $y = 0$, machine A can manufacture $4/1 = 4$ units of x. For other machines:

Machine B When $x = 0$, $y = 24/8 = 3$ and when $y = 0$ $x = 24/3 = 8$

Machine C When $x = 0$, $y = 35/10 = 3.5$ and when $y = 0$, $x = 35 / 7 = 5$.

These values we can plot on a graph, taking product X on x-axis and product Y on y- axis.

First let us draw the graph for machine A. In figure 2. 1 we get line 1 which represents $1x + 1y = 4$. The point P on Y axis shows that the company can manufacture 4 units of Y only when does not want to manufacture X. Similarly the point Q on X axis shows that the company can manufacture 4 units of X, when does not want to manufacture Y. In fact triangle POQ is the capacity of machine A and the line PQ is the boundary line for capacity of machine A.

Similarly figure 2.2 show the Capacity line RS for machine B. and the triangle ROS shows the capacity of machine B $i.e.$, the machine B can manufacture 3 units of product Y alone or 8 units of product X alone.
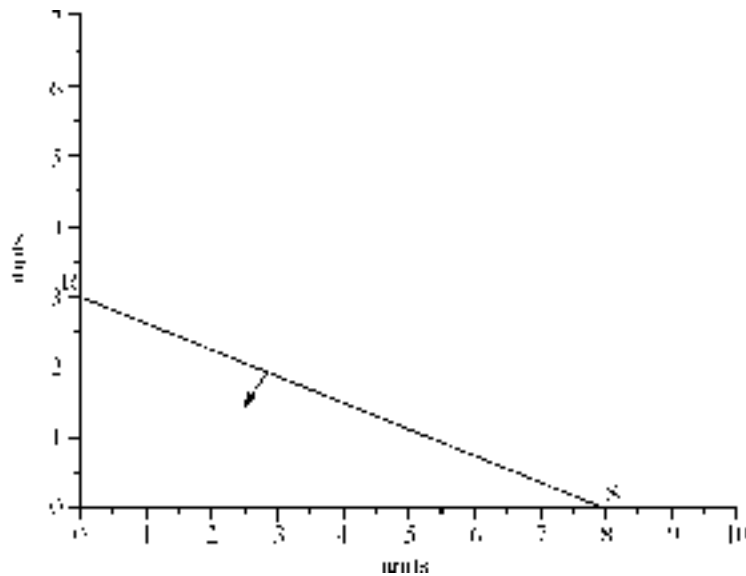
Figure 2.2. Graph for machine B

The graph 2.3 shows that the machine C has a capacity to manufacture 5 units of Y alone or 3.5 units of X alone. Line TU is the boundary line and the triangle TOU is the capacity of machine C.

The graph is the combined graph for machine A and machine B. Lines PQ and RS intersect at M. The area covered by both the lines indicates the products (X and Y) that can be manufactured by using both machines. This area is the feasible area, which satisfies the conditions of inequalities of machine A and machine B. As X and Y are processed on A and B the number of units that can be manufactured will vary and the there will be some idle capacities on both machines. The idle capacities of machine A and machine B are shown in the figure 2.4.

Linear Programming Models (Resource Allocation Models)
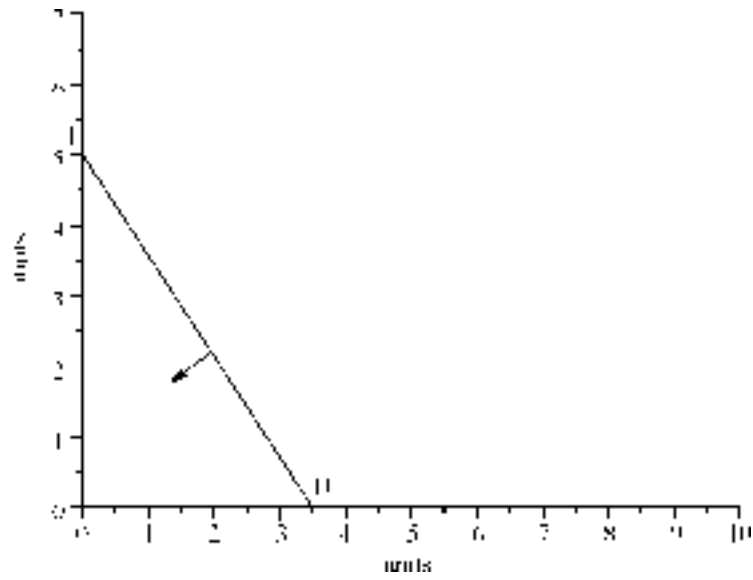
Figure 2.3. Graph for machine C
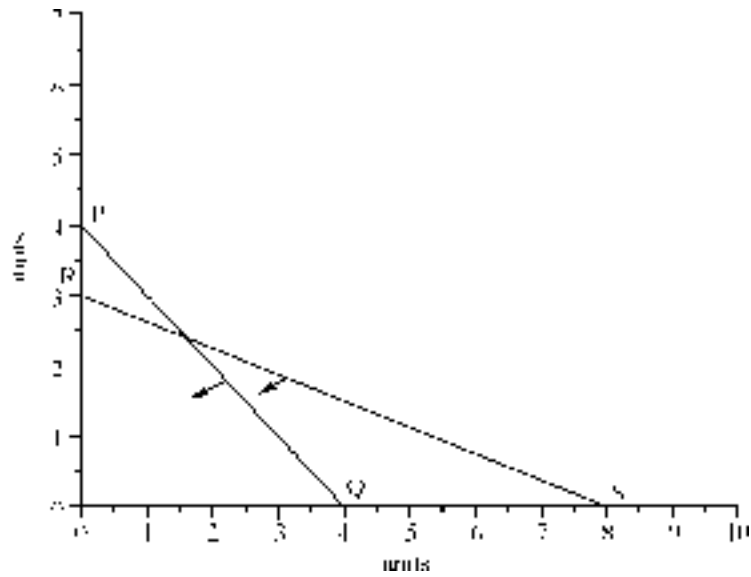
Figure 2.4. Graph of Machines A and B

Figure 2.5 shows the feasible area for all the three machines combined. This is the fact because a products *X* and *Y* are complete when they are processed on machine *A*, *B*, and *C*. The area covered by all the three lines *PQ*. *RS*, and *TU* form a closed polygon *ROUVW*. This polygon is the feasible area for the three machines. This means that all the points on the lines of polygon and any point within the polygon satisfies the inequality conditions of all the three machines. To find the optimal solution, we have two methods.
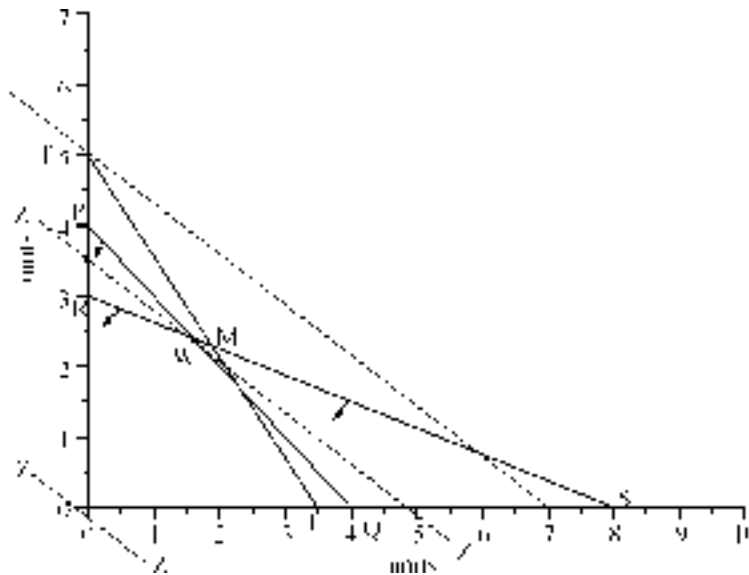
Figure 2.5. Graph for machine A, B and C combined

Method 1. Here we find the co-ordinates of corners of the closed polygon *ROUVW* and substitute the values in the objective function. In maximisaton problem, we select the co-ordinates giving maximum value. And in minimisaton problem, we select the co-ordinates, which gives minimum value.
In the problem the co-ordinates of the corners are:

R = (0, 3.5), O = (0,0), U = (3.5,0), V = (2.5, 1.5) and W = (1.6,2.4). Substituting these values in objective function:

$$Z_{(0,3.5)} = 5 \times 0 + 7 \times 3.5 = \text{Rs. } 24.50, \text{ at point } R$$
$$Z_{(0,0)} = 5 \times 0 + 7 \times 0 = \text{Rs. } 00.00, \text{ at point } O$$
$$Z_{(3.5,0)} = 5 \times 3.5 + 7 \times 0 = \text{Rs. } 17.5 \text{ at point } U$$
$$Z_{(2.5, 1.5)} = 5 \times 2.5 + 7 \times 1.5 = \text{Rs. } 23.00 \text{ at point } V$$
$$\mathbf{Z_{(1.6, 2.4)} = 5 \times 1.6 + 7 \times 2.4 = Rs. \ 24.80 \ \text{at point } W}$$

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product *X* and 2.4 units of product *Y*, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

Method 2. Isoprofit Line Method: Isoprofit line, a line on the graph drawn as per the objective function, assuming certain profit. On this line any point showing the values of x and y will yield same profit. For example in the given problem, the objective function is Maximise $Z = 5x + 7y$. If we assume a profit of Rs. 35/-, to get Rs. 35, the company has to manufacture either 7 units of *X* or 5 units of *Y*.

Hence, we draw line *ZZ* (preferably dotted line) for $5x + 7y = 35$. Then draw parallel line to this line *ZZ* at origin. The line at origin indicates zero rupees profit. No company will be willing to earn zero rupees profit. Hence slowly move this line away from origin. Each movement shows a certain profit, which is greater than Rs.0.00. While moving it touches corners of the polygon showing certain higher profit. Finally, it touches the farthermost corner covering all the area of the closed polygon. This point where the line passes (farthermost point) is the **OPTIMAL SOLUTION** of the problem. In the figure 2.6. the line *ZZ* passing through point *W* covers the entire area of the polygon, hence it is the point that yields highest profit. Now point *W* has co-ordinates (1.6, 2.4). **Now Optimal profit $Z = 5 \times 1.6 + 7 \times 2.4 =$ Rs. 24.80.**

Points to be Noted:

   (*i*) **In case lsoprofit line passes through more than one point, then it means that the problem has more than one optimal solution, *i.e*., alternate solutions all giving the same profit. This helps the manager to take a particular solution depending on the demand position in the market. He has options.**

  (*ii*) **If the lsoprofit line passes through single point, it means to say that the problem has unique solution.**

 (*iii*) **If the Isoprofit line coincides any one line of the polygon, then all the points on the line are solutions, yielding the same profit. Hence the problem has innumerable solutions.**

 (*iv*) **If the line do not pass through any point (in case of open polygons), then the problem do not have solution, and we say that the problem is UNBOUND.**
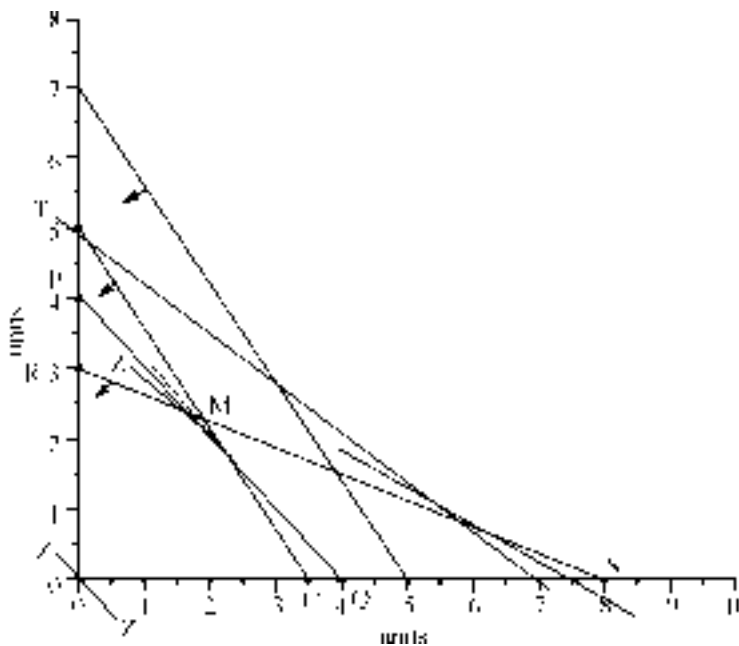
Figure 2.6. ISO profit line method.

Now let us consider some problems, which are of mathematical interest. Such problems may not exist in real world situation, but they are of mathematical interest and the student can understand the mechanism of graphical solution.

Problem 2.7. Solve graphically the given linear programming problem. (Minimization Problem).

$$\text{Minimize } Z = 3a + 5b \text{ S.T}$$
$$-3a + 4b \leq 12$$
$$2a - 1b \geq -2$$
$$2a + 3b \geq 12$$
$$1a + 0b \geq 4$$
$$0a + 1b \geq 2$$
$$\text{And both } a \text{ and } b \text{ are } \geq 0.$$

Points to be Noted:

(*i*) **In inequality $-3a + 4b \leq 12$, product/the candidate/activity requires -3 units of the resource. It does not give any meaning (or by manufacturing the product A the manufacturer can save 3 units of resource No.1 or one has to consume -3 units of A. (All these do not give any meaning as far as the practical problems or real world problems are concerned).**

(*ii*) **In the second inequality, on the right hand side we have -2. This means that -2 units of resource is available. It is absolutely wrong. Hence in solving a l.p.p. problem, one must see that the right hand side we must have always a positive integer. Hence the inequality is to be multiplied by -1 so that the inequality sign also changes. In the present case it becomes: $-2a + 1b \leq 2$.**

Solution: Now the problem can be written as:

Minimize $Z = 3a + 5b$ S.T.

When converted into equations they can be written as Min. $Z = 3a + 5b$ *S.T.*

$-3a + 4b \leq 12$	$-3a + 4b = 12$
$-2a + 1b \leq 2$	$-2a + 1b = 2$
$2a - 3b \geq 12$	$2a - 3b = 12$
$1a + 0b \leq 4$	$1a + 0b = 4$

$0a + 1b \geq 2$ and both a and b are $\geq = 0.0a + 1b \geq 2$ and both a and b are ≥ 0.

The lines for inequalities $-3a + 4b \leq 12$ and $-2a + 1b \leq 2$ starts from quadrant 2 and they are to be extended into quadrant 1. Figure 2.7 shows the graph, with Isocost line.

Isocost line is a line, the points on the line gives the same cost in Rupees. We write Isocost line at a far off place, away from the origin by assuming very high cost in objective function. Then we move line parallel towards the origin (in search of least cost) until it passes through a single corner of the closed polygon, which is nearer to the origin, (Unique Solution), or passes through more than one point, which are nearer to the origin (more than one solution) or coincides with a line nearer to the origin and the side of the polygon (innumerable solution).

The solution for the problem is the point P (3,2,) and the **Minimum cost is Rs. $3 \times 3 + 2 \times 5 =$ Rs. 19/-**

Problem 2.8. The cost of materials A and B is Re.1/- per unit respectively. We have to manufacture an alloy by mixing these to materials. The process of preparing the alloy is carried out on three facilities X, Y and Z. Facilities X and Z are machines, whose capacities are limited. Y is a furnace, where heat treatment takes place and the material must use a minimum given time (even if it uses more than the required, there is no harm). Material A requires 5 hours of machine X and it does not require processing on machine Z. Material B requires 10 hours of machine X and 1 hour of machine Z. Both A and B are to be heat treated at last one hour in furnace Y. The available capacities of X, Y and Z are 50 hours, 1 hour and 4 hours respectively. Find how much of A and B are mixed so as to minimize the cost.
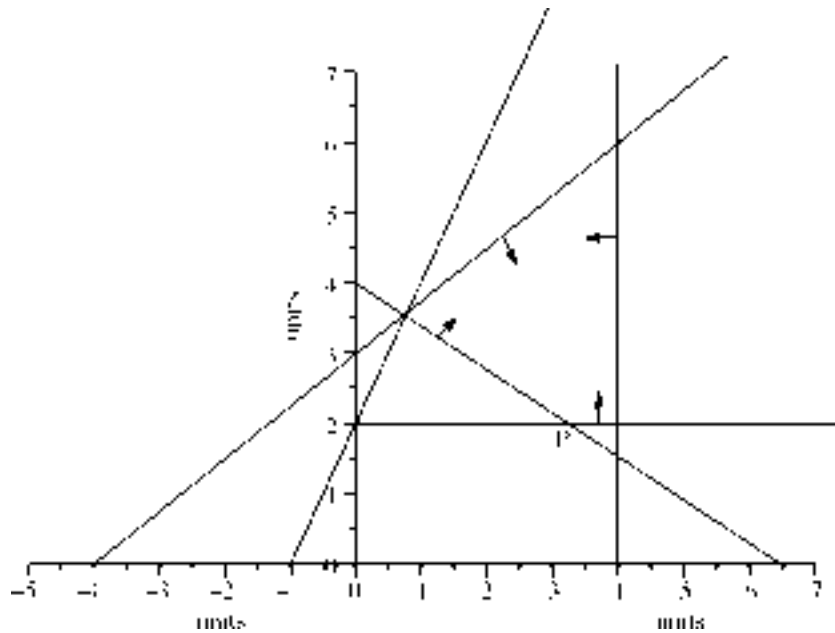
Figure 2.7. Graph for the problem 2.7

Solution: The l.p.p. model is:
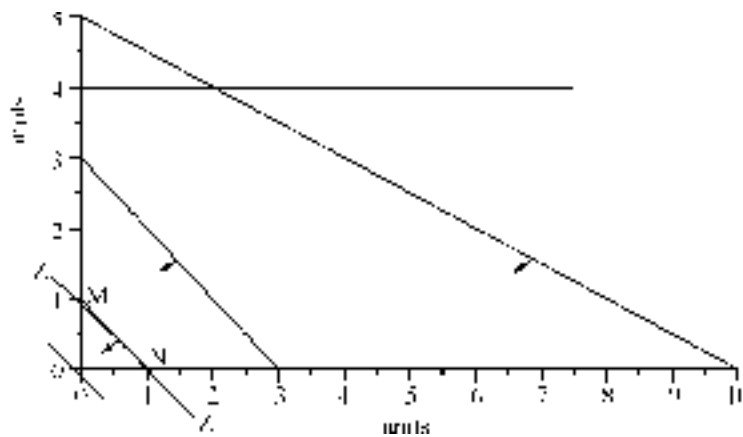
Figure 2.8. Graph for the problem 2.8

Minimize $Z = 1a + 1b$ S.T.   Equations are: Minimise $Z = 1a + 1b$ S.T

$5a + 10b \leq 50$, $5a + 10b = 50$

$1a + 1b \geq 1$ $1a + 1b = 1$

$0a + 1b \leq 4$ and both a and b are ≥ 0. $0a + 1b = 4$ and both a and b are ≥ 0.

Figure 2.8 shows the graph. Here Isocost line coincides with side of the polygon, *i.e.*, the line MN. Hence the problem has innumerable solutions. Any value on line (1,1) will give same cost. **Optimal cost is Re.1/-**

Problem 2.9. Maximise $Z = 0.75\ a + 1b$ S.T.

$1a + 1b \geq 0$

$-0.5\ a + 1b \leq 1$ and both a and b are ≥ 0.

Solution: Writing the inequalities as equations,

$1a + 1b = 0$ *i.e.*, $a = b = 1$ which is a line passing through origin at $45°$

$0.5\ a + 1\ b = 1$ and both a and b are ≥ 0. Referring to figure 2.9.

The polygon is not closed one *i.e.*, the feasible area is unbound. When Isoprofit line is drawn, it passes through open side of the polygon and it does not coincide with any corner or any line. Hence the line can be moved indefinitely, still containing a part of the feasible area. Thus there is no finite maximum value of Z. That the value of Z can be increased indefinitely. **When the value of Z can be increased indefinitely, the problem is said to have an UNBOUND solution.**
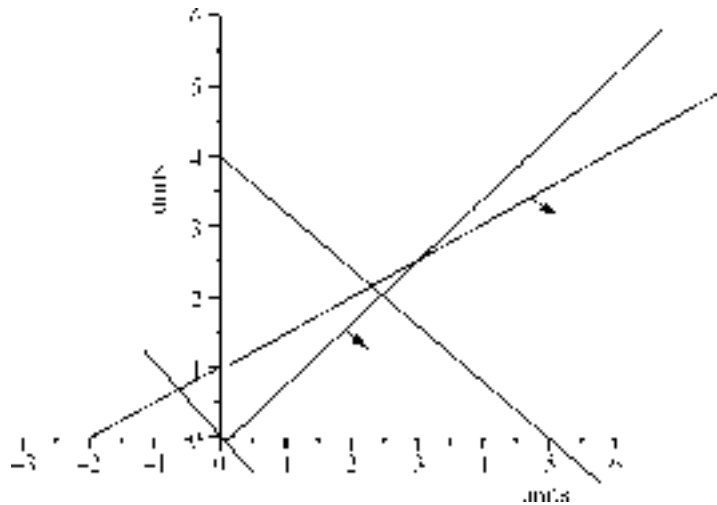
Figure 2.9. Graph for the problem 2.9

Problem 2.10. A company manufactures two products X and Y on two facilities A and B. The data collected by the analyst is presented in the form of inequalities. Find the optimal product mix for maximising the profit.

Maximise $Z = 6x - 2y$ S.T.   Writing in the equation form: Maximise $Z = 6x - 2y$ S.T.

$2x - 1y \leq 2$ $2x - 1y = 2$

$1x + 0y \leq 3$ and both x and y are ≥ 0 $1x + 0y = 3$ and both x and y are ≥ 0

Solution: The straight line for $2x - 1y = 2$ starts in 4th quadrant and is to be extended into first quadrant. The polygon is not a closed one and the feasible area is unbound. But when an Isoprofit line is drawn it passes through a corner of the feasible area that is the corner M of the open polygon. The (figure 2.10) coordinates of M are (3, 4) and the **maximum Z = Rs. 10/-**
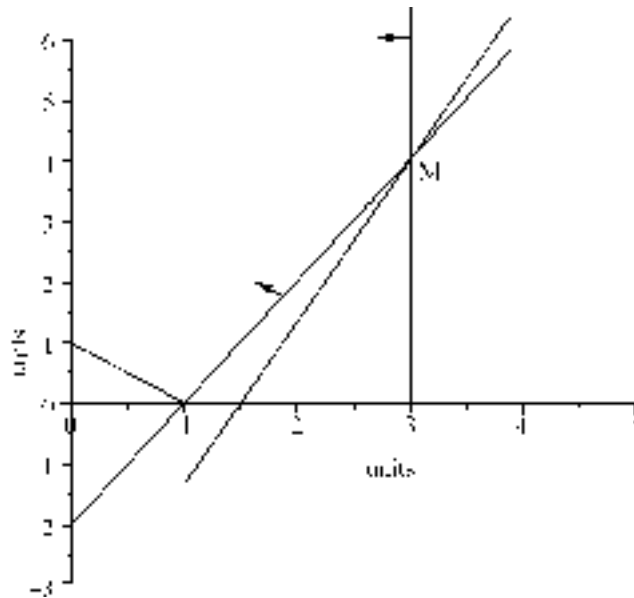
Figure 2.10. Graph for the problem 2.10

Problem 2.11. A company manufactures two products X and Y. The profit contribution of X and Y are Rs.3/- and Rs. 4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly the requirement of product Y is 4, 5, 5, and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximise the profit.

Solution: Enter the given data in the table below:

	products		
Machines	*X*	*Y*	*Availability in hours.*
	(Time in hours)		
A	5	4	200
B	3	5	150
C	5	4	100
D	8	4	80
Profit in Rs. Per unit:	3	4	

The inequalities and equations for the above data will be as follows. Let the company manufactures x units of X and y units of Y. (Refer figure 2.11)

Maximise Z $3x + 4y$ S.T.   Maximise $Z = 3x + 4y$ S.T.

$5x + 4y \leq 200$ $5x + 4y = 200$

$3x + 5y \leq 150$ $3x + 5y = 150$

$5x + 4y \leq 100$ $5x + 4y = 100$

$8x + 4y \leq 80$ $8x + 4y = 80$

And both x and y are ≥ 0 And both x and y are ≥ 0

In the graph the line representing the equation $8x + 4y$ is out side the feasible area and hence it is a redundant equation. It does not affect the solution. The Isoprofit line passes through corner T of the polygon and is the point of maximum profit. Therefore $Z_T = Z_{(32,10)} = 3 \times 32 + 4 \times 10 = $ Rs. 136/.

Problem 2.12. This problem is of mathematical interest.

Maximise $Z = 3a + 4b$ S.T. Converting and writing in the form of equations,

$1a - 1b \leq -1.$ Maximise $Z = 3a + 4b$ S.T

$-1a + 1b \leq 0$ $1a - 1b = 0$

And both a and b are ≥ 0 $-1a + 1b = 0$

And both a and b are ≥ 0

Referring to figure 2.11, the straight line for equation 1 starts in second quadrant and extended in to first quadrant. The line for equation 2 passes through the origin. We see that there is no point, which satisfies both the constraints simultaneously. Hence there is no feasible solution. **Given l.p.p. has no feasible solution.**
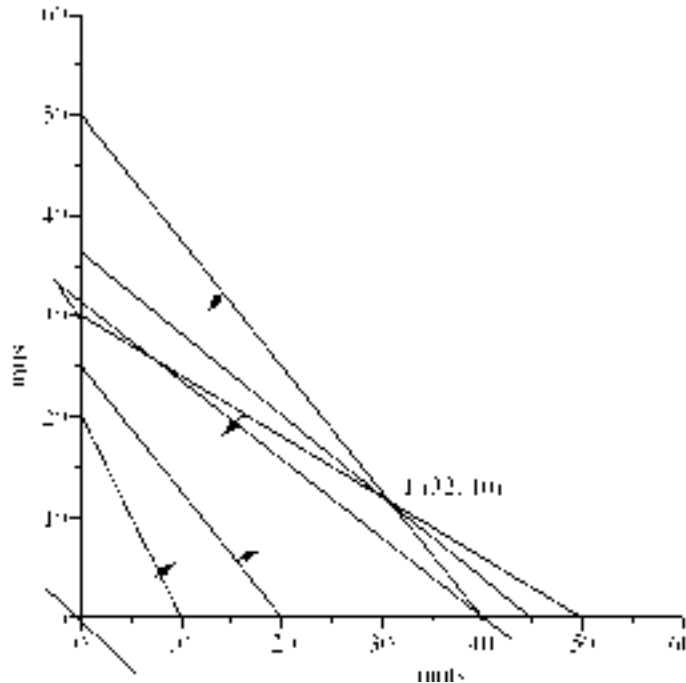
Figure 2.11. Graph for the problem 2.11
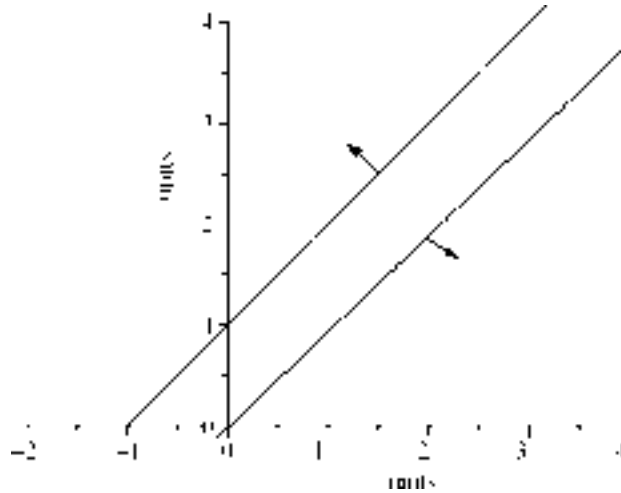
Figure 2.12. Graph for the problem 2.12

Problem 2.13. Solve the l.p.p. by graphical method.

Maximise $Z = 3a + 2b$ S.T.

$1a + 1b \leq 4$

$1a - 1b \leq 2$ and both a and b are ≥ 0.

Solution: The figure 2.13 shows the graph of the equations.

Equations are: Maximise $Z = 3a + 2b$ S.T.

$1a + 1b = 4$

$1a - 1b = 2$ and both a and b are ≥ 0.

In the figure the Isoprofit line passes through the point N (3,1). Hence **optimal Profit $Z = 3 \times 3 + 2 \times 1 =$ Rs.11 /-**
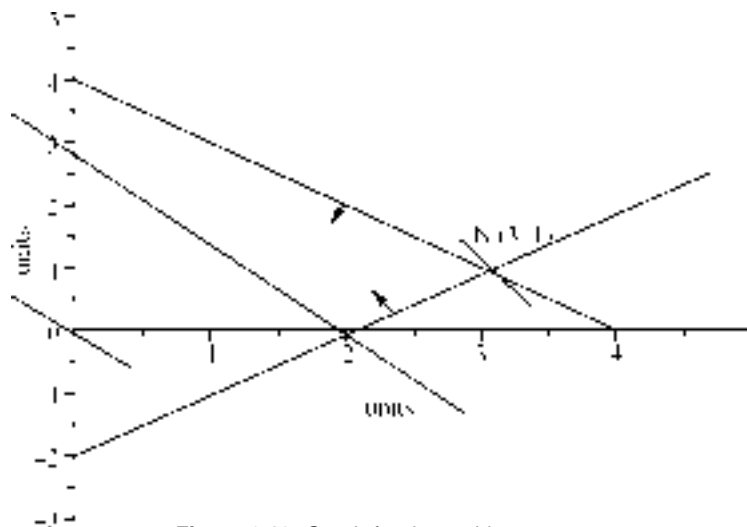
Figure 2.13. Graph for the problem 2.13

Problem 2.14: Formulate the l.p.p. and solve the below given problem graphically.

Old hens can be bought for Rs.2.00 each but young ones costs Rs. 5.00 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week. Each egg costs Rs. 0.30. A hen costs Rs.1.00 per week to feed. If the financial constraint is to spend Rs.80.00 per week for hens and the capacity constraint is that total number of hens cannot exceed 20 hens and the objective is to earn a profit more than Rs.6.00 per week, find the optimal combination of hens.

Solution: Let x be the number of old hens and y be the number of young hens to be bought. Now the old hens lay 3 eggs and the young one lays 5 eggs per week. Hence total number of eggs one get is $3x + 5y$.

Total revenues from the sale of eggs per week is Rs. 0.30 $(3x + 5y)$ *i.e.*, 0.90 x + 1.5 y

Now the total expenses per week for feeding hens is Re.1 $(1x + 1y)$ *i.e.*, $1x + 1y$.

Hence the net income = Revenue – Cost = $(0.90 x + 1.5 y) – (1 x + 1 y) = - 0.1 x + 0.5 y$ or $0.5y - 0.1 x$. Hence the desired l.p.p. is

Maximise $Z = 0.5 y - 0.1 \times$ S.T.

$2 x + 5 y \leq 80$

$1x + 1y \leq 20$ and both x and y are ≥ 0

The equations are:

Maximise $Z = 0.5 y - 0.1 \times$ S.T.

$2 x + 5 y = 80$

$1x + 1y = 20$ and both x and y are ≥ 0

In the figure 2.13, which shows the graph for the problem, the isoprofit line passes through the point C. Hence $Zc = Z (0,16) = $ Rs.8.00. Hence, one has to buy 16 young hens and his weekly profit will be Rs.8.00
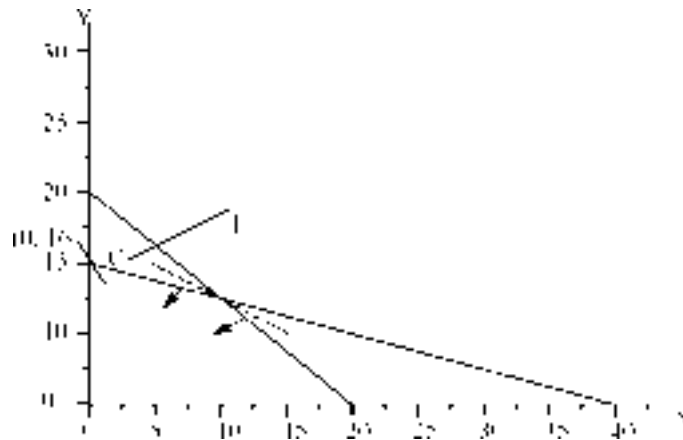
Figure 2.14. Graph for the problem 2.14

Point to Note: **In case in a graphical solution, after getting the optimal solution, one more constraint is added, we may come across following situations.**

(*i*) **The feasible area may reduce or increase and the optimal solution point may be shifted depending the shape of the polygon leading to decrease or increase in optimal value of the objective function.**

(*ii*) **Some times the new line for the new constraint may remain as redundant and imposes no extra restrictions on the feasible area and hence the optimal value will not change.**

(*iii*) **Depending on the position of line for the new constraint, there may not be any point in the feasible area and hence there may not be a solution. OR the isoprofit line may coincide with a line and the problem may have innumerable number of solutions.**

SUMMARY

1. The graphical method for solution is used when the problem deals with 2 variables.

2. The inequalities are assumed to be equations. As the problem deals with 2 variables, it is easy to write straight lines as the relationship between the variables and constraints are linear. In case the problem deals with three variables, then instead of lines we have to draw planes and it will become very difficult to visualize the feasible area.

3. If at all there is a feasible solution (feasible area of polygon) exists then, the feasible area region has an important property known as **convexity Property** in geometry. (**Convexity** means: Convex polygon consists of a set points having the property that the segment joining any two points in the set is *entirely* in the convex set. There is a mathematical theorem, which states, "The points which are simulations solutions of a system of inequalities of the $\leq$ type form a polygonal convex set".

   The regions will not have any holes in them, *i.e.*, they are solids and the boundary will not have any breaks. This can be clearly stated that **joining any two points in the region also lies in the region.**

4. The boundaries of the regions are lines or planes.

5. There will be corners or extreme points on the boundary and there will be edges joining the various corners. The closed figure is known as **polygon**.

6. The objective function of a maximisation is represented by assuming suitable outcome (revenue) and is known as **Isoprofit line**. In case of minimisation problem, assuming suitable cost, a line, known as **Isocost line**, represents the objective function.

7. If isoprofit or isocost line coincides with one corner, then the problem has **unique** solution. In case it coincides with more than one point, then the problem has **alternate** solutions. If

the isoprofit or isocost line coincides with a line, then the problem will have **innumerable number** of solutions.

8. The different situation was found when the objective function could be made arbitrarily large. Of course, no corner was optimal in that case.

QUESTIONS

1. An aviation fuel manufacturer sells two types of fuel A and B. Type A fuel is 25 % grade 1 gasoline, 25 % of grade 2 gasoline and 50 % of grade 3 gasoline. Type B fuel is 50 % of grade 2 gasoline and 50 % of grade 3 gasoline. Available for production are 500 liters per hour grade 1 and 200 liters per hour of grade 2 and grade 3 each. Costs are 60 paise per liter for grade 1, 120 paise for grade 2 and 100 paise for grade 3. Type A can be sold at Rs. 7.50 per liter and B can be sold at Rs. 9.00 per liter. How much of each fuel should be made and sold to maximise the profit.

2. A company manufactures two products X_1 and X_2 on three machines A, B, and C. X_1 require 1 hour on machine A and 1 hour on machine B and yields a revenue of Rs.3/-. Product X_2 requires 2 hours on machine A and 1 hour on machine B and 1 hour on machine C and yields revenue of Rs. 5/-. In the coming planning period the available time of three machines A, B, and C are 2000 hours, 1500 hours and 600 hours respectively. Find the optimal product mix.

3. Maximize $Z = 1\ x + 1\ y$ S.T.

   $1\ x + 2\ y \leq 2000$

   $1\ x + 1\ y \leq 1500$

   $0\ x + 1\ y \leq 600$ and both x and y are ≥ 0.

4. Maximise $Z = 8000a + 7000b$ S.T.

   $3\ a + 1\ b \leq 66$

   $1\ a + 1\ b \leq 45$

   $1\ a + 0\ b \leq 20$

   $0\ a + 1\ b \leq 40$ and both a and b are ≥ 0.

5. Minimise $Z = 1.5\ x + 2.5\ y$ S.T.

   $1\ x + 3\ y \geq 3$

   $1\ x + 6\ y \geq 2$ and both x and $y \geq 0$

6. Maximise $Z = 3\ a + 2\ b$ S.T.

   $1a - 1\ b \leq 1$

   $1\ a + 1b \geq 3$ and both x and y are ≥ 0

7. Maximise $Z = -3\ x + 2\ y$ S.T.

   $1\ x + 0\ y \leq 3$

   $1\ x - 1\ y \leq 0$ and both x and y are ≥ 0

Linear Programming Models (Resource Allocation Models)

8. Maximize $Z = -1\ a + 2b$ S.T.
   $-1\ a + 1\ b \le 1$
   $-1\ a + 2\ b \le 4$ and both a and b are ≥ 0.

9. Maximise $Z = 3\ x - 2\ y$ S.T.
   $1x + 1\ y \le 1$
   $2\ x + 2\ y \ge 4$ and both x and y are ≥ 0

10. Maximize $Z = 1x + 1\ y$ S.T.
    $1\ x - 1y \ge 0$
    $-3x + 1\ y \ge 3$ and both x and $y \ge 0$

Linear Programming Models:

(Solution by Simplex Method)
Resource Allocation Model – Maximisation Case

3.1. INTRODUCTION

As discussed earlier, there are many methods to solve the Linear Programming Problem, such as Graphical Method, Trial and Error method, Vector method and Simplex Method. Though we use graphical method for solution when we have two problem variables, the other method can be used when there are more than two decision variables in the problem. Among all the methods, **SIMPLEX METHOD** is most powerful method. It deals with iterative process, which consists of first designing a **Basic Feasible Solution** or a **Programme** and proceed towards the **OPTIMAL SOLUTION** and testing each feasible solution for **Optimality** to know whether the solution on hand is optimal or not. If not an optimal solution, redesign the programme, and test for optimality until the test confirms **OPTIMALITY.** Hence we can say that the Simplex Method depends on two concepts known as **Feasibility** and **optimality**.

The simplex method is based on the property that the optimal solution to a linear programming problem, if it exists, can always be found in one of the basic feasible solution. The simplex method is quite simple and mechanical in nature. The iterative steps of the simplex method are repeated until a finite optimal solution, if exists, is found. If no optimal solution, the method indicates that no finite solution exists.

3.2. COMPARISION BETWEEN GRAPHICAL AND SIMPLEX METHODS

1. The graphical method is used when we have two decision variables in the problem. Whereas in Simplex method, the problem may have any number of decision variables.

2. In graphical method, the inequalities are assumed to be equations, so as to enable to draw straight lines. But in Simplex method, the inequalities are converted into equations by:

   (*i*) Adding a **SLACK VARIABLE** in maximisation problem and subtracting a **SURPLUS VARIABLE** in case of minimisation problem.

3. In graphical solution the **Isoprofit** line moves away from the origin to towards the far off point in maximisation problem and in minimisation problem, the **Isocost** line moves from far off distance towards origin to reach the nearest point to origin.

4. In graphical method, the areas outside the feasible area (area covered by all the lines of constraints in the problem) indicates idle capacity of resource where as in Simplex method, the presence of slack variable indicates the idle capacity of the resources.

5. In graphical solution, if the isoprofit line coincides with more than one point of the feasible polygon, then the problem has second alternate solution. In case of Simplex method the net-evaluation row has zero for non-basis variable the problem has alternate solution. (If two alternative optimum solutions can be obtained, the infinite number of optimum, solutions can be obtained).

However, as discussed in the forth coming discussion, the beauty of the simplex method lies in the fact that the relative exchange profitabilities of all the non -basis variables (vectors) can be determined simultaneously and easily; the replacement process is such that the new basis does not violate the feasibility of the solution.

3.3. MAXIMISATION CASE

Problem 3.1: A factory manufactures two products A and B on three machines X, Y, and Z. Product A requires 10 hours of machine X and 5 hours of machine Y a one our of machine Z. The requirement of product B is 6 hours, 10 hours and 2 hours of machine X, Y and Z respectively. The profit contribution of products A and B are Rs. 23/– per unit and Rs. 32 /– per unit respectively. In the coming planning period the available capacity of machines X, Y and Z are 2500 hours, 2000 hours and 500 hours respectively. Find the optimal product mix for maximizing the profit.
Solution:

The given data is:

	Products		
Machines	A Hrs.	B Hrs.	Capacity in hours
X	10	6	2500
Y	5	10	2000
Z	1	2	500
Profit/unit Rs.	23	32	—

Let the company manufactures a units of A and b units of B. Then the inequalities of the constraints (machine capacities) are:

Maximise $Z = 23\,a + 32\,b$ S.T.————▶OBJECTIVE FUNCTION

$10\,a + 6\,b \le 2500$

$5\,a + 10\,b \le 2000$ ——————————▶ STRUCTURAL CONSTRAINTS.

$1\,a + 2\,b \le 500$

And both a and b are ≥ 0.————▶ NON-NEGATIVITY CONSTRAINT.

Now the above inequalities are to be converted into equations.

Take machine X: One unit of product A requires 10 hours of machine X and one unit of product B require 6 units. But company is manufacturing a units of A and b units of B, hence both put together must be **less than or equal to** 2,500 hours. Suppose a = 10 and b = 10 then the total consumption is $10 \times 10 + 6 \times 10 = 160$ hours. That is out of 2,500 hours, 160 hours are consumed, and 2,340 hours

are still remaining idle. So if we want to convert it into an equation then $100 + 60 + 2,340 = 2,500$. As we do not know the exact values of decision variables a and b how much to add to convert the inequality into an equation. For this we represent the idle capacity by means of a **SLACK VARIABLE** represented by **S**. Slack variable for first inequality is S_1, that of second one is S_2 and that of 'n'$^{\text{th}}$ inequality is S_n.

Regarding the objective function, if we sell one unit of A it will fetch the company Rs. 23/– per unit and that of B is Rs. 32/– per unit. If company does not manufacture A or B, all resources remain idle. Hence the profit will be Zero rupees. This clearly shows that the profit contribution of each hour of idle resource is zero. In Linear Programming language, we can say that the company has capacity of manufacturing 2,500 units of S_1, *i.e.*, S_1 is an **imaginary product**, which require one hour of machine X alone. Similarly, S_2 is an **imaginary product** requires one hour of machine Y alone and S_3 is an imaginary product, which requires one hour of machine Z alone. In simplex language S_1, S_2 and S_3 are idle resources. The profit earned by keeping all the machines idle is Rs.0/–. Hence the profit contributions of S_1, S_2 and S_3 are Rs.0/– per unit. By using this concept, the inequalities are converted into equations as shown below:

Maximise $Z = 23\ a + 32\ b + 0S_1 + 0S_2 + 0S_3$ S.T.

$$10\ a + 6\ b + 1S_1 = 2500$$
$$5\ a + 10\ b + 1S_2 = 2000$$
$$1\ a + 2\ b + 1S_3 = 500 \text{ and } a, b, S_1, S_2 \text{ and } S_3 \text{ all } \geq 0.$$

In Simplex version, all variables must be available in all equations. Hence the Simplex format of the model is:

Maximise $Z = 23\ a + 32\ b + 0S_1 + 0S_2 + 0S_3$ S.T.

$$10\ a + 6\ b + 1S_1 + 0S_2 + 0S_3 = 2500$$
$$5\ a + 6\ b + 0S_1 + 1S_2 + 0S_3 = 2000$$
$$1\ a + 2\ b + 0S_1 + 0S_2 + 1S_3 = 500 \text{ and } a, b, S_1, S_2 \text{ and } S_3 \text{ all } \geq 0.$$

The above data is entered in a table known as **simplex table (or tableau)**. There are many versions of table but in this book only one type is used.

In Graphical method, while finding the profit by Isoprofit line, we use to draw Isoprofit line at origin and use to move that line to reach the far off point from the origin. This is because starting from zero rupees profit; we want to move towards the maximum profit. Here also, first we start with zero rupees profit, *i.e.*, considering the slack variables as the basis variables (problem variables) in the initial programme and then improve the programme step by step until we get the optimal profit. Let us start the first programme or initial programme by rewriting the entries as shown in the above simplex table.

Linear Programming Models : Solution by Simplex Method

This column shows Basic or Problem variables.

This column shows objective co-efficients corresponding to basic variables in the programme

This column shows the values of the basic variables, the value of such non basic variable is $= 0$

This row shows C_j above each variable, the respective objective coefficient.

Variable row lists all the variable in the problem.

The numbers under non-basic variables represent substitution Ratios.

Every simplex tableau contains an identity Matrix under the basic variables.

Programme Variable or Basic variable	Profit per unit in Rs.	Quantity or Capacity	23 a	32 b	0 S_1	0 S_2	0 S_3
S_1	0	2500	10	6	1	0	0
S_2	0	2000	5	10	0	1	1
S_3	0	500	1	2	0	0	1
Z_j			0	0	0	0	0
Net Evaluation $C_j - Z_j$			23	32	0	0	0

The number in Z_j row under each column variable gives the total gross amount of **outgoing profit** when we consider the exchange between one unit of the column variable and the basic Variable.

The numbers in the net-evaluation row, under each column represent the **opportunity cost** of not having one unit of the respective column variables in the solution. In other words, the number represent the potential improvement in the objective function that will result by introducing into the programme one unit of the respective column variable.

Table: 1. Initial Programme

Solution: $a = 0$, $b = 0$, $S_1 = 2500$, $S_2 = 2000$ and $S_3 = 500$ and $Z =$ Rs. 0.00.

Programme (Basic variables)	Profit per unit In Rs. C_b	Quantity in Units.	C_j 23 a	32 b	0 S_1	0 S_2	S_3	Replacement Ratio.
S_1	0	2500	10	6	1	0	0	2500/6 = 416.7
S_2	0	2000	5	**10**	0	1	0	2000/10 = 200
S_3	0	500	1	2	0	0	1	500/2 = 250
Z_j			0	0	0	0	0	
$C_j - Z_j$ = Opportunity cost in Rs. Net evaluation row.			23	32	0	0	0	

The interpretation of the elements in the first table

1. In the first column, programme column, are the problem variables or basis variables that are included in the solution or the company is producing at the initial stage. These are S_1, S_2 and S_3, which are known as **basic variables.**

2. The second column, labeled as Profit per unit in Rupees shows the profit co-efficient of the basic variables *i.e.*, C_b. In this column we enter the profit co-efficient of the variables in the program. In table 1, we have S_1, S_2 and S_3 as the basic variables having Rs.0.00 as the profit and the same is entered in the programme.

3. In the quantity column, that is 3rd column, the values of the basic variables in the programme or solution *i.e.*, quantities of the units currently being produced are entered. In this table, S_1, S_2 and S_3 are being produced and the units being produced (available idle time) is entered *i.e.*, 2500, 2000 and 500 respectively for S_1, S_2 and S_3. The variables that are not present in this column are known as **non-basic variables**. The values of non-basis variables are zero; this is shown at the top of the table (solution row).

4. In any programme, the profit contribution, resulting from manufacturing the quantities of basic variables in the quantity column is **the sum of product of quantity column element and the profit column element.**

    In the present table the total profit is $Z = 2500 \times 0 + 2000 \times 0 + 500 \times 0 = $ Rs. 0.00.

5. The elements under column of non-basic variables, *i.e.*, *a* and *b* (or the main body of the matrix) are interpreted to mean **physical ratio of distribution** if the programme consists of only slack variables as the basic variables. Physical ratio of distribution means, at this stage, if company manufactures one unit of '*a*' then 10 units of S_1, 5 units of S_2 and 1 unit of S_3 **will be reduced or will go out or to be scarified**. By sacrificing the basic variables, the company will lose the profit to an extent the sum of product of quantity column element and the profit column element. At the same time it earns a profit to an extent of **product of profit co-efficient of incoming variable and the number in the quantity column against the just entered (in coming) variable.**

6. Coming to the entries in the identity matrix, the elements under the variables, S_1, S_2 and S_3 are unit vectors, hence we apply the principle of **physical ratio of distribution**, one unit of S_1 replaces one unit of S_1 and so on. Ultimately the profit is zero only. In fact while doing successive modifications in the programme towards getting optimal; solution, finally the unit matrix transfers to the main body. **This method is very much similar with G.J. method (Gauss Jordan) method in matrices, where we solve simultaneous equations by writing in the form of matrix. The only difference is that in G.J method, the values of variables may be negative, positive or zero. But in Simplex method as there is non-negativity constraint, the negative values for variables are not accepted.**

7. C_j at the top of the columns of all the variables represent the coefficients of the respective variables *I* the objective function.

8. The number in the Z_j row under each variable gives the total gross amount of outgoing profit when we consider the exchange between one unit of column, variable and the basic variables.

9. The number in the **net evaluation row, $C_j - Z_j$** row gives the **net effect** of exchange between **one unit** of each variable and **basic variables**. This they are zeros under columns of S_1, S_2 and S_3. **A point of interest to note here is the net evaluation element of any basis variable (or problem variable) is ZERO only. Suppose variable '*a*' becomes basis**

variable, the entry in net evaluation row under *'a'* is zero and so on. Generally the entry in net evaluation row is known as **OPPORTUNITY COST. Opportunity cost means for not including a particular profitable variable in the programme, the manufacturer has to lose the amount equivalent to the profit contribution of the variable**. In the present problem the net evaluation under the variable *'a'* is Rs. 23 per unit and that of *'b'* is Rs, 32 per unit. Hence the if the company does not manufacture *'a'* at this stage it has to face a penalty of Rs. 23/– for every unit of *'a'* for not manufacturing and the same of product variable *'b'* is Rs. 32/–. Hence the opportunity cost of product *'b'* is higher than that of *'a'*, hence *'b'* will be the incoming variable. **In general, select the variable, which is having higher opportunity cost as the incoming variable (or select the variable, which is having highest positive number in the net evaluation row**.

In this problem, variable *'b'* is having higher opportunity cost; hence it is the incoming variable. This should be marked by an arrow ($\uparrow$) at the bottom of the column and enclose the elements of the column in a rectangle this column is known as **KEY COLUMN**. The elements of the key column show the **substitution ratios**, *i.e.*, how many units of slack variable goes out when the variable enters the programme.

Divide the capacity column elements by key column numbers to get REPLACEMENT RATIO COLUMN ELEMENTS, which show that how much of variable *'b'* can be manufactured in each department, without violating the given constraints. Select the lowest replacement ratio and mark a tick ($\sqrt{}$) at the end of the row, which indicates **OUT GOING VARIABLE**. Enclose the elements of this column in a rectangle, which indicates **KEY ROW**, indicating out going variable. We have to select the lowest element because this is the **limiting ratio**, so that, that much of quantity of product can be manufactured on all machines or in all departments as the case may be. In the problem 200 units is the limiting ratio, which falls against S_2, *i.e.*, S_2 is the outgoing variable. This means that the entire capacity of machine *Y* is utilized. By manufacturing 200 units of *'b'*, $6 \times 200 = 1200$ hours of machine *X* is consumed and $2 \times 200 = 400$ hours of machine *Z* is consumed. Still 2500 – 1200 = 1300 hours of machine *X* and 500 – 400 = 100 units of machine *Z* remains idle. This is what exactly we see in graphical solution when two lines of the machines are superimposed.

The element at the intersection of key column and key row is known as **KEY NUMBER**. This is known as key number because with this number we have to get the next table.

For getting the **revised programme**, we have to transfer the rows of table 1 to table 2. To do this the following procedure is used.

Step 1: To Write the incoming variable *'b'* in place of out going variable S_2. Enter the profit of *'b'* in profit column. Do not alter S_1 and S_3. While doing so DO NOT ALTER THE POSITION OF THE ROWS.

Step 2: DIVIDING THE ELEMENTS OF OLD COLUMN BY KEY COLUMN ELEMENTS obtains capacity column elements.

Step 3: Transfer of key row: DIVIDE ALL ELEMENTS OF KEY ROW BY RESPECTIVE KEY COLUMN NUMBER.

Step 4: Transfer of Non-Key rows: NEW ROW NUMBER = (old row number – corresponding key row number) × fixed ratio.

Fixed ratio = Key column number of the row/key number.

Step 5: Elements of Net evaluation row are obtained by:

Objective row element at the top of the row – Σ key column element × profit column element.

Step 6: Select the highest positive element in net evaluation row or highest opportunity cost and mark the column by an arrow to indicate key column (incoming variable).

Step 7: Find the replacement ratios by dividing the capacity column element in the row by key column element of the same row and write the ratios in replacement ratio column. Select the limiting (lowest) ratio and mark with a tick mark to indicate key row (out going variable).

The element at the intersection of key column and key row is known as key number.

Continue these steps until we get:

(*i*) For maximisation problem all elements of net evaluation row must be either zeros or negative elements.

(*ii*) For Minimisation problem, the elements of net evaluation row must be either zeros or positive elements.

Table: 2.

Solution: $S_1 = 1,300$, $S_2 = 0$, $S_3 = 100$, $a = 0$, $b = 200$, $Z = 32 \times 200 = $ Rs. 6400.

Problem variable.	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement Ratio (R.R)
S_1	0	1,300	**7**	0	1	-0.6	0	1300/7 = 185.7
b	32	200	0.5	1	0	0.10	0	400
S_3	0	100	0	0	0	-0.5	1	--
Z_j			16	32	0	3.2	0	
C_j- Z_j	= net evaluation		7	0	0	- 3.2	0	

1. Transfer of Key row: 2000/10 , 5 /10, 10 /10, 0 /10, 1 / 10, 0 / 10
2. Transfer of Non key rows:

Rule: (Old row Number – corresponding key row number) – key column number / key number = new row no.

1st row.
2500 – 2000 × 6/10 = 1300
10 – 10 × 6/10 = 0
6 – 10 × 6/10 = 0
1 – 0 × 6/10 = 1
0 – 1 × 6/10 = – 0.6
0 – 0 × 6/10 = 0

2nd row:
500 – 2000 × 2/10 = 100
1 – 5 × 2/10 = 0
2 – 10 × 2/10 = 0
0 – 0 × 2/10 = 0
0 – 1 × 2/10 = – 0.2
1 – 0 × 2 / 10 = 1

Replacement ratios: 1300/7 = 185.7, 200/0.5 = 400, 100/0 = Infinity.

Net evaluation row elements =

Column under 'a' = 23 − (7 × 0 + 0.5 × 32 + 0 × 0) = 23 − 16 = 7

'b' = 32 − (0 × 0 + 1 × 32 + 0 × 0) = 32 − 32 = 0

S_1 = 0 − (1 × 0 + 0 × 32 + 0 × 0) = 0

S_2 = 0 − (− 0.6 × 0 + 0.1 × 32 + −0.2 × 0) = − 3.2

S_3 = 0 − (0 × 0 + 0 × 32 + 1 × 0) = 0

In the above table, the net evaluation under S_2 is − **3.2.** This resource is completely utilized to manufacture product B. The profit earned by manufacturing B is Rs. 6400/−. As per the law of economics, the **worth of resources** used must be equal to the profit earned. Hence the element **3.2** (ignore negative sign) is known as **economic worth or artificial accounting price (technically it can be taken as MACHINE HOUR RATE) of the resources or shadow price of the resource. (In fact all the elements of reevaluation row under slack variables are shadow prices of respective resources). This concept is used to check whether the problem is done correctly or not.   To do this MULTIPLY THE ELEMENTS IN NET EVALUATION ROW UNDER SLACK VARIABLES WITH THE ORIGINAL CAPACITY CONSTRAINTS GIVEN IN THE PROBLEM AND FIND THE SUM OF THE SAME. THIS SUM MUST BE EQUAL TO THE PROFIT EARNED BY MANUFACTRUING THE PRODUCT.**

Σ Shadow prices of resources used must be equal to the profit earned.

Table: 3.

Problem variable	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement ratio
a	23	185.7	1	0	0.143	− 0.086	0	
b	32	107.14	0	1	− 0.07	0.143	0	
S_3	0	100	0	0	0	− 0.02	1	
Z_j			23	32	1	2.6	0	
$C_j − Z_j$	Net evaluation.		0	0	−1.0	− 2.6	0	

Transfer of key row: 1300/7 = 185.7, 7/7 = 1, 0/7 = 0, 1/7 = 0.143, −3 /5 = − 0.086 0/7 = 0

Row No. 2   Row No.3

200 − 1300 × 1/14 = 107.14   As the fixed ratio will be zero for this row

0.5 − 7 × 1/14 = 0   the row elements will not change.

1 − 0 × 1/14 = 1

0 − 1 × 1/14 = − 0.07

0.1 − (− 0.6) × 1/14 = 0.143

0 − 0 × 1/14 = 0

Net evaluation row elements:

For 'a' = 23 − 1 × 23 + 0 × 32 + 0 × 0 = 0

For 'b' = 32 − 0 × 23 + 1 × 32 +0 × 0 = 0

For S_1 = 0 − 0.143 × 23 + (− 0.07 × 32) + 0 × 0 = −1

For S_2 = 0 − (−0.086 × 23) + 0.143 × 32 + (− 0.02 × 0) = − 2.6

For $S_3 = 0 - 0 \times 23 + 0 \times 32 + 1 \times 0 = 0$

Profit $Z = 185.7 \times 23 + 107.14 \times 32 = $ Rs. 7,700

Shadow price $= 1 \times 2500 + 2.6 \times 2000 = $ Rs. $2500 + 5200 = $ Rs. 7700/–

As all the elements of net evaluation row are either negative elements or zeros, the solution is optimal.

Also the profit earned is equal to the shadow price.

The answer is the company has to manufacture:

185.7 units of *A* and 107.14 units of *B* and the optimal return is *Z* = Rs. 7,700/–

The interpretation of the elements in the first table

1. In the first column, programme column, are the problem variables or basis variables that are included in the solution or the company is producing at the initial stage. These are S_1, S_2 and S_3, which are known as **basic variables.**

2. The second column, labeled as Profit per unit in Rupees shows the profit co-efficient of the basic variables *i.e.*, C_b. In this column we enter the profit co-efficient of the variables in the program. In table 1, we have S_1, S_2 and S_3 as the basic variables having Rs.0.00 as the profit and the same is entered in the programme.

3. In the quantity column, that is 3rd column, the values of the basic variables in the programme or solution *i.e.*, quantities of the units currently being produced are entered. In this table, S_1, S_2 and S_3 are being produced and the units being produced (available idle time) is entered *i.e.*, 2500, 2000 and 500 respectively for S_1, S_2 and S_3. The variables that are not present in this column are known as **non-basic variables**. The values of non-basis variables are zero; this is shown at the top of the table (solution row).

4. In any programme, the profit contribution, resulting from manufacturing the quantities of basic variables in the quantity column is **the sum of product of quantity column element and the profit column element.**

   In the present table the total profit is $Z = 2500 \times 0 + 2000 \times 0 + 500 \times 0 = $ Rs. 0.00.

5. The elements under column of non-basic variables, *i.e.*, *a* and *b* (or the main body of the matrix) are interpreted to mean **physical ratio of distribution** if the programme consists of only slack variables as the basic variables. Physical ratio of distribution means, at this stage, if company manufactures one unit of '*a*' then 10 units of S_1, 5 units of S_2 and 1 unit of S_3 **will be reduced or will go out or to be scarified**. By sacrificing the basic variables, the company will lose the profit to an extent the sum of product of quantity column element and the profit column element. At the same time it earns a profit to an extent of **product of profit co-efficient of incoming variable and the number in the quantity column against the just entered (in coming) variable.**

6. Coming to the entries in the identity matrix, the elements under the variables, S_1, S_2 and S_3 are unit vectors, hence we apply the principle of **physical ratio of distribution**, one unit of S_1 replaces one unit of S_1 and so on. Ultimately the profit is zero only. In fact while doing successive modifications in the programme towards getting optimal; solution, finally the unit matrix transfers to the main body. **This method is very much similar with G.J. method (Gauss Jordan) method in matrices, where we solve simultaneous equations by writing in the form of matrix. The only difference is that in G.J method, the values of variables may be negative, positive or zero. But in Simplex method as there is non-negativity constraint, the negative values for variables are not accepted.**

7. C_j at the top of the columns of all the variables represent the coefficients of the respective variables *I* the objective function.

8. The number in the Z_j row under each variable gives the total gross amount of outgoing profit when we consider the exchange between one unit of column, variable and the basic variables.

9. The number in the **net evaluation row, $C_j - Z_j$** row gives the **net effect** of exchange between **one unit** of each variable and **basic variables**. This they are zeros under columns of S_1, S_2 and S_3. **A point of interest to note here is the net evaluation element of any basis variable (or problem variable) is ZERO only. Suppose variable '*a*' becomes basis**

variable, the entry in net evaluation row under *'a'* is zero and so on. Generally the entry in net evaluation row is known as **OPPORTUNITY COST. Opportunity cost means for not including a particular profitable variable in the programme, the manufacturer has to lose the amount equivalent to the profit contribution of the variable**. In the present problem the net evaluation under the variable *'a'* is Rs. 23 per unit and that of *'b'* is Rs, 32 per unit. Hence the if the company does not manufacture *'a'* at this stage it has to face a penalty of Rs. 23/– for every unit of *'a'* for not manufacturing and the same of product variable *'b'* is Rs. 32/–. Hence the opportunity cost of product *'b'* is higher than that of *'a'*, hence *'b'* will be the incoming variable. **In general, select the variable, which is having higher opportunity cost as the incoming variable (or select the variable, which is having highest positive number in the net evaluation row**.

In this problem, variable *'b'* is having higher opportunity cost; hence it is the incoming variable. This should be marked by an arrow ($\uparrow$) at the bottom of the column and enclose the elements of the column in a rectangle this column is known as **KEY COLUMN**. The elements of the key column show the **substitution ratios**, *i.e.*, how many units of slack variable goes out when the variable enters the programme.

Divide the capacity column elements by key column numbers to get REPLACEMENT RATIO COLUMN ELEMENTS, which show that how much of variable *'b'* can be manufactured in each department, without violating the given constraints. Select the lowest replacement ratio and mark a tick ($\sqrt{}$) at the end of the row, which indicates **OUT GOING VARIABLE**. Enclose the elements of this column in a rectangle, which indicates **KEY ROW**, indicating out going variable. We have to select the lowest element because this is the **limiting ratio**, so that, that much of quantity of product can be manufactured on all machines or in all departments as the case may be. In the problem 200 units is the limiting ratio, which falls against S_2, *i.e.*, S_2 is the outgoing variable. This means that the entire capacity of machine *Y* is utilized. By manufacturing 200 units of *'b'*, $6 \times 200 = 1200$ hours of machine *X* is consumed and $2 \times 200 = 400$ hours of machine *Z* is consumed. Still $2500 – 1200 = 1300$ hours of machine *X* and $500 – 400 = 100$ units of machine *Z* remains idle. This is what exactly we see in graphical solution when two lines of the machines are superimposed.

The element at the intersection of key column and key row is known as **KEY NUMBER**. This is known as key number because with this number we have to get the next table.

For getting the **revised programme**, we have to transfer the rows of table 1 to table 2. To do this the following procedure is used.

Step 1: To Write the incoming variable *'b'* in place of out going variable S_2. Enter the profit of *'b'* in profit column. Do not alter S_1 and S_3. While doing so DO NOT ALTER THE POSITION OF THE ROWS.

Step 2: DIVIDING THE ELEMENTS OF OLD COLUMN BY KEY COLUMN ELEMENTS obtains capacity column elements.

Step 3: Transfer of key row: DIVIDE ALL ELEMENTS OF KEY ROW BY RESPECTIVE KEY COLUMN NUMBER.

Step 4: Transfer of Non-Key rows: NEW ROW NUMBER = (old row number – corresponding key row number) × fixed ratio.
Fixed ratio = Key column number of the row/key number.

Step 5: Elements of Net evaluation row are obtained by:
Objective row element at the top of the row – Σ key column element × profit column element.

Step 6: Select the highest positive element in net evaluation row or highest opportunity cost and mark the column by an arrow to indicate key column (incoming variable).

Step 7: Find the replacement ratios by dividing the capacity column element in the row by key column element of the same row and write the ratios in replacement ratio column. Select the limiting (lowest) ratio and mark with a tick mark to indicate key row (out going variable).

The element at the intersection of key column and key row is known as key number.

Continue these steps until we get:

  (*i*) For maximisation problem all elements of net evaluation row must be either zeros or negative elements.

  (*ii*) For Minimisation problem, the elements of net evaluation row must be either zeros or positive elements.

Table: 2.

Solution: $S_1 = 1,300$, $S_2 = 0$, $S_3 = 100$, $a = 0$, $b = 200$, $Z = 32 \times 200 =$ Rs. 6400.

Problem variable.	*Profit in Rs.*	*Capacity*	*C_j 23 a*	*32 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement Ratio (R.R)*
S_1	0	1,300	**7**	0	1	-0.6	0	1300/7 = 185.7
b	32	200	0.5	1	0	0.10	0	400
S_3	0	100	0	0	0	-0.5	1	--
Z_j			16	32	0	3.2	0	
C_j- Z_j	= net evaluation		7	0	0	- 3.2	0	

1. Transfer of Key row: 2000/10 , 5 /10, 10 /10, 0 /10, 1 / 10, 0 / 10
2. Transfer of Non key rows:

Rule: (Old row Number – corresponding key row number) – key column number / key number = new row no.

1st row. $2500 – 2000 \times 6/10 = 1300$ 2nd row: $500 – 2000 \times 2/10 = 100$
$10 – 10 \times 6/10 = 0$
$6 – 10 \times 6/10 = 0$ $1 – 5 \times 2/10 = 0$
$1 – 0 \times 6/10 = 1$ $2 – 10 \times 2/10 = 0$
$0 – 1 \times 6/10 = – 0.6$ $0 – 0 \times 2/10 = 0$
$0 – 0 \times 6/10 = 0$ $0 – 1 \times 2/10 = – 0.2$
                                                      $1 – 0 \times 2 / 10 = 1$

Replacement ratios: 1300/7 = 185.7, 200/0.5 = 400, 100/0 = Infinity.

Net evaluation row elements =

Column under 'a' = $23 - (7 \times 0 + 0.5 \times 32 + 0 \times 0) = 23 - 16 = 7$

$\qquad$ 'b' = $32 - (0 \times 0 + 1 \times 32 + 0 \times 0) = 32 - 32 = 0$

$\qquad$ $S_1 = 0 - (1 \times 0 + 0 \times 32 + 0 \times 0) = 0$

$\qquad$ $S_2 = 0 - (-0.6 \times 0 + 0.1 \times 32 + -0.2 \times 0) = -3.2$

$\qquad$ $S_3 = 0 - (0 \times 0 + 0 \times 32 + 1 \times 0) = 0$

In the above table, the net evaluation under S_2 is **– 3.2.** This resource is completely utilized to manufacture product B. The profit earned by manufacturing B is Rs. 6400/–. As per the law of economics, the **worth of resources** used must be equal to the profit earned. Hence the element **3.2** (ignore negative sign) is known as **economic worth or artificial accounting price (technically it can be taken as MACHINE HOUR RATE) of the resources or shadow price of the resource. (In fact all the elements of reevaluation row under slack variables are shadow prices of respective resources). This concept is used to check whether the problem is done correctly or not. To do this MULTIPLY THE ELEMENTS IN NET EVALUATION ROW UNDER SLACK VARIABLES WITH THE ORIGINAL CAPACITY CONSTRAINTS GIVEN IN THE PROBLEM AND FIND THE SUM OF THE SAME. THIS SUM MUST BE EQUAL TO THE PROFIT EARNED BY MANUFACTRUING THE PRODUCT.**

$\qquad$ **Σ Shadow prices of resources used must be equal to the profit earned.**

Table: 3.

Problem variable	Profit in Rs.	Capacity	C_j 23 a	32 b	0 S_1	0 S_2	0 S_3	Replacement ratio
a	23	185.7	1	0	0.143	– 0.086	0	
b	32	107.14	0	1	– 0.07	0.143	0	
S_3	0	100	0	0	0	– 0.02	1	
Z_j			23	32	1	2.6	0	
$C_j - Z_j$	Net evaluation.		0	0	–1.0	– 2.6	0	

Transfer of key row: $1300/7 = 185.7$, $7/7 = 1$, $0/7 = 0$, $1/7 = 0.143$, $-3/5 = -0.086$ $0/7 = 0$

Row No. 2 $\qquad\qquad\qquad\qquad\qquad$ Row No.3

$200 - 1300 \times 1/14 = 107.14$ $\qquad\qquad$ As the fixed ratio will be zero for this row

$0.5 - 7 \times 1/14 = 0$ $\qquad\qquad\qquad\quad$ the row elements will not change.

$1 - 0 \times 1/14 = 1$

$0 - 1 \times 1/14 = -0.07$

$0.1 - (-0.6) \times 1/14 = 0.143$

$0 - 0 \times 1/14 = 0$

Net evaluation row elements:

For 'a' = $23 - 1 \times 23 + 0 \times 32 + 0 \times 0 = 0$

For 'b' = $32 - 0 \times 23 + 1 \times 32 + 0 \times 0 = 0$

For $S_1 = 0 - 0.143 \times 23 + (-0.07 \times 32) + 0 \times 0 = -1$

For $S_2 = 0 - (-0.086 \times 23) + 0.143 \times 32 + (-0.02 \times 0) = -2.6$

For $S_3 = 0 - 0 \times 23 + 0 \times 32 + 1 \times 0 = 0$

Profit $Z = 185.7 \times 23 + 107.14 \times 32 =$ Rs. 7,700

Shadow price $= 1 \times 2500 + 2.6 \times 2000 =$ Rs. $2500 + 5200 =$ Rs. 7700/–

As all the elements of net evaluation row are either negative elements or zeros, the solution is optimal.

Also the profit earned is equal to the shadow price.

The answer is the company has to manufacture:

185.7 units of *A* and 107.14 units of *B* and the optimal return is *Z* = Rs. 7,700/–

Simplex method

Simplex method is the method to solve (LPP) models which contain two or more decision variables.

Basic variables:

Are the variables which coefficients <u>One</u> in the equations and <u>Zero</u> in the other equations.

Non-Basic variables:

Are the variables which coefficients are taking any of the values, whether positive or negative or zero.

Slack, surplus & artificial variables:

a) If the inequality be $\leq$ (less than or equal, then we add a slack variable **+ S** to change $\leq$ to =.
b) If the inequality be $\geq$ (greater than or equal, then we subtract a surplus variable **- S** to change $\geq$ to =.
c) If we have = we use artificial variables.

The steps of the simplex method:

Step 1:
Determine a starting basic feasible solution.
Step 2:
Select an entering variable using the optimality condition. Stop if there is no entering variable.
Step 3:
Select a leaving variable using the feasibility condition.

Optimality condition:

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the Z-row.

The optimum is reached at the iteration where all the Z-row coefficient of the non-basic variables are non-negative (non-positive).

Feasibility condition:

For both maximization and minimization problems the leaving variable is the basic associated with the smallest non-negative ratio (with strictly positive denominator).

Pivot row:

a) Replace the leaving variable in the basic column with the entering variable.

b) New pivot row equal to current pivot row divided by pivot element.

c) All other rows:

New row=current row - (pivot column coefficient) *new pivot row.

Example 1:

Use the simplex method to solve the (LP) model:

$$max\ Z = 5x_1 + 4x_2$$

Subject to

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Solution:

$$max\ Z - 5x_1 + 4x_2 = 0$$

Subject to

$$6x_1 + 4x_2 + S_1 = 24$$

$$x_1 + 2x_2 + S_2 = 6$$

$$-x_1 + x_2 + S_3 = 1$$

$$x_2 + S_4 = 2$$

Table 1:

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
S_1	6	4	1	0	0	0	24
S_2	1	2	0	1	0	0	6
S_3	-1	1	0	0	1	0	1
S_4	0	1	0	0	0	1	2
Max Z	-5	-4	0	0	0	0	0

$\frac{24}{6} = 4$

$\frac{6}{1} = 6$

$\frac{1}{-1} = -1$ (ignore)

$\frac{2}{0} = \infty$ (ignore)

The entering variable is x_1 and S_1 is a leaving variable.

Table 2:

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
x_1	1	2/3	1/6	0	0	0	4
S_2	0	4/3	-1/6	1	0	0	2
S_3	0	5/3	1/6	0	1	0	5
S_4	0	1	0	0	0	1	2
Max Z	0	-2/3	5/6	0	0	0	20

■ Pivot row or new x_1-row$=\frac{1}{6}$ [current S_1 −row]

$$= \frac{1}{6}[6 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0 \quad 24]$$

$$= [1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4]$$

- **New S_2-row=[current S_2 −row]-(1)[new x_1 −row]**

   =[1 2 0 1 0 0 6]- (1)[1 2/3 1/6 0 0 0 0 4]

   =[0 4/3 -1/6 1 0 0 2]

- **New S_3-row=[current S_3 −row]-(1)[new x_1 −row]**

   =[-1 1 0 0 1 0 1]- (1)[1 2/3 1/6 0 0 0 0 4]

   =[0 5/3 1/6 0 1 0 5]

- **New S_4-row=[current S_4 −row]-(0)[new x_1 −row]**

   =[0 1 0 0 0 1 2]- (0)[1 2/3 1/6 0 0 0 0 4]

   =[0 1 0 0 0 1 2]

- **New Z-row=[current Z −row]-(-5)[new x_1 −row]**

   =[-5 -4 0 0 0 0 0]-(-5)[1 2/3 1/6 0 0 0 0 4]

   =[0 -2/3 5/6 0 0 0 20]

Now:

$$\frac{4}{\frac{2}{3}} = 6$$

$$\frac{2}{\frac{4}{3}} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

$$\frac{5}{\frac{5}{3}} = 3$$

$$\frac{2}{1} = 2$$

The entering variable is x_2 and S_2 is a leaving variable.

Table 3: (optimal solution):

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
S_3	0	0	3/8	-5/4	1	0	5/2
S_4	0	0	1/8	-3/4	0	1	1/2
Max Z	0	0	5/6	1/2	0	0	21

■ **Pivot row or new x_2-row=$\dfrac{1}{\frac{4}{3}}$ [current S_2 –row]**

$=\dfrac{1}{\frac{4}{3}}$[0   4/3   -1/6   1   0   0   2]

$=$[0   1   -1/8   ¾   0   0   3/2]

- **New x_1-row=[current x_1 –row]-(2/3)[new x_2 –row]**

=[1   2/3   1/6   0   0   0   4]- (2/3)[0   1   -1/8   ¾   0   0   3/2]

=[1   0   ¼   -1/2   0   0   3]

- **New S_3-row=[current S_3 –row]-(5/2)[new x_2 –row]**

=[0   5/3   1/6   0   1   0   5]-(5/3)[0   1   -1/8   ¾   0   0   3/2]

=[0   0   3/8   -5/4   1   0   5/3]

- **New S_4-row=[current S_4 –row]-(1)[new x_2 –row]**

=[0   1   0   0   0   1   2]-(1)[0   1   -1/8   ¾   0   0   3/2]

=[0   0   1/8   -3/4   0   1   ½]

New Z-row=[current Z –row]-(-2/3)[new x_2 –row]

=[0   -2/3   5/6   0   0   0   20]-(-2/3)[0   1   -1/8   ¾   0   0 3/2]

=[0   0   ¾   ½   0   0   21]

Then the solution is:

$$x_1 = 3 \ \& \ x_2 = \frac{3}{2} \ \& \ S_3 = \frac{5}{2} \ \& \ S_4 = \frac{1}{2}$$
$$S_1 = 0, \ S_2 = 0$$

Example 2:

Use the simplex method to solve the (LP) model:
$$max \ Z = 2x_1 + 3x_2$$

Subject to

$$0.25x_1 + 0.5x_2 \le 40$$

$$0.4x_1 + 0.2x_2 \quad \le 40$$

$$0.8x_2 \quad \le 40$$

$$x_1, x_2 \quad \ge 0$$

Solution:

$$max \ Z - 2x_1 + 3x_2 = 0$$

Subject to

$$0.25x_1 + 0.5x_2 + S_1 = 40$$

$$0.4x_1 + 0.2x_2 + S_2 = 40$$

$$0.8x_2 + S_3 = 40$$

$$x_1, x_2, +S_1, +S_2, +S_3 \ \ge 0$$

Table 1:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	0.25	0.5	1	0	0	40
S_2	0.4	0.2	0	1	0	40
S_3	0	0.8	0	0	1	40
Max Z	-2	-3	0	0	0	0

$$\frac{40}{0.5} = 80$$

$$\frac{40}{0.2} = 200$$

$$\frac{40}{0.8} = \boxed{50}$$

- ■ **Pivot row or new S_3-row** $= \frac{1}{0.8}$ [0   0.8   0   0   1   40]

  = [0   1   0   0   1.25   50]

New S_1-row = [current S_1 −row]-(0.5)[new x_2 −row]

= [0.25   0.5   1   0   0   40]-(0.5)[0   1   0   0   1.25   50]

= [0   0.5   0   0   -0.625   15]

New S_2-row = [current S_2 −row]-(0.2)[new x_2 −row]

= [0.4   0.2   0   1   0   40]-(0.2)[0   1   0   0   1.25   50]

[0.4   0   0   1   -0.25   30]

New Z-row = [current Z −row]-(-3)[new x_2 −row]

= [-2   -3   0   0   0   0]-(-3)[0   1   0   0   1.25   50]

= [-2   0   0   0   3.75   150]

Table 2:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	0.25	0	1	0	-0.625	15
S_2	0.4	0	0	1	-0.25	30
x_2	0	1	0	0	1.25	50
Max Z	-2	0	0	0	3.75	150

$$\frac{15}{0.25} = 60$$

$$\frac{30}{0.4} = 75$$

$$\frac{50}{0} = \infty \quad \text{(ignore)}$$

Pivot row or new S_1-row=$\frac{1}{0.25}$ [0.25 0 1 0 -0.625 15]

=[1 0 4 0 -2.5 60]

New S_2-row=[current S_2 –row]-(0.4)[new x_1 –row]

=[0.4 0 0 0 -0.25 30]-(0.4)[1 0 4 0 -2. 5 60]

[0 0 -1.6 0 -0.75 6]

New x_2-row=[0 1 0 0 1.25 50]-(0)[1 0 4 0 -2. 5 60]

=[0 1 0 0 1.25 50]

New Z-row=[current Z –row]-(-2)[1 0 4 0 -2.5 60]

=[-2 0 0 0 3.75 150]-(-2)[1 0 4 0 -2. 5 60]

[0 0 8 0 -1.25 270]

Table 3:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
x_1	1	0	4	0	-2.5	60
S_2	0	0	-1.6	1	0.75	6
x_2	0	1	0	0	1.25	50
Max Z	0	0	8	0	-1.25	270

$$\frac{60}{-2.5} = -24 \text{ (ignore)}$$

$$\frac{6}{0.75} = 8$$

$$\frac{50}{1.25} = 40$$

New S_2-row=$\frac{1}{0.75}$ =[current S_2-row] =$\frac{1}{0.75}$ [0 0 -1.6 0 0.75 6]

$\qquad$ =[0 0 -2.133 0 1 8]

New x_1-row= [1 0 4 0 -2.5 60]-(-2.5)[0 0 -2.133 0 1 8]

$\qquad$ =[1 0 -1.333 0 0 80]

New x_2-row= [0 1 0 0 1.25 50]-(-1.25)[0 0 -2.133 0 1 8]

$\qquad$ =[0 1 -2.76 0 0 40]

New Z-row= [0 0 8 0 -1.25 270]-(-2.5)[0 0 -2.133 0 1 8]

$\qquad$ =[0 0 5.33 0 0 280]

Table 3: (optimal solution):

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
x_1	1	0	-1.333	0	0	80
S_3	0	0	-2.133	0	1	8
x_2	0	1	-2.67	0	0	40
Max Z	0	0	5.33	0	0	280

The optimal solution :

x_1=80   , $x_2 = 40$, $S_1 = 0$ & $S_2 = 0$ // Z=280

Example 3:

Use the simplex method to solve the (LP) model:
$$min \ Z = -6x_1 - 10x_2 - 4x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 1000$$

$$x_1 + x_2 \qquad \leq 500$$

$$x_1 + 2x_2 \qquad \leq 700$$

$$x_1, x_2, x_3 \qquad \geq 0$$

Solution:

$$min \ Z + 6x_1 + 10x_2 + 4x_3 = 0$$

Subject to

$$x_1 + x_2 + x_3 + S_1 = 1000$$

$$x_1 + x_2 \quad + S_2 \qquad = 500$$

$$x_1 + 2x_2 \quad + S_3 \quad = 700$$

$$x_1, x_2, x_3 , S_1, S_2 \ S_3 \ \geq 0$$

Table 1:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
S_1	1	1	1	1	0	0	1000
S_2	1	1	0	0	1	0	500
S_3	1	2	0	0	1	1	700
Max Z	6	10	4	0	0	0	0

$\frac{1000}{1} = 1000$

$\frac{500}{1} = 500$

$\frac{700}{2} = \boxed{350}$

New S_3-row or x_2-row $= \frac{1}{2}$ [1 2 0 0 0 1   700]

$= [\frac{1}{2} \ 1 \ 0 \ 0 \ 0 \ \frac{1}{2} \ 350]$

New S_1-row = [1 1 1 1 0 0   1000]-(1)[$\frac{1}{2}$ 1 0 0 0 $\frac{1}{2}$ 350]

$= [\frac{1}{2} \ 0 \ 1 \ 1 \ 0 \ -\frac{1}{2} \ 650]$

New S_2-row = [1 1 0 0 1 0   500]-(1)[$\frac{1}{2}$ 1 0 0 0 $\frac{1}{2}$ 350]

$= [\frac{1}{2} \ 0 \ 0 \ 0 \ 1 \ -\frac{1}{2} \ 150]$

New Z-row = $[6\ \ 10\ \ 4\ \ 0\ \ 0\ \ 0\ \ \ 0]-(10)[\frac{1}{2}\ \ 1\ 0\ \ 0\ \ 0\ \ \frac{1}{2}\ \ \ 350]$

$\qquad\qquad\qquad =[1\ \ \ 0\ \ \ 4\ \ \ 0\ \ \ 0\ \ -5\ \ \ \ -3500]$

Table 2:

↓

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
S_1	1/2	0	①1	1	0	-1/2	650
S_2	1/2	0	0	0	1	-1/2	150
x_2	1/2	1	0	0	0	1/2	350
Max Z	1	0	④4	0	0	-5	-3500

$\boxed{\dfrac{650}{1} = \boxed{650}}$

$\dfrac{150}{0} = \infty$ **(ignore)**

$\dfrac{350}{0} = \infty$ **(ignore)**

New S_1-row or x_3-row $=1[\frac{1}{2}\ \ 0\ \ 1\ \ \ 1\ \ 0\ \ -\frac{1}{2}\ \ \ 650]$

$\qquad\qquad\qquad\qquad =[\frac{1}{2}\ \ 0\ \ 1\ \ \ 1\ \ 0\ \ -\frac{1}{2}\ \ \ 650]$

New S_2-row $= [\frac{1}{2}\ \ 0\ \ 0\ \ 0\ \ 1\ -\frac{1}{2}\ \ \ 150]-(0)[\frac{1}{2}\ \ 0\ \ 1\ \ \ 1\ \ 0\ \ -\frac{1}{2}\ \ \ 650]$

$\qquad\qquad\quad =[\frac{1}{2}\ \ 0\ \ 0\ \ 0\ \ 1\ -\frac{1}{2}\ \ \ 150]$

New x_2-row $= [\frac{1}{2}\ \ 1\ \ 0\ \ 0\ \ 0\ \ \ \frac{1}{2}\ \ \ 350]-(0)[\frac{1}{2}\ \ 0\ \ 1\ \ \ 1\ \ 0\ \ -\frac{1}{2}\ \ \ 650]$

$\qquad\qquad\quad =[\frac{1}{2}\ \ 1\ \ 0\ \ 0\ \ 0\ \ \ \frac{1}{2}\ \ \ 350]$

New Z-row $= [1\ \ 0\ \ 4\ \ 0\ \ 0\ \ -5\ \ \ -3500]-(4)[\frac{1}{2}\ \ 0\ \ 1\ \ \ 1\ \ 0\ \ -\frac{1}{2}\ \ \ 650]$

$\qquad\qquad\quad =[-1\ \ 0\ \ 0\ \ -4\ \ 0\ \ \ -3\ \ \ -6100]$

Table 3: (optimal solution):

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
x_3	1/2	0	1	1	0	-1/2	650
S_2	1/2	0	0	0	1	-1/2	150
x_2	1/2	1	0	0	0	1/2	350
Max Z	-1	0	0	-4	0	-3	-6100

The optimal solution :

x_3=650 , $x_2 = 350$, $S_1 = 0$ S_3 & $S_2 = 150$ $x_1 = 0$ // Z=280

Example 4:

Use the simplex method to solve the (LP) model:

$$max\ Z = 4x_1 - x_2$$

Subject to

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + 3x_2 \leq 12$$

$$x_1 - x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Solution:

$$max\ Z - 4x_1 + x_2 = 0$$

Subject to

$$x_1 + 2x_2 + S_1 = 4$$

$$2x_1 + 3x_2 + S_2 = 12$$

$$x_1 - x_2 + S_3 = 3$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

Table 1:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	1	2	1	0	0	4
S_2	2	3	0	1	0	12
S_3	1	-1	0	0	1	3
Max Z	-4	1	0	0	0	0

$\frac{4}{1} = 4$

$\frac{12}{2} = 6$

$\frac{3}{1} = 3$

New S_3-row or x_1-row = 1[1 -1 0 0 1 3]

=[1 -1 0 0 1 3]

New S_1-row = [1 2 1 0 0 4]-(1)[1 − 1 0 0 1 3]

=[0 3 1 0 -1 1]

New S_2-row = [2 3 0 1 0 12]-(2)[1 − 1 0 0 1 3]

=[0 5 0 1 -2 6]

New Z-row = [-4 1 0 0 0 0]-(-4)[1 − 1 0 0 1 3]

=[0 -3 0 0 4 12]

Table 2:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	0	3	1	0	-1	1
S_2	0	5	0	1	-2	6
x_1	1	-1	0	0	1	3
Max Z	0	-3	0	0	4	12

$$\frac{1}{3} = \frac{1}{3}$$

$$\frac{6}{5} = \frac{6}{5}$$

$$\frac{3}{-1} = -3 \quad \text{(ignore)}$$

New S_1-row or x_2-row $= (\frac{1}{3})[0\ \ 3\ 1\ 0\ \ -1\ \ 1]$

$\qquad\qquad\qquad\qquad = [0\ \ 1\ 1/3\ 0\ \ -1/3\ \ 1/3]$

New S_2-row $= [0\ 5\ 0\ 1\ -2\ \ 6]-(5)[\ 0\ \ 1\ 1/3\ \ 0 - 1/3\ \ 1/3]$

$\qquad\qquad\qquad = [0\ \ 0\ \ -2/3\ 1\ \ 11/3\ \ 13/3]$

New x_1-row $= [1\ -1\ 0\ 0\ 1\ \ 3]-(-1)[\ 0\ \ 1\ 1/3\ \ 0 - 1/3\ \ 1/3]$

$\qquad\qquad\qquad = [1\ \ 0\ \ 1/3\ 0\ 2/3\ \ 10/3]$

New Z-row $= [0\ -3\ 0\ 0\ 4\ \ 12]-(-3)[\ 0\ \ 1\ 1/3\ \ 0 - 1/3\ \ 1/3]$

$\qquad\qquad\qquad = [0\ \ 0\ \ 1\ \ 0\ \ 3\ \ \ 13]$

Table 3: (optimal solution):

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
x_2	0	1	1/3	0	-1/3	1/3
S_2	0	0	-2/3	1	11/3	13/3
x_1	1	0	1/3	0	2/3	10/3
Max Z	0	0	1	0	3	13

The optimal solution :

$x_1 = 10/3\quad,\quad x_2 = 1/3\ , S_2 = 13/3\ \ S_1\ \&\ S_3 = 0\ //\ \text{Z=13}$

Example 5:

Use the simplex method to solve the (LP) model:

$$max\ Z = 16x_1 + 17x_2 + 10x_3$$

Subject to

$$x_1 + 2x_2 + 4x_3 \le 2000$$

$$2x_1 + x_2 + x_3 \le 3600$$

$$x_1 + 2x_2 + 2x_3 \le 2400$$

$$x_1 \le 30$$

$$x_1, x_2, x_3 \ge 0$$

Solution:

$$max\ Z - 16x_1 - 17x_2 - 10x_3 = 0$$

Subject to

$$x_1 + 2x_2 + 4x_3 + S_1 = 2000$$

$$2x_1 + x_2 + x_3 + S_2 = 3600$$

$$x_1 + 2x_2 + 2x_3 + S_3 = 2400$$

$$x_1 + S_4 = 30$$

$$x_1, x_2, x_3 \ge 0, \quad S_1, S_2, S_3, S_4 \ge 0$$

Table 1:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Sol.
S_1	1	2	4	1	0	0	0	2000
S_2	2	1	1	0	1	0	0	3600
S_3	1	2	2	0	0	1	0	2400
S_4	1	0	0	0	0	0	1	30
Max Z	-16	-17	-10	0	0	0	0	0

$$\frac{2000}{2} = \boxed{1000}$$

$$\frac{3600}{1} = 3600$$

$$\frac{2400}{2} = 1200$$

$$\frac{30}{0} = \infty \quad \text{(ignore)}$$

New S_1-row or x_1-row $= (\frac{1}{2})[1 \quad 2 \ 4 \ 1 \quad 0 \ 0 \ 0 \quad 2000]$

$\qquad\qquad\qquad\qquad = [1/2 \ 1 \ 2 \ 1/2 \quad 0 \ 0 \ 0 \ 1000]$

New S_2-row $= [2 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 3600]$

$\qquad\qquad\quad -(1)[\ 1/2 \ 1 \ 2 \quad 1/2 \ 0 \quad 0 \quad 0 \ 1000]$

$\quad = [3/2 \ 0 \ -1 \ -1/2 \ 1 \ 0 \ 0 \quad 2600]$

New S_3-row $= [1 \quad 2 \ 2 \ 0 \ 0 \ 1 \ 0 \ 2400]$

$\qquad\qquad\quad -(2)[\ 1/2 \ 1 \ 2 \quad 1/2 \ 0 \quad 0 \quad 0 \ 1000]$

$\quad = [0 \ \ 0 \ -2 \ -1 \ 0 \ 1 \ 0 \quad 400]$

New S_4-row $= [1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 30]$

$\qquad\qquad\quad -(0)[\ 1/2 \ 1 \ 2 \quad 1/2 \ 0 \quad 0 \quad 0 \ 1000]$

$\quad = [1 \ \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 30]$

New Z-row $= [-16 \quad -17 \quad -10 \ 0 \ 0 \ 0 \ 0 \ 0]$

$\qquad\qquad\quad -(-17)[\ 1/2 \ 1 \ 2 \quad 1/2 \ 0 \quad 0 \quad 0 \ 1000]$

$\quad = [15/2 \ 0 \ 24 \ 17/2 \ 0 \ 0 \ 0 \ 17000]$

Table 2: (optimal solution):

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Sol.
x_2	1/2	1	2	1/2	0	0	0	1000
S_2	3/2	0	-1	-1/2	1	0	0	2600
S_3	0	0	-2	-1	0	1	0	400
S_4	1	0	0	0	0	0	1	30
Max Z	15/2	0	24	17/2	0	0	0	17000

The optimal solution :

$x_2 = 1000$, $S_2 = 2600$, $S_3 = 400, S_4 = 30$ $x_1, x_2 S_1 = 0 /$

Z=17000

Example 6:

Use the simplex method to solve the (LP) model:

$$max\ Z = 3x_1 + 5x_2 + 4x_3$$

Subject to

$$2x_1 + 3x_2 \leq 8$$

$$2x_1 + 5x_2 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$x_1, x_2, \ x_3 \geq 0$$

Solution:

$$max\ Z - 3x_1 - 5x_2 - 4x_3 = 0$$

Subject to

$$2x_1 + 3x_2 + S_1 \leq 8$$

$$2x_1 + 5x_2 + S_2 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 + S_3 \leq 15$$

$$x_1, x_2, \ x_{3,} S_1, \ S_2, S_3 \geq 0$$

Table 1:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
S_1	2	3	0	1	0	0	8
S_2	2	5	0	0	1	0	10
S_3	3	2	4	0	0	1	15
Max Z	-3	-5	-4	0	0	0	0

$\frac{8}{3} = 2.7$

$\frac{10}{5} = 2$

$\frac{15}{2} = 7.5$

New S_2-row or x_2-row $= (\frac{1}{5})$[2 5 0 0   1 0 10]

$\qquad\qquad\qquad\qquad$ =[2/5 1 0 0   1/5 0 2]

New S_1-row = [2   3   0   1   0   0   8]

$\qquad\qquad$ -(3)[2/5   1   0   0 1/5   0   2]

$\qquad\qquad$ =[4/5   0   0   1   -3/5 0   2]

New S_3-row = [3   2   4   0   0   1   15]

$\qquad\qquad$ -(2)[2/5   1   0   0 1/5   0   2]

$\qquad\qquad$ =[11/5   0   4   0   -2/5 1   11]

New Z-row = [-3   -5   -4   0   0   0   0]

$\qquad\qquad$ -(-5)[2/5   1   0   0 1/5   0   2]

$\qquad\qquad$ =[-1   0   -4   0   1 0   10]

Table 2:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
S_1	4/5	0	0	1	-3/5	0	2
x_2	2/5	1	0	0	1/5	0	2
S_3	11/5	1	4	0	-2/5	1	11
Max Z	-1	0	-4	0	1	0	10

New S_3-row or x_3-row $= (\frac{1}{4})[11/5 \ \ 0 \ 4 \ 0 \ \ -2/5 \ 1 \ \ 11]$

$$= [11/20 \ \ 0 \ 1 \ 0 \ \ -1/10 \ \ 1/4 \ \ 11/4]$$

New S_1-row $= [4/5 \ \ 0 \ \ 0 \ \ 1 \ -3/5 \ \ 0 \ \ 2 \ ]$

$$-(0)[\ 11/20 \ \ 0 \ 1 \ 0 \ -1/10 \ \ 1/4 \ \ 11/4]$$

$$= [4/5 \ \ 0 \ \ 0 \ \ 1 \ -3/5 \ 0 \ \ 2]$$

New x_2-row $= [2/5 \ \ 1 \ \ 0 \ \ 0 \ \ 1/5 \ \ 0 \ \ 2 \ ]$

New Z-row $= [-1 \ \ 0 \ -4 \ \ 0 \ \ 1 \ \ 0 \ \ 10 \ ]$

$$-(-4)[\ 11/20 \ \ 0 \ 1 \ \ 0 \ -1/10 \ \ 1/4 \ \ 11/4]$$

$$= [6/5 \ \ 0 \ \ 0 \ \ 0 \ \ 3/5 \ 1 \ \ 21]$$

Table 3: (optimal solution):

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
S_1	4/5	0	0	1	-3/5	0	2
x_2	2/5	1	0	0	1/5	0	2
x_3	11/20	0	1	0	-1/10	1/4	11/4
Max Z	6/5	0	0	0	3/5	1	21

The optimal solution :

$x_2 = 2$,

$\quad x_3 = 11/4$,

$S_1 = 2$,

$\quad$ Z=21

$x_1 = 0$, $S_2 = 0$, $S_3 = 0$,

Unit 2
The Big M Method

Learning outcomes

- **The Big M Method to solve a linear programming problem.**

In the previous discussions of the Simplex algorithm I have seen that the method must start with a **basic feasible solution**. In my examples so far, I have looked at problems that, when put into standard LP form, conveniently have an **all slack** starting solution. An all slack solution is only a possibility when all of the constraints in the problem have <= inequalities. Today, we are going to look at methods for dealing with LPs having other constraint types.

Remember that simplex needs a place to start – it must start from a basic feasible solution then move to another basic feasible solution to improve the objective value.
With these assumptions, I can obtain an initial basic feasible solution /dictionary by letting all slack variables be basic, all original variables be non basic
Obviously, these assumptions do not hold for every LP. What do I do when they don't?
When a basic feasible solution is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M Method

If an LP has any $\geq$ or = constraints, a starting basic feasible solution may not be readily apparent.
The Big M method is a version of the Simplex Algorithm that first finds a basic feasible solution by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps
1. **Modify the constraints** so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a $\leq$, $\geq$, or = constraint. **2. Convert each inequality constraint to standard form** (If constraint i is a $\leq$ constraint, we add a

slack variable s_i; and if constraint i is a $\geq$ constraint, we subtract an excess variable e_i).

3. **Add an artificial variable a_i to the constraints identified as $\geq$ or = constraints at the end of Step 1.** Also add the sign restriction $a_i \geq 0$.

4. **If the LP is a max problem, add (for each artificial variable) -Ma_i to the objective function** where M denote a very large positive number.

5. **If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function.**

6. **Solve the transformed problem by the simplex**. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!
Let's look at an example.

Example 1

Minimize $z =$ $4x_1 + x_2$
Subject to:

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 >= 6$$
$$x_1 + 2x_2 <= 4$$

$$x_1, x_2 >= 0$$

By **introducing a surplus in the second constraint and a slack in the third** we get the following LP in standard form:

Minimize $z =$ $4x_1 + x_2$
Subject to:

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - S_2 = 6$$
$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, S_2, s_3 >= 0$$

Neither of the first two constraint equations has a slack variable or other variable that we can use to be basic in a feasible starting solution so we

must use artificial variables. If we introduce the artificial variables R_1 and R_2 into the first two constraints, respectively, and $MR_1 + MR_2$ into the objective function, we obtain:

Minimize   z = $4x_1 + x_2 + MR_1 + MR_2$
Subject to:

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - S_2 + R_2 = 6$$
$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, S_2, s_3, R_1, R_2 >= 0$$

We can now set x_1, x_2 and S_2 to zero and **use R_1, R_2 and s_3 as the starting basic feasible solution.**

In tableau form we have:

Basic	z	x_1	x_2	S_2	R_1	R_2	s_3	Solution
z	1	-4	-1	0	-M	-M	0	0
R_1	0	3	1	0	1	0	0	3
R_2	0	4	3	-1	0	1	0	6
s_3	0	1	2	0	0	0	1	4

At this point, we have our starting solution in place but we **must adjust our z-row to reflect the fact that we have introduced the variables R_1 and R_2 with non-zero coefficients (M)**.

We can see that if we substitute 3 and 6 into the objective function for R_1 and R_2, respectively, that z = 3M + 6M = 9M. In our tableau, however, **z is shown to be equal to 0**. We can **eliminate this inconsistency by substituting out R_1 and R_2 in the z-row**. Because each artificial variable's column contains exactly one 1, we can **accomplish this by multiplying each of the first two constraint rows by M and adding them both to the current z-row.**

New z-row = Old z-row + M*R_1-row + M*R_2-row

Old z-row: (1   -4   -1   0   **-M**   **-M**   0   0)
+ M*R_1-row: (0   3M   M   0   **M**   0   0   **3M**)
+ M*R_2-row: (0   4M   3M   –M   0   **M**   0   **6M**)

New z-row: (1 -4+7M –1+4M –M   **0**   **0**   0   **9M**)

Our tableau now becomes

Basic	z	x_1	x_2	S_2	R_1	R_2	S_3	Solution
z	1	-4+7M	-1+4M	-M	0	0	0	9M
R_1	0	3	1	0	1	0	0	3
R_2	0	4	3	-1	0	1	0	6
S_3	0	1	2	0	0	0	1	4

Now we have the expected form for our starting solution.

We now **apply the simplex method as before**. Since this is a **minimization problem** we **select the entering variable with the most *positive* objective row coefficient**. In this case, **that is x_1**. Calculating the intercept ratios we get:

$R_1 - 3/3 = 1$
$R_2 - 6/4 = 1.5$
$S_3 -- 4/1 =$

So we **select R_1 as our leaving variable**.

Performing the Gauss-Jordan row operations, we obtain the new tableau:

Basic	z	x_1	x_2	S_2	R_1	R_2	S_3	Solution
z	1	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
x_1	0	1	1/3	0	1/3	0	0	1
R_2	0	0	5/3	-1	-4/3	1	0	2
S_3	0	0	5/3	0	-1/3	0	1	3

In this tableau, we can see that **x_2 will be our next entering variable and R_2 will leave**.

We can thus see that **the simplex algorithm will quickly remove both R_1 and R_2 from the solution just as we intended** when we assigned them the coefficient of M in the objective function. If we continue to apply the simplex algorithm, we will find that **the optimal solution is**:

$x_1 = 2/5$
$x_2 = 9/5$
$S_2 = 1$

with z = 17/5

Two important considerations accompany use of the M method.

The use of the penalty **M may not always force the artificial variable to zero level by the final iteration**. This can occur in the case where **the given LP has no feasible solution**. If **any artificial variable is positive in the final iteration than the LP has no feasible solution space**.

Theoretically, the application of the M technique requires that **M approaches infinity** but to **computerize the solution algorithm**, **M must be finite while being "sufficiently large."** The pitfall in this case is, however, **if M is too large it can lead to substantial round-off error yielding an incorrect optimal solution**. For this reason, **most commercial LP solvers do not apply the M-method** but use, rather, an artificial variable method called the two-phase method. For educational purposes, TORA, allows the implementation of the M-method with a user selected value for M where M is sufficiently large to allow solution of the problem. **The definition of the term "sufficiently large" is dependent upon the problem in question and requires some judgment for implementation.**

Example 2 Minimise $z = 2x_1 - 3x_2 + x_3$
subject to
$$3x_1 - 2x_2 + x_3 \leq 5,$$
$$x_1 + 3x_2 - 4x_3 \leq 9,$$
$$x_2 + 5x_3 \geq 1,$$
$$x_1 + x_2 + x_3 = 6,$$
$$x_1, x_2, x_3 \geq 0.$$

Solution We obtain the linear programming problem: minimise x_8 subject to

$3x_1 - 2x_2 + x_3 + x_4 = 5,$
$x_1 + 3x_2 - 4x_3 + x_5 = 9,$
$x_1 + x_2 + x_3 + x_6 = 6,$
$- x_2 - 5x_3 + x_7 = -1,$
$-2x_1 + 3x_2 - x_3 + x_8 = 0,$
$x_1, ..., x_7 \geq 0,$

- where x_6 is an artificial variable. In tableau T_1 of Table 1, pivoting about y_{33} (= 1) removes $\mathbf{a}_6$ from the basis. The rows of tableau T_2 are then rearranged to give tableau T_3 so that the bad row is below the others, and column $\mathbf{a}_6$ is ignored from here on. Pivoting in T_3 about y_{12} (= 7) gives tableau T_4 which has the basic feasible solution (0, 33/7, 9/7, 92/7, 0, 0, 71/7, - 90/7). This has $x_6 = 0$ and is an optimal solution, so $x_1 = 0$, $x_2 = $ 33/7, $x_3 = 9/7$ is an optimal solution of the original problem with optimal value -90/7.

Min z = $2x_1 + 3x_2$

s.t.

$1/2x_1 + 1/4x_2 \leq 4$…………………1

$x_1 + 3x_2 \geq 20$………………………2

$x_1 + x_2 = 10$…………………………3

$x_1, x_2 \geq 0$

Step1: Make the right hand side of all constraints positive.

We don't have any negative right hand side.

Step2:Identify each constraint which is $\geq$ or=.

Constraints 2 and 3 apply the above conditions.

Step3: For each<= constraint add a slack variable and for each constraint subtract an excess variable to make them equalities.

1…………………………$1/2x_1 + 1/4x_2 + s_1 = 4$

2…………………………..$x_1 + 3x_2 - e_1 = 20$

Step4:For each>=or = constraint add an artificial variable $a_i(a_i\text{'s}>0)$,which is to be chosen in the starting bfs.

2……………………….$x_1 + 3x_2 - e_1 + a_2 = 20$

3……………………….$x_1 + x_2 + + a_3 = 10$

Step5: If the LP is a min add $+Ma_i$ to the objective function. If it is a max add $-Ma_i$ to the objective function. Here M represents a very big number such that in the min problem $+Ma_i$ is arbitrarily large so that a_i the artificial variable is best to be chosen as zero, which we require . Similar reasoning applies in the max problem.

Min z =$2x_1 + 3x_2 + Ma_2 + Ma_3$

Step6: Choose those artificial variables in the starting bfs and proceed to find the Optimal Tableau. If in the end, artificial variables

are zero we find the solution, but if they are not equal to zero then we don't have a optimal solution. Thus, original LP is infeasible.

After these steps we have the LP:

$$\text{Min } z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

st

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_1 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

After all, we have the table:

Entering variable ↓

Basis	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	2M-2	4M-3	0	-M	0	0	30M
s_1	0	1/4	1/4	1	0	0	0	4
Leaving var. ← a_2	0	1	3	0	-1	1	0	20
a_3	0	1	1	0	0	0	1	10
z	1	(2M-3)/3	0	0	(M-3)/3	(3-4M)/3	0	60+10M
s_1	0	5/12	0	1	1/12	-1/12	0	7/3
x_2	0	1/3	1	0	-1/3	1/3	0	20/3
Leaving var ← a_3	0	2/3	0	0	1/3	-1/3	1	10/3
z	1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25
s_1	0	0	0	1	-1/8	1/8	-5/8	1/4
x_2	0	0	1	0	-1/2	1/2	-1/2	5
x_1	0	1	0	0	1/2	-1/2	3/2	5

In the optimal solution, we have: $\{z, s_1, x_2, x_1\} = \{25, 1/4, 5, 5\}$. Since we don't have any of artificial variables a_1 and a_2 in the optimal solution, the solution is feasible. If any of the artificial variables a_1, a_2 were not equal to zero then we would have infeasibility as described below:

$$\text{Min } z = x_1 + 3x_2$$

s.t.

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4$$

$$x_1 + 3x_2 \geq 10$$

$$x_1 + x_2 \quad = 10$$

After going through all the steps described above we end with the optimal tableau:

Basis	z	x_1	s_2	s_1	e_2	a_2	a_3	RHS
z	1	1-2M	0	0	-M	0	3-4M	30+6M
s_1	0	¼	0	1	0	0	-1/4	3/2
a_2	0	-2	0	0	-1	1	-3	6
s_2	0	1	1	0	0	0	1	10

In the above example, we have the optimal tableau since reduced costs of all non basic variables are non positive. However note that the optimal solution contains the very big number M, which should not have been the case for a min problem, thus we say that our original LP was infeasible. We also say that from the fact that we have the artificial variablea2 in the basic variables, which shouldn't have been the case for a feasible LP.

Example 3
Maximize Z = x$_1$ + 5x$_2$
Subject to:
$3x_1 + 4x_2 \pounds 6$
$x_1 + 3x_2 \ ^3 \ 2$
Where $x_1, x_2 \ ^3 \ 0$

Solution

Introducing _**slack**_ and _**surplus**_ variables

$3x_1 + 4x_2 + x_3 = 6$
$x_1 + 3x_2 - x_4 = 2$

Where:
x_3 is a slack variable
x_4 is a surplus variable.

The surplus variable x_4 represents the extra units.

Now if we let x_1 and x_2 equal to zero in the initial solution, we will have $x_3 = 6$ and $x_4 = -2$, which is not possible because a surplus variable cannot be negative. Therefore, we need _**artificial variables**_.

Maximize $x_1 + 5x_2 - MA_1$

Subject to:

$3x_1 + 4x_2 + x_3 = 6$
$x_1 + 3x_2 - x_4 + A_1 = 2$

Where:
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, A_1 \geq 0$

C_B	Basic variables B	C_j	1	5	0	0	−M	Solution values b (= X_B)
			x_1	x_2	x_3	x_4	A_1	
0	x_3		3	4	1	0	0	6
−M	A_1		1	3	0	−1	1	2
$Z_j - C_j$			−M − 1	−3M − 5	0	M	0	

Here, $a_{11} = 3$, $a_{12} = 4$, $a_{13} = 1$, $a_{14} = 0$, $a_{15} = 0$, $b_1 = 6$
$a_{21} = 1$, $a_{22} = -3$, $a_{23} = 0$, $a_{24} = -1$, $a_{25} = 1$, $b_2 = 2$

Calculating $Z_j - C_j$

First column = $0 * 3 + (-M) * 1 - 1 = -M - 1$
Second column = $0 * 4 + (-M) * 3 - 5 = -3M - 5$
Third column = $0 * 1 + (-M) * 0 - 0 = 0$
Fourth column = $0 * 0 + (-M) * (-1) - 0 = M$
Fifth column = $0 * 0 + (-M) * 1 - (-M) = 0$

Choose the smallest negative value from Zj − Cj.
Substitute M = 0
Smallest negative value is −5. So second column is the element column.
Now find out the minimum positive value.

Minimum (6 / 4, 2 / 3) = 2 / 3
So second row is the element row.
Here, the pivot (key) element = 3.
Therefore, A1 departs and x_2 enters.

Calculating values for table 2

Calculating values for first row

a_{11} = 3 − 1 * 4 / 3 = 5 /3
a_{12} = 4 − 3 * 4 / 3 = 0
a_{13} = 1 − 0 * 4 / 3 = 1
a_{14} = 0 − (−1) * 4 / 3 = 4 / 3
b_1 = 6 − 2 * 4 / 3 = 10 / 3

Calculating values for key row

a_{21} = 1 / 3
a_{22} = 3 / 3 =1
a_{23} = 0 / 3 = 0
a_{24} = −1 / 3
b_2 = 2 / 3

Table 2

	C_j	1	5	0	0	
C_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (= X_B)
0	x_3	5 / 3	0	1	4 / 3	10 / 3
5	x_2	1 / 3	1	0	−1 / 3	2 / 3
$Z_j − C_j$		2 / 3	0	0	−5 / 3	

Table 3

	C_j	1	5	0	0	
C_B	Basic variables B	x_1	X_2	x_3	x_4	Solution values b (= X_B)
0	x_4	5 / 4	0	3 / 4	1	5 / 2
5	x_2	3 / 4	1	1 / 4	0	3/2

$Z_j - C_j$		$11/4$	0	$5/4$	0	

Since all the values of $Z_j - C_j$ are positive, *this is the **optimal solution**.*

$x_1 = 0$, $x_2 = 3/2$

$Z = 0 + 5 * 3/2 = 15/2$

The Two-Phase Simplex Method

The LP we solved in the previous lecture allowed us to find an initial BFS very easily. In cases where such an obvious candidate for an initial BFS does not exist, we can solve a different LP to find an initial BFS. We will refer to this as phase I. In phase II we then proceed as in the previous lecture.

Consider the LP to

$$
\begin{aligned}
\text{minimize} \quad & 6x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 1 \\
& 2x_1 - x_2 \geq 1 \\
& 3x_2 \leq 2 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

We change from minimization to maximization and introduce slack variables to obtain the following equivalent problem:

$$
\begin{aligned}
\text{maximize} \quad & -6x_1 - 3x_2 \\
\text{subject to} \quad & x_1 + x_2 - z_1 = 1 \\
& 2x_1 - x_2 - z_2 = 1 \\
& 3x_2 + z_3 = 2 \\
& x_1, x_2, z_1, z_2, z_3 \geq 0.
\end{aligned}
$$

Unfortunately, the basic solution with $x_1 = x_2 = 0$, $z_1 = z_2 = -1$, and $z_3 = 2$ is not feasible. We can, however, add an *artificial variable* to the left-hand side of each constraint where the slack variable and the right-hand side have opposite signs, and then minimize the sum of the artificial variables starting from the obvious BFS where the artificial variables are non-zero instead of the corresponding slack variables. In the example, we

$$
\begin{aligned}
\text{minimize} \quad & y_1 + y_2 \\
\text{subject to} \quad & x_1 + x_2 - z_1 + y_1 = 1 \\
& 2x_1 - x_2 - z_2 + y_2 = 1 \\
& 3x_2 + z_3 = 2 \\
& x_1, x_2, z_1, z_2, z_3, y_1, y_2 \geq 0,
\end{aligned}
$$

and the goal of phase I is to solve this LP starting from the BFS where $x_1 = x_2 = z_1 = z_2 = 0$, $y_1 = y_2 = 1$, and $z_3 = 2$. If the original problem is feasible, we will be able to find a BFS where $y_1 = y_2 = 0$. This automatically gives us an initial BFS for the original problem.

In summary, the two-phase simplex method proceeds as follows:

1. Bring the constraints into equality form. For each constraint in which the slack variable and the right-hand side have opposite signs, or in which there is no slack variable, add a new artificial variable that has the same sign as the right-hand side.

2. Phase I: minimize the sum of the artificial variables, starting from the BFS where the absolute value of the artificial variable for each constraint, or of the slack variable in case there is no artificial variable, is equal to that of the right-hand side.

3. If some artificial variable has a positive value in the optimal solution, the original problem is infeasible; stop.

4. Phase II: solve the original problem, starting from the BFS found in phase I.

While the original objective is not needed for phase I, it is useful to carry it along as an extra row in the tableau, because it will then be in the appropriate form at the beginning of phase II. In the example, phase I therefore starts with the following tableau:

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
y_1	1	1	−1	0	0	1	0	1
y_2	2	−1	0	−1	0	0	1	1
z_3	0	3	0	0	1	0	0	2
II	−6	−3	0	0	0	0	0	0
I	3	0	−1	−1	0	0	0	2

Note that the objective for phase I is written in terms of the *non-basic* variables. This can be achieved by first writing it in terms of y_1 and y_2, such that we have −1 in the columns for y_1 and y_2 and 0 in all other columns because we are *maximizing* $-y_1 - y_2$, and then adding the first and second row to make the entries for all variables in the basis equal to zero.

Phase I now proceeds by pivoting on a_{21} to get

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$
	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	0	3	0	0	1	0	0	2
II	0	−6	0	−3	0	0	3	3
I	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	0	$-\frac{3}{2}$	$\frac{1}{2}$

and on a_{14} to get

x_1	x_2	z_1	z_2	z_3	y_1	y_2	
0	3	-2	1	0	2	-1	1
1	1	-1	0	0	1	0	1
0	3	0	0	1	0	0	2
II 0	3	-6	0	0	6	0	6
I 0	0	0	0	0	-1	-1	0

Note that we could have chosen a_{12} as the pivot element in the second step, and would have obtained the same result.

This ends phase I as $y_1 = y_2 = 0$, and we have found a BFS for the original problem with $x_1 = z_2 = 1$, $z_3 = 2$, and $x_2 = z_1 = 0$. After dropping the columns for y_1 and y_2 and the row corresponding to the objective for phase I, the tableau is in the right form for phase II:

x_1	x_2	z_1	z_2	z_3	
0	3	-2	1	0	1
1	1	-1	0	0	1
0	3	0	0	1	2
0	3	-6	0	0	6

By pivoting on a_{12} we obtain the following tableau, corresponding to an optimal solution of the original problem with $x_1 = 2/3$, $x_2 = 1/3$, and value -5:

x_1	x_2	z_1	z_2	z_3	
0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
0	0	2	-1	1	1
0	0	-4	-1	0	5

It is worth noting that the problem we have just solved is the dual of the LP in Example 1.1, which we solved in the previous lecture, augmented by the constraint $3x_2 \leqslant 2$. Ignoring the column and row corresponding to z_3, the slack variable for this new constraint, the final tableau is essentially the negative of the transpose of the final tableau we obtained in the previous lecture. This makes sense because the additional constraint is not tight in the optimal solution, as we can see from the fact that $z_3 \neq 0$.

Duality in Linear Programming

In the preceding chapter on sensitivity analysis, we saw that the shadow-price interpretation of the optimal simplex multipliers is a very useful concept. First, these shadow prices give us directly the marginal worth of an additional unit of any of the resources. Second, when an activity is ''priced out'' using these shadow prices, the opportunity cost of allocating resources to that activity relative to other activities is determined. Duality in linear programming is essentially a unifying theory that develops the relationships between a given linear program and another related linear program stated in terms of variables with this shadow-price interpretation. The importance of duality is twofold. First, fully understanding the shadow-price interpretation of the optimal simplex multipliers can prove very useful in understanding the implications of a particular linear-programming model. Second, it is often possible to solve the related linear program with the shadow prices as the variables in place of, or in conjunction with, the original linear program, thereby taking advantage of some computational efficiencies. The importance of duality for computational procedures will become more apparent in later chapters on network-flow problems and large-scale systems.

4.1 A PREVIEW OF DUALITY

We can motivate our discussion of duality in linear programming by considering again the simple example given in Chapter 2 involving the firm producing three types of automobile trailers. Recall that the decision variables are:

$$x_1 = \text{number of flat-bed trailers produced per month,}$$

$$x_2 = \text{number of economy trailers produced per month,}$$

$$x_3 = \text{number of luxury trailers produced per month.}$$

The constraining resources of the production operation are the metalworking and woodworking capacities measured in days per month. The linear program to maximize contribution to the firm's overhead (in hundreds of dollars) is:

Maximize $z = 6x_1 + 14x_2 + 13x_3,$

subject to:

$$
\begin{aligned}
\tfrac{1}{2}x_1 + 2x_2 + x_3 &\le 24, \\
x_1 + 2x_2 + 4x_3 &\le 60, \\
x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 &\ge 0.
\end{aligned}
\tag{1}
$$

After adding slack variables, the initial tableau is stated in canonical form in Tableau 1.

In Chapter 2, the example was solved in detail by the simplex method, resulting in the final tableau, repeated here as Tableau 2.

Tableau 1

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	2	1	1	
x_5	60	1	2	4		1
$(-z)$	0	6	14	13		

Tableau 2

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

As we saw in Chapter 3, the shadow prices, y_1 for metalworking capacity and y_2 for woodworking capacity, can be determined from the final tableau as the negative of the reduced costs associated with the slack variables x_4 and x_5. Thus these shadow prices are $y_1 = 11$ and $y_2 = \frac{1}{2}$, respectively.

We can interpret the shadow prices in the usual way. One additional day of metalworking capacity is worth $1100, while one additional day of woodworking capacity is worth only $50. These values can be viewed as the breakeven rents that the firm could pay per day for additional capacity of each type. If additional capacity could be rented for *less* than its corresponding shadow price, it would be profitable to expand capacity in this way. Hence, in allocating the scarce resources to the production activities, we have determined shadow prices for the resources, which are the values imputed to these resources at the margin.

Let us examine some of the economic properties of the shadow prices associated with the resources. Recall, from Chapter 3, Eq. (11), that the reduced costs are given in terms of the shadow prices as follows:

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad (j = 1, 2, \ldots, n).$$

Since a_{ij} is the amount of resource i used per unit of activity j, and y_i is the imputed value of that resource, the term

$$\sum_{i=1}^{m} a_{ij} y_i$$

is the total value of the resources used per unit of activity j. It is thus the marginal resource cost for using that activity. If we think of the objective coefficients c_j as being marginal revenues, the reduced costs $\bar{c}_j$ are simply net marginal revenues (i.e., marginal revenue minus marginal cost).

For the basic variables x_1 and x_3, the reduced costs are zero,

$$\bar{c}_1 = 6 - 11(\tfrac{1}{2}) - \tfrac{1}{2}(1) = 0,$$
$$\bar{c}_3 = 13 - 11(1) - \tfrac{1}{2}(4) = 0.$$

The values imputed to the resources are such that the net marginal revenue is zero on those activities operated at a positive level. That is, for any production activity at positive level, *marginal revenue must equal marginal cost*.

The situation is much the same for the nonbasic variables x_2, x_4, and x_5, with corresponding reduced costs:

$$\bar{c}_2 = 14 - 11(2) - \tfrac{1}{2}(2) = -9,$$
$$\bar{c}_4 = 0 - 11(1) - \tfrac{1}{2}(0) = -11,$$
$$\bar{c}_5 = 0 - 11(0) - \tfrac{1}{2}(1) = -\tfrac{1}{2}.$$

The reduced costs for all nonbasic variables are negative. The interpretation is that, for the values imputed to the scarce resources, *marginal revenue is less than marginal cost* for these activities, so they should not be pursued. In the case of x_2, this simply means that we should not produce *any* economy trailers. The cases of x_4 and x_5 are somewhat different, since slack variables represent unused capacity. Since the marginal revenue of a slack activity is zero, its reduced cost equals *minus its marginal cost*, which is just the shadow price of the corresponding capacity constraint, as we have seen before.

The above conditions interpreted for the reduced costs of the decision variables are the familiar optimality conditions of the simplex method. Economically we can see why they must hold. If marginal revenue exceeds marginal cost for any activity, then the firm would improve its contribution to overhead by *increasing* that activity. If, however, marginal cost exceeds marginal revenue for an activity operated at a positive level, then the firm would increase its contribution by *decreasing* that activity. In either case, a new solution could be found that is an improvement on the current solution. Finally, as we have seen in Chapter 3, those nonbasic variables with zero reduced costs represent possible alternative optimal solutions.

Until now we have used the shadow prices mainly to impute the marginal resource cost associated with each activity. We then selected the best activities for the firm to pursue by comparing the marginal revenue of an activity with its marginal resource cost. In this case, the shadow prices are interpreted as the opportunity costs associated with consuming the firm's resources. If we now value the firm's total resources at these prices, we find their value,

$$v = 11(24) + \tfrac{1}{2}(60) = 294,$$

is exactly equal to the optimal value of the objective function of the firm's decision problem. The implication of this valuation scheme is that the firm's metalworking and woodworking capacities have an imputed worth of \$264 and \$30, respectively. Essentially then, the shadow prices constitute an *internal* pricing system for the firm's resources that:

1. permits the firm to select which activity to pursue by considering only the marginal profitability of its activities; and
2. allocates the contribution of the firm to its resources at the margin.

Suppose that we consider trying to determine directly the shadow prices that satisfy these conditions, without solving the firm's production-decision problem. The shadow prices must satisfy the requirement that marginal revenue be less than or equal to marginal cost for all activities. Further, they must be non-negative since they are associated with less-than-or-equal-to constraints in a maximization decision problem. Therefore, the unknown shadow prices y_1 on metalworking capacity and y_2 on woodworking capacity must satisfy:

$$
\begin{aligned}
\tfrac{1}{2}y_1 + \ \ y_2 &\geq\ \ 6, \\
2y_1 + 2y_2 &\geq\ 14, \\
y_1 + 4y_2 &\geq\ 13, \\
y_1 \geq 0, \qquad y_2 &\geq 0.
\end{aligned}
$$

These constraints require that the shadow prices be chosen so that the net marginal revenue for each activity is nonpositive. If this were not the case, an improvement in the firm's total contribution could be made by changing the choice of production activities.

Recall that the shadow prices were interpreted as breakeven rents for capacity at the margin. Imagine for the moment that the firm does not own its productive capacity but has to rent it. Now consider any values for the shadow prices, or rental rates, that satisfy the above constraints, say $y_1 = 4$ and $y_2 = 4$. The total worth of the rented capacities evaluated at these shadow prices is $v = 24(4) + 60(4) = 336$, which is greater than the maximum contribution of the firm. Since the imputed value of the firm's resources is derived solely from allocating the firm's contribution to its resources, $v = 336$ is too high a total worth to impute to the firm's resources. The firm clearly could not break even if it had to rent its production capacity at such rates.

If we think of the firm as renting all of its resources, then surely it should try to rent them at least cost. This suggests that we might determine the appropriate values of the shadow prices by minimizing the total rental cost of the resources, subject to the above constraints. That is, solve the following linear program:

$$\text{Minimize } v = 24y_1 + 60y_2,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}y_1 + \quad y_2 &\geq 6, \\ 2y_1 + 2y_2 &\geq 14, \\ y_1 + 4y_2 &\geq 13, \\ y_1 \geq 0, \qquad y_2 &\geq 0. \end{aligned} \tag{2}$$

If we solve this linear program by the simplex method, the resulting optimal solution is $y_1 = 11$, $y_2 = \frac{1}{2}$, and $v = 294$. These are exactly the desired values of the shadow prices, and the value of v reflects that the firm's contribution is fully allocated to its resources. Essentially, the linear program (2), in terms of the shadow prices, determines rents for the resources that would allow the firm to break even, in the sense that its total contribution would exactly equal the total rental value of its resources. However, the firm in fact owns its resources, and so the shadow prices are interpreted as the breakeven rates for renting *additional* capacity.

Thus, we have observed that, by solving (2), we can determine the shadow prices of (1) directly. Problem (2) is called the *dual* of Problem (1). Since Problem (2) has a name, it is helpful to have a generic name for the original linear program. Problem (1) has come to be called the *primal*.

In solving any linear program by the simplex method, we also determine the shadow prices associated with the constraints. In solving (2), the shadow prices associated with its constraints are $u_1 = 36$, $u_2 = 0$, and $u_3 = 6$. However, these shadow prices for the constraints of (2) are exactly the optimal values of the decision variables of the firm's allocation problem. Hence, in solving the dual (2) by the simplex method, we apparently have solved the primal (1) as well. As we will see later, this will always be the case since "the dual of the dual is the primal." This is an important result since it implies that the dual may be solved instead of the primal whenever there are computational advantages.

Let us further emphasize the implications of solving these problems by the simplex method. The optimality conditions of the simplex method require that the reduced costs of basic variables be zero. Hence,

$$\begin{aligned} \text{if } \hat{x}_1 > 0, \quad &\text{then } \bar{c}_1 = 6 - \tfrac{1}{2}\hat{y}_1 - \hat{y}_2 = 0; \\ \text{if } \hat{x}_3 > 0, \quad &\text{then } \bar{c}_3 = 13 - \hat{y}_1 - 4\hat{y}_2 = 0. \end{aligned}$$

These equations state that, if a decision variable of the primal is positive, then the corresponding constraint in the dual must hold with equality. Further, the optimality conditions require that the nonbasic variables be zero (at least for those variables with negative reduced costs); that is,

$$\text{if } \bar{c}_2 = 14 - 2\hat{y}_1 - 2\hat{y}_2 < 0, \quad \text{then } \hat{x}_2 = 0.$$

These observations are often referred to as *complementary slackness* conditions since, if a variable is positive, its corresponding (complementary) dual constraint holds with equality while, if a dual constraint holds with strict inequality, then the corresponding (complementary) primal variable must be zero.

These results are analogous to what we have seen in Chapter 3. If some shadow price is positive, then the corresponding constraint must hold with equality; that is,

$$\text{if } \hat{y}_1 > 0, \quad \text{then } \tfrac{1}{2}\hat{x}_1 + 2\hat{x}_2 + \hat{x}_3 = 24;$$
$$\text{if } \hat{y}_2 > 0, \quad \text{then } \hat{x}_1 + 2\hat{x}_2 + 4\hat{x}_3 = 60.$$

Further, if a constraint of the primal is not binding, then its corresponding shadow price must be zero. In our simple example there do not happen to be any nonbinding constraints, other than the implicit nonnegativity constraints. However, the reduced costs have the interpretation of shadow prices on the nonnegativity constraints, and we see that the reduced costs of x_1 and x_3 are appropriately zero.

In this chapter we develop these ideas further by presenting the general theory of duality in linear programming.

4.2   DEFINITION OF THE DUAL PROBLEM

The duality principles we have illustrated in the previous sections can be stated formally in general terms. Let the primal problem be:

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m), \tag{3}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Associated with this primal problem there is a corresponding dual problem given by:

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{4}$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

These primal and dual relationships can be conveniently summarized as in Fig. 4.1.

Without the variables $y_1, y_2, \ldots, y_m$, this tableau is essentially the tableau form utilized in Chapters 2 and 3 for a linear program. The first m rows of the tableau correspond to the constraints of the primal problem, while the last row corresponds to the objective function of the primal problem. If the variables $x_1, x_2, \ldots x_n$, are ignored, the columns of the tableau have a similar interpretation for the dual problem. The first n columns of the tableau correspond to the constraints of the dual problem, while the last column corresponds to the objective function of the dual problem. Note that there is one dual variable for each explicit constraint in the primal, and one primal variable for each explicit constraint in the dual. Moreover, the dual constraints are the familiar optimality condition of ''pricing out'' a column. They state that, at optimality, no activity should appear to be profitable from the standpoint of its reduced cost; that is,

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \leq 0.$$

Figure 4.1   Primal and dual relationships.

To illustrate some of these relationships, let us consider an example formulated in Chapter 1. Recall the portfolio-selection problem, where the decision variables are the amounts to invest in each security type:

$$\text{Maximize } z = 0.043x_A + 0.027x_B + 0.025x_C + 0.022x_D + 0.045x_E,$$

subject to:

$$
\begin{array}{lrcl}
\textit{Cash} & x_A + x_B + x_C + x_D + x_E & \leq & 10, \\
\textit{Governments} & x_B + x_C + x_D & \geq & 4, \\
\textit{Quality} & 0.6x_A + 0.6x_B - 0.4x_C - 0.4x_D + 3.6x_E & \leq & 0, \\
\textit{Maturity} & 4x_A + 10x_B - x_C - 2x_D - 3x_E & \leq & 0,
\end{array}
$$

$$x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E \geq 0.$$

The dual of this problem can be found easily by converting it to the standard primal formulation given in (3). This is accomplished by multiplying the second constraint by -1, thus changing the ''greater than or equal to'' constraint to a ''less than or equal to'' constraint. The resulting primal problem becomes:

$$\text{Maximize } z = 0.043x_A + 0.027x_B + 0.025x_C + 0.022x_D + 0.045x_E,$$

subject to:

$$
\begin{array}{rcl}
x_A + x_B + x_C + x_D + x_E & \leq & 10, \\
- x_B - x_C - x_D & \leq & -4, \\
0.6x_A + 0.6x_B - 0.4x_C - 0.4x_D + 3.6x_E & \leq & 0, \\
4x_A + 10x_B - x_C - 2x_D - 3x_E & \leq & 0,
\end{array}
$$

$$x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \quad x_D \geq 0, \quad x_E \geq 0.$$

According to expression (4), the corresponding dual problem is:

$$\text{Minimize } v = 10y_1 - 4y_2,$$

subject to:

$$
\begin{array}{rcl}
y_1 + 0.6y_3 + 4y_4 & \geq & 0.043, \\
y_1 - y_2 + 0.6y_3 + 10y_4 & \geq & 0.027, \\
y_1 - y_2 - 0.4y_3 - y_4 & \geq & 0.025, \\
y_1 - y_2 - 0.4y_3 - 2y_4 & \geq & 0.022, \\
y_1 + 3.6y_3 - 3y_4 & \geq & 0.045,
\end{array}
$$

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0, \qquad y_4 \geq 0.$$

By applying the simplex method, the optimal solution to both primal and dual problems can be found to be:[*]

Primal: $x_A = 3.36$, $\quad x_B = 0$, $\quad x_C = 0$, $\quad x_D = 6.48$,

$\qquad x_E = 0.16$, $\quad$ and $z = 0.294$;

Dual: $\quad y_1 = 0.0294$, $\quad y_2 = 0$, $\quad y_3 = 0.00636$,

$\qquad y_4 = 0.00244$, $\quad$ and $v = 0.294$.

As we have seen before, the optimal values of the objective functions of the primal and dual solutions are equal. Furthermore, an optimal dual variable is nonzero only if its associated constraint in the primal is binding. This should be intuitively clear, since the optimal dual variables are the shadow prices associated with the constraints. These shadow prices can be interpreted as values imputed to the scarce resources (binding constraints), so that the value of these resources equals the value of the primal objective function.

To further develop that the optimal dual variables are the shadow prices discussed in Chapter 3, we note that they satisfy the optimality conditions of the simplex method. In the final tableau of the simplex method, the reduced costs of the basic variables must be zero. As an example, consider basic variable x_A. The reduced cost of x_A in the final tableau can be determined as follows:

$$\bar{c}_A = c_A - \sum_{i=1}^{5} a_{iA} y_i$$
$$= 0.043 - 1(0.0294) - 0(0) - 0.6(0.00636) - 4(0.00244) = 0.$$

For nonbasic variables, the reduced cost in the final tableau must be nonpositive in order to ensure that no improvements can be made. Consider nonbasic variable x_B, whose reduced cost is determined as follows:

$$\bar{c}_B = c_B - \sum_{i=1}^{5} a_{iB} y_i$$
$$= 0.027 - 1(0.0294) - 1(0) - 0.6(0.00636) - 10(0.00244) = -0.0306.$$

The remaining basic and nonbasic variables also satisfy these optimality conditions for the simplex method. Therefore, the optimal dual variables must be the shadow prices associated with an optimal solution.

Since any linear program can be put in the form of (3) by making simple transformations similar to those used in this example, then any linear program must have a dual linear program. In fact, since the dual problem (4) is a linear program, it must also have a dual. For completeness, one would hope that the dual of the dual is the primal (3), which is indeed the case. To show this we need only change the dual (4) into the form of (3) and apply the definition of the dual problem. The dual may be reformulated as a maximization problem with less-than-or-equal-to constraints, as follows:

$$\text{Maximize } v' = \sum_{i=1}^{m} -b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} -a_{ij} y_i \leq -c_j \qquad (j = 1, 2, \ldots, n), \tag{5}$$
$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

---

[*] Excel spreadsheet available at `http://web.mit.edu/15.053/www/Sect4.2_Primal_Dual.xls`

Applying the definition of the dual problem and letting the dual variables be x_j, $j = 1, 2, \ldots, n$, we have

$$\text{Minimize } z' = \sum_{j=1}^{n} -c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} - a_{ij} x_j \geq -b_i \qquad (i = 1, 2, \ldots, m), \tag{6}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

which, by multiplying the constraints by minus one and converting the objective function to maximization, is clearly equivalent to the primal problem. Thus, the *dual of the dual is the primal*.

4.3 FINDING THE DUAL IN GENERAL

Very often linear programs are encountered in equality form with nonnegative variables. For example, the canonical form, which is used for computing a solution by the simplex method, is in equality form. It is of interest, then, to find the dual of the equality form:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, m), \tag{7}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

A problem in equality form can be transformed into inequality form by replacing each equation by two inequalities. Formulation (7) can be rewritten as

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\left.\begin{array}{c} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i \\[2em] \displaystyle\sum_{j=1}^{n} -a_{ij} x_j \leq -b_i \end{array}\right\} \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

The dual of the equality form can then be found by applying the definition of the dual to this problem. Letting y_i^+ and y_i^- $(i = 1, 2, \ldots, m)$ be the dual variables associated with the first m and second m constraints, respectively, from expression (4), we find the dual to be:

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i^+ + \sum_{i=1}^{m} -b_i y_i^-,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i^+ + \sum_{i=1}^{m} -a_{ij} y_i^- \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i^+ \geq 0, \quad y_i^- \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Collecting terms, we have:

$$\text{Minimize } v = \sum_{i=1}^{m} b_i (y_i^+ - y_i^-),$$

subject to:

$$\sum_{i=1}^{m} a_{ij}(y_i^+ - y_i^-) \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i^+ \geq 0, \quad y_i^- \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Letting $y_i = y_i^+ - y_i^-$, and noting that y_i is unrestricted in sign, gives us the dual of the equality form (7):

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{8}$$

$$y_i \text{ unrestricted} \qquad (i = 1, 2, \ldots, m).$$

Note that the dual variables associated with equality constraints are unrestricted.

There are a number of relationships between primal and dual, depending upon whether the primal problem is a maximization or a minimization problem and upon the types of constraints and restrictions on the variables. To illustrate some of these relationships, let us consider a general maximization problem as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m'),$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (i = m' + 1, m' + 2, \ldots, m''), \tag{9}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = m'' + 1, m'' + 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

We can change the general primal problem to equality form by adding the appropriate slack and surplus variables, as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} \quad = b_i \qquad (i = 1, 2, \ldots, m'),$$

$$\sum_{j=1}^{n} a_{ij}x_j \quad - x_{n+i} = b_i \qquad (i = m' + 1, m' + 2, \ldots, m''),$$

$$\sum_{j=1}^{n} a_{ij}x_j \quad = b_i \qquad (i = m'' + 1, m'' + 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n + m'').$$

Letting y_i, y_i', and y_i'' be dual variables associated respectively with the three sets of equations, the dual of (9) is then

$$\text{Minimize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y_i' + \sum_{i=m''+1}^{m} b_i y_i'',$$

subject to:

$$\sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y_i' + \sum_{i=m''+1}^{m} a_{ij} y_i'' \geq c_j \qquad (10)$$

$$y_i \qquad\qquad \geq 0,$$
$$-y_i' \qquad\qquad \geq 0,$$

where y_i'' is unrestricted in sign and the last inequality could be written $y_i' \leq 0$. Thus, if the primal problem is a maximization problem, the dual variables associated with the less-than-or-equal-to constraints are non-negative, the dual variables associated with the greater-than-or-equal-to constraints are nonpositive, and the dual variables associated with the equality constraints are unrestricted in sign.

These conventions reflect the interpretation of the dual variables as shadow prices of the primal problem. A less-than-or-equal-to constraint, normally representing a scarce resource, has a positive shadow price, since the expansion of that resource generates additional profits. On the other hand, a greater-than-or-equal-to constraint usually represents an external requirement (e.g., demand for a given commodity). If that requirement increases, the problem becomes more constrained; this produces a decrease in the objective function and thus the corresponding constraint has a negative shadow price. Finally, changes in the righthand side of an equality constraint might produce either negative or positive changes in the value of the objective function. This explains the unrestricted nature of the corresponding dual variable.

Let us now investigate the duality relationships when the primal problem is cast in *minimization*, rather than maximization form:

$$\text{Minimize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad (i = 1, 2, \ldots, m'),$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad (i = m' + 1, m' + 2, \ldots, m''), \tag{11}$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = m'' + 1, m'' + 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Since the dual of the dual is the primal, we know that the dual of a minimization problem will be a maximization problem. The dual of (11) can be found by performing transformations similar to those conducted previously. The resulting dual of (11) is the following maximization problem:

$$\text{Maximize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y_i' + \sum_{i=m''+1}^{m} b_i y_i'',$$

subject to:

$$\sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y_i' + \sum_{i=m''+1}^{m} a_{ij} y_i'' \geq c_j \tag{12}$$

$$y_i \leq 0 \qquad (i = 1, 2, \ldots, m'),$$

$$y_i' \geq 0 \qquad (i = m' + 1, m' + 2, \ldots, m'').$$

Observe that now the sign of the dual variables associated with the inequality constraints has changed, as might be expected from the shadow-price interpretation. In a cost-minimization problem, increasing the available resources will tend to decrease the total cost, since the constraint has been relaxed. As a result, the dual variable associated with a less-than-or-equal-to constraint in a minimization problem is nonpositive. On the other hand, increasing requirements could only generate a cost increase. Thus, a greater-than-or-equal-to constraint in a minimization problem has an associated nonnegative dual variable.

The primal and dual problems that we have just developed illustrate one further duality correspondence. If (12) is considered as the primal problem and (11) as its dual, then unrestricted variables in the primal are associated with equality constraints in the dual.

We now can summarize the general duality relationships. Basically we note that equality constraints in the primal correspond to unrestricted variables in the dual, while inequality constraints in the primal correspond to restricted variables in the dual, where the sign of the restriction in the dual depends upon the combination of objective-function type and constraint relation in the primal. These various correspondences are summarized in Table 4.1. The table is based on the assumption that the primal is a maximization problem. Since the dual of the dual is the primal, we can interchange the words primal and dual in Table 4.1 and the correspondences will still hold.

4.4   THE FUNDAMENTAL DUALITY PROPERTIES

In the previous sections of this chapter we have illustrated many duality properties for linear programming. In this section we formalize some of the assertions we have already made. The reader not interested in the theory should skip over this section entirely, or merely read the statements of the properties and ignore the proofs. We consider the primal problem in inequality form so that the primal and dual problems are symmetric. Thus,

Table 4.1

Primal (Maximize)	Dual (Minimize)
ith constraint $\leq$	ith variable ≥ 0
ith constraint $\geq$	ith variable ≤ 0
ith constraint $=$	ith variable unrestricted
jth variable ≥ 0	jth constraint $\geq$
jth variable ≤ 0	jth constraint $\leq$
jth variable unrestricted	jth constraint $=$

any statement that is made about the primal problem immediately has an analog for the dual problem, and conversely. For convenience, we restate the primal and dual problems.

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m), \tag{13}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{14}$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

The first property is referred to as ''weak duality'' and provides a bound on the optimal value of the objective function of either the primal or the dual. Simply stated, the value of the objective function for any feasible solution to the primal maximization problem is bounded from above by the value of the objective function for any feasible solution to its dual. Similarly, the value of the objective function for its dual is bounded from below by the value of the objective function of the primal. Pictorially, we might represent the situation as follows:

$$
\begin{array}{l|l}
\text{Dual} & \downarrow \quad v \text{ decreasing} \\
\text{feasible} & \\
\hline
\text{Primal} & \uparrow \quad z \text{ increasing} \\
\text{feasible} &
\end{array}
$$
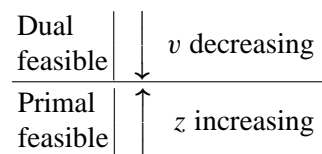
The sequence of properties to be developed will lead us to the ''strong duality'' property, which states that the optimal values of the primal and dual problems are in fact equal. Further, in developing this result, we show how the solution of one of these problems is readily available from the solution of the other.

Weak Duality Property.   If $\overline{x}_j$, $j = 1, 2, \ldots, n$, is a feasible solution to the primal problem and $\overline{y}_i$, $i = 1, 2, \ldots, m$, is a feasible solution to the dual problem, then

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i.$$

The weak duality property follows immediately from the respective feasibility of the two solutions. Primal feasibility implies:

$$\sum_{j=1}^{n} a_{ij} \overline{x}_j \leq b_i \quad (i = 1, 2, \ldots, m) \quad \text{and} \quad \overline{x}_j \geq 0 \quad (j = 1, 2, \ldots, n),$$

while dual feasibility implies

$$\sum_{i=1}^{m} a_{ij} \overline{y}_i \geq c_j \quad (j = 1, 2, \ldots, n) \quad \text{and} \quad \overline{y}_i \geq 0 \quad (i = 1, 2, \ldots, m).$$

Hence, multiplying the ith primal constraint by $\overline{y}_i$ and adding yields:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \overline{x}_j \overline{y}_i \leq \sum_{i=1}^{m} b_i \overline{y}_i,$$

while multiplying the jth dual constraint by $\overline{x}_j$ and adding yields:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \overline{y}_i \overline{x}_j \geq \sum_{j=1}^{n} c_j \overline{x}_j.$$

Since the lefthand sides of these two inequalities are equal, together they imply the desired result that

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i.$$

There are a number of direct consequences of the weak duality property. If we have feasible solutions to the primal and dual problems such that their respective objective functions are equal, then these solutions are optimal to their respective problems. This result follows immediately from the weak duality property, since a dual feasible solution is an upper bound on the optimal primal solution and this bound is attained by the given feasible primal solution. The argument for the dual problem is analogous. Hence, we have an optimality property of dual linear programs.

Optimality Property.   If $\hat{x}_j$, $j = 1, 2, \ldots, n$, is a feasible solution to the primal problem and $\hat{y}_i$, $i = 1, 2, \ldots, m$, is a feasible solution to the dual problem, and, further,

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i,$$

then $\hat{x}_j$, $j = 1, 2, \ldots, n$, is an optimal solution to the primal problem and $\hat{y}_i$, $i = 1, 2, \ldots, m$, is an optimal solution to the dual problem.

Furthermore, if one problem has an unbounded solution, then the dual of that problem is infeasible. This must be true for the primal since any feasible solution to the dual would provide an upper bound on the primal objective function by the weak duality theorem; this contradicts the fact that the primal problem is unbounded. Again, the argument for the dual problem is analogous. Hence, we have an unboundedness property of dual linear programs.

Unboundedness Property.   If the primal (dual) problem has an unbounded solution, then the dual (primal) problem is infeasible.

We are now in a position to give the main result of this section, the ''strong duality'' property. The importance of this property is that it indicates that we may in fact solve the dual problem in place of or in conjunction with the primal problem. The proof of this result depends merely on observing that the shadow prices determined by solving the primal problem by the simplex method give a dual feasible solution, satisfying the optimality property given above.

Strong Duality Property.   If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem, and these two values are equal. That is, $\hat{z} = \hat{v}$ where

$$\hat{z} = \text{Max} \sum_{j=1}^{n} c_j x_j, \qquad\qquad \hat{v} = \text{Min} \sum_{i=1}^{m} b_i y_i,$$

subject to: subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad\qquad \sum_{i=1}^{m} a_{ij} y_i \ge c_j,$$

$$x_j \ge 0; \qquad\qquad\qquad\qquad y_i \ge 0.$$

Let us see how to establish this property. We can convert the primal problem to the equivalent equality form by adding slack variables as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \ge 0 \qquad (j = 1, 2, \ldots, n + m).$$

Suppose that we have applied the simplex method to the linear program and $\hat{x}_j$, $j = 1, 2, \ldots, n$, is the resulting optimal solution. Let $\hat{y}_i$, $i = 1, 2, \ldots, m$, be the shadow prices associated with the optimal solution. Recall that the shadow prices associated with the original constraints are the multiples of those constraints which, when subtracted from the original form of the objective function, yield the form of the objective function in the final tableau [Section 3.2, expression (11)]. Thus the following condition holds:

$$-z + \sum_{j=1}^{n} \overline{c}_j x_j = -\sum_{i=1}^{m} b_i \hat{y}_i, \qquad\qquad (15)$$

where, due to the optimality criterion of the simplex method, the reduced costs satisfy:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \hat{y}_i \le 0 \qquad (j = 1, 2, \ldots, n), \qquad\qquad (16)$$

and

$$\overline{c}_j = 0 - \hat{y}_i \le 0 \qquad (j = n + 1, n + 2, \ldots, n + m). \qquad\qquad (17)$$

Conditions (16) and (17) imply that $\hat{y}_i$, for $i = 1, 2, \ldots, m$, constitutes a feasible solution to the dual problem. When x_j is replaced by the optimal value $\hat{x}_j$ in expression (15), the term

$$\sum_{j=1}^{n} \overline{c}_j \hat{x}_j$$

is equal to zero, since $\overline{c}_j = 0$ when $\hat{x}_j$ is basic, and $\hat{x}_j = 0$ when $\hat{x}_j$ is nonbasic. Therefore, the maximum value of z, say $\hat{z}$, is given by:

$$-\hat{z} = -\sum_{i=1}^{m} b_i \hat{y}_i.$$

Moreover, since $\hat{x}_j$, for $j = 1, 2, \ldots, n$, is an optimal solution to the primal problem,

$$\sum_{j=1}^{n} c_j \hat{x}_j = \hat{z} = \sum_{i=1}^{m} b_i \hat{y}_i.$$

This is the *optimality property* for the primal feasible solution $\hat{x}_j$, $j = 1, 2, \ldots, n$, and the dual feasible solution $\hat{y}_i$, $i = 1, 2, \ldots, m$, so they are optimal for their respective problems. (The argument in terms of the dual problem is analogous.)

   It should be pointed out that it is *not* true that if the primal problem is infeasible, then the dual problem is unbounded. In this case the dual problem may be either unbounded or infeasible. All four combinations of feasibility and infeasibility for primal and dual problems may take place. An example of each is indicated in Table 4.2.

   In example (2) of Table 4.2 it should be clear that x_2 may be increased indefinitely without violating feasibility while at the same time making the objective function arbitrarily large. The constraints of the dual problem for this example are clearly infeasible since $y_1 + y_2 \geq 2$ and $y_1 + y_2 \leq -1$ cannot simultaneously hold. A similar observation is made for example (3), except that the primal problem is now infeasible while the dual variable y_1 may be increased indefinitely. In example (4), the fact that neither primal nor dual problem is feasible can be checked easily by multiplying the first constraint of each by minus one.

4.5 COMPLEMENTARY SLACKNESS

We have remarked that the duality theory developed in the previous section is a unifying theory relating the optimal solution of a linear program to the optimal solution of the dual linear program, which involves the shadow prices of the primal as decision variables. In this section we make this relationship more precise by defining the concept of complementary slackness relating the two problems.

   Complementary Slackness Property.   If, in an optimal solution of a linear program, the value of the dual variable (shadow price) associated with a constraint is nonzero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

   For the primal linear program posed as a maximization problem with less-than-or-equal-to constraints, this means:

i)   if $\hat{y}_i > 0$,   then $\displaystyle\sum_{j=1}^{n} a_{ij} \hat{x}_j = b_i$;

ii)   if $\displaystyle\sum_{j=1}^{n} a_{ij} \hat{x}_j < b_i$,   then $\hat{y}_i = 0$.

Table 4.2

1	*Primal feasible*	*Dual feasible*
	Maximize $z = 2x_1 + x_2$,	Minimize $v = 4y_1 + 2y_2$,
	subject to:	subject to:
	$x_1 + x_2 \leq 4$,	$y_1 + y_2 \geq 2$,
	$x_1 - x_2 \leq 2$,	$y_1 - y_2 \geq 1$,
	$x_1 \geq 0, \quad x_2 \geq 0$.	$y_1 \geq 0, \qquad y_2 \geq 0$.
2	*Primal feasible and unbounded*	*Dual infeasible*
	Maximize $z = 2x_1 + x_2$,	Minimize $v = 4y_1 + 2y_2$,
	subject to:	subject to:
	$x_1 - x_2 \leq 4$,	$y_1 + y_2 \geq 2$,
	$x_1 - x_2 \leq 2$,	$-y_1 - y_2 \geq 1$,
	$x_1 \geq 0, \quad x_2 \geq 0$.	$y_1 \geq 0, \qquad y_2 \geq 0$.
3	*Primal infeasible*	*Dual feasible and unbounded*
	Maximize $z = 2x_1 + x_2$,	Minimize $v = -4y_1 + 2y_2$,
	subject to:	subject to:
	$-x_1 - x_2 \leq -4$,	$-y_1 + y_2 \geq 2$,
	$x_1 + x_2 \leq 2$,	$-y_1 + y_2 \geq 1$,
	$x_1 \geq 0, \quad x_2 \geq 0$.	$y_1 \geq 0, \qquad y_2 \geq 0$.
4	*Primal infeasible*	*Dual infeasible*
	Maximize $z = 2x_1 + x_2$,	Minimize $v = -4y_1 + 2y_2$,
	subject to:	subject to:
	$-x_1 + x_2 \leq -4$,	$-y_1 + y_2 \geq 2$,
	$x_1 - x_2 \leq 2$,	$y_1 - y_2 \geq 1$,
	$x_1 \geq 0, \quad x_2 \geq 0$.	$y_1 \geq 0, \qquad y_2 \geq 0$.

We can show that the complementary-slackness conditions follow directly from the strong duality property just presented. Recall that, in demonstrating the weak duality property, we used the fact that:

$$\sum_{j=1}^{n} c_j \hat{x}_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \hat{x}_j \hat{y}_i \leq \sum_{i=1}^{m} b_i \hat{y}_i \tag{18}$$

for any $\hat{x}_j, \ j = 1, 2, \ldots, n$, and $\hat{y}_i, \ i = 1, 2, \ldots, m$, feasible to the primal and dual problems, respectively. Now, since these solutions are not only feasible but optimal to these problems, equality must hold throughout. Hence, considering the righthand relationship in (18), we have:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \hat{x}_j \hat{y}_i = \sum_{i=1}^{m} b_i \hat{y}_i,$$

which implies:

$$\sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij} \hat{x}_j - b_i \right] \hat{y}_i = 0.$$

Since the dual variables $\hat{y}_i$ are nonnegative and their coefficients

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j - b_i$$

are nonpositive by primal feasibility, this condition can hold only if each of its terms is equal to zero; that is,

$$\left[\sum_{j=1}^{n} a_{ij}\hat{x}_j - b_i\right]\hat{y}_i = 0 \qquad (i = 1, 2, \ldots, m).$$

These latter conditions are clearly equivalent to (i) and (ii) above.

For the dual linear program posed as a minimization problem with greater-than-or-equal-to constraints, the complementary-slackness conditions are the following:

iii)   if $\hat{x}_j > 0$,   then $\displaystyle\sum_{i=1}^{m} a_{ij}y_i = c_j$,

iv)   if $\displaystyle\sum_{i=1}^{m} a_{ij}\hat{y}_i > c_j$,   then $\hat{x}_j = 0$.

These conditions also follow directly from the strong duality property by an argument similar to that given above. By considering the lefthand relationship in (18), we can easily show that

$$\left[\sum_{i=1}^{m} a_{ij}y_i - c_j\right]x_j = 0 \qquad (j = 1, 2, \ldots, n),$$

which is equivalent to (iii) and (iv).

The complementary-slackness conditions of the primal problem have a fundamental economic interpretation. If the shadow price of the ith resource (constraint) is strictly positive in the optimal solution $\hat{y}_i > 0$, then we should require that all of this resource be consumed by the optimal program; that is,

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j = b_i.$$

If, on the other hand, the ith resource is not fully used; that is,

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j < b_i,$$

then its shadow price should be zero, $\hat{y}_i = 0$.

The complementary-slackness conditions of the dual problem are merely the optimality conditions for the simplex method, where the reduced cost $\bar{c}_j$ associated with any variable must be nonpositive and is given by

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i \leq 0 \qquad (j = 1, 2, \ldots, n).$$

If $\hat{x}_j > 0$, then $\hat{x}_j$ must be a basic variable and its reduced cost is defined to be zero. Thus,

$$c_j = \sum_{i=1}^{m} a_{ij}\hat{y}_i.$$

If, on the other hand,

$$c_j - \sum_{i=1}^{m} a_{ij} \hat{y}_i < 0,$$

then $\hat{x}_j$ must be nonbasic and set equal to zero in the optimal solution; $\hat{x}_j = 0$.

We have shown that the strong duality property implies that the complementary-slackness conditions must hold for both the primal and dual problems. The converse of this also is true. If the complementary-slackness conditions hold for both problems, then the strong duality property holds. To see this, let $\hat{x}_j$, $j = 1, 2, \ldots, n$, and $\hat{y}_i$, $i = 1, 2, \ldots, m$, be feasible solutions to the primal and dual problems, respectively. The complementary-slackness conditions (i) and (ii) for the primal problem imply:

$$\sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij} \hat{x}_j - b_i \right] \hat{y}_i = 0,$$

while the complementary-slackness conditions (iii) and (iv) for the dual problem imply:

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{m} a_{ij} \hat{y}_i - c_j \right] \hat{x}_j = 0.$$

These two equations together imply:

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \hat{x}_j \hat{y}_i = \sum_{i=1}^{m} b_i \hat{y}_i,$$

and hence the values of the primal and dual objective functions are equal. Since these solutions are feasible to the primal and dual problems respectively, the *optimality property* implies that these solutions are optimal to the primal and dual problems. We have, in essence, shown that the complementary-slackness conditions holding for both the primal and dual problems is equivalent to the *strong duality property*. For this reason, the complementary-slackness conditions are often referred to as the *optimality conditions*.

> ***Optimality Conditions.***   If $\hat{x}_j$, $j = 1, 2, \ldots, n$, and $\hat{y}_i, i = 1, 2, \ldots, m$, are feasible solutions to the primal and dual problems, respectively, then they are optimal solutions to these problems if, and only if, the complementary-slackness conditions hold for both the primal and the dual problems.

4.6   THE DUAL SIMPLEX METHOD

One of the most important impacts of the general duality theory presented in the previous section has been on computational procedures for linear programming. First, we have established that the dual can be solved in place of the primal whenever there are advantages to doing so. For example, if the number of constraints of a problem is much greater than the number of variables, it is usually wise to solve the dual instead of the primal since the solution time increases much more rapidly with the number of constraints in the problem than with the number of variables. Second, new algorithms have been developed that take advantage of the duality theory in more subtle ways. In this section we present the dual simplex method. We have already seen the essence of the dual simplex method in Section 3.5, on righthand-side ranging, where the variable transitions at the boundaries of each range are essentially computed by the dual simplex method. Further, in Section 3.8, on parametric programming of the righthand side, the dual simplex method is the cornerstone of the computational approach. In this section we formalize the general algorithm.

Recall the canonical form employed in the simplex method:

$$
\begin{aligned}
x_1 &&&+ \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1,n}x_n &= \bar{b}_1, \\
&x_2 &&+ \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2,n}x_n &= \bar{b}_2, \\
&&\ddots &\hspace{2em}\vdots \hspace{4em} \vdots & \vdots \\
&&x_m &+ \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{m,n}x_n &= \bar{b}_m, \\
(-z) &&&+ \quad \bar{c}_{m+1}x_{m+1} + \cdots + \quad \bar{c}_n x_n &= -\bar{z}_0.
\end{aligned}
$$

The conditions for $x_1, x_2, \ldots, x_m$ to constitute an optimal basis for a maximization problem are:

i) $\qquad\qquad\qquad\qquad \bar{c}_j \leq 0 \qquad (j = 1, 2, \ldots, n),$

ii) $\qquad\qquad\qquad\qquad \bar{b}_i \geq 0 \qquad (i = 1, 2, \ldots, m).$

We could refer to condition (i) as primal optimality (or equivalently, dual feasibility) and condition (ii) as primal feasibility. In the primal simplex method, we move from basic feasible solution to adjacent basic feasible solution, increasing (not decreasing) the objective function at each iteration. Termination occurs when the primal optimality conditions are satisfied. Alternatively, we could maintain primal optimality (dual feasibility) by imposing (i) and terminating when the primal feasibility conditions (ii) are satisfied. This latter procedure is referred to as the *dual simplex method* and in fact results from applying the simplex method to the dual problem. In Chapter 3, on sensitivity analysis, we gave a preview of the dual simplex method when we determined the variable to leave the basis at the boundary of a righthand-side range. In fact, the dual simplex method is most useful in applications when a problem has been solved and a subsequent change on the righthand side makes the optimal solution no longer primal feasible, as in the case of parametric programming of the righthand-side values.

The rules of the dual simplex method are identical to those of the primal simplex algorithm, except for the selection of the variable to leave and enter the basis. At each iteration of the dual simplex method, we require that:

$$
\bar{c}_j = c_j - \sum_{i=1}^{m} y_i a_{ij} \leq 0;
$$

and since $y_i \geq 0$ for $i = 1, 2, \ldots, m$, these variables are a dual feasible solution. Further, at each iteration of the dual simplex method, the most negative $\bar{b}_i$ is chosen to determine the pivot row, corresponding to choosing the most positive $\bar{c}_j$ to determine the pivot column in the primal simplex method.

Prior to giving the formal rules of the dual simplex method, we present a simple example to illustrate the essence of the procedure. Consider the following maximization problem with nonnegative variables given in Tableau 3. This problem is in "dual canonical form" since the optimality conditions are satisfied, but the basic variables are not yet nonnegative.

Table 4.3

Basic variables	Current values	x_1	x_2	x_3	x_4
x_3	-1	-1	-1	1	
x_4	-2	-2	-3		1
$(-z)$	0	-3	-1		

In the dual simplex algorithm, we are attempting to make all variables nonnegative. The procedure is the opposite of the primal method in that it first selects the variable to drop from the basis and then the new variable to introduce into the basis in its place. The variable to drop is the basic variable associated with the constraint with the most negative righthand-side value; in this case x_4. Next we have to determine the entering variable. We select from only those nonbasic variables that have a negative coefficient in the pivot

row, since then, after pivoting, the righthand side becomes positive, thus eliminating the primal infeasibility in the constraint. If all coefficients are positive, then the primal problem is clearly infeasible, because the sum of nonnegative terms can never equal the negative righthand side.

In this instance, both nonbasic variables x_1 and x_2 are candidates. Pivoting will subtract some multiple, say t, of the pivot row containing x_4 from the objective function, to give:

$$(-3 + 2t)x_1 + (-1 + 3t)x_2 - tx_4 - z = 2t.$$

Since we wish to maintain the optimality conditions, we want the coefficient of each variable to remain nonpositive, so that:

$$
\begin{aligned}
-3 + 2t &\le 0 & \text{(that is, } t \le \tfrac{3}{2}\text{)}, \\
-1 + 3t &\le 0 & \text{(that is, } t \le \tfrac{1}{3}\text{)}, \\
-t &\le 0 & \text{(that is, } t \ge 0\text{)}.
\end{aligned}
$$

Setting $t = \tfrac{1}{3}$ preserves the optimality conditions and identifies the new basic variable with a zero coefficient in the objective function, as required to maintain the dual canonical form.

These steps can be summarized as follows: the variable to leave the basis is chosen by:

$$\bar{b}_r = \operatorname*{Min}_i \{\bar{b}_i\} = \operatorname{Min} \{-1, -2\} = \bar{b}_2.$$

The variable to enter the basis is determined by the dual ratio test:

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \operatorname*{Min}_j \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} \,\middle|\, \bar{a}_{rj} < 0 \right\} = \operatorname{Min} \left\{ \frac{-3}{-2}, \frac{-1}{-3} \right\} = \frac{\bar{c}_2}{\bar{a}_{22}}.$$

Pivoting in x_2 in the second constraint gives the new canonical form in Tableau 4.

Table 4.4

Basic variables	Current values	x_1	x_2	x_3	x_4
x_3	$-\frac{1}{3}$	$-\frac{1}{3}$		1	$-\frac{1}{3}$
x_2	$\frac{2}{3}$	$\frac{2}{3}$	1		$-\frac{1}{3}$
$(-z)$	$\frac{2}{3}$	$-\frac{7}{3}$			$-\frac{1}{3}$

Clearly, x_3 is the leaving variable since $\bar{b}_1 = -\frac{1}{3}$ is the only negative righthand-side coefficient; and x_4 is the entering variable since $\bar{c}_4 / \bar{a}_{14}$ is the minimum dualratio in the first row. After pivoting, the new canonical form is given in Tableau 5

Table 4.5

Basic variables	Current values	x_1	x_2	x_3	x_4
x_4	1	1		-3	1
x_2	1	1	1	-1	
$(-z)$	1	-2		-1	

Since $\bar{b}_i \ge 0$ for $i = 1, 2$, and $\bar{c}_j \le 0$ for $j = 1, 2, \ldots, 4$, we have the optimal solution.

Following is a formal statement of the procedure. The proof of it is analogous to that of the primal simplex method and is omitted.

Dual Simplex Algorithm

STEP (0) The problem is initially in canonical form and all $\bar{c}_j \leq 0$.

STEP (1) If $\bar{b}_i \geq 0$, $i = 1, 2, \ldots, m$, then *stop*, we are optimal. If we continue, then there exists some $\bar{b}_i < 0$.

STEP (2) Choose the row to pivot in (i.e., variable to drop from the basis) by:

$$\bar{b}_r = \underset{i}{\text{Min}} \left\{ \bar{b}_i | \bar{b}_i < 0 \right\}.$$

If $\bar{a}_{rj} \geq 0$, $j = 1, 2, \ldots, n$, then *stop*; the primal problem is infeasible (dual unbounded). If we continue, then there exists $\bar{a}_{rj} < 0$ for some $j = 1, 2, \ldots, n$.

STEP (3) Choose column s to enter the basis by:

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \underset{j}{\text{Min}} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} \middle| \bar{a}_{rj} < 0 \right\}.$$

STEP (4) Replace the basic variable in row r with variable s and reestablish the canonical form (i.e., pivot on the coefficient $\bar{a}_{rs}$).

STEP (5) Go to Step (1).

Step (3) is designed to maintain $\bar{c}_j \leq 0$ at each iteration, and Step (2) finds the most promising candidate for an improvement in feasibility.

4.7 PRIMAL-DUAL ALGORITHMS

There are many other variants of the simplex method. Thus far we have discussed only the primal and dual methods. There are obvious generalizations that combine these two methods. Algorithms that perform both primal and dual steps are referred to as primal-dual algorithms and there are a number of such algorithms. We present here one simple algorithm of this form called the *parametric primal-dual*. It is most easily discussed in an example.

$$x_1 \geq 0, \quad x_2 \geq 0,$$
$$x_1 + x_2 \leq 6,$$
$$-x_1 + 2x_2 \leq -\tfrac{1}{2},$$
$$x_1 - 3x_2 \leq -1,$$
$$-2x_1 + 3x_2 = z(\text{max}).$$

The above example can easily be put in canonical form by addition of slack variables. However, neither primal feasibility nor primal optimality conditions will be satisfied. We will arbitrarily consider the above example as a function of the parameter θ in Tableau 6.

Clearly, if we choose θ large enough, this system of equations satisfies the primal feasibility and primal optimality conditions. The idea of the parametric primal-dual algorithm is to choose θ large initially so that these conditions are satisfied, and attempt to reduce θ to zero through a sequence of pivot operations. If we start with $\theta = 4$ and let θ approach zero, primal optimality is violated when $\theta < 3$. If we were to reduce θ below 3, the objective-function coefficient of x_2 would become positive. Therefore we perform a primal simplex pivot by introducing x_2 into the basis. We determine the variable to leave the basis by the minimum-ratio rule of the primal simplex method:

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \underset{i}{\text{Min}} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \middle| \bar{a}_{is} > 0 \right\} = \left\{ \frac{6}{1}, \frac{2\frac{1}{2}}{2} \right\} = \frac{\bar{b}_2}{\bar{a}_{22}}.$$

x_4 leaves the basis and the new canonical form is then shown in Tableau 7.

Tableau 6

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	6	1	1	1		
x_4	$-\frac{1}{2}+\theta$	-1	2		1	
x_5	$-1+\theta$	1	-3			1
$(-z)$	0	-2	$(3-\theta)$			

Tableau 7

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	$6\frac{1}{4}-\frac{1}{2}\theta$	$\frac{3}{2}$		1	$-\frac{1}{2}$	
x_2	$-\frac{1}{4}+\frac{1}{2}\theta$	$-\frac{1}{2}$	1		$\frac{1}{2}$	
x_5	$-\frac{7}{4}+\frac{5}{2}\theta$	$-\frac{1}{2}$			$\frac{3}{2}$	1
$(-z)$	$-(3-\theta)(-\frac{1}{4}+\frac{1}{2}\theta)$	$(-\frac{1}{2}-\frac{1}{2}\theta)$			$(-\frac{3}{2}+\frac{1}{2}\theta)$	

The optimality conditions are satisfied for $\frac{7}{10}\le\theta\le 3$. If we were to reduce θ below $\frac{7}{10}$, the righthand-side value of the third constraint would become negative. Therefore, we perform a dual simplex pivot by dropping x_5 from the basis. We determine the variable to enter the basis by the rules of the dual simplex method:

$$\operatorname*{Min}_{j}\left\{\frac{\bar{c}_j}{\bar{a}_{3j}}\,\middle|\,\bar{a}_{3j}<0\right\}=\left\{\frac{-\frac{1}{2}+-\frac{1}{2}\theta}{-\frac{1}{2}}\right\}=\frac{\bar{c}_1}{a_{31}}.$$

After the pivot is performed the new canonical form is given in Tableau 8.

Tableau 8

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	$1+7\theta$			1	4	3
x_2	$\frac{3}{2}-2\theta$		1		-1	-1
x_1	$\frac{7}{2}-5\theta$	1			-3	-2
$(-z)$	$-(3-\theta)(-\frac{1}{4}+\frac{1}{2}\theta)$ $+(\frac{1}{2}+\frac{1}{2}\theta)(\frac{7}{2}-5\theta)$				$(-3-\theta)$	$(-1-\theta)$

As we continue to decrease θ to zero, the optimality conditions remain satisfied. Thus the optimal final tableau for this example is given by setting θ equal to zero.

Primal-dual algorithms are useful when simultaneous changes of both righthand-side and cost coefficients are imposed on a previous optimal solution, and a new optimal solution is to be recovered. When parametric programming of both the objective-function coefficients and the righthand-side values is performed simultaneously, a variation of this algorithm is used. This type of parametric programming is usually referred to as the *rim problem*. (See Appendix B for further details.)

3.14.3. Dual Simplex Method

We remember that while getting optimal solution for a linear programming problem, by using simplex method, we start with initial basic feasible solution (with slack variables in the programme for maximization problem and artificial variables in the programme for minimization problem) and through stages of iteration along simplex algorithm we improve the solution till we get optimal solution. An optimal solution is one, in terms of algorithm for maximization, in which, the net evaluation row are either negative elements are zeros, *i.e.*, the dual variables are feasible. Now with the knowledge of Primal and Dual relationship, we know that the net evaluation row elements are the values of dual variables and hence it suggests dual feasible solution. There are situations, where the primal solution may be infeasible, but corresponding variables indicate that dual is feasible. Thus, solution of primal is infeasible but optimum. In a dual simplex method initial solution is infeasible but optimum, and through iteration it reaches feasibility at which stages it also reaches true optimum.

Problem 3.39:

Minimize $Z = 2a + 1b$ s.t.

$3a + 1b \geq 3$

$4a + 3b \geq 6$

$1a + 2b \leq 3$ and both a and b are ≥ 0.

The problem can be written as:

Maximize $Z = -2a - 1b$ s.t.

$-3a - 1b \leq -3$

$-4a - 3b \leq -6$

$1a + 2b \leq 3$ and both a and b are ≥ 0.

The linear programming version is:

Maximize $Z = -2a - 1b + 0S_1 + 0S_2 + 0S_3$ s.t

$-3a - 1b + 1S_1 + 0S_2 + 0S_3 = -3$

$-4a - 3b + 0S_1 + 1S_2 + 0S_3 = -6$

$1a + 2b + 0S_1 + 0S_2 + 1S_3 = 3$

And $a, b, S_1, S_2,$ and S_3 all ≥ 0.

Table: I. $a = 0$, $b = 0$, $S_1 = -3$, $S_2 = -6$, $S_3 = 3$ and $Z = $ Rs.0/–

Problem variable	Profit Rs.	C_j Capacity units	-2 a	-1 b	0 S_1	0 S_2	0 S_3	Replacement ratio
S_1	0	-3	-3	-1	1	0	0	3
S_2	0	-6	-4	-3	0	1	0	1/2
S_3	0	3	1	2	0	0	1	3/2
		Net evaluation	-2	-1	0	0	0	

Quotient Row: $\qquad\qquad$ –4/–2 $\quad$ –1/–3

Now observe here, both $S_1 = -3$ and $S_2 = -6$ are negative and $S_3 = 3$ positive. This shows that the basic variables of primal (S_1 and S_3) are infeasible. Moreover the net evaluation row elements are negative or zeros, the solution is optimal. Now let us change the solution towards feasibility without disturbing optimality. That is deciding the incoming variable and outgoing variable. **In regular simplex method, we first decide incoming variable. In dual simplex we first decide outgoing variable, *i.e.*, key row.**

$\quad$ (*a*) **Criterion for out going variable:**

The row, which has got the largest negative value (highest number with negative sign) in the capacity column, becomes the key row and the variable having a solution in that row becomes out going variable. If all the values in the capacity column are non-negative and if all net evaluation row elements are negative or zeros, the solution is optimal basic feasible solution. In the given example, the row containing S_2 is having highest number with negative sign; hence S_2 is the out going variable.

$\quad$ (*b*) **Criterion for incoming variable:**

Divide the net evaluation row elements by corresponding coefficients (if negative) of the key row. The column for which the coefficient is smallest becomes key column and the variable in that column becomes entering variable. (If all the matrix coefficients in the key row are positive, the problem has no feasible solution).

In the problem given, variable satisfies the condition, hence variable '*b*' is the incoming variable. The rest of the operation is similar to regular simplex method.

Table: II. $a = 0$, $b = 2$, $S_1 = -1$, $S_2 = 0$, $S_3 = -1$ and $Z = $ Rs. -2

Problem variable	Profit Rs.	C_j Capacity units	-2 a	-1 b	0 S_1	0 S_2	0 S_3	Replacement ratio
S_1	0	-1	$-5/3$	0	1	$-1/3$	0	3/5
b	-1	2	4/3	1	0	$-1/3$	0	6/4
S_3	0	-1	$-5/3$	0	0	2/3	1	3/5
		Net evaluation quotient:	$-2/3$ $-1/5$	0 $-$	0 $-$	$-1/3$ -1	$-$	$-$

The solution is infeasible. As net evaluation row elements are negative the solution remains optimal and as basic variables are negative, it is infeasible.

Now both rows of S_1 and S_3 are having –1 in capacity column, any one of them becomes key row. Let us select first row as key row. The quotient row shows that '*a*' as the incoming variable.

Table: III. $a = 3/5$, $b = 6/5$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $Z = -$Rs. 12/5

Problem variable	Profit Rs.	C_j Capacity units	-2 a	-1 b	0 S_1	0 S_2	0 S_3	Replacement ratio
a	-2	3/5	1	0	$-3/5$	1/5	0	
b	-1	6/5	0	1	4/5	$-3/5$	0	
S_3	0	0	0	0	-1	1	1	
		Net evaluation	0	0	$-2/5$	$-1/5$	0	

As the net evaluation row elements are negatives or zeros, the solution is optimal and feasible. Answer is $a = 3/5$ and $b = 6/5$ and profit $Z = -$ Rs. 12/5. That is for minimization problem the minimum cost is Rs. 12/5.

To Summarize:

  (*i*) **The solution associated with a basis is optimal if all basic variables are ≥ 0.**

  (*ii*) **The basic variable having the largest negative value (highest number with negative sign), is the outgoing variable and the row containing it is the key row.**

  (*iii*) **If α is the matrix coefficient of the key row and is < 0, the variable for which (index row coefficient/α) is numerically smallest will indicate incoming variable.**

  (*iv*) **If $\alpha > 0$, for all the variables in the key row the problem is infeasible.**

Problem 3.40:

Minimize $Z = 20x + 16y$ s.t.

$1x + 1y \geq 12$

$2x + 1y \geq 17$

$2x + 0y \geq 5$

$0x + 1y \geq 6$ and

Both x and y are ≥ 0

The problem can be written as:

Maximize $Z = -20x - 16y + 0S_1 + 0S_2 + 0S_3 + 0S_4$ s.t.

$-1x - 1y + 1S_1 + 0S_2 + 0S_3 + 0S_4 = -12$

$-2x - 1y + 0S_1 + 1S_2 + 0S_3 + 0S_4 = -17$

$-2x + 0y + 0S_1 + 0S_2 + 1S_3 + 0S_4 = -5$

$0x - 1y + 0S_1 + 0S_2 + 0S_3 + 1S_4 = -6$

and x, y, S_1, S_2, S_3 and S_4 all ≥ 0.

Solution:

Table: I. $x = 0$, $y = 0$, $S_1 = -12$, $S_2 = -17$, $S_3 = -6$, $S_4 -6$, $Z = $ Rs.0/–

Problem variable	Profit Rs.	C_j Capacity units	-20 x	-16 y	0 S_1	0 S_2	0 S_3	0 S_4	Replacement ratio
S_1	0	-12	-1	-1	1	0	0	0	12
S_2	0	-17	-2	-1	0	1	0	0	8.5
S_3	0	-5	-2	0	0	0	1	0	2.5
S_4	0	-6	0	-1	0	0	0	1	0
		Net evaluation	-20	-16	0	0	0	0	
		Quotient	10	16					

As all the elements of net evaluation row are negative, the solution is optimal but as slack variables have –ve values, the solution is infeasible. The row with highest number with negative sign becomes outgoing variable. Here S_2 is out going variable *i.e.*, it becomes key row. By dividing net evaluation row elements by corresponding key row elements, quotient row elements are obtained, which show that x is the incoming variable (lowest number).

Table: II. $x = 17/2$, $y = 0$, $S_1 = -7/2$, $S_2 = 0$, $S_3 = 12$, $S_4 = -6$ and $Z = -$ Rs. 170/–

Problem variable	Profit Rs.	C_j Capacity units	-20 x	-16 y	0 S_1	0 S_2	0 S_3	0 S_4	Replacement ratio
S_1	0	$-7/2$	0	$-1/2$	1	$-1/2$	0	0	7
x	-20	$17/2$	1	$1/2$	0	$-1/2$	0	0	Negative
S_3	0	12	0	1	0	-1	1	0	12
S_4	0	-6	0	-1	0	0	0	1	6
		Net evaluation	0	-6	0	-10	0	0	
		Quotient	$-$	6	$-$	$-$	$-$	$-$	

Solution is optimal and infeasible as net evaluation row elements are negative and slack variables are negative.

Table: III. $x = 11/2$, $y = 6$, $S_1 = -1/2$, $S_2 = 0$, $S_3 = 6$, $S_4 = 0$, $Z = -$ Rs.206/–

Problem variable	Profit Rs.	C_j Capacity units	-20 x	-16 y	0 S_1	0 S_2	0 S_3	0 S_4	Replacement ratio
S_1	0	$-1/2$	0	0	**1**	$-1/2$	0	$-1/2$	Negative
x	-20	$11/2$	1	0	0	$-1/2$	0	$1/2$	
S_3	0	6	0	0	0	-1	1	1	
y	-16	6	0	1	0	0	0	-1	
		Net evaluation	0	0	0	-10	0	-6	
		Quotient				20		12	

Table: IV. $X = 5$, $y = 7$, and $Z = -$ Rs. 212/–

Problem variable	Profit Rs.	C_j Capacity units	-20 x	-16 y	0 S_1	0 S_2	0 S_3	0 S_4	Replacement ratio
S_4	0	1	0	0	-2	1	0	1	
x	-20	5	1	0	1	-1	0	0	
S_3	0	5	0	0	2	-2	1	0	
y	-16	7	0	1	-2	1	0	0	
		Net evaluation	0	0	-12	-4	0	0	
		Quotient							

As all the elements of net evaluation row are either negative or zeros the solution is optimal and as all the slack variables and basic variables have positive values the solution is feasible.

$x = 5$, $y = 7$, $S_1 = 0$, $S_2 = 0$. $S_3 = 5$, $S_4 = 1$ and $Z = -$ Rs. 212/– *i.e* minimum optimal is Rs.212/

Problem 3.41:

Minimize $Z = 10a + 6b + 2c$ s.t.

$-1a + 1b + 1c \geq 1$

$3a + 1b - 1c \geq 2$

And all $a, b, c, \geq 0$.

The problem can be written as:

Maximize $Z = -10a - 6b - 2c + 0S_1 + 0S_2$

$1a - 1b - 1c + 1S_1 + 0S_2 = -1$

$-3a - 1b + 1c + 0S_1 + 1S_2 = -2$

And a, b, c, S_1, S_2 all ≥ 0

Solution:

Table: I. $a = 0$, $b = 0$, $X = 0$, $S_1 = -1$, $S_2 = -2$, $Z = $ Rs. 0/–

Problem variable	Profit Rs.	C_j Capacity units	-10 a	-6 b	-2 c	0 S_1	0 S_2	Replacement ratio
S_1	0	-1	1	-1	-1	1	0	-1
S_2	0	-2	-3	-1	1	0	1	2/3
		Net evaluation	-10	-6	-2	0	0	
		Quotient	10/3	6				

Net evaluation row elements are negative hence solution is optimal but slack variables are negative, hence the solution is infeasible. Variable 'a' has lowest quotient hence incoming variable and S_2 has got highest element with negative sign, it is out going variable and the row having S_2 is the key row.

Table: II. $a = 2/3$, $b = 0$, $z = 0$, $S_1 = -5/3$, $S_2 = 0$, $Z = -$ Rs. 20/3.

Problem variable	Profit Rs.	C_j Capacity units	-10 a	-6 b	-2 c	0 S_1	0 S_2	Replacement ratio
S_1	0	$-5/3$	0	$-4/3$	$-2/3$	1	1/3	Negative
a	-10	2/3	1	1/3	$-1/3$	0	$-1/3$	
		Net evaluation	0	$-8/3$	$-16/3$	0	$-10/3$	
		Quotient		2	8			

Variable 'b' has lowest positive quotient, it is incoming variable, S_1 has highest number with negative sign, it is the out going variable. The solution is infeasible optimal.

Table: III. $a = 1/4$, $b = 5/4$, $c = 0$, $S_1 = 0$, $S_2 = 0$, $Z = -$ Rs. 10/–

Problem Variable	Profit Rs.	C_j Capacity units	-10 a	-6 b	-2 c	0 S_1	0 S_2	Replacement ratio
b	-6	5/4	0	1	1/2	$-3/4$	$-1/4$	
a	-10	1/4	1	0	$-1/2$	1/4	$-1/4$	
		Net evaluation	0	0	-4	-2	-4	

The solution is optimal and feasible. $a = ¼$, $b = 5/4$, $c = 0$ and $Z = -$ Rs. 10/–. Hence the minimum cost is Rs. 10/–

UNIT - 3

Linear Programming: II

Transportation Model

4.1. INTRODUCTION

In operations Research Linear programming is one of the model in mathematical programming, which is very broad and vast. Mathematical programming includes many more optimization models known as Non - linear Programming, Stochastic programming, Integer Programming and Dynamic Programming - each one of them is an efficient optimization technique to solve the problem with a specific structure, which depends on the assumptions made in formulating the model. We can remember that the general linear programming model is based on the assumptions:

(*a*) Certainty

The resources available and the requirement of resources by competing candidates, the profit coefficients of each variable are assumed to remain unchanged and they are certain in nature.

(*b*) Linearity

The objective function and structural constraints are assumed to be linear.

(*c*) Divisibility

All variables are assumed to be continuous; hence they can assume integer or fractional values.

(*d*) Single stage

The model is static and constrained to one decision only. And planning period is assumed to be fixed.

(*e*) Non-negativity

A non-negativity constraint exists in the problem, so that the values of all variables are to be ≥ 0, *i.e.* the lower limit is zero and the upper limit may be any positive number.

(*f*) Fixed technology

Production requirements are assumed to be fixed during the planning period.

(*g*) Constant profit or cost per unit

Regardless of the production schedules profit or cost remain constant.

Now let us examine the applicability of linear programming model for **transportation** and **assignment models**.

4.2. TRANSPORTATION MODEL

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.**

To understand the problem more clearly, let us take an example and discuss the rationale of transportation problem. Three factories *A*, *B* and *C* manufactures sugar and are located in different regions. Factory *A* manufactures, b_1 tons of sugar per year and *B* manufactures b_2 tons of sugar per year and *C* manufactures b_3 tons of sugar. The sugar is required by four markets *W*, *X*, *Y* and *Z*. The requirement of the four markets is as follows: Demand for sugar in Markets *W*, *X*, *Y*and *Z* is d_1, d_2, d_3 and d_4 tons respectively. The transportation cost of one ton of sugar from each factory to market is given in the matrix below. The objective is to transport sugar from factories to the markets at a minimum total transportation cost.

	Markets	*Transportation cost per ton in Rs.*				*Availability in tons*
		W	*X*	*Y*	*Z*	
Factories	*A*	c_{11}	c_{12}	c_{13}	c_{14}	b_1
	B	c_{21}	c_{22}	c_{23}	c_{24}	b_2
	C	c_{31}	c_{32}	c_{33}	c_{34}	b_3
Demand in Tons.		d_1	d_2	d_3	d_4	$\Sigma b_j / \Sigma d_j$

For the data given above, the mathematical model will be:

Minimize $Z = c_{11} x_{11} + c_{12} x_{12} + c_{13} x_{13} + c_{14} x_{14} + c_{21} x_{21} + c_{22} x_{22} + c_{23} x_{23} + c_{24} x_{24} + c_{31} x_{31} + c_{32} x_{32} + c_{33} x_{33} + c_{34} x_{34}$ subject to a condition: ⟶ **OBJECTIVE FUNCTION.**

$a_{11} x_{11} + a_{12} x_{12} + a_{13} x_{13} + a_{14} x_{14} \leq b_1$ (because the sum must be less than or equal to the
available capacity)

$a_{21} x_{21} + a_{22} x_{22} + a_{23} x_{23} + a_{24} x_{24} \leq b_2$
$a_{31} x_{31} + a_{32} x_{32} + a_{33} x_{33} + a_{34} x_{34} \leq b_3$ ⟶ **MIXED STRUCTURAL CONSTRAINTS.**

$a_{11} x_{11} + a_{21} x_{21} + a_{31} x_{31} \geq d_1$
(because the sum must be greater than or equal to the demand
$a_{12} x_{12} + a_{22} x_{22} + a_{32} x_{32} \geq d_2$ of the market. We cannot send less than what is required)
$a_{13} x_{13} + a_{23} x_{23} + a_{33} x_{33} \geq d_3$
$a_{14} x_{14} + a_{24} x_{24} + a_{34} x_{34} \geq d_4$ and
All x_{ij} and x_{ji} are ≥ 0 where $i = 1,2,3$ and $j = 1,2,3,4$. (This is because we cannot
supply negative elements). ⟶ **NON-NEGATIVITY CONSTRAINT.**

The above problem has got the following properties:

1. It has an objective function.
2. It has structural constraints.
3. It has a non-negativity constraint.
4. The relationship between the variables and the constraints are linear.

We know very well that these are the properties of **a linear programming problem**. Hence the transportation model is also a **linear programming problem**. But a **special type of linear programming problem.**

Once we say that the problem has got the characteristics of linear programming model, and then we can solve it by simplex method. Hence we can solve the transportation problem by using the simplex method. As we see in the above given transportation model, the structural constraints are of mixed type. That is some of them are of ≤ type and some of them are of ≥ type. When we start solving the transportation problem by simplex method, it takes more time and laborious. Hence we use **transportation algorithm or transportation method** to solve the problem. Before we discuss the transportation algorithm, let us see how a general model for transportation problem appears. The general problem will have '*m*' rows and '*n*' columns *i.e.*, $m \times n$ matrix.

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \, x_j \quad \text{s.t. where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

$$\sum_{i=1}^{m} a_{ij} \, x_{ij} \le b_i \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

$$\sum_{j=1}^{n} a_{ij} \, x_{ji} \ge d_j \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

4.3. COMPARISON BETWEEN TRANSPORTATION MODEL AND GENERAL LINEAR PROGRAMMING MODEL

Similarities

1. Both have objective function.
2. Both have linear objective function.
3. Both have non - negativity constraints.
4. Both can be solved by simplex method. In transportation model it is laborious.
5. A general linear programming problem can be reduced to a transportation problem if (*a*) the $a_{ij}'s$ (coefficients of the structural variables in the constraints) are restricted to the values 0 and/or 1 and (*b*) There exists homogeneity of units among the constraints.

Differences

1. Transportation model is basically a minimization model; where as general linear programming model may be of maximization type or minimization type.

2. The resources, for which, the structural constraints are built up is homogeneous in transportation model; where as in general linear programming model they are different. That is one of the constraint may relate to machine hours and next one may relate to man-hours etc. In transportation problem, all the constraints are related to one particular resource or commodity, which is manufactured by the factories and demanded by the market points.

3. The transportation problem is solved by transportation algorithm; where as the general linear programming problem is solved by simplex method.

4. The values of structural coefficients (*i.e.* x_{ij}) are not restricted to any value in general linear programming model, where as it is restricted to values either 0 or 1 in transportation problem. Say for example:

   Let one of the constraints in general linear programming model is: $2x - 3y + 10z \le 20$. Here the coefficients of structural variables *x*, *y* and *z* may negative numbers or positive numbers of zeros. Where as in transportation model, say for example $x_{11} + x_{12} + x_{13} + x_{14} = b_i = 20$. Suppose the value of variables x_{11}, and x_{14} are 10 each, then $10 + 0. x_{12} + 0. x_{13} + 10 = 20$. Hence the coefficients of x_{11} and x_{14} are 1 and that of x_{12} and x_{13} are zero.

4.4. APPROACH TO SOLUTION TO A TRANSPORTATION PROBLEM BY USING TRANSPORTATION ALGORITHM

The steps used in getting a solution to a transportation problem is given below:

4.4.1. Initial Basic Feasible Solution

Step 1. Balancing the given problem. Balancing means check whether sum of availability constraints must be equals to sum of requirement constraints. That is $\Sigma b_i = \Sigma d_j$. Once they are equal, go to step two. If not by opening a *Dummy row* or *Dummy column* balance the problem. The *cost coefficients of dummy cells are zero*. If Σb_i is greater than Σd_j, then open a dummy column, whose requirement constraint is equals to $\Sigma b_i - \Sigma d_j$ and the cost coefficient of the cells are zeros. In case if Σd_j is greater than Σb_i, then open a dummy row, whose availability constraint will be equals to $\Sigma d_j - \Sigma b_i$ and the cost coefficient of the cells are zeros. Once the balancing is over, then go to second step. Remember while solving general linear programming problem to convert an inequality into an equation, we add (for maximization problem) a slack variable. In transportation problem, the dummy row or dummy column, exactly similar to a slack variable.

Step II. A .Basic feasible solution can be obtained by three methods, they are

(*a*) North - west corner method.

(*b*) Least - cost cell method. (Or Inspection method Or Matrix minimum - row minimum - column minimum method)

(*c*) Vogel's Approximation Method, generally known as VAM.

After getting the basic feasible solution (*b.f.s.*) give **optimality test** to check whether the solution is optimal or not.

There are two methods of giving optimality test:

(*a*) Stepping Stone Method.

(*b*) Modified Distribution Method, generally known as **MODI** method.

4.4.2.Properties of a Basic feasible Solution

1. The allocation made must satisfy the rim requirements, *i.e.,* it must satisfy availability constraints and requirement constraints.
2. It should satisfy non negativity constraint.
3. Total number of allocations must be equal to $(m + n - 1)$, where '*m*' is the number of rows and '*n*' is the number of columns. Consider a value of $m = 4$ and $n = 3$, *i.e.* 4×3 matrix. This will have four constraints of $\le$ type and three constraints of $\ge$ type. Totally it will have $4 + 3$ (*i.e* $m + n$) inequalities. If we consider them as equations, for solution purpose, we will have 7 equations. In case, if we use simplex method to solve the problem, only six rather than seen structural constraints need to be specified. In view of the fact that the sum of the origin capacities (availability constraint) equals to the destination requirements (requirement constraint) *i.e.,* $\Sigma b_i = \Sigma d_j$, any solution satisfying six of the seven constraints will automatically satisfy the last constraint. In general, therefore, if there are '*m*' rows and '*n*' columns, in a given transportation problem, we can state the problem completely with $m + n - 1$ equations. This means that one of the rows of the simplex tableau represents a redundant constraint and, hence, can be deleted. This also means that a basic feasible solution of a transportation problem has only $m + n - 1$ positive components. If $\Sigma b_i = \Sigma d_j$, it is always possible to get a basic feasible solution by North-west corner method, Least Cost cell method or by VAM.

4.4.3. Basic Feasible Solution by North - West corner Method

Let us take a numerical example and discuss the process of getting basic feasible solution by various methods.

Example 4.1. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations. Find the Optimal Solution for least cost transportation cost.

Factories.	Cost in Rs. per ton (× 100) Markets.			Availability in tons.
	X	Y	Z	
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma\ b = 29,\ \Sigma\ d = 23$

Here Σb is greater than Σd hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

a) North- west corner method

(*i*) Balance the problem. That is see whether $\Sigma b_i = \Sigma d_j$. If not open a dummy column or dummy row as the case may be and balance the problem.

(*ii*) Start from the left hand side top corner or cell and make allocations depending on the availability and requirement constraint. If the availability constraint is less than the requirement constraint, then for that cell make allocation in units which is equal to the availability constraint. In general, verify which is the smallest among the availability and requirement and allocate the smallest one to the cell under question. Then proceed allocating either sidewise or down-ward to satisfy the rim requirement. Continue this until all the allocations are over.

(*iii*) Once all the allocations are over, *i.e.,* both rim requirement (column and row *i.e.,* availability and requirement constraints) are satisfied, write allocations and calculate the cost of transportation.

Solution by North-west corner method:

For cell *AX* the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell *AX*. Next $10 - 7 = 3$, this is allocated to cell *AY* to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to $m + n - 1$, then the solution is ***basic feasible solution***. The solution, we got have **7** allocations which is $= 4 + 4 - 1 = 7$. Hence the solution is basic feasible solution.
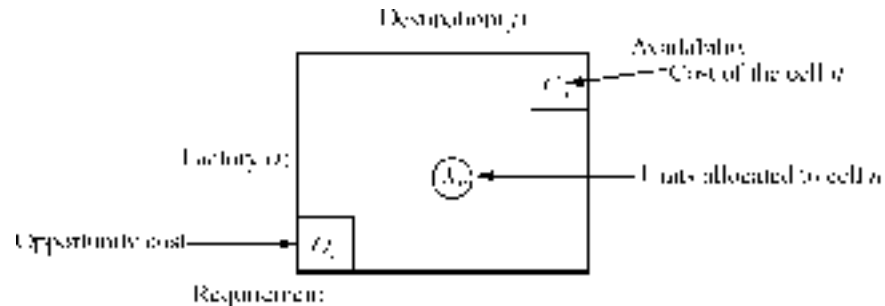
Now allocations are:

From	To	Units in tons	Cost in Rs.
A	X	7	$7 \times 4 = 28$
A	Y	3	$3 \times 3 = 09$
B	Y	8	$8 \times 6 = 48$
C	Y	1	$1 \times 4 = 04$
C	Z	4	$4 \times 3 = 12$
D	Z	1	$1 \times 4 = 04$
D	DUMMY	5	$5 \times 0 = 00$
	Total in Rs.		105

4.4.4. Solution by Least cost cell (or inspection) Method: (Matrix Minimum method)

(*i*) Identify the lowest cost cell in the given matrix. In this particular example it is $= 0$. Four cells of dummy column are having zero. When more than one cell has the same cost, then both the cells are competing for allocation. This situation in transportation problem is known as **tie**. To break the tie, select any one cell of your choice for allocation. Make allocations to this cell either to satisfy availability constraint or requirement constraint. Once one of these is satisfied, then mark crosses ($\times$) in all the cells in the row or column which ever has completely allocated. Next search for lowest cost cell. In the given problem it is cell **BZ** which is having cost of Re.1/- Make allocations for this cell in similar manner and mark crosses to the cells in row or column which has allocated completely. Proceed this way until all allocations are made. Then write allocations and find the cost of transportation. As the total number of allocations are **7** which is equals to $4 + 4 - 1 = 7$, the solution is basic feasible solution.

(**Note:** The numbers under and side of rim requirements shows the sequence of allocation and the units remaining after allocation)

Allocations are:

From	To	Units in tons	Cost in Rs.
A	Y	8	$8 \times 3 = 24$
A	Z	2	$2 \times 2 = 04$
B	Z	3	$3 \times 1 = 03$
B	DUMMY	5	$5 \times 0 = 00$
C	X	1	$1 \times 6 = 06$
C	Y	4	$4 \times 4 = 16$
D	X	6	$6 \times 3 = 18$
		Total in Rs.	**71**

4.4.5.   Solution by Vogel's Approximation Method: (Opportunity cost method)

(*i*)   In this method, we use concept of **opportunity cost**. Opportunity cost is the penalty for not taking correct decision. ***To find the row opportunity cost in the given matrix deduct the smallest element in the row from the next highest element. Similarly to calculate the column opportunity cost, deduct smallest element in the column from the next highest element. Write row opportunity costs of each row just by the side of availability constraint and similarly write the column opportunity cost of each column just below the requirement constraints. These are known as penalty column and penalty row.***

The rationale in deducting the smallest element form the next highest element is:

Let us say the smallest element is 3 and the next highest element is 6. If we transport one unit

through the cell having cost Rs.3/-, the cost of transportation per unit will be Rs. 3/-. Instead we transport through the cell having cost of Rs.6/-, then the cost of transportation will be Rs.6/- per unit. That is for not taking correct decision; we are spending Rs.3/- more (Rs.6 – Rs.3 = Rs.3/-). *This is the penalty for not taking correct decision and hence the opportunity cost. This is the lowest opportunity cost in that particular row or column as we are deducting the smallest element form the next highest element.*

Note: If the smallest element is three and the row or column having one more three, then we have to take next highest element as three and not any other element. Then the opportunity cost will be zero. In general, if the row has two elements of the same magnitude as the smallest element then the opportunity cost of that row or column is zero.

(*ii*) Write row opportunity costs and column opportunity costs as described above.

(*iii*) Identify the highest opportunity cost among all the opportunity costs and write a tick (√) mark at that element.

(*iv*) If there are two or more of the opportunity costs which of same magnitude, then select any one of them, to break the tie. While doing so, see that both availability constraint and requirement constraint are simultaneously satisfied. If this happens, we may not get basic feasible solution *i.e* solution with $m + n – 1$ allocations. As far as possible see that both are not satisfied simultaneously. In case if inevitable, proceed with allocations. We may not get a solution with, $m + n – 1$ allocations. For this we can allocate a small element epsilon ($\in$) to any one of the empty cells. This situation in transportation problem is known as degeneracy. (This will be discussed once again when we discuss about optimal solution).

In transportation matrix, all the cells, which have allocation, are known as **loaded cells** and those, which have no allocation, are known as **empty cells**.

(**Note:** All the allocations shown in matrix 1 to 6 are tabulated in the matrix given below:)

Consider matrix (1), showing cost of transportation and availability and requirement constraints. In the first row of the matrix, the lowest cost element is 0, for the cell A-Dummy and next highest element is 2, for the cell AZ. The difference is 2 – 0 = 2. The meaning of this is, if we transport the load through the cell A-Dummy, whose cost element is 0, the cost of transportation will be = Rs.0/- for

each unit transported. Instead, if we transport the load through the cell, AZ whose cost element is Rs. 2/- the transportation cost is = Rs.2/- for each unit we transport. This means to say if we take decision to send the goods through the cell AZ, whose cost element is Rs.2/- then the management is going to loose Rs. 2/- for every unit it transport through *AZ*. Suppose, if the management decide to send load through the cell *AX*, Whose cost element is Rs.4/-, then the penalty or the opportunity cost is *Rs*.4/-. We write the minimum opportunity cost of the row outside the matrix. Here it is shown in brackets. Similarly, we find the column opportunity costs for each column and write at the bottom of each corresponding row (in brackets). After writing all the opportunity costs, then we select the highest among them. In the given matrix it is Rs.3/- for the rows *D* and *C*. **This situation is known as tie.** When tie exists, select any of the rows of your choice. At present, let us select the row *D*. Now in that row select the lowest cost cell for allocation. This is because; our objective is to minimize the transportation cost. For the problem, it is *D*-dummy, whose cost is zero. For this cell examine what is available and what is required? Availability is 6 tons and requirement is 5 tons. Hence allocate 5 tons to this cell and cancel the dummy row from the problem. Now the matrix is reduced to 3×4. Continue the above procedure and for every allocation the matrix goes on reducing, finally we get all allocations are over. Once the allocations are over, count them, if there are $m + n - 1$ allocations, then the solution is basic feasible solution. Otherwise, the **degeneracy** occurs in the problem. To solve degeneracy, we have to **add epsilon** $(\in)$, a small element to one of the empty cells. This we shall discuss, when we come to discuss optimal solution. Now for the problem the allocations are:

From	To	Load	Cost in Rs.
A	X	3	$3 \times 4 = 12$
A	Y	7	$7 \times 3 = 21$
B	X	3	$3 \times 5 = 15$
B	Z	5	$5 \times 1 = 05$
C	Y	5	$5 \times 4 = 20$
D	X	1	$1 \times 3 = 03$
D	DUMMY	5	$5 \times 0 = 00$
		Total Rs.	**76**

Now let us compare the three methods of getting basic feasible solution:

North – west corner method.	Inspection or least cost cell method	Vogel's Approximation Method.
1. The allocation is made from the left hand side top corner irrespective of the cost of the cell.	The allocations are made depending on the cost of the cell. Lowest cost is first selected and then next highest etc.	The allocations are made depending on the opportunity cost of the cell.
2. As no consideration is given to the cost of the cell, naturally the total transportation cost will be higher than the other methods.	As the cost of the cell is considered while making allocations, the total cost of transportation will be comparatively less.	As the allocations are made depending on the opportunity cost of the cell, the basic feasible solution obtained will be very nearer to optimal solution.
3. It takes less time. This method is suitable to get basic feasible solution quickly.	The basic feasible solution, we get will be very nearer to optimal solution. It takes more time than northwest coroner method.	It takes more time for getting basic Feasible solution. But the solution we get will be very nearer to Optimal solution.
4. When basic feasible solution alone is asked, it is better to go for northwest corner method.	When optimal solution is asked, better to go for inspection method for basic feasible solution and MODI for optimal solution.	VAM and MODI is the best option to get optimal solution.

In the problem given, the total cost of transportation for Northwest corner method is Rs. 101/-. The total cost of transportation for Inspection method is Rs. 71/- and that of VAM is Rs. 76/-. The total cost got by inspection method appears to be less. That of Northwest coroner method is highest. The cost got by VAM is in between.

Now let us discuss the method of getting optimal solution or methods of giving optimality test for basic feasible solution.

4.4.6. Optimality Test: (Approach to Optimal Solution)

Once, we get the basic feasible solution for a transportation problem, the next duty is to test whether the solution got is an optimal one or not? This can be done by two methods. (*a*) By Stepping Stone Method, and (*b*) By Modified Distribution Method, or MODI method.

(*a*) **Stepping stone method of optimality test**

To give an optimality test to the solution obtained, we have to find the opportunity cost of empty cells. As the transportation problem involves decision making under certainty, we know that an optimal solution must not incur any positive opportunity cost. Thus, we have to determine whether any positive opportunity cost is associated with a given progarmme, *i.e.,* for empty cells. **Once the opportunity cost of all empty cells are negative, the solution is said to be optimal.** In case any one cell has got positive opportunity cost, then the solution is to be modified. The Stepping stone method is used for finding the opportunity costs of empty cells. Every empty cell is to be evaluated for its opportunity cost. To do this the methodology is:

   1. Put a small '+' mark in the empty cell.

2. Starting from that cell draw a loop moving horizontally and vertically from loaded cell to loaded cell. Remember, there should not be any diagonal movement. We have to take turn only at loaded cells and move to vertically downward or upward or horizontally to reach another loaded cell. In between, if we have a loaded cell, where we cannot take a turn, ignore that and proceed to next loaded cell in that row or column.
3. After completing the loop, mark minus (–) and plus (+) signs alternatively.
4. Identify the lowest load in the cells marked with negative sign.
5. This number is to be added to the cells where plus sign is marked and subtract from the load of the cell where negative sign is marked.
6. Do not alter the loaded cells, which are not in the loop.
7. The process of adding and subtracting at each turn or corner is necessary to see that rim requirements are satisfied.
8. Construct a table of empty cells and work out the cost change for a shift of load from loaded cell to loaded cell.
9. If the cost change is positive, it means that if we include the evaluated cell in the programme, the cost will increase. If the cost change is negative, the total cost will decrease, by including the evaluated cell in the programme.
10. The negative of cost change is the opportunity cost. Hence, in the optimal solution of transportation problem empty cells should not have positive opportunity cost.
11. Once all the empty cells have negative opportunity cost, the solution is said to be optimal.

One of the drawbacks of stepping stone method is that we have to write a loop for every empty cell. Hence it is tedious and time consuming. Hence, for optimality test we use MODI method rather than the stepping stone method.

Let us take the basic feasible solution we got by Vogel's Approximation method and give optimality test to it by stepping stone method.

Basic Feasible Solution obtained by VAM:

Table showing the cost change and opportunity costs of empty cells:

Table I.

S.No.	Empty Cell	Evalution Loop formation	Cost change in Rs.	Opportunity cost -(Cost change)
1.	AZ	+AZ – AX + BX – BZ	+2 – 4 + 5 – 1 = + 2	–2
2	A Dummy	+ A DUMMY – AX + BX – B DUMMY	+0 – 4 + 3 – 0 = – 1	+1
3	BY	+ BY – AY + AX – BX	+6 – 3 + 4 – 5 = +2	-2
4	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+0 – 5 +3 – 0 = –2	+2
5	CX	+CX – CY + AX – AY	6 – 4 + 3 – 4 = +1	– 1
6	CZ	+CZ – BZ + BX –AX + AY – CY	+2 –1 +5 – 4 +5 – 4 =+1	– 1
7	C DUMMY	+ C DUMMY – D DUMMY + DX – AX + AY – CY	+ 0 – 0 +3 – 4 +3 – 4 = –2	+2
8	DY	+DY – DX + AX – AY	+5 – 3 +4 – 3 = +3	– 3
9	DZ	+DZ – DX +BX – BZ	+4 – 3 + 5 – 1 = +5	– 5

In the table 1 cells A DUMMY, B DUMMY, C DUMMY are the cells which are having positive opportunity cost. Between these two cells B DUMMY and C DUMMY are the cells, which are having higher opportunity cost i.e Rs. 2/ - each. Let us select any one of them to include in the improvement of the present programme. Let us select C DUMMY.

Table II.

S.No.	Empty Cell	Evalution Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX –DX + D DUMMY – C DUMMY + CY – AY	+ 4 – 3 + 0 – 0 + 4 – 3 = + 2	–2
2	AX	AZ – AY + CY – C DUMMY + D DUMMY – DX+ BX – BZ	+ 2 – 3 + 4 – 0 + 0 – 3 + 3 – 0 = + 4	–4
3	ADUMMY	+ A DUMMY – AY + DX – D DUMMY	+ 0 – 4 + 3 – 0 = – 1	+1
4	BY	+BY – BX + DX – D DUMMY + C DUMMY – CY	+ 6 –5 + 3 –0 + 0 – 4 = 0	0
5	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+ 0 – 5 + 3 – 0 = –2	+2
6	CX	+ CX – DX + D DUMMY – C DUMMY	+ 6 –3 + 0 – 0 = +3	–3
7	CZ	+ CZ – C DUMMY + D DUMMY – DX + BX – BZ	+ 2 – 0 + 0 – 3 + 5 – 1 = + 3	–3
8	DY	DY – CY + C DUMMY – D DUMMY	+ 5 – 4 + 0 – 0 = 1	–1
9	DZ	+ DZ – DX + BX – BZ	+ 4 – 3 +5 – 1 = + 5	–5

Cells A DUMMY and B DUMMY are having positive opportunity costs. The cell B DUMMY is having higher opportunity cost. Hence let us include this cell in the next programme to improve the solution.

Table III.

S.No.	Empty Cell	Evaluation Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX – AY + CY – C DUMMY + B DUMMY – BX	+4 – 3 + 4 – 0 + 0 – 5 = 0	0
2	AZ	+ AZ – BZ + B DUMMY – C DUMMY + CX – AX	+2 –1 + 0 – 0 + 4 – 3 =+ 2	–2
3	A DUMMY	+ A DUMMY – C DUMMY + CY – AY	+0 – 0 + 4 – 3 = +1	–1
4	BY	+ BY – B DUMMY + C DUMMY – CY	+6 – 0 + 0 – 4 = + 2	–2
5	CX	+ CX – BX + B DUMMY – C DUMMY	+6 – 5 + 0 – 0 = +1	–1
6	CZ	+ CZ – BZ + B DUMMY – C DUMMY	+2 – 1 + 0 – 0 = +1	–1
7	DY	+DY – CY + C DUMMY – B DUMMY + BX – DX	+5 – 4 + 0 – 0 + 5 – 3 = +3	–3
8	DZ	+ DZ – BZ + BX – DX	+4 – 1 + 5 – 3 = +5	–5
9	D DUMMY	+ D DUMMY – DX + BX – B DUMMY	+ 0 – 3 + 5 – 0 = +2	–2

All the empty cells have negative opportunity cost hence the solution is optimal. The **allocations are:**

S.No	Loaded cell	Load	Cost in Rs.
1	AY	10	$10 \times 3 = 30$
2	BX	01	$01 \times 5 = 05$
3	BZ	05	$05 \times 1 = 05$
4	B DUMMY	02	$02 \times 0 = 00$
5	CY	02	$02 \times 4 = 08$
6	C DUMMY	03	$03 \times 0 = 00$
7	DX	06	$06 \times 3 = 18$
	Total in Rs.		66

Total minimum transportation cost is Rs. 66/-

Optimal allocation.

(*b*) **Modified Distribution Method of Optimality test**

In stepping stone method, we have seen that to get the opportunity cost of empty cells, for every cell we have to write a loop and evaluate the cell, which is a laborious process. In MODI (Modified DIstribution method, we can get the opportunity costs of empty cells without writing the loop. After

getting the opportunity cost of all the cells, we have to select the cell with highest positive opportunity cost for including it in the modified solution.

Steps in MODI method:

1. Select row element (u_i) and Column element (v_j) for each row and column, such that $u_i + v_j$ = the actual cost of loaded cell. In MODI method we can evaluate empty cells simultaneously and get the opportunity cost of the cell by using the formula $(u_i + v_j) – C_{ij}$, where C_{ij} is the actual cost of the cell.

2. In resource allocation problem (maximization or minimization method), we have seen that once any variable becomes basis variable, *i.e.*, the variable enters the programme; its opportunity cost or net evaluation will be zero. Here, in transportation problem also, once any cell is loaded, its opportunity cost will be zero. Now the opportunity cost is given by $(u_i + v_j) – C_{ij}$, which is, equals to zero for a loaded cell.

   i.e. $(u_i + v_j) – C_{ij} = 0$ which means, $(u_i + v_j) = C_{ij}$. Here $(u_i + v_j)$ is known as **implied cost of the cell. For any loaded cell the implied cost is equals to actual cost of the cell as its opportunity cost is zero. For any empty cell, (implied cost – actual cost) will give opportunity cost.**

3. How to select u_i and v_j? The answer is:

   (*a*) Write arbitrarily any one of them against a row or against a column. The written u_i or vj may be any whole number *i.e* u_i or v_j may be $\leq$ or $\geq$ to zero. By using the formula $(u_i + v_j) = C_{ij}$ for a loaded cell, we can write the other row or column element. For example, if the actual cost of the cell $C_{ij} = 5$ and arbitrarily we have selected $u_i = 0$, then v_j is given by $u_i + v_j = 0 + v_j = 5$. Hence $v_j = –5$. Like this, we can go from loaded cell to loaded cell and complete entering of all u_i s and v_j s.

   (*b*) Once we get all u_i s and v_j s, we can evaluate empty cells by using the formula $(u_i + v_j)$ – Actual cost of the cell = opportunity cost of the cell, and write the opportunity cost of each empty cell at left hand bottom corner.

   > (*c*) **Once the opportunity costs of all empty cells are negative, the solution is said to be optimal. In case any cell is having the positive opportunity cost, the programme is to be modified.**
   > **Remember the formula that IMPLIED COST OF A CELL = $u_i + v_j$**
   > **Opportunity cost of loaded cell is zero *i.e* $(u_i + v_j)$ = Actual cost of the cell.**
   > **Opportunity cost of an empty cell = implied cost – actual cost of the cell = $(u_i + v_j) – C_{ij}$**
   >
   > (*d*) **In case of degeneracy, *i.e.* in a basic feasible solution, if the number of loaded cells are not equals to $m + n – 1$, then we have to add a small element *epsilon* ($\in$), to any empty cell to make the number of loaded cells equals to $m + n – 1$. While adding '$\in$' we must be careful enough to see that this $\in$ should not form a closed loop when we draw horizontal and vertical lines from loaded cell to loaded cell. In case the cell to which we have added $\in$ forms a closed loop, then if we cannot write all u_i s and v_j s.**
   > $\in$ **is such a small element such that $a + \in = a$ or $a – \in = a$ and $\in – \in = 0$.**

Implied cost	Actual cost	Action
$u_i + v_j >$	C_{ij}	A better programme can be designed by including this cell in the solution.
$u_i + v_j =$	C_{ij}	Indifferent; however, an alternative programme with same total cost can be written by including this cell in the programme.
$u_i + v_j <$	C_{ij}	Do not include this cell in the programme.

Now let us take the basic feasible solution obtained by VAM method and apply MODI method of optimality test.

Basic feasible solution got by VAM method.

The cell C DUMMY is having a positive opportunity cost. Hence we have to include this cell in the programme. The solution has $m + n - 1$ allocations.

The cell B DUMMY is having a positive opportunity cost. Thïs is to be included in the modified programme.

As the opportunity cost of all empty cells are negative, the solution is optimal. The solution has $m + n - 1$ allocations.

The allocations are:

S.No	Loaded Cell	Load	Cost in Rs.
1	AY	10	$10 \times 3 = 30$
2.	BX	01	$01 \times 5 = 05$
3.	BZ	05	$05 \times 1 = 05$
4.	B DUMMY	02	$02 \times 0 = 00$
5.	CY	02	$02 \times 4 = 08$
6.	C DUMMY	03	$03 \times 0 = 00$
7.	CX	06	$06 \times 3 = 18$
	Total Cost in Rs.		66

Readers can verify the optimal solution got by Stepping stone method and the MODI method they are same. And they can also verify the opportunity costs of empty cells they are also same. This is the advantage of using MODI method to give optimality test. Hence the combination of VAM and MODI can be conveniently used to solve the transportation problem when optimal solution is asked.

4.4.7. Alternate Solutions

By principle, we know that the opportunity cost of a loaded cell or a problem variable is always equals to zero. In case any empty cell of the optimal solution of a transportation problem got zero as the opportunity cost, it should be understood that it is equivalent to a loaded cell. Hence by including that cell, we can derive another solution, which will have same total opportunity cost, but different allocations. Once one alternate solution exists, we can write any number of alternate solutions. The methodology is:

1. Let the Optimal solution is matrix A with one or more empty cells having zero as the opportunity cost.

2. By including the cell having zero as the opportunity cost, derive one more optimal solution, let it be the matrix B.

3. The new matrix C is obtained by the formula: $C = d\ A + (1-\ d)\ B,$ where 'd' is a positive fraction less than 1.

It is better to take always $d = 1/2$, so that $C = 1/2\ A + 1/2\ B.$

Now we shall take the optimal solution of the problem above and write the alternate optimal solutions.

Matrix A (First optimal Solution).

The cell AX, having zero opportunity cost is included in revised solution. The loop is:

$+ AX - BX + B\ DUMMY - C\ DUMMY + CY - AY = + 4 - 5 + 0 - 0 + 4 - 3 = 0$

Allocation:

S.No	Loaded Cell	Load	Cost in Rs.
1.	AX	01	$01 \times 4 = 04$
2.	AY	09	$09 \times 3 = 18$
3.	BZ	05	$05 \times 1 = 05$
4.	B Dummy	03	$03 \times 0 = 00$
5.	CY	03	$03 \times 4 = 12$
6.	C Dummy	02	$02 \times 0 = 00$
7.	DX	06	$06 \times 3 = 18$
	Total cost in Rs.		66

Matrix B (First alternative solution):

Matrix C (Second alternate solution)

The total cost is $0.5 \times 4 + 9.5 \times 3 + 0.5 \times 5 + 5 \times 1 + 2.5 \times 0 + 2.5 \times 0 + 2.5 \times 0 + 6 \times 3 = $ Rs. 66/-

Once we get one alternate solution we can go on writing any number of alternate solutions until we get the first optimal solution.

4.5. MAXIMIZATION CASE OF TRANSPORTATION PROBLEM

Basically, the transportation problem is a minimization problem, as the objective function is to minimize the total cost of transportation. Hence, when we would like to maximize the objective function. There are two methods.

(*i*) The given matrix is to be multiplied by –1, so that the problem becomes maximization problem. Or ii) Subtract all the elements in the matrix from the highest element in the matrix. Then the problem becomes maximization problem. Then onwards follow all the steps of maximization problem to get the solution. Let us consider the same problem solved above.

Problem 4.2. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the returns the factory can get, by selling the sugar in each market. Formulate a transportation problem and solve for maximizing the returns.

	Profit in Rs. per ton ($\times$ 100) Markets.			*Availability in tons.*
	X	*Y*	*Z*	
Factories.				
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma\ b = 29,\ \Sigma\ d = 23$

Here $\Sigma\ b$ is greater than $\Sigma\ d$ hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by VAM and then give optimality test by MODI method. The balanced matrix of the transportation problem is:

By multiplying the matrix by –1, we can convert it into a maximisation problem. Now in VAM we have to find the row opportunity cost and column opportunity costs. In minimisation problem, we use to subtract the smallest element in the row from next highest element in that row for finding row opportunity cost. Similarly, we use to subtract smallest element in the column by next highest element

in that column to get column opportunity cost. Here as we have multiplied the matrix by –1 the highest element will become lowest element. Hence subtract the lowest element from the next highest element as usual. Otherwise, instead of multiplying by –1 simply find the difference between highest element and the next lowest element and take it as opportunity cost of that row or column. For example in the given problem in the row A, the highest element is 4 and the next lowest element is 3 and hence the opportunity cost is 4 – 3 = 1. (Or smallest element is –4 and the next highest element is –3 and the opportunity cost is –3 – (– 4) = –3 + 4 = 1). Similarly, we can write all opportunity costs. Once we find the opportunity costs, rest of the procedure is same. That is, we have to select highest opportunity cost and select the highest profit element in that row or column for allocation. Obtain the basic feasible solution. As usual the basic feasible solution must have $m + n – 1$ allocations. If the allocations are not equal to $m + n – 1$, the problem degenerate. In that case, add $\in$ to an empty cell, which do not form loop with other loaded cells. Once we have basic feasible solution, the optimality test by MODI method, is followed. Here, once the opportunity costs of all the cells are positive, (as we have converted the maximistion problem into minimisation problem) the solution is said to be optimal.

In the given problem as the opportunity costs of all empty cells are positive, the solution is optimal. And the optimal return to the company is Rs. 125/-.

Allocations:

S.No	Loaded Cell	Load	Cost in Rs.
1.	AX	02	$02 \times 4 = 08$
2.	AY	03	$03 \times 3 = 09$
3.	A Dmy	05	$05 \times 0 = 00$
4.	BY	08	$08 \times 6 = 48$
5.	CX	05	$05 \times 6 = 30$
6.	DY	01	$01 \times 5 = 05$
7.	DZ	05	$05 \times 4 = 20$
	Total returns in Rs.		125

(1)

(2)

	X	Y	Z	MaY		
A	4	3	2	··	10	1
B	··	··	1	1	8	
F	3	··	4 5	··	··	1
	7	12	5	3	24	
	·	1	2	··		

↑

(3)

	X	Y	DMY		
A	4	3	··	10	1
B	5	6	··	8	1
D	1	5 1	··	·	2 ←
	1	12	5	19	
	1	1	··		

(4)

	X	Y	DMY		
A	4	3	··	10	1
B	5	5 8	11	5	·
	2	11	5	18	
	1	3	··		

↑

(5)

	X	Y	DMY	
A	4 2	3 3	11 5	10
	2	3	5	10

4.6. DEGENERACY IN TRANSPORTATION PROBLEM

Earlier, it is mentioned that the basic feasible solution of a transportation problem must have $(m + n – 1)$ basis variables or allocations. This means to say that the number of occupied cells or loaded cells in a given transportation problem is 1 less than the sum of number of rows and columns in the transportation matrix. Whenever the number of occupied cells is less than $(m + n – 1)$, the transportation problem is said to be degenerate.

Degeneracy in transportation problem can develop in two ways. First, the problem becomes degenerate when the initial programme is designed by northwest corner or inspection or VAM, *i.e.* at the stage of initial allocation only.

To solve degeneracy at this stage, we can allocate extremely small amount of goods (very close to zero) to one or more of the empty cells depending on the shortage, so that the total occupied cells becomes $m + n – 1$. The cell to which small element (load) is allocated is considered to be an occupied cell. In transportation problems, Greek letter '$\in$' represents the small amount. One must be careful enough to see that the smallest element epsilon is added to such an empty cell, which will enable us to write row number 'u_i' and column number 'v_j' without any difficulty while giving optimality test to the basic feasible solution by MODI method. That is care must be taken to see that the epsilon is added to such a cell, which will not make a **closed loop**, when we move horizontally and vertically from loaded cell to loaded cell.

(Note: Epsilon is so small so that if it is added or subtracted from any number, it does not change the numerical value of the number for which it added or from which it is subtracted.).

Secondly, the transportation problem may become degenerate during the solution stages. This happens when the inclusion of a most favorable empty cell *i.e.* cell having highest opportunity cost results in simultaneous vacating of two or more of the currently occupied cells. Here also, to solve degeneracy, add epsilon to one or more of the empty cells to make the number of occupied cells equals to $(m + n – 1)$.

To understand the procedure let us solve one or two problems.

Problem. 4.3. Solve the transportation problem given below

Solution by Northwest corner method:

Initial allocation show that the solution is not having ($m+n-1$) allocations. Hence degeneracy occurs.

Cost in Rs. per unit

Destinations

Origins	A	B	C	Available capacity	Row number u_i
X	(20) [2]	[1]	[2]	20	
Y	[3]	(15) [4]	(25) [1]	40	
Requirement	20	15	25	60	

Cost in Rs. per unit

Destinations

Origins	A	B	C	Available capacity	Row number u_i
X	(30) [2]	(∈) [1]	[2]	20	
Y	[3]	(15) [4]	(25) [1]	40	
Requirement	20	15	25	60	
Column elements					

The smallest load $\in$ is added to cell *XB* which does not make loop with other loaded cells.

Cost in Rs. per unit

Destinations

Origins	A	B	C	Available capacity	Row number u_i
X	(30) [2]	(∈) [1]	[2]	20	
Y	[3]	(15) [4]	(25) [1]	40	
Requirement	20	15	25	60	
Column elements					

Shifting of load by drawing loops to cell YA.

The basic feasible solution is having four loaded cells. As the number of columns is 3 and number of rows is 2 the total number of allocations must be $2 + 3 - 1 = 4$. The solution got has four allocations. Hence the basic feasible solution. Now let us give optimality test by MODI method.

Row numbers u_i s and column numbers v_j s are written in the matrix and opportunity cost of empty cells are evaluated. As the opportunity cost of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	Loaded Cell	Load	Cost in Rs.
1.	XA	05	$05 \times 2 = 50$
2.	XB	15	$15 \times 1 = 15$
3.	YA	15	$15 \times 3 = 45$
4.	YC	25	$25 \times 1 = 25$
	Total cost in Rs.		135

Problem. 4.4. Solve the transportation problem given below:

Cost in Rs. per unit

To → From ↑	D_1	D_2	D_3	D_4	D_5	Availability	u
O_1	4 ㉚	3 ⑩	1	2	6	40	0
O_2	5	2 ⑳	3 ⑩	4	5	30	–1
O_3	3	5	6 ⑤	3 ⑮	2	20	2
O_4	2	4	4	3 ⑤	3 ⑤	10	4
Requirement	30	30	15	20	5	100	
v	4	3	4	1	1		

Let us make initial assignment by using Northwest corner method. To modify the solution we include the cell O_1D_3 in the programme, as it is having highest opportunity cost.

Improved solution:

Cost in Rs. per unit

To → From ↑	D_1	D_2	D_3	D_4	D_5	Availability	u
O_1	4 ㉚	3 ⓪	1 ⑩	2	6	40	0
O_2	5	2 ㉚	3	4	5	30	0
O_3	3	5	6 ⑤	3 ⑮	2	20	5
O_4	2	4	4	3 ⑤	3 ⑤	10	7
Requirement	30	30	15	20	5	100	
v	4	2	1	2	4		

Total number of allocations are less than $m + n - 1$. Hence we have to add one epsilon to an empty cell. Remember, in transportation problem, which has minimization of cost as its objective function a, we have to add epsilon to recently vacated cell, which is having lowest shipping cost. We have a tie between two cells, *i.e.* O_1D_2 and O_2D_3. Let us select O_1D_2 to add epsilon. To improve the solution, let us take empty cell O_4D_1 in the programme.

Improved Programme: The solution is not having $m + n - 1$ allocations. We have to add epsilon; in the programme epsilon is added to cell 0_4D_4

Revised Programme.

Cost in Rs. per unit

The epsilon is shifted to an empty cell. The improved solution is having 8 allocations. Hence a feasible solution.

As the cell O_1D_4 having positive opportunity cost, let us include and revise the programme.

Revised programme. Cell 0_3D_5 having positive opportunity cost is included in revised programme.

Cost in Rs. per unit

Revised programme: Cell O_3D_1 having positive opportunity cost is included in the revised programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u
O_1	㉔ [4]	② [3]	⑮ [1]	⑤ [2]	[6]	40	0
O_2	[5]	㉚ [2]	[3]	[4]	[5]	50	−1
O_3	+ [3]	[5]	[6]	⑱ [1]	② [2]	20	1
O_4	⑩ [2]	[1]	[1]	[5]	[3]	10	−2
Requirement	32	30	15	20	3	100	
v	4	2	1	2	1		

Revised Programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u
O_1	⑤ [4]	② [3]	⑮ [1]	⑳ [2]	[6] −1	40	0
O_2	[5] −1	㉚ [2]	[3] −3	[4] −1	[5] −1	50	−1
O_3	⑮ [3]	[5] 1	[6] 6	[1] 2	⑤ [2]	20	1
O_4	⑩ [2]	[1] −1	[1] 6	[5] 6	[3] 3	10	−3
Requirement	32	30	15	20	3	100	
v	4	3	1	2	3		

As the opportunity costs of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	Loaded Cell	Load	Cost in Rs.
1.	O_1D_1	5	$5 \times 4 = 20$
2.	O_1D_2	ε	----
3.	O_1D_3	15	$15 \times 1 = 15$
4.	O_1D_4	20	$20 \times 2 = 40$
5.	O_2D_2	30	$30 \times 2 = 60$
6.	O_3D_1	15	$15 \times 3 = 45$
7.	O_3D_5	5	$5 \times 2 = 10$
8.	O_4D_1	10	$10 \times 2 = 20$
	Total Cost in Rs.		210/-

The same problem, if we solve by VAM, the very first allocation will be feasible and optimality test shows that the solution is optimal.

Roc: Row opportunity cost, COC= Column opportunity cost, Avail: Availability, Req: Requirement.

	D_1	D_2	D_3		D_4	D_5	Avail	ROC
O_1	4	3	1	**15**	2	6	40	1
O_2	5	2	3		4	5	30	1
O_3	3	5	6		3	2	20	1
O_4	2	4	4		5	3	10	1
REQ	30	30	15		20	5	100	
COC	1	1	2		1	1		

	D_1	D_2		D_4	D_5	Avail	ROC
O_1	4	3		2	6	25	1
O_2	5	2	**30**	4	5	30	2
O_3	3	5		3		20	1
O_4	2	4		5	3	10	1
REQ	30	30		20	5	85	
COC	1	1		1	1		

	D_1	D_4		D_5	Avail	ROC
O_1	4	2	**20**	6	25	2
O_3	3	3		2	20	1
O_4	2	5		3	10	1
REQ	30	20		5	55	
ROC	1	1		1		

	D_1		D_5	Avail	ROC	
O_1	4	5	6	5	2	←
O_3	3		2	20	1	
O_4	2		3	10	1	
REQ	30		5	35		
COC	1		1			

	D_1		D_5	Avail	ROC	
O_3	3		2	20	1	
O_4	2	10	3	10	1	←
REQ	25		5	30		
COC	1		1			

	D_1		D_5		AVAIL
O_3	3	15	2	5	20
REQ	15		5		20

Allocation by VAM:

Cost in Rs. per unit

   Allocations are same as in the optimal solution got by northwest corner method. All opportunity costs of empty cells are negative. Hence the total transportation cost is Rs. 210/-

4.7. TIME MINIMISATION MODEL OR LEAST TIME MODEL OF TRANSPORTATION TIME.

It is well known fact that the transportation problem is cost minimization model, *i.e* we have to find the least cost transportation schedule for the given problem. Some times the cost will become secondary factor when the time required for transportation is considered. This type of situation we see in military operation. When the army want to send weapons or food packets or medicine to the war front, then the time is important than the money. They have to think of what is the least time required to transport the goods than the least cost of transportation. Here the given matrix gives the time elements, *i.e.* time required to reach from one origin to a destination than the cost of transportation of one unit from one origin to a destination. A usual, we can get the basic feasible solution by Northwest corner method or by least time method or by VAM. To optimize the basic feasible solution, we have to identify the highest time element in the allocated cells, and try to eliminate it from the schedule by drawing loops and encouraging to take the cell, which is having the time element less than the highest one. Let us take a problem and work out the solution. Many a time, when we use VAM for basic feasible solution, the chance of getting an optimal solution is more. Hence, the basic feasible solution is obtained by Northwest corner method.

Problem 4.5. The matrix given below shows the time required to shift a load from origins to destinations. Formulate a least time schedule. Time given in hours.

Roc: Row opportunity cost, Coc: Column opportunity cost, Avail: Availability, Req: Requirement.

1. Initial assignment by Northwest corner method: The Maximum time of allocated cell is 17 hours. Any cell having time element greater than 17 hours is cancelled, so that it will not in the programme.

By drawing loops, let us try to avoid 17 hours cell and include a cell, which is having time element less than 17 hours. The basic feasible solution is having $m + n - 1$ allocations.

Here also the maximum time of transport is 17 hours.

In this allocation highest time element is 11 hours. Let us try to reduce the same.

In this allocation also the maximum time element is 11 hours. Let us try to avoid this cell.

No more reduction of time is possible. Hence the solution is optimal and the time required for completing the transportation is 10 Hours. T_{max} = 10 hours.

4.8. PURCHASE AND SELL PROBLEM: (TRADER PROBLEM)

Problem. 4.7 M/S Epsilon traders purchase a certain type of product from three manufacturing units in different places and sell the same to five market segments. The cost of purchasing and the cost of transport from the traders place to market centers in Rs. per 100 units is given below:

Place of Manufacture.	Availability In units x 10000.	Manufacturing cost in Rs. per unit	Market Segments. (Transportation cost in Rs.per 100 units).				
			1	2	3	4	5
Bangalore (B)	10	40	40	30	20	25	35
Chennai (C)	15	50	30	50	70	25	40
Hyderabad (H)	5	30	50	30	60	55	40
	Requirement in units × 10000		6	6	8	8	4

The trader wants to decide which manufacturer should be asked to supply how many to which market segment so that the total cost of transportation and purchase is minimized.

Solution

Here availability is 300000 units and the total requirement is 320000 units. Hence a dummy row (D) is to be opened. The following matrix shows the cost of transportation and purchase per unit in Rs. from manufacturer to the market centers directly.

	1	2	3	4	5	Availability
B	4040	4030	4020	4025	4035	10
C	5030	5050	5070	5025	5040	15
H	3050	3030	3060	3055	3040	5
D	0	0	0	0	0	2
Requirement.	6	6	8	8	4	32

Let us multiply the matrix by 100 to avoid decimal numbers and get the basic feasible solution by VAM.

Table. Avail: Availability. Req: Requirement, Roc: Row opportunity cost, Coc: Column opportunity cost.

Tableau. I Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail	Roc
H	4800	4850	4820	4825	4835	10	
C	5800	5850	5820	5825	5840	15	
H	3850	3830	3840	3855	3800	5	
IF	0	0	0	0	0	2	
Req	6	6	8	8	4	32	
Coc							

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail	Roc
H	4800	4850	4820	4025	4035	10	5
C	5000	5850	5020	5025	5040	15	5
H	3050	3830	3041	3055	3000	5	10
IF	0	0	②0	0	0	2	0
Req	6	6	8	8	4		
Coc							

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail	Roc
H	4800	4850	4820	4025	4035	10	5
C	5000	5850	5020	5025	5040	15	5
H	3050	③3830	3041	3055	3000	5	10
IF	0	0	②0	0	0	2	
Req	6	6	6	8	4	30	
Coc							

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail	Row
H	4042	4030	⑤ 4020	4025	4035	10	5
C	5050	5050	5070	5025	5040	15	5
11	3050	⑤ 3030	3040	3055	3050	5	5
D	0	0	② 0	0	0	2	0
Req	6	1	6	8	4	27	
Coc	1055	1020	1055	1055	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail	Row
H	4045	① 4030	⑥ 4020	4025	4035	1	5
C	5050	5050	5070	5025	5040	15	5
11	3050	⑤ 3030	3040	3055	3050		
D	0	0	② 0	0	0		
Req	6	1		8	4	19	
Coc	995	1020		1055	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail	Row
H	1045	① 4030	⑥ 4020	4025	1055 3	3	5
C	5070	5050	5070	5025	5040	15	5
11	3050	⑤ 3030	3040	3055	3050		5
D	0	0	2 0	0	0		0
Req	6			8	4	18	
Coc	995			1055	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail
B	4005 ①	4020 ⑥	4025 ③	4035	20
C	5030 ⑥	5050	5070	5025 ⑧	5040 ①	18
H	3005	3020 ⑤	3070	3065	3010	5
D	0	0	0 ₂	0	1	2
Req	6	6	8	8	4	32

Final Allocation by MODI method.

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail
B	4005 ①	4020 ⑧	4025 ①	4035	20
C	5030 ⑥	5050	5070	5025 ⑧	5040 ①	18
H	3005	3020 ⑤	3070	3065	3010	5
D	0	0	0	0	1 ②	2
Req	6	6	8	8	4	32

Allocation:

From	To	Load	Cost in Rs.
Bangalore	2	10,000	4,03,000
Bangalore	3	80,000	32, 16,000
Bangalore	5	10,000	4, 03,000
Chennai	1	60,000	30, 18,000
Chennai	4	80,000	40, 20,000
Chennai	5	10,000	5, 04,000
Hyderabad	2	50,000	15, 15,000
Total cost in Rs.			1,30, 79,000

4.9. MAXIMISATION PROBLEM: (PRODUCTION AND TRANSPORTATION SCHEDULE FOR MAXIMIZATION)

This type of problems will arise when a company having many units manufacturing the same product and wants to satisfy the needs of various market centers. The production manager has to work out for transport of goods to various market centers to cater the needs. Depending on the production schedules and transportation costs, he can arrange for transport of goods from manufacturing units to the market centers, so that his costs will be kept at minimum. At the same time, this problem also helps him to prepare schedules to aim at maximizing his returns.

Problem.4.8. A company has three manufacturing units at X, Y and Z which are manufacturing certain product and the company supplies warehouses at A, B, C, D, and E. Monthly regular capacities for regular production are 300, 400 and 600 units respectively for X, Y and Z units. The cost of production per unit being Rs.40, Rs.30 and Rs. 40 respectively at units X, Y and Z. By working overtime it is possible to have additional production of 100, 150 and 200 units, with incremental cost of Rs.5, Rs.9 and Rs.8 respectively. If the cost of transportation per unit in rupees as given in table below, find the allocation for the total minimum production cum transportation cost. Under what circumstances one factory may have to work overtime while another may work at under capacity?

Transportation cost in Rs.   To

From	A	B	C	D	E
X	12	14	18	13	16
Y	11	16	15	11	12
Z	16	17	19	16	14
REQ	400	400	200	200	300

(*a*) If the sales price per unit at all warehouses is Rs. 70/- what would be the allocation for maximum profit? Is it necessary to obtain a new solution or the solution obtained above holds valid?

(*b*) If the sales prices are Rs.70/-, Rs. 80/-, Rs. 72/-, Rs. 68/- and Rs. 65/- at A, B, C, D and E respectively what should be the allocation for maximum profit?

Solution: Total production including the overtime production is 1750 units and the total requirement by warehouses is 1500 units. Hence the problem is unbalanced. This can be balance by opening a Dummy Row (*DR*), with cost coefficients equal to zero and the requirement of units is 250. The cost coefficients of all other cells are got by adding production and transportation costs. The production cum transportation matrix is given below:

	A	B	C	D	E	DC	Availability
X	52	54	58	53	56	0	300
Y	41	46	45	41	42	0	400
Z	56	57	59	56	54	0	600
XOT	57	59	63	58	61	0	100
YOT	50	55	54	50	51	0	150
ZOT	64	65	67	64	62	0	200
Requirement:	400	400	200	200	300	250	1750

Initial Basic feasible solution by VAM:

	A	B	C	D	E	DR	Avail	u_i
X	52 (300)	54	58	55	56	0	300	52
		0	2	0	–1	7		
Y	41	42	40	41 (100)	42 (300)	0	400	40
	4	1				17		
Z	56	57 (400)	57 (100)	56 (100)	54	0	600	55
	–				–	–2		
XOT	57 (50)	56	63	58	61	0	100	57
		0	2	0	4	50		
YOT	50 (50)	55	51 (100)	50	51	0	50	50
		–3		–1	–1	–7		
ZOT	64	63	67	64	62	0 (200)	200	57
	7	6	6	6	5			
REQ	400	400	200	200	300	250	1750	
v_j	0	2	4	1	0	–57		

As we have $m + n - 1$ (= 11) allocations, the solution is feasible and all the opportunity costs of empty cells are negative, the solution is optimal.

Allocations:

Cell	Load	Cost in Rs.
XA	300	$300 \times 52 = 15,600$
YD	100	$100 \times 41 = 4,100$
YE	300	$300 \times 40 = 12,000$
ZB	400	$400 \times 54 = 21,000$
ZC	100	$100 \times 59 = 5,900$
ZD	100	$100 \times 56 = 5,600$
XOT A	50	$50 \times 57 = 2,850$
XOT DR	50	$50 \times 0 = 0$
YOT A	50	$50 \times 50 = 5,500$
YOT C	100	$100 \times 54 = 5,400$
ZOT DR	50	$50 \times 0 = 0$
Total Cost in Rs.		$75,550$

Allocation by VAM:

(1)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0	100	50
YOT	50	55	54	50	51	0	150	50
ZOT	64	65	67	64	62	0 **(200)**	2 00	62	←
REQ	400	400	2 00	2 00	300	2 50	1750	
COC	9	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0 **(50)**	100	57	←
YOT	50	55	54	50	51	0	150	50
REQ	400	400	2 00	2 00	300	2 50	15 50	
COC	9	8	9	9	9	0		

(3)

	A	*B*	*C*	*D*	*E*	*AVAIL*	*ROC*
X	52	54	58	53	56	300	1
Y	41	46	45	41	42 **(300)**	400	0
Z	56	57	59	56	54	600	2
XOT	57	59	63	58	61	50	2
YOT	50	55	54	50	51	150	0
REQ	400	400	2 00	2 00	300	1500	
COC	9	8	9	9	9		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two ∈ s to two empty cells.

(4)

	A	B	C	D	AVAIL	ROC
X	52	54	58	53	300	1
Y	41	46	45	41 **(100)**	100	0
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54	50	150	0
REQ	400	400	2 00	2 00	1200	
COC	9	8	9	9		

↑

(5)

	A	B	C	D	Avail	Roc
X	52	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54 **(150)**	50	150	0
Req	400	400	200	100	1100	
Coc	2	1	4	3		

↑

(6)

	A	B	C	D	Avail	Roc
X	52 **(300)**	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
Req	400	400	50	100	950	
Coc	4	3	1	3		

↑

(6)

	A	B	C	D	Avail	Roc
Z	56	57	59 **(50)**	56	550	0
XOT	57	59	63	58	50	1
Req	100	400	50	100	600	
Coc	1	2	4	2		

↑

(7)

	A	*B*	*D*		*Avail*	*Roc*
Z	56	57	56	**(100)**	550	0
XOT	57	59	58		50	1
Req	100	400	100		600	
Coc	1	2	2			

(8)

	A	*B*		*Avail*	*Roc*
Z	56	57	**(400)**	450	1
XOT	57	59		50	2
Req	100	400		500	
Coc	1	2			

(9)

	A	*Avail*
Z	56 **(50)**	50
XOT	57 **(50)**	50
	100	

In the table showing optimal solution, we can understand that the company *X* has to work 50% of its over time capacity, and company *Y* has to work 100% of its overtime capacity and company Z will not utilize its overtime capacity.

(*a*) Here the total profit or return that the trading company gets is equals to Sales revenue – total expenses, which include manufacturing cost and transportation cost. Hence,

Profit = (Total Sales Revenue) – (Manufacturing cost + transportation cost).

In the question given the sales price is same in all market segments, hence, the profit calculated is independent of sales price. Hence the programme, which minimizes the total cost will, maximizes the total profit. Hence the same solution will hold good. We need not work a separate schedule for maximization of profit.

(*b*) Here sales price in market segments will differ. Hence we have to calculate the total profit by the formula given above for all the markets and work for solution to maximise the profit.

The matrix showing the total profit earned by the company:

	A	B	C	D	E	IX	Avail	u_i
X	19	26 (300)	14	15	9	11 500		
	)	5	3	8	(			
Y	29 (300)	31	27 (ε)	27	25	0 400		14
	16		1	0	11			
Z	14	23	13 (200)	12 (50)	11 (300)	11 400 (50)		
	1	ε						
XOT	3	21 (100)	9	19	4	11 100		1
	3	5	3	4	1			
YOT	2	25	18	18 (150)	14	11 150		
	1	1		3	6			
ZOT	6	15	5	4	3	11 200 (200)		
	9	5	8	8	8			
Req	400	100 200	200	300	250	1750		
Coe	19	23 13	12	11	11			

As all the opportunity cost of empty cells are positive (maximization problem), the solution is optimal.

The allocations are:

Cell	Load	Cost in Rs.
XB	300	$300 \times 26 = 7,800$
YA	400	$400 \times 29 = 11,600$
ZC	200	$200 \times 13 = 2,600$
ZD	50	$50 \times 12 = 600$
ZE	300	$300 \times 11 = 3,300$
Z DR	50	$50 \times 0 = 0$
XOT B	100	$100 \times 21 = 2,100$
YOT D	150	$150 \times 18 = 2,700$
ZOT DR	200	$200 \times 0 = 0$
Profit in Rs.		$= 30,700$

	A	B	C	D	E	DC	Avail	Coc
X	18	26	14	15	9	0	300	8
Y	29 **400**	34	27	27	25	0	400	5
Z	14	23	13	12	11	0	600	9
XOT	13	21	9	10	4	0	100	8
YOT	20	25	18	18	14	0	150	5
ZOT	6	15	5	4	3	0	200	9
Req	400	400	200	200	300	250	1750	
Coc	11	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	B	C	D	E	DC	Avail	Coc
X	26 **300**	14	15	9	0	300	11
Z	23	13	12	11	0	600	10
XOT	21	9	10	4	0	100	11
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	400	200	200	300	250	1350	
Coc	1	4	3	5	0		

(3)

	B	C	D	E	DC	Avail	Coc
Z	23	13	12	11	0	600	10
XOT	21 **100**	9	10	4	0	100	11
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	100	200	200	300	250	1050	
Coc	2	5	6	3	0		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two ∈ s to two empty cells.

(4)

	C	D	E	DC	Avail	Coc
Z	13	12	11	0	600	1
YOT	18	18 **150**	14	0	150	0
ZOT	5	4	3	0	200	1
Req	200	200	300	250	950	
Coc	5	6	3	0		

(5)

	C	D	E	DC	Avail	Coc
Z	13 **200**	12	11	0	600	1
ZOT	5	4	3	0	200	1
Req	200	50	300	250	800	
Coc	8	8	8	0		

(6)

	D	E	DC	Avail	Coc
Z	12	11 **300**	0	400	1
ZOT	4	3	0	200	1
Req	50	300	250	600	
Coc	8	8	0		

(7)

(8)

Problem. 4.9. A company has booked the orders for its consignment for the months of April, May, June and July as given below:

April: 900 units, May: 800 units, June: 900 units and July: 600 units. The company can produce 750 units per month in regular shift, at a cost of Rs. 80/- per unit and can produce 300 units per month by overtime production at a cost of Rs. 100/- per unit. Decide how much the company has to produce in which shift for minimizing the cost of production. It is given that there is no holding cost of inventory.

Solution: Remember here the production of April is available to meet the orders of April and subsequent months. But the production of May cannot be available to meet the demand of April. Similarly, the production of June is not available to meet the demand of April, May, but it can meet the demand of June and subsequent months and so on. Hence very high cost of production is allocated to the cells (Infinity or any highest number greater than the costs given in the problem), which cannot meet the demands of previous months (*i.e.* back ordering is not allowed). Here total availability is 4200 units and the total demand is for 3200 units. Hence we have to open a dummy column (DC), with cost coefficients equal to zero. The balanced matrix is shown below. Let us find the initial basic feasible solution by Northwest corner method and apply optimality test by MODI method.

A: April, M: May, J: June, Jl: July, AT: April Over time, MT: May overtime, JT: June overtime, JLT: July over time. DC : Dummy column.

Tableau 1.

Here the cell JT DC is having highest opportunity cost. Hence let us include the cell in the revised programme. To find the opportunity costs of empty cells, the row number u_i and column number v_j are

shown. The cells marked with (*X*) are avoided from the programme. We can also allocate very high cost for these cells, so that they will not enter into the programme.

Tableau II. Revised programme.

As the opportunity costs of all empty cells are either zeros or negative elements, the solution is optimal. As many empty cells are having zero as the opportunity cost, they can be included in the solution and get alternate solution.

Allocations:

Demand month.	Production of the month	Load	Cost in Rs.
April	April regular	750	$750 \times 80 = 60,000$
April	April over time	150	$150 \times 100 = 15,000$
May	April over time	150	$150 \times 100 = 15,000$
May	May regular	650	$650 \times 80 = 52,000$
June	May regular	100	$100 \times 80 = 8,000$
June	May over time	300	$300 \times 100 = 30,000$
June	June Regular	500	$500 \times 80 = 40,000$
July	June regular	250	$250 \times 80 = 20,000$
July	July regular	350	$350 \times 80 = 28,000$
Dummy column	June over time	300	300×0
Dummy Column	July regular	300	300×0
Dummy column	July over time	300	300×0
	Total cost in Rs.:		2,68,000

Problem: 4.10. Let us slightly change the details given in the problem 4.9. It is given that production of a month could be stored and delivered in next month without extra costs. Let us now consider that there is a cost associated with inventory holding or inventory carrying cost. Let the inventory carrying cost is Rs. 20 per month decide the new allocation.

Solution: In the cost matrix, for regular production, the cost is Rs. 80/-, for overtime production, the cost is Rs. 100 and for the stock held the inventory carrying cost is Rs. 20/ per month. If the stock is held for two months the inventory carrying cost is Rs. 40/-. That is if the production of April is supplied in June the cost will be Rs. 80/- + Rs. 40/- =

Rs. 120/- and do on. The initial basic feasible solution is obtained by Northwest corner method.

Cell AOT DC is having highest positive opportunity cost. Hence we have to include this in the revised programme.

Production cost in Rs.

	April	May	June	July	IDC	Avail.	u_i	
April	80 ┐ 100 (750) 0		120 –20	140 ┐ 0 –80 –20		750	20	
AOM	30 ┐ 120 (150) (50) –20	40 –20	140 ┐ 0 150 –80			300	0	
May	X	80 ┐ (750) –20	40 –20	120 ┐ 0 –40		750	–10	
MOM	X 20	100 ┐	20 (150) –40	140 ┐ 0 150		300	0	
June	X	X	80 ┐ (750) 60	100 ┐ 0 10		750	10	
JOJ	X	X	20 50	120 ┐ 0 (300) –40		300	0	
July	X	X	X	80 ┐ 0 600 (150)		750	0	
JOJ	X	X	X	100 ┐ 0 (300) –20		300	0	
Req	900	800	900	600 1500		4200		
v_j	100 120	120	80	0				

In the above matrix, two cells, MO M and JO J are having positive opportunity costs = 20. Hence, they may be included in the revised programme. If we include them in the programme, the final optimal solution will be as follows:

Production cost in Rs.

	April	May	June	July	IBC	Avail.	u
April	80 ⑦⑤⓪	1??	.2u	.1? 0	0	75?	2?
		-2?	-4u	-5?	-2u		
Aust	1?? ①⑤⓪	12?	.1u	2??	0 ①⑤⓪	3??	0
		-2?	-4u	-8			
May	X	8? ⑦⑤⓪	?? 2u	2? -8?	0 -2u	75u	-2?
Mu?	X	1?? ⑤⓪	.2u -2u	.1? -8?	0 ②⑤⓪	3u?	0
June	X	X	8u ⑦⑤⓪ 2u	?? 2u	0	75?	2?
J??	X	X	?? ①⑤⓪ -u	2? ①⑤⓪	0	3??	0
July	X	X	X	8u ⑥⓪⓪ -2?	0 ①⑤⓪	75u	0
J???	X	X	X	?? ③⓪⓪	0	3u?	0
Req	9??	8u?	9??	6??	15?u	42u?	
v	1??	1??	1??	8?	0		

As all the opportunity costs of empty cells are negative, the solution is optimal. The optimal allocations are:

Month of demand	Month of production	load	cost in Rs.		Rs.
April	April regular	750	750 × 80	=	60,000
April	April over time	150	150 × 100	=	15,000
Dummy Col	April over time	150	150 × 0	=	0
May	May regular	750	750 × 80	=	60,000
May	May over time	50	50 × 100	=	5,000
Dummy Column	May over time	250	250 × 0	=	0
June	June regular	750	750 × 80	=	60,000
June	June over time	150	150 × 100	=	15,000
Dummy column	June over time	150	150 × 0	=	0
July	July regular	600	600 × 80	=	48,000
Dummy column	July regular	150	150 × 0	=	0
Dummy column	July overtime	300	300 × 0	=	0
	Total cost in Rs.				2,63,000

4.10. TRANSSHIPMENT PROBLEM

We may come across a certain situation, that a company (or companies) may be producing the product to their capacity, but the demand arises to these products during certain period in the year or the demand may reach the peak point in a certain period of the year. This is particularly true that products like Cool drinks, Textbooks, Notebooks and Crackers, etc. The normal demand for such products will exist, throughout the year, but the demand may reach peak points during certain months in the year. It may not possible for all the companies put together to satisfy the demand during peak months. It is not possible to produce beyond the capacity of the plant. Hence many companies have their regular production throughout the year, and after satisfying the existing demand, they stock the excess production in a warehouse and satisfy the peak demand during the peak period by releasing the stock from the warehouse. This is quite common in the business world. Only thing that we have to observe the inventory carrying charges of the goods for the months for which it is stocked is to be charged to the consumer. Take for example crackers; though their production cost is very much less, they are sold at very high prices, because of inventory carrying charges. When a company stocks its goods in warehouse and then sends the goods from warehouse to the market, the problem is known as **Transshipment problem**. Let us work one problem and see the methodology of solving the Transshipment Problem.

Problem. 4.11. A company has three factories *X*, *Y* and *Z* producing product *P* and two warehouses to stock the goods and the goods are to be sent to four market centers *A*, *B*, *C* and *D* when the demand arises. The figure given below shows the cost of transportation from factories to warehouses and from warehouses to the market centers, the capacities of the factories, and the demands of the market centers. Formulate a transportation matrix and solve the problem for minimizing the total transportation cost.
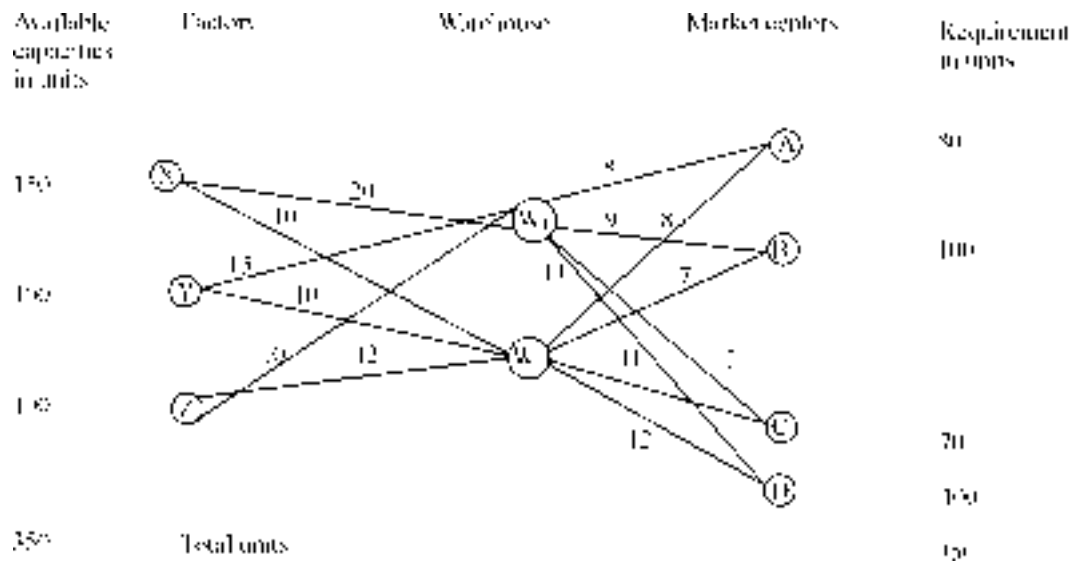
Solution:

To formulate a transportation problem for three factories and four market centers, we have to find out the cost coefficients of cells. For this, if we want the cost of the cell *XA*, the cost of transportation from *X* to warehouse W_1 + Cost transportation from W_1 to market center *A* are calculated and as our objective is to minimize the cost, the least of the above should be entered as the cost coefficient of cell *XA*. Similarly, we have to workout the costs and enter in the respective cells.

Cell *XA*: Route X-W_1-A and X- W_2- A minimum of these two (28 and 18) i.*e* 18
Cell *XB* Route X - W_1 - B and X - W_2 - B Minimum of the two is (29, 17) *i.e* 17
Cell *XC* Route X - W_1 - C and X- W_2 - C Minimum of the two is (27, 11) *i.e* 11
Cell *XD* Route X- W_1 - D and X- W_2 - D Minimum of the two is (34, 22) *i.e* 22

Similarly we can calculate for other cells and enter in the matrix. The required transportation problem is:

	A	*B*	*C*	*D*	*Available*
X	18	17	21	22	150
Y	18	17	21	22	100
Z	18	19	17	24	100
Required.	80	100	70	100	350

Basic Feasible Solution by VAM:

As the opportunity costs of all empty cells are negative, the solution is optimal. The optimal allocation is:

Cell	Route	Load	Cost in Rs.		Rs.	
XA	X-W_2-A	50	50×18	=	900	(The answer shows that the
XB	X - W_2 - B	100	100×17	=	1700	capacity of W_2 is 250 units and
YB	Y- W_2 - B	ε	---	=	----	capacity of W_1 is100 units).
YD	Y - W_2- D	100	100×22	=	2200	
ZA	Z- W_1- A	30	30×18	=	540	
ZC	Z - W_1 - C	70	70×17	=	1190	
			Total Cost in Rs.		6530	

(1)

	A	B	C	D	Avail	Roc
X	18	17	21	22	150	0
Y	18	17	21	22	100	0
Z	18	19	17 **70**	24	100	2
Req.	80	100	70	100	350	
Coc	18	17	15	22		

(2)

	A	B	D	Avail	Roc
X	18	17	22	150	1
Y	**80** 18	17	22	100	1
Z	18	19	24	30	1
Req.	80	100	100	280	
Coc	0	0	0		

(3)

	B	D	Avail	Roc
X	17 **70**	22	70	5
Y	17	22	100	5
Z	19	24	30	5
Req.	100	100	200	
Coc	0	0		

(4)

	B	D	Avail	Roc	
Y	17 **30**	22	100	5	←
Z	19	24	30	5	
Req.	30	100	200		
Coc	2	2			

	D	Avail	Roc	
Y	22 **70**	70		←
Z	24 **30**	30		
Req.	100	100		
Coc				

Problem 4.12.
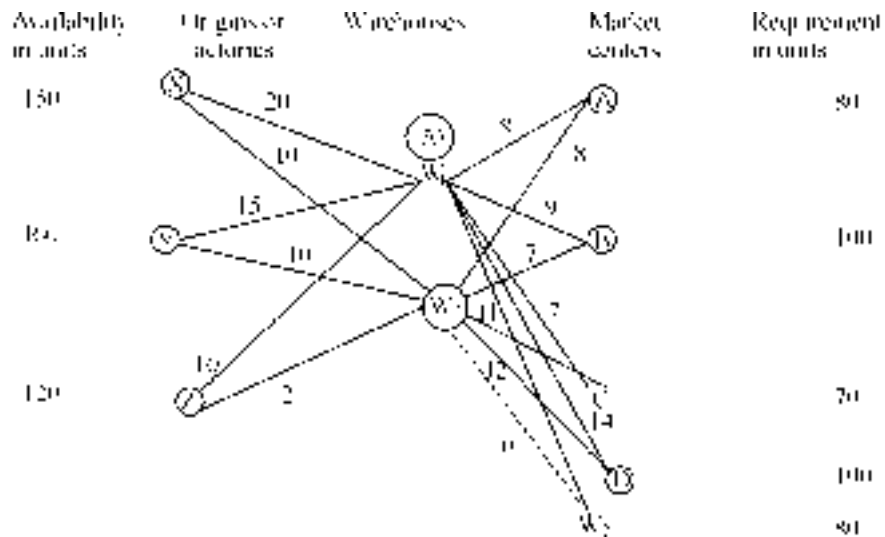
In the above some restrictions are imposed. The restrictions are:

Let warehouse W_1 be pure transshipment warehouse and W_2 is transshipment as well as distribution point.

(*i*)   The capacity limitation on $W_1 = 70$ units.

(*ii*)   The warehouse W_2 also deals with direct distribution of 80 units.

As per the given conditions, the following discussion will hold good.

Solution:

1.   As a source and intermediate transshipment node, W_1 has the capacity limitations of 70 units. Hence, availability of W_1 and requirement of destination W_1 is 70 units.

2. W_2 has no capacity limitation. However, it deals partial direct distribution of 80 units. Therefore, as a source its availability should be the difference between the total availability from all factories *i.e* X, Y and Z less its own direct distribution. 430 – 80 = 350.

3. As an intermediate destination, it should have the capacity to route entire production *i.e.* 430 units.

4. Unit cost of transportation from X, Y, and Z to destinations A, B, C and D, through W_1 and W_2 can be had from figure given, this can be entered in the table- 1 showing the initial transportation matrix.

5. There is no direct transportation from X,Y, and Z to destinations A, B, C and D. To avoid this direct routes we can allocate very high cost of transportation costs for these cells or we can avoid these cells by crossing them, *i.e.* eliminating them from the programme.

6. W_1 as source giving to W_1 as warehouse or sink, and W_2 as a source giving to W_2 as warehouse or sink will have zero cost.

	W_1	W_2	A	B	C	D	Avail.	u_i
X	20 / –12	1.0 / (150)	X	X	X	X	150	10
Y	15 / –7	10 / (160)	X	X	X	X	160	10
Z	10 / (70)	12 / (50)	X	X	X	X	120	12
W_1	0 / –2	X	ε	4	7 / (70)	1 / –2	70	0
W_2	X	0 / (70)	8 / (80)	7 / (100)	11	12 / (100)	350	0
Dem	70	400	80	100	70	100	850	
v_j		0	8	7	11	12		

As the total number of allocations are $m + n – 1$ after allocating $\in$ to cell W_1A, the solution is a basic feasible solution. By giving the optimality test by MODI method, we see that all the opportunity costs of empty cells are negative and hence the solution is optimal.

The allocation:

Cell	Load	Cost in Rs.		Rs.
XW_2	150	150×19	=	1500
YW_2	160	160×10	=	1600
ZW_1	70	70×10	=	700
ZW_2	50	50×12	=	600
W_1A	ε	-----	--	-----
W_1C	70	70×7	=	140
W_2W_2	70	70×0	=	0
W_2A	80	80×8	=	160
W_2B	100	100×7	=	700
W_2D	100	100×12	=	1200
		Total Cost in Rs.		6,600

VAM:

(1)

	W_1	W_2	A	B	C	D	Avail	ROC
X	20	10	X	X	X	X	150	10
Y	15	10	X	X	X	X	160	5
Z	10	12	X	X	X	X	120	2
	70							
W_1	0	X	8	9	7	14	70	1
W_2	X	0	8	7	11	12	350	1
Req.	70	430	80	100	70	100	850	
COC	10	10	0	2	4	2		

(2)

	W_2	A	B	C	D	Avail	ROC
X	10	X	X	X	X	150	10
Y	10	X	X	X	X	160	10
Z	12	X	X	X	X	50	12
	50						
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	430	80	100	70	100	780	
COC	10	0	2	4	2		

(3)

	W_2	A	B	C	D	Avail	ROC
X	10	X	X	X	X	150	10
	150						
Y	10	X	X	X	X	160	10
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	380	80	100	70	100	730	
COC	10	0	2	4	2		

(4)

	W_2	A	B	C	D	Avail	ROC	
Y	10 **160**	X	X	X	X	160	10	←
W_1	X	8	9	7	14	70	1	
W_2	0	8	7	11	12	350	7	
Req.	230	80	100	70	100	580		
COC	10	0	2	4	2			

(5)

	W_2	A	B	C	D	Avail	ROC
W_1	X	8	9	7	14	70	1
W_2	0 **70**	8	7	11	12	350	7
Req.	70	80	100	70	100	420	
COC	INF	0	2	4	2		

(6)

	A	B	C	D	Avail	ROC
W_1	8	9	7 **70**	14	70	1
W_2	8	7	11	12	280	1
Req.	80	100	70	100	350	
COC	0	2	4	2		

(7)

	A	B	D	Avail	ROC
W_2	8 **80**	7 **100**	12 **100**	280	
Req.	80	100	100	280	
COC					

4.12. REDUNDANCY IN TRANSPORTATION PROBLEMS

Some times, it may very rarely happen or while writing the alternate solution it may happen or during modifying the basic feasible solution it may happen that the number of occupied cells of basic feasible solution or some times the optimal solution may be greater than $m + n - 1$. This is called **redundancy in transportation problem**. This type of situation is very helpful to the manager who is looking about shipping of available loads to various destinations. This is as good as having more number of independent simultaneous equations than the number of unknowns. It may fail to give unique values of unknowns

as far as mathematical principles are concerned. But for a transportation manager, it enables him to plan for more than one orthogonal path for an or several cells to evaluate penalty costs, which obviously will be different for different paths.

4.13. SENSITIVITY ANALYSIS

(a) Non - basic variables

While discussing MODI method for getting optimal solution, we have discussed significance of **implied cost**, which fixes the upper limit of cost of the empty cell to entertain the cell in the next programme. Now let us discuss the influence of variations in present parameters on the optimum solution i.e **sensitivity of optimal solution for the variations in the costs of empty cells and loaded cells**. If unit cost of transportation of a particular non-basic variable changes, at what value of the cost of present optimum will no longer remain optimum? To answer this question, in the first instance, it is obvious that as the empty cell is not in the solution, any increase in its unit transportation cost will to qualify it for entering variable. But if the unit cost of empty cells is reduced the chances of changing the optimum value may be examined. Let us take an optimum solution and examine the above statement.

In the solution shown above as all the opportunity costs of empty cells are negative. Consider empty cell *XA*. Its opportunity cost is Rs. -3/- This means to say that the units cost of transportation of cell *XA* decreases by Rs.3/- or more *i.e* Rs.10/- the unit cost of transportation of the empty cell *XA* minus 3 = 7, or less than 7 the optimal solution changes, *i.e.* the cell *XA* will become eligible for entering into solution. Hence this cost, which shows the limit of the unit cost of empty cell, is known as implied cost in transportation problems. We can see that the opportunity cost of empty cell *ZE* is zero. This shows that the cell ZE is as good as a loaded cell and hence we can write alternate solutions by taking the cell *ZE* into consideration. (**Note: No unit cost of transportation is given for the cell ZC. Hence that cell should not be included in the programme. For this purpose, we can cross the cell or allocate very high unit cost of transportation for the cell. In case zero or any negative element is given as the unit cost of transportation for a cell, the value can be taken for further treatment.**)

5. If the problem is maximization one convert that into a minimization problem by multiplying the matrix by –1 or by subtracting all the elements of the matrix form the highest element in the matrix.

6. Find the basic feasible solution. The characteristics of the basic feasible solution are it must have $(m + n – 1)$ allocations, where **m** is the number of rows and **n** is the number of columns.

7. The basic feasible solution may be obtained by (*a*) Northwest corner method, (*b*) Least Cost method or Matrix minimum method, or (*c*) Vogel's approximation method or Opportunity cost method.

8. If initial allocations are equal to $(m + m – 1)$ proceed to next step. If it is not equal to $(m + n - 1)$ it is known as degenerate solution.

9. To solve degeneracy, add a small and negligible element $\in$ to empty cells. Take care to see that the $\in$ loaded cell do not make closed loop with other loaded cells when lines are drawn from epsilon loaded cells to other loaded cells by travelling vertically and horizontally by taking turns at loaded cells.

10. Write allocations and calculate the total cost of transportation.

11. Give optimality test to the basic feasible solution. Optimality test can be given by (*a*) Stepping stone method or (*b*) Modified distributing method or **MODI** method.

13. The characteristic of optimal solution is the opportunity costs of all empty cells are either negatives or zeros.

14. Remember if any empty cell has zero as its opportunity cost, then we can write alternate optimal solutions.

15. Write the allocations and calculate total transportation cost.

16. In case, the unit cost of transportation of any cell is zero or negative elements, take the same into considerations for further calculations. Suppose nothing is given in the cell as the unit cost of transportation, then presume that the route connecting the origin and the destination through that cell is not existing and cancel that cell and do not consider it at all while solving the problem, or else allocate very high cost of unit cost of transportation (infinity or any number which is greater than all the elements in the matrix), so that that cell will not enter into programme. (In maximization problem allocate a negative profit or return to the cell).

Problem 4.13. A company has three factories *X*, *Y*, and *Z* and four warehouses *A*, *B*, *C*, and *D*. It is required to schedule factory production and shipments from factories to warehouses in such a manner so as to minimize total cost of shipment and production. Unit variable manufacturing costs (UVMC) and factory capacities and warehouse requirements are given below:

From	*UVMC*	*To warehouses*				*Capacity in units per month.*
Factories.	Rs.	Unit shipping costs in Rs.				
		A	B	C	D	
X	10	0	1	1	2	75
Y	11	1	2	3	1	32
Z	12	4	3	3	6	67
Requirement:		65	24	16	15	

Find the optimal production and transportation schedule.

Solution: We have to optimize production and shipment cost. Hence the transportation matrix elements are the total of manufacturing cost plus transportation cost. For example, the manufacturing cost of factory *X* is Rs. 10. Hence the transportation and shipment cost will be equal to 10 + 0, 10 +1, 10 +1 and 10 +2 respectively for warehouses *A*, *B*, *C* and *D* respectively. As the total available is 174 units and the total demand is 120 units we have to open a dummy column with requirement of 54 units. The production cum transportation matrix is given below:

Production cum transportation cost per unit in Rs.

(1) Initial basic feasible solution by VAM :

	A	*B*	*C*	*D*	D*C*	*Avail*	*Roc*
X	10	11	11	11	0	75	10
Y	12	13	14	12	0	32	12
Z	16	15	15	18	0	67	15
					54		
Req	65	24	16	15	54	174	
Coc	2	2	3	1	0		

(2)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	10	11	11 **16**	11	75	0
Y	12	13	14	12	32	0
Z	16	15	15	18	13	0
Req	65	24	16	15	124	
Coc	2	2	3	0		

(3)

	A	*B*	*D*	*Avail*	*Roc*
X	10	11 **24**	11	75	0
Y	12	13	12	32	1
Z	16	15	18	13	0
Req	65	24	15	104	
Coc	2	2	0		

(4)

	A	*D*	*Avail*	*Roc*
X	10 **35**	11	35	2
Y	12	12	32	0
Z	16	18	13	2
Req	65	15	80	
Coc	2	0		

(5)

	A	*D*	*Avail*	*Roc*
Y	12	12 **15**	32	0
Z	16	18	13	2
Req	30	15	45	
Coc	4	6		

(6)

	A	*Avail*	*Roc*
Y	12 **17**	17	0
Z	16 **13**	13	2
Req	30	30	
Coc	4		

Production cum transportation cost per unit in Rs.

	A	B	C	D	DC	Avail	u
X	10	11	11	11	10	75	0
	㊽	㉔	㉙	−6	−2		
Y	12	13	14	12	10	32	2
	17	11	−1	⑮	2		
Z	16	5	15	14	10	67	1
	−2	0	⑬	㊴			
		2		4			
Req	65	24	16	15	54	174	
v	10	11	11	10	4		

As there are $m + n - 1$ allocations and all the opportunity costs of empty cells are negative, the solution is optimal.

The optimal allocations are:

Cell	Load	Cost in Rs.		Rs.
XA	48	48×10	=	480
XB	24	24×11	=	264
XC	29	29×11	=	319
YA	17	17×12	=	204
YB	15	15×12	=	180
ZC	13	13×15	=	195
Z DC	54	54×0	=	0
		Total cost in RS.	=	1642

QUESTIONS

1. Explain the process of solving a transportation problem.

2. List out the differences and similarities between Resource allocation model and Transportation model in linear programming.

3. Explain the procedure of getting basic feasible solution by using VAM.

4. Explain what are degeneracy and redundancy in transportation problem. How do you solve degeneracy in transportation problem? Distinguish between tie and degeneracy in linear programming problem.

5. Is transportation problem is of maximization type or minimization type problem? If it is one of the two, how do you solve the other version of the transportation model?

6. How do you say that a transportation model has an alternate solution? In case it has an alternate optimal solution, how do you arrive at alternate solution?

7. What is transshipment problem? In what way it differs from general transportation problem?

8. Explain the terms: (*a*) Opportunity cost, (*b*) Implied cost, (*c*) Row opportunity cost, (*d*) Column opportunity cost.

9. The DREAM - DRINK Company has to work out a minimum cost transportation schedule to distribute crates of drinks from three of its factories *X*, *Y*, and *Z* to its three warehouses *A*, *B*, and *C*. The required particulars are given below. Find the least cost transportation schedule.

Transportation cost in Rs per crate.

From / To	A	B	C	Crates Available.
X	75	50	50	1040
Y	50	25	75	975
Z	25	125	25	715
Crates required.	1300	910	520	2730

10. The demand pattern for a product at for consumer centers, *A*, *B*, *C* and *D* are 5000 units, 7000 units, 4000 units and 2000 units respectively. The supply for these centers is from three factories *X*, *Y* and *Z*. The capacities for the factories are 3000 units, 6000 units and 9000 units respectively. The unit transportation cost in rupees from a factory to consumer center is given below in the matrix. Develop an optimal transportation schedule and find the optimal cost.

From:	To			
	A	B	C	D
X	8	9	12	8
Y	3	4	3	2
Z	5	3	7	4

11. From three warehouses, *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points *X*, *Y*, and *Z*. Find the least cost transportation schedule with relevant information given below:

From Warehouses	To demand points (Transportation cost in Rs. per units).			Availability in units.
	X	Y	Z	
A	5	10	2	100
B	3	7	5	25
C	6	8	4	75
Units demand	105	30	90	

12. From three warehouses *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points 1, 2, 3, 4 and 5 monthly. The relevant information is given below:

Warehouses	Demand points (Transportation cost in Rs per unit.					Availability in units.
	1	2	3	4	5	
A	4	1	2	6	9	100
B	6	4	3	5	7	120
C	5	2	6	4	8	120
Units demand:	40	50	70	90	90	

During certain month a bridge on the road-connecting warehouse *B* to demand point 3 is closed for traffic. Modify the problem suitably and find the least cost transportation schedule. (The demand must be complied with).

13. A tin box company has four factories that supply to 5 warehouses. The variable cost of manufacturing and shipment of one ton of product from each factory to each warehouse are shown in the matrix given below, Factory capacities and warehouse requirements are shown in the margin. After several iterations the solution obtained is also shown.

Warehouses Cost in Rs per unit

Factories	A	B	C	D	T	DMF	Capacity
W	17	9	14 **25**	19 **30**	.1	9 **20**	75
X	13 **10**	6 **20**	11 **15**	11	.2	11	45
Y	6 **30**	.7	9	12	12	9	30
Z	15	20	11 **10**	14	5 **40**	11	50
Req	40	20	50	30	40	20	200

(*a*) Is this an optimal solution? How do you know?

(*b*) Is there an alternate solution? If so find it.

(*c*) Suppose some new equipment was installed that reduces the variable operation cost by Rs. 2/- per ton in factory *X*, is the shipping schedule remain optimum? If not what is the new optimum?

(*d*) Suppose the freight charges from *W* to A were reduced by Rs.2/- would this change the shipping schedule? If so what is the new optimum?

(*e*) How much would the manufacturing cost have to be reduced in W before production would be increased beyond 55 tons?

14. A company has a current shipping schedule, which is being questioned by the management as to whether or not it is optimal. The firm has three factories and five warehouses. The necessary data in terms of transportation costs in Rs. per unit from a factory to a destination and factory capacities and warehouse requirements are as follows:

Factories. (Transportation costs in Rs. per unit.)

Warehouses.	X	Y	Z	Requirement of warehouses in units.
A	5	4	8	400
B	8	7	4	400
C	6	7	6	500
D	6	6	6	400
E	3	5	4	800
Factory capacities.	800	600	1100	

Solve for an optimal shipping schedule in terms of lowest possible shipping costs.

15. Solve the following transportation problem.

Destination

Source	A	B	C	D	E	Supply
W	20	19	14	21	16	40
X	15	20	13	19	16	60
Y	18	15	18	20		70
Z	0	0	0	0	0	50
Demand.	30	40	50	40	60	

(Note: Nothing is given in cell *YE*. So you have to ignore it).

16. A manufacturing organization has 3 factories located at *X*, *Y* and *Z*. The centralized planning cell has to decide on allocation of 4 orders over the 3 factories with a view to minimizing the total cost to the organization, Demand and capacity and cost details are given as under:

Customer	Demand per month in units.
A	960
B	380
C	420
D	240

Capacities and Costs (Rs.).

Factories	Capacity units per month	Overhead costs in Rs, per month	Direct cost in Rs. per unit.
X	400	400	2.50
Y	900	720	3.00
Z	640	320	3.50

Shipping cost in Paise per unit dispatch.

To

From	A	B	C	D
X	50	70	40	35
Y	45	75	40	55
Z	70	65	60	75

It is also possible to produce 25% higher than the capacity in each factory by working overtime at 50% higher in direct costs.

    (*a*)   Build a transportation model so that the total demand is met with.

    (*b*)   Do the allocation of factory capacity by minimum cost allocation and check the solution for optimality.

16.   In a transportation problem the distribution given in the table below was suggested as an optimal solution. The capacities and requirement are given. The number in bold are allocations. The transportation costs given in Rs, per unit from a source to a destination.

    (*a*)   Test whether the given distribution is optimal?

    (*b*)   If not optimal obtain all basic optimal solution.

Source \ Destination	A	B	C	D	Capacity
X	9 **12**	8 **14**	12	14 **10**	36
Y	9 **16**	10 **28**	12	14	44
Z	8	9	11 **52**	12	32
Demand	12				

17.   A department stores wishes to purchase 7,500 purses of which 2,500 are of style *X*, 2,500 are of style *Y* and 2,500 are of style *Z*. Four manufacturers *A*, *B*, *C* and *D* bid to supply not more than the following quantities, all styles combined. *A* = 1,000, *B* = 3,000, *C* = 2, 100 and *D* = 1,900. The following table gives the cost per purse of each style of the bidders in Rs. per purse.

MANUFACTURER.

Style	A	B	C	D
X	10	4	9	5
Y	6	7	8	7
Z	3	8	6	9

(*a*) How should orders to be placed by the department store to minimize the total cost?

(*b*) If the store were to introduce a new style W, which manufacturer can supply it? How many of W can he supply?

MULTIPLE CHOICE QUESTIONS

1. Transportation problem is basically a
    (*a*) Maximization model
    (*b*) Minimization model
    (*c*) Transshipment problem
    (*d*) Iconic model   ( )

2. The column, which is introduced in the matrix to balance the rim requirements, is known as:
    (*a*) Key column
    (*b*) Idle column
    (*c*) Slack column
    (*d*) Dummy Column   ( )

3. The row, which is introduced in the matrix to balance the rim requirement, is known as:
    (*a*) Key row
    (*b*) Idle row
    (*c*) Dummy row
    (*d*) Slack row   ( )

4. One of the differences between the Resource allocation model and Transportation Model is:
    (*a*) The coefficients of problem variables in Resource allocation model may be any number and in transportation model it must be either zeros or ones.
    (*b*) The coefficients of problem variable in Resource allocation model must be either zeros or ones and in Transportation model they may be any number.
    (*c*) In both models they must be either zeros or ones only.
    (*d*) In both models they may be any number.   ( )

5. To convert the transportation problem into a maximization model we have to
    (*a*) To write the inverse of the matrix
    (*b*) To multiply the rim requirements by –1
    (*c*) To multiply the matrix by –1
    (*d*) We cannot convert the transportation problem in to a maximization problem, as it is basically a minimization problem.   ( )

6. In a transportation problem where the demand or requirement is equals to the available resource is known as
    (*a*) Balanced transportation problem,
    (*b*) Regular transportation problem,
    (*c*) Resource allocation transportation problem
    (*d*) Simple transportation model.   ( )

7. The total number of allocation in a basic feasible solution of transportation problem of $m \times n$ size is equal to:
   (*a*) $m \times n$ (*b*) $(m / n) - 1$
   (*c*) $m + n + 1$ (*d*) $m + n - 1$ (   )

8. When the total allocations in a transportation model of $m \times n$ size is not equals to $m + n - 1$ the situation is known as:
   (*a*)   Unbalanced situation   (*b*) Tie situation
   (*c*)   Degeneracy   (*d*) None of the above   (   )

9. The opportunity cost of a row in a transportation problem is obtained by:
   (*a*)   Deducting the smallest element in the row from all other elements of the row,
   (*b*)   Adding the smallest element in the row to all other elements of the row,
   (*c*)   Deducting the smallest element in the row from the next highest element of the row
   (*d*)   Deducting the smallest element in the row from the highest element in that row.

   (   )

10. In Northwest corner method the allocations are made
    (*a*)   Starting from the left hand side top corner,
    (*b*)   Starting from the right hand side top corner
    (*c*)   Starting from the lowest cost cell
    (*d*)   Starting from the lowest requirement and satisfying first.   (   )

11. VAM stands for:
    (*a*)   Value added method   (*b*) Value assessment method
    (*c*)   Vogel Adam method,   (*d*) Vogel's approximation method.   (   )

12. MODI stands for
    (*a*)   Modern distribution,   (*b*) Mendel's distribution method
    (*c*)   Modified distribution method   (*d*) Model index method   (   )

13. In the optimal solution, more than one empty cell have their opportunity cost as zero, it indicates
    (*a*)   The solution is not optimal   (*b*) The problem has alternate solution
    (*c*)   Something wrong in the solution   (*d*) The problem will cycle.   (   )

14. In case the cost elements of one or two cells are not given in the problem, it means:
    (*a*)   The given problem is wrong
    (*b*)   We can allocate zeros to those cells
    (*c*)   Allocate very high cost element to those cells
    (*d*)   To assume that the route connected by those cells are not available.   (   )

15. To solve degeneracy in the transportation problem we have to:
    (*a*)   Put allocation in one of the empty cell as zero
    (*b*)   Put a small element epsilon in any one of the empty cell

(*c*) Allocate the smallest element epsilon in such a cell, which will not form a closed loop with other loaded cells.

(*d*) Allocate the smallest element epsilon in such a cell, which will form a closed loop with other loaded cells.   (   )

16. A problem where the produce of a factory is stored in warehouses and then they are transported to various demand point as and when the demand arises is known as:

(*a*) Transshipment problem

(*b*) Warehouse problem

(*c*) Storing and transport problem

(*d*) None of the above   (   )

17. Implied Cost in transportation problem sets (in the existing program):

(*a*) The lowest limit for the empty cell beyond which it is not advisable to include in the programme,

(*b*) The highest limit for the empty cell beyond which it is not advisable to include in the programme,

(*c*) The opportunity cost of the empty cell,

(*d*) None of the above.   (   )

18. In transportation model, the opportunity cost is given by

(*a*) Implied cost + Actual cost of the cell

(*b*) Actual cost of the cell – Implied cost,

(*c*) Implied cost – Actual cost of the cell

(*d*) Implied cost × Actual cost of the cell   (   )

19. If ui and v_j are row and column numbers respectively, then the implied cost is given by:

(*a*) $u_i + v_j$ (*b*) $u_i - v_j$

(*c*) $u_i \times v_j$ (*d*) u_i / v_j (   )

20. If a transportation problem has an alternate solution, then the other alternate solutions are derived by:

(Given that the two matricides of alternate solutions are A and B, and d is any positive fraction number)

(*a*) $A + (1 - d) \times B$ (*b*) $A(1 - d) + B$

(*c*) $dA + dB$ (*d*) $dA + (1 - d) \times B$ (   )

ANSWERS

1. (*b*)	2. (*d*)	3. (*d*)	4. (*c*)
5. (*a*)	6. (*c*)	7. (*a*)	8. (*d*)
9. (*c*)	10. (*a*)	11. (*d*)	12. (*a*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*b*)	18. (*b*)	19. (*a*)	20. (*a*)

Linear Programming : III
Assignment Model

5.1. INTRODUCTION

In earlier discussion in chapter 3 and 4, we have dealt with two types of linear programming problems, *i.e*. Resource allocation method and Transportation model. We have seen that though we can use simplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them. Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given the name **ASSIGNMENT MODEL**. Basically assignment model is a minimization model. If we want to maximize the objective function, then there are two methods. One is to subtract all the elements of the matrix from the highest element in the matrix or to multiply the entire matrix by –1 and continue with the procedure. For solving the assignment problem we use Assignment technique or Hungarian method or Flood's technique. All are one and the same. Above, it is mentioned that one origin is to be assigned to one destination. This feature implies the existence of two specific characteristics in linear programming problems, which when present, give rise to an assignment problem. The first one being **the pay of matrix for a given problem is a square matrix** and the second is the optimum solution (or any solution with given constraints) for the problem is such that there can be **one and only one assignment in a given row or column of the given payoff matrix.** The transportation model is a special case of linear programming model (Resource allocation model) and assignment problem is a special case of transportation model, therefore it is also a special case of linear programming model. Hence it must have all the properties of linear programming model. That is it must have: (*i*) an objective function, (*ii*) it must have structural constraints, (*iii*) It must have non-negativity constraint and (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

5.2. The Problem

There are some types in assignment problem. They are:

(*i*) Assigning the jobs to machines when the problem has square matrix to minimize the time required to complete the jobs. Here the number of rows *i.e.* jobs are equals to the number of machines *i.e.* columns. The procedure of solving will be discussed in detail in this section.

(*ii*) The second type is maximization type of assignment problem. Here we have to assign certain jobs to certain facilities to maximize the returns or maximise the effectiveness. This is also discussed in problem number 5.2.

(*iii*) Assignment problem having non-square matrix. Here by adding a dummy row or dummy columns as the case may be, we can convert a non-square matrix into a square matrix and proceed further to solve the problem. This is done in problem number.5.9.

(*iv*) Assignment problem with restrictions. Here restrictions such as a job cannot be done on a certain machine or a job cannot be allocated to a certain facility may be specified. In such cases, we should neglect such cell or give a high penalty to that cell to avoid that cell to enter into the programme.

(*v*) Traveling sales man problem (cyclic type). Here a salesman must tour certain cities starting from his hometown and come back to his hometown after visiting all cities. This type of problem can be solved by Assignment technique and is solved in problem 5.14.

Let us take that there are 4 jobs, *W*, *X*, *Y* and *Z* which are to be assigned to four machines, *A*, *B*, *C* and *D*. Here all the jobs have got capacities to machine all the jobs. Say for example that the job *W* is to drill a half and inch hole in a Wooden plank, Job *X* is to drill one inch hole in an Aluminum plate and Job *Y* is to drill half an inch hole in a Steel plate and job *Z* is to drill half an inch hole in a Brass plate. The machine *A* is a Pillar type of drilling machine, the machine *B* is Bench type of drilling machine, Machine *C* is radial drilling machine and machine *D* is an automatic drilling machine. This gives an understanding that all machines can do all the jobs or all jobs can be done on any machine. The cost or time of doing the job on a particular machine will differ from that of another machine, because of overhead expenses and machining and tooling charges. The objective is to minimize the time or cost of manufacturing all the jobs by allocating one job to one machine. Because of this character, *i.e.* one to one allocation, the assignment matrix is always a square matrix. If it is not a square matrix, then the problem is unbalanced. Balance the problem, by opening a dummy row or dummy column with its cost or time coefficients as zero. Once the matrix is square, we can use assignment algorithm or Flood's technique or Hungarian method to solve the problem.

Jobs	Machines (Time in hours)				Availability
	A	*B*	*C*	*D*	
W	C_{11}	C_{12}	C_{13}	C_{14}	1
X	C_{21}	C_{22}	C_{23}	C_{24}	1
Y	C_{31}	C_{32}	C_{33}	C_{34}	1
Z	C_{41}	C_{42}	C_{43}	C_{44}	1
Requirement:	1	1	1	1	

Mathematical Model:

$$\text{Minimize } \mathbf{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \times_{ij} \quad \longrightarrow \quad \textbf{Objective Constraint.}$$

Subject to: $X_{ij} = (\mathbf{X}_{ij})^2$ **i and j = 1 to n**

$$\sum_{J=1}^{n} X_{ij} = 1 \ (d_j) \text{ and } \sum_{i=1}^{n} X_{ij} = 1 \ (b_i) \rightarrow \textbf{Structural Constraints.}$$

For i and j = 1 to n

(Each machine to one job only)   (Each job to one machine only)

And

$\mathbf{X}_{ij} = \mathbf{0}$ **for all values of** j **and** $i.$ $\rightarrow$ **Non-negativity constraint.**

5.3. COMPARISION BETWEEN TRANSPORTATION PROBLEM AND ASSIGNMENT PROBLEM

Now let us see what are the similarities and differences between Transportation problem and Assignment Problem.

Similarities

1. Both are special types of linear programming problems.
2. Both have objective function, structural constraints, and non-negativity constraints. And the relationship between variables and constraints are linear.
3. The coefficients of variables in the solution will be either 1 or zero in both cases.
4. Both are basically minimization problems. For converting them into maximization problem same procedure is used.

Differences

Transportation Problem	*Assignment Problem.*
1. The problem may have rectangular matrix or square matrix. 2. The rows and columns may have any number of allocations depending on the rim conditions. 3. The basic feasible solution is obtained by northwest corner method or matrix minimum method or VAM 4. The optimality test is given by stepping stone method or by MODI method. 5. The basic feasible solution must have m + n – 1 allocations. 6. The rim requirement may have any numbers (positive numbers). 7. In transportation problem, the problem deals with one commodity being moved from various origins to various destinations.	1. The matrix of the problem must be a square matrix. 2. The rows and columns must have one to one allocation. Because of this property, the matrix must be a square matrix. 3. The basic feasible solution is obtained by Hungarian method or Flood's technique or by Assignment algorithm. 4. Optimality test is given by drawing minimum number of horizontal and vertical lines to cover all the zeros in the matrix. 5. Every column and row must have at least one zero. And one machine is assigned to one job and vice versa. 6. The rim requirements are always 1 each for every row and one each for every column. 7. Here row represents jobs or machines and columns represents machines or jobs.

5.4. APPROACH TO SOLUTION

Let us consider a simple example and try to understand the approach to solution and then discuss complicated problems.

1. Solution by visual method

In this method, first allocation is made to the cell having lowest element. (In case of maximization method, first allocation is made to the cell having highest element). If there is more than one cell having smallest element, tie exists and allocation may be made to any one of them first and then second one is selected. In such cases, there is a possibility of getting alternate solution to the problem. This method is suitable for a matrix of size 3×4 or 4×4. More than that, we may face difficulty in allocating.

Problem 5.1.

There are 3 jobs *A*, *B*, and *C* and three machines *X*, *Y*, and *Z*. All the jobs can be processed on all machines. The time required for processing job on a machine is given below in the form of matrix. Make allocation to minimize the total processing time.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Allocation: *A* to *X, B* to *Y* and *C* to *Z* and the total time = 11 + 13 +12 = 36 hours. (Since 11 is least, Allocate *A* to *X,* 12 is the next least, Allocate *C* to *Z*)

2. Solving the assignment problem by enumeration

Let us take the same problem and workout the solution.

Machines (time in hours)

C	13	15	12
Jobs	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

S.No	Assignment	Total cost in Rs.
1	AX BY CZ	11 + 13 + 12 = 36
2	AX BZ CY	11 + 17 + 15 = 43
3	AY BX CZ	16 + 20 + 12 = 48
4	AY BZ CX	16 + 17 + 13 = 46
5	AZ BY CX	21 + 13 + 13 = 47
6	AZ BX CY	21 + 20 + 15 = 56

Like this we have to write all allocations and calculate the cost and select the lowest one. If more than one assignment has same lowest cost then the problem has alternate solutions.

3. Solution by Transportation method

Let us take the same example and get the solution and see the difference between transportation problem and assignment problem. The rim requirements are 1 each because of one to one allocation.

Machines (Time in hours)

Jobs	X	Y	Z	Available
A	11	16	21	1
B	20	13	17	1
C	13	15	12	1
Req	1	1	1	3

By using northwest corner method the assignments are:

Machines (Time in hours)

Jobs	X	Y	Z	Available
A	1	E		1
B		1	$\in$	1
C			1	1
Req	1	1	1	3

As the basic feasible solution must have $m + n - 1$ allocations, we have to add 2 epsilons. Next we have to apply optimality test by MODI to get the optimal answer.

This is a time consuming method. Hence it is better to go for assignment algorithm to get the solution for an assignment problem.

4. Hungarian Method / Flood's technique / Assignment algorithm: (opportunity cost method)

Let us once again take the same example to workout with assignment algorithm.

Machines (time in hours)

Jobs	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

Step 1.   Deduct the smallest element in each row from the other elements of the row. The matrix thus got is known as **Row opportunity cost matrix (ROCM). The logic here is if we assign the job to any machine having higher cost or time, then we have to bear the penalty. If we subtract smallest element in the row or from all other element of the row, there will be at least one cell having zero, i.e zero opportunity cost or zero penalty. Hence that cell is more competent one for assignment.**

Step 2. Deduct the smallest element in each column from other elements of the column. The matrix thus got is known as **Column opportunity cost matrix (COCM). Here also by creating a zero by subtracting smallest element from all other elements we can see the penalty that one has to bear. Zero opportunity cell is more competent for assignment.**

Step 3. Add COCM and ROCM to get the **Total opportunity cost matrix (TOCM).**

Step 4. (modified): Total opportunity cost matrix can be got by simplify doing row operation on Column opportunity matrix or column operation on row opportunity cost matrix. This method is simple one and saves time. (Doing row operation on column opportunity matrix means: Deduct the smallest element in the row from all other elements in the row in column opportunity matrix and vice versa).

The property of total opportunity cost matrix is that it will have at least one zero in every row and column. All the cells, which have zero as the opportunity cost, are eligible for assignment.

Step 5. Once we get the total opportunity **cost matrix, cover all the zeros by MINIMUM NUMBER OF HORIZONTAL AND VERTICAL LINES**. (First cover row or column, which is having maximum number of zeros and then next row or column having next highest number of zeros and so on until all zeros are covered. Remember, only horizontal and vertical lines are to be drawn.

Step 6. If the lines thus drawn are equal to the number of rows or columns (because of square matrix), we can make assignment. If lines drawn are not equal to the number of rows or columns go to step 7.

Step 7. **To make assignment**: Search for a single zero either row wise or column wise. If you start row wise, proceed row by row in search of single zero. Once you find a single zero; assign that cell by enclosing the element of the cell by a **square**. Once all the rows are over, then start column wise and once you find single zero assign that cell and enclose the element of the one cell in a square. Once the assignment is made, then all the zeros in the row and column corresponding to the assigned cell should be cancelled. Continue this procedure until all assignments are made. Some times we may not find single zero and find more than one zero in a row or column. It indicates, that the problem has an alternate solution. We can write alternate solutions. (The situation is known as a **TIE** in assignment problem).

Step 8. If the lines drawn are less than the number of rows or columns, then we cannot make assignment. Hence the following procedure is to be followed:

The cells covered by the lines are known as **Covered cells**. The cells, which are not covered by lines, are known as **uncovered cells**. The cells at the intersection of horizontal line and vertical lines are known as **Crossed cells.**

  (*a*) Identify the smallest element in the uncovered cells.

   (*i*) Subtract this element from the elements of all other uncovered cells.

   (*ii*) Add this element to the elements of the crossed cells.

   (*iii*) Do not alter the elements of covered cells.

  (*b*) Once again cover all the zeros by **minimum number of horizontal and vertical lines.**

  (*c*) Once the lines drawn are equal to the number of rows or columns, assignment can be made as said in step (6).

(*d*)   If the lines are not equal to number of rows or columns, repeat the steps 7 (a) and 7 (b) until we get the number of horizontal and vertical lines drawn are equal to the number of rows or columns and make allocations as explained in step (6).

Note: For maximization same procedure is adopted, once we convert the maximization problem into minimization problem by multiplying the matrix by (-1) or by subtracting all the elements of the matrix from highest element in the matrix. Once we do this, the entries in the matrix gives us **relative costs**, hence the problem becomes minimisaton problem. Once we get the optimal assignment, the total value of the original pay off measure can be found by adding the individual original entries for those cells to which assignment have been made.

Now let us take the problem given above and solve.

Solution

Machines (time in hours)

Jobs	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

Step1: To find ROCM.

Machines (time in hours)

Jobs	X	Y	Z
A	0	5	10
B	7	0	4
C	1	3	0

Step 2. To find TOCM (do column operation in ROCM)

Machines (time in hours)

Jobs	X	Y	Z
A	0	5	10
B	7	0	4
C	1	3	0

Because in each column, zero is the lowest element, the matrix remains unchanged, *i.e.* The COCM itself TOCM.

Step 3. To cover all the zeros by minimum number of horizontal and vertical lines.

Machines (time in hours)

Jobs	X	Y	Z
A	0	5	10
B	7	0	4
C	1	3	0

Assignment is:

Machines (time in hours)

Jobs	X	Y	Z
A	**0**	5	10
B	7	**0**	4
C	1	3	**0**

Assignment	Time in hours.
A TO X	11
B TO Y	13
C TO Z	12
Total:	36 hours.

Problem 5.2.

A company has five jobs *V*, *W*, *X*, *Y* and *Z* and five machines *A*, *B*, *C*, *D* and E. The given matrix shows the return in Rs. of assigning a job to a machine. Assign the jobs to machines so as to maximize the total returns.

Machines.
Returns in Rs.

Jobs	A	B	C	D	E
V	5	11	10	12	4
W	2	4	6	3	5
X	3	12	5	14	6
Y	6	14	4	11	7
Z	7	9	8	12	5

Solution

As the objective function is to maximize the returns, we have to convert the given problem into minimization problem.
Method 1. Here highest element in the matrix is 14, hence subtract all the element form 14 and write the relative costs. (Transformed matrix).

Machines
Returns in Rs.

Jobs	A	B	C	D	E
V	9	3	4	2	10
W	12	10	8	11	9
X	11	2	9	0	8
Y	8	0	10	3	7
Z	7	5	6	2	9

ROCM:

**Machines
Returns in Rs.**

Jobs	A	B	C	D	E
V	7	1	2	0	8
W	4	2	0	3	1
X	11	2	9	0	8
Y	8	0	10	3	7
Z	5	3	4	0	7

By doing column operation on ROCM, we get the total opportunity cost matrix.
TOCM:

**Machines
Returns in Rs.**

Jobs	A	B	C	D	E
V	3	1	2	0	7
W	0	2	0	3	0
X	7	2	9	0	7
Y	4	0	10	3	6
Z	1	3	4	0	6

Only three lines are there. So we have to go to step 7. The lowest element in uncovered cell is 1, hence subtract 1 from all uncovered cells and add this element to crossed cells and write the matrix. The resultant matrix is:

**Machines
Return in Rs.**

Jobs	A	B	C	D	E
V	2	0	1	0	6
W	0	3	0	4	0
X	6	1	8	0	6
Y	4	0	10	4	6
Z	0	2	3	0	5

Only foor lines are there, hence repeat the step 7 until we get 5 lines.

**Machines
Return in Rs.**

Jobs	A	B	C	D	E
V	1	0	0	0	5
W	0	3	0	5	0
X	5	1	7	0	5
Y	3	0	9	4	5
Z	0	3	3	1	5

All zeros are covered by 5 lines, Hence assignment can be made. Start row wise or column wise and go on making assignment, until all assignments are over.

**Machines
Return in Rs.**

Jobs	A	B	C	D	E
V	2	1	**0**	x0	5
W	1	4	0x	5	**0**
X	6	2	7	**0**	5
Y	3	0	8	3	4
Z	**0**	3	2	0x	4

Job	Machine	Return in Rs.
V	C	10
W	E	5
X	D	14
Y	B	14
Z	A	7
Total in Rs.		50

Problem 5.3.

Five jobs are to be assigned to 5 machines to minimize the total time required to process the jobs on machines. The times in hours for processing each job on each machine are given in the matrix below. By using assignment algorithm make the assignment for minimizing the time of processing.

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

Solution

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

COCM

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	0	0	0	0	0
B	5	0	3	3	0
C	0	5	5	5	0
D	6	2	9	2	0
E	0	4	2	3	4

As the COCM has at least one zero in every column and row, this itself can be considered as TOCM, because as the zero is the lowest number in each column, the matrix remains unchanged. If we cover all the zeros by drawing horizontal and vertical lines, we get only four lines. Applying step 7 we get the following matrix.

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	2	0	0	0	2
B	7	0	3	3	2
C	0	3	3	3	0
D	6	0	7	0	0
E	0	2	0	1	4

As there are five lines that cover all zeros, we can make assignment.

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	2	✗0	**0**	✗0	2
B	7	**0**	3	3	2
C	✗0	3	3	3	**0**
D	6	✗0	7	**0**	✗0
E	**0**	2	✗0	1	4

Alternate solution:

Machines (time in hours)

Jobs	V	W	X	Y	Z
A	2	0̸	0̸	**0**	2
B	7	**0**	3	3	2
C	**0**	3	3	3	0̸
D	6	0̸	7	0̸	**0**
E	0̸	2	**0**	1	4

First Solution: *A* to *X*, *B* to *W*, *C* to *Z*, *D* to *Y* and *E* to *V* Cost is: 3 + 4 + 4 + 7 + 2 = 20 hours.

Second Solution: *A* to *Y*, *B* to *W*, *C* to *V*, *D* to *Z* and *E* to *X*. Cost is: 5 + 4 + 2 + 4 + 5 = 20 Hours.

When there is a tie, make assignment arbitrarily first to one of the zeros and then proceed, we will get the assignment. When there is a tie, there exists an alternate solution.

Problem 5.4.

A manager has 4 jobs on hand to be assigned to 3 of his clerical staff. Clerical staff differs in efficiency. The efficiency is a measure of time taken by them to do various jobs. The manager wants to assign the duty to his staff, so that the total time taken by the staff should be minimum. The matrix given below shows the time taken by each person to do a particular job. Help the manager in assigning the jobs to the personnel.

Jobs.	Men (time taken to do job in hours).		
	X	Y	Z
A	10	27	16
B	14	28	7
C	36	21	16
D	19	31	21

Solution

The given matrix is unbalanced. To balance the matrix, open a dummy column with time coefficients as zero.

(DC = Dummy column).

Men (Time taken in hours)

	X	Y	Z	DC
A	10	27	16	0
B	14	28	7	0
C	36	21	16	0
D	19	31	21	0

As every row has a zero, we can consider it as ROCM and by doing column operation, we can write TOCM. Now apply step 7.

Men (Time taken in hours).

Jobs	X	Y	Z	DC
A	0	6	9	0
B	4	7	0	0
C	26	0	9	0
D	9	10	14	0

Men (Time taken in hours).

Jobs	X	Y	Z	DC
A	**0**	6	9	0̶
B	4	7	**0**	0̶
C	26	**0**	9	0̶
D	9	10	14	**0**

The assignment is: *A* to *X*, *B* to *Z*, and *C* to *Y* and *D* is not assigned.
Total time required is: $10 + 7 + 21 = 38$ Hours.

Problem 5.5.

A company has four market segments open and four salesmen are to be assigned one to each segment to maximize the expected total sales. The salesmen differ in their ability and the segments also differ in their sales potential. The details regarding the expected sales in each segment by a typical salesman under most favourable condition are given below.

Segment *A* = Rs. 60,000, Segment *B* = Rs. 50,000, Segment *C* = Rs. 40,000 and Segment *D* = Rs. 30,000. It is estimated that working under same condition, the ability of salesmen in terms of proportional yearly sales would be as below:
Salesman *W* = 7, Salesman *X* = 5, Salesman *Y* = 5 and Salesman *Z* = 4.
Assign segments to salesmen for maximizing the total expected sales.

Solution

To simplify the calculations, let us consider sales of Rs.10, 000/- as one unit of sale, then salesman *W* 's annual sales in four segments are:

His proportionate sale is seven out of 21 $(7 + 5 + 5 + 4 = 21)$. In case the annual sales is 6 units

(Rs.60, 000), then his proportional sales would be $(7 / 21) \times 6 = 42 / 21$ similarly his sales in all the segments would be $(7/21) \times 6$, $(7/21) \times 5$, $(7/21) \times 5$, and $(7/21) \times 4$ *i.e.* 42/21, 35/21, 35/21 and 28/21. Like wise we can calculate the proportional sales of all salesmen and write the matrix showing the sales of each salesman in different market segments. The matrix is given below:

Market segments.

Sales (x1000)		6	5	4	3
Salesproportion	Salesmen	W	X	Y	Z
7	W	42/21	3 5/21	28/21	21/21
5	X	30/21	25/21	20/21	15/21
5	Y	30/21	25/21	20/21	15/21
4	Z	24/21	20/21	16/21	12/21

Multiply the matrix by21 to avoid the denominator. As the problem is maximization one, convert the problem into minimization problem by multiplying by (-1) (Second method). The resultant matrix is:

Market segments.

SalesMen	A	B	C	D
W	–42	–35	–28	–21
X	–30	–25	–20	–15
Y	–30	–25	–20	–15
Z	–24	–20	–16	–12

ROCM:

Market segments.

SalesMen	A	B	C	D
W	0	7	14	21
X	0	5	10	15
Y	0	5	10	15
Z	0	4	8	12

TOCM:

Market segments.

SalesMen	A	B	C	D
W	0	3	6	9
X	0	1	2	3
Y	0	1	2	3
Z	0	0	0	0

TOCM:

Market segments.

SalesMen	A	B	C	D
W	0	2	5	8
X	0	0	1	2
Y	0	0	1	2
Z	1	0	0	0

TOCM:

Market segments.

SalesMen	A	B	C	D
W	0	2	4	7
X	0	0	0	1
Y	0	0	0	1
Z	2	1	0	0

Assignment (First solution)

Market segments.

SalesMen	A	B	C	D
W	**0**	2	4	7
X	0x	**0**	0x	1
Y	0x	0x	**0**	1
Z	2	1	0x	**0**

(Alternate Solution)

Market segments.

SalesMen	A	B	C	D
W	**0**	2	4	7
X	0x	0x	**0**	1
Y	0x	**0**	0x	1
Z	2	1	0x	0

Solution I: W to A, X to B, Y to C and Z to D. Sales: 42 + 25 + 20 + 12 = Rs. 99 x 10, 000
Solution II: W to A, X to C, Y to B and Z to D Sales: 42 + 20 + 25 + 12 = Rs. 99 x 10,000

Problem 5.6.

The city post office has five major counters namely, Registration (*R*), Savings (*S*), Money – Order (*M*), Postal stationary (*P*) and Insurance / license (*I*). The postmaster has to assign five counters

to five clerks *A*, *B*, *C*, *D* and *E* one for each counter. Considering the experience and ability of these clerks he rates their suitability on a certain 10 - point scale of effectiveness of performance for accomplishing different counter duties, as listed below. Assign the counters to the clerks for maximum effective performance.

Clerks (effective performance)

Counters	A	B	C	D	E
R	6	6	4	6	7
S	5	4	3	6	8
M	7	6	3	5	5
P	7	5	6	8	8
I	4	3	6	7	6

Convert the problem into minimization problem. (We can deduct all other elements form highest element).

Note : As every row has a zero, we can consider it as Row Opportunity Cost Matrix.

ROCM

Clerks (effective performance)

Counters	A	B	C	D	E
R	1	1	3	1	0
S	3	4	5	2	0
M	0	1	4	2	2
P	1	3	2	0	0
I	3	4	1	0	1

TOCM:

Clerks (effective performance)

Counters	A	B	C	D	E
R	1	0	2	1	0
S	3	3	4	2	0
M	0	0	3	2	0
P	1	2	1	0	0
I	3	3	0	0	1

As five lines are there we can make assignment.

Clerks (effective performance)

Counters ↓	A	B	C	D	E
R	1	**0**	2	1	0x
S	3	3	4	2	**0**
M	**0**	0x	3	2	2
P	1	2	1	**0**	0x
I	3	3	0	0x	1

Assignment: *R* to *B*, *S* to *E*, *M* to *A*, *P* to *D* and *I* to *C*. Total effectiveness: 6 + 8 + 7 + 8 + 6 = 35 points.

Problem 5.7.

There are 5 jobs namely, *A*, *B*, *C*, *D*, and *E*. These are to be assigned to 5 machines *P*, *Q*, *R*, *S* and *T* to minimize the cost of production. The cost matrix is given below. Assign the jobs to machine on one to one basis.

Jobs (Cost in Rs.)

Machines ↓	A	B	C	D	E
P	8	7	4	11	6
Q	10	5	5	13	7
R	6	9	8	7	12
S	6	7	2	3	2
T	7	8	8	1 0	5

ROCM:

Jobs (Cost in Rs.)

Machines ↓	A	B	C	D	E
P	4	3	0	7	2
Q	5	0	0	8	2
R	0	3	2	1	6
S	4	5	0	1	0
T	2	3	3	5	0

TOCM:

Jobs (Cost in Rs.)

Machines ↓	A	B	C	D	E
P	4	3	0	6	2
Q	5	0	0	7	2
R	0	3	2	0	6
S	4	5	0	0	0
T	2	3	3	4	0

There are five lines and hence we can make assignment.

Jobs (Cost in Rs.)

Machines ↓	A	B	C	D	E
P	4	3	**0**	6	2
Q	5	**0**	⊠	7	2
R	**0**	3	2	⊠	6
S	4	5	⊠	**0**	⊠
T	2	3	3	4	**0**

Assignment: *P* to *C*, *Q* to *B*, *R* to *A*, and *S* to *D* and T to *E*. Total cost = 4 + 5 + 6 + 3 + 5 = Rs.23/-

Problem 5.8.

Four different jobs are to be done on four machines, one job on each machine, as set up costs and times are too high to permit a job being worked on more than one machine. The matrix given below gives the times of producing jobs on different machines. Assign the jobs to machine so that total time of production is minimized.

Machines (time in hours)

Jobs ↓	A	B	C	D
P	10	14	22	12
Q	16	10	18	12
R	8	14	20	14
S	20	8	16	6

Solution

ROCM:

Machines (time in hours)

Jobs↓	A	B	C	D
P	0	4	12	2
Q	6	0	8	2
R	0	6	12	6
S	14	2	10	0

TOCM:

Machines (time in hours)

Jobs↓	A	B	C	D
P	0	4	4	2
Q	6	0	0	2
R	0	6	4	6
S	14	2	2	0

TOCM:

Machines (time in hours)

Jobs↓	A	B	C	D
P	0	4	4	2
Q	6	0	0	2
R	0	6	4	6
S	14	2	6	0

TOCM:

Machines (time in hours)

Jobs↓	A	B	C	D
P	0	2	2	0
Q	8	0	0	2
R	0	4	2	4
S	16	2	4	0

TOCM:

Machines (time in hours)

Jobs	A	B	C	D
P	0	0	0	0
Q	10	0	0	4
R	0	2	0	0
S	16	0	0	0

Four lines are there hence we can make assignment. As there is a tie, we have more than one solution.

Solution I.

TOCM:

Machines (time in hours)

Jobs	A	B	C	D
P	**0**	0	0	0
Q	10	**0**	0	4
R	0	2	**0**	0
S	14	0	4	**0**

Assignment: *P* to *A*, *Q* to *B*, *R* to *C* and *S* to *D*.
Time: 10 + 10 + 20 + 06 = 46 hours.

Solution II.

TOCM:

Machines (time in hours)

Jobs	A	B	C	D
P	0	**0**	0	0
Q	10	0	**0**	4
R	**0**	2	0	0
S	14	0	4	**0**

Assignment: *P* to *B*, *Q* to *C*, *R* to *A* and *S* to *D*.
Time: 14 + 18 + 8 + 6 = 46 hours.

We can write many alternate solutions.

Problem 5.9.

On a given day District head quarter has the information that one ambulance van is stationed at each of the five locations *A*, *B*, *C*, *D* and *E*. The district quarter is to be issued for the ambulance van to reach 6 locations namely, *P*, *Q*, *R*, *S*, *T* and *U*, one each. The distances in Km. between present locations of ambulance vans and destinations are given in the matrix below. Decide the assignment of vans for minimum total distance, and also state which destination should not expect ambulance van to arrive.

To (distance in Km.)

From	*P*	*Q*	*R*	*S*	*T*	*U*
A	18	21	31	17	26	29
B	16	20	18	16	21	31
C	30	25	27	26	18	19
D	25	33	45	16	32	20
E	36	30	18	15	31	30

Solution

As the given matrix is not square matrix, balance the same by opening one dummy row (DR), with zero as the elements of the cells.

To (distance in Km.)

From	*P*	*Q*	*R*	*S*	*T*	*U*
A	18	21	31	17	26	29
B	16	20	18	16	21	31
C	30	25	27	26	18	19
D	25	33	45	16	32	20
E	36	30	18	15	31	30
DR	0	0	0	0	0	0

As every column has got one zero, we can take it as COCM. Now doing row operation on COCM, we get TOCM.

TOCM

	P	*Q*	*R*	*S*	*T*	*U*
A	1	4	14	0	9	12
B	0	4	2	0	5	15
C	12	7	9	8	0	1
D	9	17	29	0	16	4
E	21	15	3	0	16	15
DR	0	0	0	0	0	0

Assignment: *A* to *P*, *B* to *R*, *C* to *T*, *D* to *U*, *E* to *S* and DR to *Q* i.e the van at *Q* will not go to any destination.

Total Distance: 18 + 18 + 18 + 20 + 15 = 89 Km.

Other alternative assignments are:

From:	A	B	C	D	E	DR	Station for which no van	Total Distance in Km.
To:	Q	P	T	S	U	R	R	89
	S	P	T	U	R	Q	Q	89

Brain tonic:

    a) In case the cost of dispatching an ambulance is 3 times the distance, determine the assignment of ambulances to destinations.

    (*b*) In case the operating cost of a van is proportional to the square of the distance decide the assignment.

(Note: a) By multiplying the entire matrix by 3 we get the cost matrix. This does not have any effect on the final solution. Hence the same solution will hold good.

    (*b*) We have to write the elements by squaring the elements of the original matrix and make fresh assignment.)

Problem 5.10.

A job order company has to work out the assignment of 5 different jobs on five different machines. The cost of machining per unit of job and set up cost of the job on a machine are as given in the matrix *A* and *B* given below. The jobs are to be made in bathe sizes show against them. Set up cost is independent of previous set up.

Matrix A. (Operating cost in Rs)

Jobs (machining cost in Rs)

Machines.	A	B	C	D	E
P	0.80	1.10	0.70	1.60	6.20
Q	1.20	0.90	1 .20	0.80	5.40
R	2.10	2.00	1.00	2.20	4.90
S	---	1.60	2.00	1.90	3.60
T	3.20	2.00	2.00	2.00	2.60
Batch size in units.	100	100	150	100	50

Matrix B (Set up cost in Rs)

Jobs (cost in Rs)

Machines	A	B	C	D	E
P	60	70	70	30	40
Q	40	50	50	20	80
R	30	40	40	40	100
S	---	90	60	50	60
T	80	100	80	60	60

Solution

Multiply the Matrix *A* by 100 and add it to the matrix *B* we get the matrix given below. For the element *SA* as nothing is given, we can eliminate it for further consideration or assign a very high cost for the element so as to avoid it from further calculations.

Jobs (combined setup and processing cost in Rs)

Machines ↓	A	B	C	D	E
P	140	180	175	190	350
Q	160	140	230	100	350
R	240	240	190	260	255
S	1000	250	360	240	240
T	400	300	380	260	190

ROCM:

Machines ↓	A	B	C	D	E
P	0	40	35	50	210
Q	60	40	130	0	250
R	50	50	0	70	65
S	760	10	120	0	0
T	210	110	190	70	0

TOCM:

Machines ↓	A	B	C	D	E
P	0	30	35	50	210
Q	60	30	130	**0**	250
R	50	40	**0**	70	65
S	760	**0**	120	0̸	0̸
T	210	100	190	70	**0**

Assignment: *P* to *A*, *Q* to *D*, *R* to *C*, *S* to *B* and *T* to *E*.
Total cost = 140 + 100 + 190 + 250 + 190 = Rs. 870/-

Problem 5.11.

There are five major projects namely, Fertiliser plants, Nuclear poser plants, Electronic park, Aircraft complex and Heavy machine tools. These five plants are to be assigned to six regions namely *A*, *B*, *C*, *D*, *E* and *F*, insisting on allocation of as many number of projects as possible in their region. The state department has evaluated the effectiveness of projects in different regions for (*a*) Employment potential, (*b*) Resource utilization potential, (*c*) Economic profitability and (*d*) Environmental degradation index as given below in

Tableau I. (The ranking is on a 20 point scale). Assign one project to one region depending on the maximum total effectiveness. (Plants are given serial numbers 1 to 5)

Tableau I.

Reg. long.	Local Employment Potential.					Resource Allocation Potential.					Economic Profitability Index.					Environmental Degradation index.				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
A	16	10	8	12	11	7	6	4	5	3	11	13	14	15	10	15	14	5	3	2
B	18	15	12	10	7	11	4	3	2	1	10	15	17	11	16	13	14	5	3	2
C	12	16	12	5	8	16	5	4	3	2	13	14	16	12	11	12	11	5	4	2
D	14	10	13	6	8	15	3	2	4	1	7	10	5	11	8	12	11	5	4	2
E	15	17	11	18	11	8	3	4	2	4	10	12	7	11	16	9	6	5	4	3
F	12	18	11	15	1 4	1 7	5	2	1	3	5	10	12	13	12	6	3	5	5	2

Solution

In this problem, for maximization of total effectiveness, the first three *i.e.* Employment potential, Resource utilization potential and economic profitability index are to be added and the environmental degradation is to be subtracted from the sum to get the total effectiveness. Once we get the effectiveness matrix, then the projects are to be assigned to the regions for maximization of total effectiveness.

The total effectiveness matrix: (Note: The matrix is of the order 5×6, hence it is to be balanced by opening a dummy column - *DC*). The first element of the matrix can be worked out as: $16 + 7 + 11 - 15 = 19$. Other elements can be worked out similarly.

Total effectiveness matrix:

Plants.

Regions ↓	1	2	3	4	5	DC
A	19	15	21	29	22	0
B	26	20	27	20	22	0
C	29	24	27	16	19	0
D	24	12	15	17	15	0
E	24	26	17	27	28	0
F	28	30	20	24	27	0

As there is a dummy column the same matrix may be considered as ROCM. By deducting all the elements of a column from the highest element of the column, we get the Total Opportunity Cost Matrix.

TOCM:

Plants.

Regions ↓	1	2	3	4	5	DC
A	10	15	6	0	6	0x
B	3	10	0	9	6	0x
C	0	6	0x	13	11	0x
D	5	18	12	12	13	0
E	5	4	10	2	0	0x
F	2	0	7	5	1	0x

Allocation: Fertilizer: C, Nuclear Plant: F, Electronic Park: B,
Aircraft Complex: A, Heavy Machine Tools: E

5.5. SCHEDULING PROBLEM

Now let us work scheduling problem. This type of problems we can see in arranging air flights or bus transport or rail transport. The peculiarities of this type of problem is that one flight / train / bus leaves form a station with some flight number / train number / bus number. After reaching the destination, the same plane / train/bus leaves that place (destination) and reaches the hometown with different number. For example plane bearing flight number as 101 leaves Bangalore and reaches Bombay and leaves Bombay as flight number 202 and reaches Bangalore. Our problem here is how to arrange a limited number of planes with crew / trains with crew / bus with crew between two places to make the trips without inconvenience, by allowing required lay over time. Lay over time means the time allowed for crew to take rest before starting.

Problem 5. 12. (Scheduling Problem).

For the following Airline time table between Banglore and Mumbai it is required to pair to and for flights for the same crew, so as to minimize the lay over time of the crew on ground away from Head quarters. It is possible to assign Banglore or Bombay as the head quarter. Decide the pairing of flights and head quarters of the concerned crew. It is stipulated that the same crew cannot undertake next flight, within one hour of the arrival. That is one hour is the layover time.

Flight No.	Departure Mumbai	Arrival Bangalore	Flight No.	Departure Bangalore	Arrival. Mumbai
101	6-30 a.m	7.45 a.m	102	7.00 a.m	8.00 a.m
103	9.00 a.m.	10.30 a.m	104	11.00 a.m.	12.15 p.m
105	1.00 p.m.	2.15 p.m.	106	3.00 p.m.	4.15 p.m.
107	4.00 p.m.	5.30 p.m	108	5.45 p.m	7.15 p.m
109	8.00 p.m	9.30 p.m.	110	8.30 p.m.	9.45 p.m.

Solution

Now let us consider the layover times separately for crew based at Mumbai and crew based at Bangalore.

Let us consider one flight and discuss how to calculate layover time. For example, flight No. 101 leaves Mumbai at 6.30 a.m and reaches Bangalore at 7.45 a.m. Unless the crew takes one our rest, they cannot fly the airplane. So if the crew cannot leave Bangalore until 8.45 a.m. So there is no chance for the crew to go for flight No. 102. But they can go as flight Nos. 103, 106, 108 and 110. As we have to minimize the flyover time, we can take the nearest flight *i.e.* 103. The flight 103 leaves Bangalore at 11.00 a.m. By 11.00 a.m the crew might have spent time at Bangalore from 7.45 a.m to 11.00 a.m. That is it has spent 3 hours and 15 minutes. If we convert 3 hours and 15 minutes in terms of quarter hours, it will become 13-quarter hours. Similarly the flight 102 which arrives at Mumbai at 8.00 a.m. wants to leave as flight 101 at 6.30 a.m. it has to leave next day morning. Hence the layover time will be 22 hours and 30 minutes. Like wise, we can workout layover time for all flights and we can write two matrices, one for crew at Mumbai and other for crew at Bangalore.

Tableau I. Lay over time for Mumbai based crew:

Flight numbers. (Quarter hours)

Flight No.	102	104	106	108	110
101	23.25	3.25	7.25	10.00	11.75
103	18.50	24.50	4.50	7.25	10.00
105	16.75	20.75	24.75	3.50	5.25
107	13.50	17.50	21.50	24.25	3.00
109	9.50	13.50	17.50	20.25	23.00

Tableau II. Lay over time for Bangalore based crew:

Layover time in quarter hours.

Flight No.	102	104	106	108	110
101	23.50	18.25	14.25	11.75	8.75
103	1.00	20.75	16.75	13.75	13.25
105	5.00	24.75	20.75	17.75	15.25
107	8.00	3.75	23.75	20.75	18.25
109	12.00	7.75	7.75	24.25	22.25

The matrices can be multiplied by four to convert decimals into whole numbers for convenience of calculations.

Tableau II. Bombay based layover times

Flight No.	102	104	106	108	110
101	93	13	29	40	47
103	74	98	14	29	40
105	67	83	97	14	21
107	54	70	86	97	12
109	38	54	70	81	92

Layover time of crew stationed at Bangalore. (*)

Flight No.	102	104	106	108	110
101	90	73	57	67	35
103	4	83	67	55	53
105	20	97	83	71	61
107	32	15	95	83	73
109	48	31	15	97	89

Now let us select the minimum elements from both the matrices and write another matrix with these elements. As our objective is to minimize the total layover time, we are selecting the lowest element between the two matrices. Also, let us mark a * for the entries of the matrix showing layover time of the crew at Bangalore.

Matrix showing the lowest layover time
(The elements marked with * are from Bangalore matrix)

Flight No.	102	104	106	108	110
101	90*	13	29	40	35*
103	4*	83*	14	29	40
105	20*	83	83*	14	21
107	32*	15*	86	83*	12
109	38	31*	15*	81	89*

ROCM: As every column has got a zero, this may be considered as TOCM and assignment can be made. Note that all zeros in the matrix are in independent position we can make assignment.

Flight No.	102	104	106	108	110
101	77*	**0**	16	27	22*
103	**0***	79*	10	25	36
105	6*	69	69*	**0**	7
107	20*	3*	74	71*	**0**
109	23	16*	**0***	65	74*

Assignment and pairing:

Flight No.	Leaves as	Crew based at.
101	104	Bombay
103	102	Bangalore
105	108	Bombay
107	110	Bombay
109	106	Bangalore.

Total Layover time is: 3.25 + 1.00 + 3.50 + 3.0 + 17.50 = 28 hours and 15 minutes.

Problem 5.13.

An airline that operates seven days a week has the timetable shown below. Crews must have a minimum layover time 5 hours between flights. Obtain the pairing of flights that minimises layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover. For each pair also mention the town where crew should be based.

Chennai - Bangalore   Bangalore - Chennai.

FlightNo.	Departure	Arrival	Flight No.	Departure	Arrival
101	7.00 a.m	8.00 a.m	201	8.00 a.m	9.00 a.m
102	8.00 a.m	9.00 a.m	202	9.00 a.m	10.00 a.m
103	1.00 p.m	2.00 p.m	203	12.00 noon	1.00 p.m.
104	6.00 p.m.	7.00 p.m	204	8.00 p.m	9.00 p.m

Let us write two matrices one for layover time of Chennai based crew and other for Bangalore based crew.

As explained in the example 5.11 the departure of the crew once it reaches the destination, should be found after taking the minimum layover time given, *i.e.* 5 hours. After words, minimum elements from both the matrices are to be selected to get the matrix showing minimum layover times. Finally, we have to make assignment for minimum layover time.

Layover time for Chennai based crew in hours.

Tableau I.

FlightNo.	201	202	203	203
101	24	25	28	12
102	23	24	27	11
103	20	19	22	6
104	13	14	17	25

Layover time for Bangalore based crew in hours.
Tableau I.

FlightNo.	201	202	203	203
101	22	21	18	10
102	23	22	19	11
103	28	27	24	16
104	9	8	5	21

Minimum of the two matrices layover time. The Bangalore based times are marked with a (*).
Tableau I.

FlightNo.	201	202	203	203
101	22*	21*	18*	10*
102	23**	22*	19*	11**
103	20	19	22	6
104	9*	8*	5*	21*

The elements with two stars (**) appear in both the matrices.

ROCM
Tableau I.

FlightNo.	201	202	203	203
101	12	11	8	0
102	12	11	8	0
103	14	13	16	0
104	4	3	0	16

TOCM:

FlightNo.	201	202	203	203
101	8	8	8	0
102	8	8	8	0
103	6	10	16	0
104	0	0	0	16

FlightNo.	201	202	203	203
101	2	2	2	0
102	2	2	2	0
103	0	4	10	0
104	0	0	0	22

FlightNo.	201	202	203	203
101	0	0	0	0
102	0	0	0	0
103	0	4	10	2
104	0	0	0	24

FlightNo.	201	202	203	204
101	0	0	0	0*
102	0	0*	0	0
103	0	4	10	2
104	0	0	0*	24

Assignment:

Flight No.	Leaves as	Based at
101	204	Bangalore
102	202	Bangalore
103	201	Chennai
104	203	Bangalore.

Total layover time: $10 + 22 + 20 + 5 = 67$ hours.

5.6. TRAVELING SALESMAN PROBLEM

Just consider how a postman delivers the post to the addressee. He arranges all the letters in an order and starts from the post office and goes from addressee to addressee and finally back to his post office. If he does not arrange the posts in an order he may have to travel a long distance to clear all the posts. Similarly, a traveling sales man has to plan his visits. Let us say, he starts from his head office and go round the branch offices and come back to his head office. While traveling he will not visit the branch already visited and he will not come back until he visits all the branches.

There are different types of traveling salesman's problems. One is **cyclic problem**. In this problem, he starts from his head quarters and after visiting all the branches, he will be back to his head quarters. The second one is **Acyclic problem**. In this case, the traveling salesman leaves his head quarters and after visiting the intermediate branches, finally reaches the last branch and stays there. The first type of the problem is solved by Hungarian method or Assignment technique. The second one is solved by Dynamic programming method.

Point to Note: The traveling salesman's problem, where we sequence the cities or branches he has to visit is a SEQUENCING PROBLEM. But the solution is got by Assignment technique. Hence basically, the traveling salesman problem is a SEQUENCING PROBLEM; the objective is to minimize the total distance traveled.

The mathematical statement of the problem is: Decide variable $x_{ij} = 1$ or 0 for all values of I and j so as to:

Minimise $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \text{ for all } i \text{ and } j = 1,2.....n \text{ Subject to}$$

$$\sum_{J=1}^{n} X_{ij} = 1 \text{ for } i = 1,2,....n \quad \text{(Depart from a city once only)}$$

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j = 1,2,....n \quad \text{(Arrive at a city once only)}$$

And all $x_{ij} \geq 0$ for all i and j

This is indeed a statement of assignment problem, which may give to or more disconnected cycles in optimum solution. This is not permitted. That is salesman is not permitted to return to the origin of his tour before visiting all other cities in his itinerary. The mathematical formulation above does not take care of this point.

A restriction like $X_{ab} + X_{bc} + X_{ca} \leq 2$ will prevent sub-cycles of cities A, B, C and back to A. It is sufficient to state at this stage that all sub- cycles can be ruled out by particular specifications of linear constraints. This part, it is easy to see that a variable $x_{ij} = 1$, has no meaning. To exclude this from solution, we attribute very large cost to it *i.e.* infinity or big M, which is very larger than all the elements in the matrix.

In our solutions big M is used.

Problem 5.14.

A salesman stationed at city A has to decide his tour plan to visit cities B, C, D, E and back to city A I the order of his choice so that total distance traveled is minimum. No sub touring is permitted. He cannot travel from city A to city A itself. The distance between cities in Kilometers is given below:

Cities	A	B	C	D	E
A	M	16	18	13	20
B	21	M	16	27	14
C	12	14	M	15	21
D	11	18	19	M	21
E	16	14	17	12	M

Instead of big M we can use infinity also. Or any element, which is sufficiently larger than all the elements in the matrix, can be used.

Solution

COCM:

Cities	A	B	C	D	E
A	M	3	5	0	7
B	7	M	2	13	0
C	0	2	M	3	9
D	0	7	8	M	10
E	4	2	5	0	M

TOCM:

Cities	A	B	C	D	E
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence modify the matrix. Smallest element in the uncovered cells is 3, deduct this from all other uncovered cells and add this to the elements at the crossed cells. Do not alter the elements in cells covered by the line.

TOCM

Cities	A	B	C	D	E
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence once again modify the matrix.

Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D*, and *D* to *A*. As there is a tie

TOCM:

Cities	A	B	C	D	E
A	M	1	**0**	0	4
B	10	M	0 x	16	**0**
C	0x	**0**	M	3	6
D	**0**	5	3	M	7
E	4	0x	0	**0**	M

Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D* and *D* to *A*. as there is a tie between the zero cells, the problem has alternate solution. The total distance traveled by the salesman is: $18 + 14 + 14 + 11 + 12 = 69$ Km.

A to *C* to *B* to *E* to *D* to *A*, the distance traveled is 69 Km.

Note: See that no city is visited twice by sales man.

Problem 5.15.

Given the set up costs below, show how to sequence the production so as to minimize the total setup cost per cycle.

Jobs	A	B	C	D	E
A	M	2	5	7	1
B	6	M	3	8	2
C	8	7	M	4	7
D	12	4	6	M	5
E	1	3	2	8	M

Solution

COCM:

Jobs	A	B	C	D	E
A	M	1	4	6	0
B	4	M	1	6	0
C	4	3	M	0	3
D	8	0	2	M	1
E	0	2	1	7	M

TOCM:

Jobs	A	B	C	D	E
A	M	1	3	6	**0**
B	4	M	**0**	6	0x
C	4	3	M	**0**	3
D	8	**0**	1	M	1
E	**0**	2	0x	7	M

We can draw five lines and make assignment. The assignment is:

From *A* to *E* and From *E* to *A* cycling starts, which is not allowed in salesman problem. Hence what we have to do is to select the next higher element than zero and make assignment with those elements. After assignment of next higher element is over, then come to zero for assignment. If we cannot finish the assignment with that higher element, then select next highest element and finish assigning those elements and come to next lower element and then to zero. Like this we have to finish all assignments. In this problem, the next highest element to zero is 1. Hence first assign all ones and then consider zero for assignment. Now we shall first assign all ones and then come to zero.

TOCM:

Jobs	A	B	C	D	E
A	M	**1**	3	6	0x
B	4	M	**0**	6	0x
C	4	3	M	**0**	3
D	8	0	1	M	**1**
E	**0**	2	0x	7	M

The assignment is *A* to *B*, *B* to *C*, *C* to *D* and *D* to *E* and *E* to *A*. (If we start with the element DC then cycling starts.

Now the total distance is $5 + 3 + 4 + 5 + 1 = 18 + 1 + 1 = 20$ Km. The ones we have assigned are to be added as penalty for violating the assignment rule of assignment algorithm.

Problem 5.16.

Solve the traveling salesman problem by using the data given below:

$C_{12} = 20$, $C_{13} = 4$, $C_{14} = 10$, $C_{23} = 5$, $C_{34} = 6$, $C_{25} = 10$, $C_{35} = 6$, $C_{45} = 20$ and $C_{ij} = C_{ji}$. And there is no route between cities '*i*' and '*j*' if a value for C_{ij} is not given in the statement of the problem. (*i* and *j* are = 1,2,..5)

Solution

Cities	1	2	3	4	5
1	M	20	4	10	M
2	20	M	5	M	10
3	4	5	M	6	6
4	10	M	6	M	20
5	M	10	6	20	M

Now let us work out COCM/ROCM and TOCM, and then make the assignment.

TOCM:

Cities.	1	2	3	4	5
1	M	12	**0**	0x	M
2	11	M	0x	M	**0**
3	0x	1	M	**0**	1
4	**0**	M	0x	M	9
5	M	**0**	0x	8	M

The sequencing is: 1 to 3, 3 to 4, 4 to 1 and 1 to 3 etc., Cycling starts. Hence we shall start assigning with 1 the next highest element and then assign zeros. Here also we will not get the sequencing. Next we have to take the highest element 8 then assign 1 and then come to zeros.

TOCM:

Cities.	1	2	3	4	5
1	M	12	**0**	0	M
2	11	M	0	M	**0**
3	0	**1**	M	0	1
4	**0**	M	0	M	9
5	M	0	0	8	M

Sequencing is: 1 to 3, 3 to 2, 2 to 5, 5 to 4 and 4 to 1.
The optimal distance is : $4 + 10 + 5 + 10 + 20 = 49 + 1 + 8 = 58$ Km.

Problem 5.17.

A tourist organization is planning to arrange a tour to 5 historical places. Starting from the head office at A then going round *B*, *C*, *D* and *E* and then come back to *A*. Their objective is to minimize the total distance covered. Help them in sequencing the cities. *A*, *B*, *C*, *D* and E as the shown in the figure. The numbers on the arrows show the distances in Km.
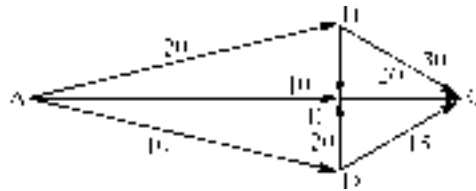
Solution

The distance matrix is as given below:

Places	A	B	C	D	E
A	M	20	M	10	10
B	20	M	30	M	35
C	M	30	M	15	20
D	10	M	15	M	20
E	10	35	20	20	M

COCM

Places	A	B	C	D	E
A	M	10	M	0	0
B	0	M	10	M	15
C	M	15	M	0	5
D	0	M	5	M	10
E	0	25	10	10	M

TOCM:

Places	A	B	C	D	E
A	M	0	M	0	0
B	0	M	5	M	15
C	M	5	M	0	5
D	0	M	0	M	10
E	0	15	5	10	M

TOCM:

Places	A	B	C	D	E
A	M	0	M	5	0
B	0	M	5	M	10
C	M	0	M	0	0
D	0	M	0	M	5
E	0	10	5	10	M

TOCM:

Places	A	B	C	D	E
A	M	0	M	5	0
B	0	M	5	M	5
C	M	0	M	0	0
D	0	M	0	M	0
E	0	5	5	5	M

Places	A	B	C	D	E
A	M	**0**	M	5	0x
B	0x	M	**0**	M	0x
C	M	0x	M	**0**	0x
D	5	M	0x	M	**0**
E	0	0x	0x	0x	M

The sequencing is: *A* to *B*, *B* to *C*, *C* to *D*, *D* to *E* and *E* to *A*.
The total distance is: 20 +30 +15 + 20 + 10 = 95 Km.

5.7. SENSITIVITY ANALYSIS

In fact there is very little scope for sensitivity analysis in Assignment Problem because of the mathematical structure of the problem. If we want to avoid high cost assigning a facility (*i* th) to a job (*j* th), then we can do it by giving a cost of assignmat say infinity or Big M to that cell so that it will not enter into programme. In case of maximisaton model, we can allocate a negative element to that cell to avoid it entering the solution. Further, if one facility (man) can do two jobs *i.e.* 2 jobs are to be assigned to the facility, then this problem can be dealt with by repeating the man's or facility's column and introducing a dummy row to maintain the square matrix. Similarly, if two similar jobs are there, write two identical rows of the two jobs separately and then solve by making a square matrix. Besides these, the addition of a constant throughout any row or column does not affect the optimal solution of the assignment problem.

QUESTIONS

1. Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed by each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

Engineers.	*Projects*			
	A	*B*	*C*	*D*
1	12	10	10	8
2	14	NOT ELIGIBLE	15	11
3	6	10	16	4
4	8	10	9	7

2. (*a*) Explain the differences and similarities between Assignment problem and Transportation problem.

(*b*) Explain why VAM or any other methods of getting basic feasible solution to a transportation problem is not used to get a solution to assignment problem. What difficulties you come across?

3. Explain briefly the procedure adopted in assignment algorithm.

4. Is traveling salesman problem is an assignment problem? If yes how? If not what are the differences between assignment problem and traveling salesman problem.

5. What do you mean by balancing an assignment problem? What steps you take to solve maximization case in assignment problem? Explain.

6. A Computer center has got three expert programmers. The center needs three application programmes to be developed. The head of the computer center, after studying carefully the programmes to be developed estimate the computer time in minutes required by the experts to the application programmes as given in the matrix below. Assign the programmers to the programmes in such a way that the total computer time is least.

Programmers.	Programme.		
	A	B	C
1	120	100	80
2	70	90	110
3	110	140	120

7. (*a*). A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows in hundreds of rupees. Assign the jobs to machines to minimize the total cost.

(*b*) If the given matrix happens to be returns to the company by assigning a particular job to a machine, then what will be the assignment? Will the same assignment hold well? If not what will you do to get the new solution.

Jobs (hundreds of rupees)

Machines.	1	2	3	4	5
1	2.5	5	1	6	1
2	2	5	1.5	7	3
3	3	6.5	2	8	3
4	3.3	7	2	9	4.5
5	4	7	3	9	6
6	6	9	5	10	6

8. Miss *A*, *B*, *C*, *D*, *E*, *F* and *G* are seven girls in a 15-member college musical extravaganza team. M/S I, II, III, IV, V, VI, VII and VIII are the male members of the team and eligible bachelors except Mr. IV. The team decides thinking in terms of matrimonial bondage amongst them that the match - making should be such as to maximize the happiness of the entire group. Fortunately for Mr. IV is already married and he is asked to devise measure of happiness, collect data and decide the pairs.

Mr. IV collects on a 20 - point scale girl's liking for different boys and calls it as *X* - factor and the boy's liking for different girls as *Y*- factor. The matrix given below shows these to factors. The elements in the brackets are *Y* - factors. Mr. IV is baffled by the pattern of emotional linkages and variations in their intensities.

He decides on his own without consulting anyone concerned, to give more weightages to *X* - factor on account of intuitional soundness of girl's soundness of judgement and their emotional steadfastness, beside flexibility in adjustments. Therefore, he takes $K = 2X + B$ as the factor of pair's matrimonial happiness. Then coded matrix of *A*, *B*, *C*, *D*, *E*, *F*, and *G* against *K*, *L*, *M*, *N*, *P*, *Q*, and *R* is given to an Operations Research student to solve it as an assignment problem. (Girls form A to G and Boys form K to R). The solution was handed over to all the concerned. Make the assignment.

Points. Factor Y (Factor X)

Boys

Girls	K	L	M	N	P	Q	R
A	11 (14)	14 (15)	15 (10)	13 (18)	16 (15)	17 (14)	12 (10)
B	16 (15)	13 (17)	18 (11)	15 (18)	16 (14)	17 (14)	14 (12)
C	10 (16)	10 (11)	10 (12)	10 (18)	10 (18)	10 (18)	10 (13)
D	16 (16)	11 (14)	10 (12)	10 (18)	11 (15)	12 (13)	14 (15)
E	7 (18)	5 (17)	12 (13)	14 (10)	6 (16)	17 (12)	15 (16)
F	9 (18)	16 (12)	12 (11)	13 (18)	12 (11)	15 (12)	16 (15)
G	15 (15)	17 (10)	16 (7)	17 (18)	15 (10)	17 (13)	16 (14)

9. Solve the traveling salesman problem given below for minimizing the total distance traveled. Distance in Km.

Cities	A	B	C	D	E
A	M	10	8	29	12
B	16	14	12	10	9
C	6	3	17	14	12
D	12	19	17	14	12
E	11	8	16	13	M

10. An airline that operates flights between Delhi and Bombay has the following timetable. Pair the flights, so as to minimize the total layover time for the crew. The plane, which reaches its destination, cannot leave that place before 4 hours of rest.

Flight No.	Departure	Arrival	Flight No.	Departure	Arrival
101	9.00 a.m	11.00 a.m	201	10.00 a.m	12.00 Nn.
102	10.00 a.m	12.00 Nn	202	12.00 Nn	2.00 p.m
103	4.00 p.m	6.00 p.m	203	3.00 p.m	5.00 p.m
104	7.00 p.m	9.00 p.m	204	8.00 p.m	10.p.m.

11. The productivity of operators *A, B, C, D*, and *E* on different machines *P, Q, R, S*, and *T* are given in the matrix below. Assign machine to operators of maximum productivity.

**Productivity
Machines.**

Operators	P	Q	R	S	T
A	9	14	10	7	12
B	8	11	12	---	13
C	10	10	8	11	---
D	12	14	11	10	7
E	13	10	12	13	10

12. In the above problem, operating costs of machines / shift are Rs.6/-, Rs.7/- Rs.15/-, Rs. 11/- and Rs. 10/- respectively, and Daily wages are Rs. 25/- , Rs. 30/- , Rs. 28/-, Rs. 26/- and Rs.20/- respectively for machine a, *B*, c, *D* and *E*. And all the operators on piece - bonus, so that for every one piece above the basic production per shift the bonus is paid at the rates are as shown on next page on different machines along with basic production per shift. Find the cost of production and the cost per unit. Assign the machines to operators for minimum cost of production per piece.

Machines.

Particulars.	P	Q	R	S	T
Basic productionPieces per shift.	8	10	8	7	7
Incentive bonusPer piece in Rs.	1.0	1.0	1.6	2.0	2.0

MULTIPLE CHOICE QUESTIONS

1. Assignment Problem is basically a
    (*a*) Maximization Problem, (*b*) Minimization Problem, (*c*) Transportation Problem
    (*d*) Primal problem   ( )

2. The Assignment Problem is solved by
    (*a*) Simplex method,   (*b*) Graphical method, (*c*) Vector method,   (*d*) Hungarian method
                                                                                       ( )

3. In Index method of solving assignment problem
    (*a*) The whole matrix is divided by smallest element, (*b*) The smallest element is subtracted
          from whole matrix   (*c*) Each row or column is divided by smallest element in that
          particular row or column,   (*d*) The whole matrix is multiplied by – 1.

4. In Hungarian method of solving assignment problem, the row opportunity cost matrix is
    obtained by:
    (*a*) Dividing each row by the elements of the row above it,
    (*b*) By subtracting the elements of the row from the elements of the row above it.
    (*c*) By subtracting the smallest element from all other elements of the row.
    (*d*) By subtracting all the elements of the row from the highest element in the matrix.
                                                                                       ( )

5. In Flood's technique of solving assignment problem the column opportunity cost matrix is
    obtained by:
    (*a*) Dividing each column by the elements of a column which is right side of the column
    (*b*) By subtracting the elements of a column from the elements of the column which is
          right side of the column
    (*c*) By subtracting the elements of the column from the highest element of the matrix.
    (*d*) By subtracting the smallest elements in the column from all other elements of the
          column.   (   )

6. The property of total opportunity cost matrix is
    (*a*) It will have zero as elements of one diagonal,
    (*b*) It will have zero as the elements of both diagonals,
    (*c*) It will have at least one zero in each column and each row
    (*d*) It will not have zeros as its elements.   ( )

7. The horizontal and vertical lines drawn to cover all zeros of total opportunity matrix must be:
   (*a*) Equal to each other,
   (*b*) Must be equal to $m \times$ n (where m and n are number of rows and columns)
   (*c*) $m + n$ (*m* and *n* are number of rows and columns)
   (*d*) Number of rows or columns. ( )

8. The assignment matrix is always is a
   (*a*) Rectangular matrix, (*b*) Square matrix (*c*) Identity matrix (*d*) None of the above.
   ( )

9. To balance the assignment matrix we have to:
   (*a*) Open a Dummy row,
   (*b*) Open a Dummy column,
   (*c*) Open either a dummy row or column depending on the situation,
   (*d*) You cannot balance the assignment matrix.
   ( )

10. In cyclic traveling salesman problem the elements of diagonal from left top to right bottom are
    (*a*) Zeros, (*b*) All negative elements, (*c*) All are infinity (*d*) all are ones. ( )

11. To convert the assignment problem into a maximization problem
    (*a*) Deduct smallest element in the matrix from all other elements.
    (*b*) All elements of the matrix are deducted from the highest element in the matrix.
    (*c*) Deduct smallest element in any row form all other elements of the row.
    (*d*) Deduct all elements of the row from highest element in that row. ( )

12. The similarity between Assignment Problem and Transportation problem is:
    (*a*) Both are rectangular matrices, (*b*) Both are square matrices,
    (*c*) Both can be solved by graphical method, (*d*) Both have objective function and non-negativity constraints. ( )

13. The following statement applies to both transportation model and assignment model
    (*a*) The inequalities of both problems are related to one type of resource.
    (*b*) Both use VAM for getting basic feasible solution
    (*c*) Both are tested by MODI method for optimality
    (*d*) Both have objective function, structural constraint and non-negativity constraints.
    ( )

14. To test whether allocations can be made or not (in assignment problem), minimum number of horizontal and vertical lines are drawn. In case the lines drawn is not equal to the number of rows (or columns), to get additional zeros, the following operation is done:
    (*a*) Add smallest element of the uncovered cells to the elements to the line
    (*b*) Subtract smallest element of uncovered rows from all other elements of uncovered cells.
    (*c*) Subtract the smallest element from the next highest number in the element.
    (*d*) Subtract the smallest element from the element at the intersection of horizontal and vertical line. ( )

15. The total opportunity cost matrix is obtained by doing:
    (*a*) Row operation on row opportunity cost matrix,
    (*b*) by doing column operation on row opportunity cost matrix,
    (*c*) By doing column operation on column opportunity cost matrix
    (*d*) None of the above   (   )
16. Flood's technique is a method used for solving
    (*a*) Transportation problem, (*b*) Resource allocation model, (*c*) Assignment mode.
    (*d*) Sequencing model   (   )
17. The assignment problem will have alternate solutions when total opportunity cost matrix has
    (*a*) At least one zero in each row and column,
    (*b*) When all rows have two zeros,
    (*c*) When there is a tie between zero opportunity cost cells,
    (*d*) If two diagonal elements are zeros.   (   )
18. The following character dictates that assignment matrix is a square matrix:
    (*a*) The allocations in assignment problem are one to one
    (*b*) Because we find row opportunity cost matrix
    (*c*) Because we find column opportunity matrix
    (*d*) Because one to make allocations, one has to draw horizontal and veridical lines. (   )
19. When we try to solve assignment problem by transportation algorithm the following difficulty arises:
    (*a*) There will be a tie while making allocations
    (*b*) The problem will get alternate solutions,
    (*c*) The problem degenerate and we have to use epsilon to solve degeneracy
    (*d*) We cannot solve the assignment problem by transportation algorithm.   (   )

ANSWERS

1. (*b*)	2. (*d*)	3. (*c*)	4. (*c*)
5. (*d*)	6. (*c*)	7. (*d*)	8. (*b*)
9. (*c*)	10. (*c*)	11. (*b*)	12. (*d*)
13. (*d*)	14. (*b*)	15. (*b*)	16. (*c*)
17. (*c*)	18. (*a*)	19. (*c*)	

UNIT 4
SEQUENCING MODELS

Syllabus: Introduction to Sequencing, Sequencing Problems, Solution to Sequencing Problem - Processing n-jobs through one machine, Processing n-jobs through two machines, Processing n-jobs through Three Machines, Processing two through m-machines, processing n-jobs through m-machines.

LEARNING OBJECTIES

After studying this chapter, you should be able to:

- Appreciate the aims to study sequencing techniques.
- Use Johnson's rule of sequencing or scheduling.
- Solve some specific problems of scheduling jobs on one machine, two or three machines.
- See how to extend Johnson's rule to more complicated problems.

INTRODUCTION

The short-term schedules show an optimal order (sequence) and time in which jobs are processed as well as show timetables for jobs, equipment, people, materials, facilities and all other resources that are needed to support the production plan. The schedules should use resources efficiently to give low costs and high utilisations. Other purpose of scheduling are, minimising customers wait time for a product, meeting promised delivery dates, keeping stock levels low, giving preferred working pattern, minimising waiting time of patients in a hospital for different types of tests and so on.

The general scheduling or sequencing problem may be described as: Let there be n jobs to be performed one at a time on each of m machines. The sequence (order) of the machines in which each job should be performed is given. The actual or expected time required by the jobs on each of the machines is also given. The general sequencing problem, therefore, is to find the sequence out of $(n!)^m$ possible sequences which minimise the total elapsed time between the start of the job in the first machine and the completion of the last job on the last machine.

In particular, if n = 3 and m= 3, then total number of possible sequences will be $(3!)^3 = 216$. Theoretically, it may be possible to find optimum sequence but it will require a big computational time. Thus, one should adopt sequencing technique.

To find optimum sequence we first calculate the total elapsed time for each of the possible sequences. As stated earlier, even if values of m and n are very small, it is difficult to get the desired sequence with total minimum elapsed time. However, due to certain rules designed by Johnson, the task of determining an optimum sequence has become quite easy.

NOTATIONS, TERMINOLOGY AND ASSUMPTIONS

Notations

t_{ij} = Processing time (time required) for job i on machine j.

T = Total elapsed time for processing all the jobs. This includes idle time, if any.

I_{ij} = Idle time on machine j from the end of job (j - 1) to the start of job i.

Terminology

- **Number of Machines:** The number of machines refer to the number of service facilities through which a job must pass before it is assumed to be completed.
- **Processing Time:** It is the time required by a job on each machine.
- **Processing Order:** It refers to the order (sequence) in which machines are required for completing the job.
- **Idle Time on a Machine:** It is the time for which a machine does not have a job to process, i.e idle time from the end of job (i-1) to the start of job i.
- **Total Elapsed Time:** It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
- **No Passing Rule:** It refers to the rule of maintaining the order in which jobs are to be processed on given machines. For example, if n jobs are to be processed on two

machines M_1 and M_2 in the order M_1 M_2, then each job should go to machine M_1 first and then to M_2.

Assumptions

1. The processing time on different machines are exactly known and are independent of the order of the jobs in which they are to be processed.

2. The time taken by the job in moving from one machine to another is negligible.

3. Once a job has begun on a machine, it must be completed before another job can begin on the same machine.

4. All jobs are known and are ready for processing before the period under consideration begins.

5. Only one job can be processed on a given machine at a time.

6. Machines to be used are of different types.

7. The order of completion of jobs are independent of the sequence of jobs.

PROCESSING n JOBS THROUGH TWO MACHINES

Let there be n jobs, each of which is to be processed through two machines, M_1 and M_2 in the order M_1 M_2, i.e. each job has to be passed through the same sequence of operations. In other words, a job is assigned on machine M_1 first and after it has been completely processed on machine M_1, it is assigned to machine M_2. If the machine M_2 is not free at the moment for processing the same job, then the job has to wait in waiting line for its turn on machine M_2, i.e. passing is not allowed.

Since passing is not allowed, therefore, machine M_1 will remain busy in processing all the n jobs one-by-one which machine M_2 may remain idle time of the second machine. This can be achieved only by determining sequence of n jobs which are to be processed on two machines M_1 and M_2. The procedure suggested by Johnson for determining the optimal sequence can be summarised as follows:

The Algorithm

Step 1   List the jobs along with their processing times on each machine in a table as shown below:

Processing Time on Machine	Job Number			
	1	2	3	n
M_1	t_{11}	t_{12}	t_{13}	t_{1n}
M_2	t_{21}	t_{22}	t_{23}	t_{2n}

Step 2   Examine the columns for processing times on machines M_1 and M_2, and find the smallest processing time in each column, i.e find out, min. (t_{1j}, t_{2j}) for all j.

Step 3(a)   If the smallest processing time is on machine M_1, then schedule the job as early as possible without moving jobs already schedules, i.e place the job in the first available position in the sequence. If the processing time is on machine M_2, then schedule the job as late as possible without moving any jobs already scheduled, i.e. place the job in the last available position in the sequence.

b. If there is a tie in selecting the minimum of all the processing times, then there may be three situations:

3

a. Minimum among all processing times is same for the machine i.e. min $(t_{1j}, t_{2j}) = t_{2k} = t_{2r}$, then process the kth job first and the rth job last.

b. If the tie for the minimum occurs among processing times t_{1j} on machine M_1 only, then select the job corresponding to the smallest job subscript first.

c. If the tie for the minimum occurs among processing times t_{2j} on machine M_2, then select the job corresponding to the largest job corresponding to the largest job subscript last.

Step 4   Remove the assigned jobs from the table. If the table is empty, stop and go to Step 5. Otherwise, got to Step 2.

Step 5   Calculate idle time for machines M_1 and M_2:

a. Idle time for machine M_1 = (Total Elapsed Time) - (Time when the last job in a sequence finishes on machine M_1)

b. Idle time for machine M_2 = Time at which the first job in a sequence finishes on machine $M_1 + \sum_{j=2}^{n}$ {(Time when the jth job in a sequence starts on machine M_2) - (Time when the (j - 1)th job in a sequence finishes on machine M_2)}.

Step 6   The total elapsed time to process all jobs through two machines is given by

Total Elapsed time = Time when the nth job in a sequence finishes on Machine M_2.

$$= \sum_{j=1}^{n} M2j + \sum_{j=1}^{n} I2j$$

where M_{2j} = Time required for processing jth job on machine M_2.

I_{2j} = Time for which machine M_2 remains idle after processing (j - 1)th job and before starting work in jth job.

EXAMPLE 1

Find the sequence that minimises the total elapsed time required to complete the following tasks on two machines:

Task	A	B	C	D	E	F	G	H	I
Machine I	2	5	4	9	6	8	7	5	4
Machine II	6	8	7	4	3	9	3	8	11

Solution:

The smallest processing time between the two machines is 2 which corresponds to task A on Machine I. Thus, task A is scheduled as early as possible to give the sequence as shown below:

A								

After the task A has been set for processing first, we are left with 8 tasks and their processing times as given below:

Task	B	C	D	E	F	G	H	I
Machine I	5	4	9	6	8	7	5	4
Machine II	8	7	4	3	9	3	8	11

The minimum processing time in this reduced problem is 3 which corresponds to task E and G both on machine II. Since the corresponding processing time of task E on machine I is less than the corresponding processing time of task G on machine I, therefore, task E will be scheduled in the last and task G shall be scheduled before it. Tasks E and G will not be considered further. Thus, current partial sequence of scheduling tasks becomes:

A							G	E

A set of processing times now gets reduced to:

Task	B	C	D	F	H	I
Machine I	5	4	9	8	5	4
Machine II	8	7	4	9	8	11

The smallest processing time in this reduced problem is 4, which corresponds to task C and I on machine I and to task D on machine II. Thus task C will be placed in the second sequence cell and task I in the third sequence cell and task D in the sequence cell before task G. The entries of the partial sequence are now:

A	C	I				D	G	E

The set of processing time now gets reduced as follows:

Task	B	F	H
Machine I	5	8	5
Machine II	8	9	8

the smallest processing time in this reduced problem is 5, which corresponds to tasks B and H both on machine I. Since the corresponding processing times of B and h on machine II is same, therefore, either of these two tasks can be placed in fourth and fifth sequence cells. Thus, it indicates an alternative optimal sequence. the optimal sequences are, therefore, given below:

A	C	I	B	H	F	D	G	E

A	C	I	H	B	F	D	G	E

The minimum elapsed time for machines I and II is calculated as shown in Table 1.

Task Sequence	Machine I		Machine II	
	Time In	Time Out	Time In	Time Out
A	0	2	2	8
C	2	6	8	15
I	6	10	15	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
G	37	44	55	58
E	44	50	58	61

In table 1, the minimum elapsed time, i.e time from start of task A to completion of last task E is 61 hours. During this time the machine I remains idle for 61 - 50 = 11 hours. The idle time for machine II is equal to the time at which the first task A in the sequence finishes on machine I plus the last task E in the sequence starts on machine II minus the last but one task G finishes on machine II, i.e 2 + 58 - 58 = 2 hours.

PROCESSING n JOBS THROUGH THREE MACHINES

Johnson provides an extension of his procedure to the case in which there are three instead of two machines. Each job is to be processed through three machines M_1, M_2 and M_3. The list of jobs with their processing times is given below. An optimal solution to this problem can be obtained if either or both of the following conditions hold good.

Processing time on Machine	Job Number			
	1	2	3	4
M_1	t_{11}	t_{12}	t_{13}	t_{1n}
M_2	t_{21}	t_{22}	t_{23}	t_{2n}
M_3	t_{31}	t_{32}	t_{33}	t_{3n}

1. The minimum processing time on machine M_1 is at least as great as the maximum processing time on machine M_2, that is,

$$\min t_{1j} \geq \max t_{1j}, \qquad j = 1, 2, 3, ...n$$

2. The minimum processing time on machine M_3 is at least as great as the maximum processing time on machine M_2, that is

$$\min t_{3j} \geq \max t_{2j}, \qquad j = 1, 2, 3, ...n$$

If either or both the above conditions hold good, then the steps of the algorithm can be summarised in the following steps:

THE ALGORITHM

Step 1: Examine processing times of given jobs on all three machines and if either one or both the above conditions hold, then go to step 2, otherwise the algorithm fails.

Step 2: Introduce two fictitious machines, say G and H with corresponding processing times given by

    i. $t_{Gj} = t_{1j} + t_{2j}$, $j = 1, 2, 3, ...n$.

that is, processing time on machine G is the sum of the processing times on machines M_1 and M_2, and

    ii. $t_{Hj} = t_{2j} + t_{3j}$, $j = 1, 2, 3, ...n$.

that is, processing time on machine H is the sum of the processing times on machines M_2 and M_3.

Step 3: Determine the optimal sequence of jobs for this n-job, two machine equivalent sequencing problem with the prescribed ordering GH in the same way as discussed earlier.

EXAMPLE

Find the sequence that minimises the total time required in performing the following jobs on three machines in the order ABC. Processing times (in hours) are given in the following table:

Job	1	2	3	4	5
Machine A	8	10	6	7	11
Machine B	5	6	2	3	4
Machine C	4	9	8	6	5

Solution: Here, min $(t_{Aj}) = 6$; Min $(t_{Cj}) = 4$; max $(t_{Bj}) = 6$. Since min $(t_{Aj}) \geq (t_{Bj})$ for all j is satisfied, the given problem can be converted into a problem of 5 jobs and two machines. The processing time on two fictitious machines G and H can be determined by the following relationships:

$t_{Gj} = t_{Aj} + t_{Bj},$ j = 1, 2, 3, ...n.

and $t_{Hj} = t_{Bj} + t_{Cj},$ j = 1, 2, 3, ...n.

The processing times for the new problem are given below:

Job	1	2	3	4	5
Machine G	8 + 5 = 13	10 + 6 = 16	6 + 2 = 8	7 + 3 = 10	11 + 4 = 15
Machine H	5 + 4 = 9	6 + 9 = 15	2 + 8 = 10	3 + 6 = 9	4 + 5 = 9

When the algorithm described for n jobs on two machines is applied to this problem, the optimal sequence so obtained is given by

3	2	5	1	4

The total minimum elapsed time is given in Table 1:

Job Sequence	Machine A		Machine B		Machine C	
	Time In	Time Out	Time In	Time Out	Time In	Time Out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
5	16	27	27	31	31	36
1	27	35	35	40	40	44
4	35	42	42	45	45	51

Table 1 indicates that the minimum total elapsed time is 51 hours. The idle time for machines A, B and C is 9 (=51 - 42) hours, 6 (= 51 - 45) hours and 9 (= 8 - 0) + (45 - 44) hours, respectively.

PROCESSING n JOBS THROUGH m MACHINES

Let there be n jobs, each of which is to be processed through m machines, say $M_1, M_2, ...M_m$ in the order $M_1, M_2, ...M_m$. The optimal solution to this problem can be obtained if either or both of the following conditions hold good.

(a)   Min $\{t_{1j}\} \geq$ Max $\{t_{1j}\}$;   j = 2, 3, ... m - 1

and or (b) Min $\{t_{mj}\} \geq$ Max $\{t_{ij}\}$;   j = 2, 3, ... m - 1

that is, the minimum processing time on machines M_1 and M_m is as great as the maximum processing time on any of the remaining (m - 1) machines.

If either or both these conditions hold good, then the steps of the algorithm can be summarised in the following steps:

Step 1: Find, Min $\{t_{1j}\}$, Min $\{t_{mj}\}$ and max $\{t_{ij}\}$ and verify above conditions. If either or both the conditions mentioned above hold, then go to step 2. Otherwise the algorithm fails.

Step 2: Convert m-machine problem into 2-machine problem by introducing two fictitious machines, say

(i) $t_{Gj} = t_{1j} + t_{2j} + t_{3j} +......+t_{m-1j}$ j = 1, 2, 3, ...n.

i.e. processing time of n-jobs on machine G is the sum of the processing times on Machines $M_1, M_2 ......M_{m-1j}$

(ii) $t_{Hj} = t_{2j} + t_{3j} + t_{4j} +......+ t_{mj}$ j = 1, 2, 3, ...n.

7

i.e. processing time of n-jobs on machine H is the sum of the processing times on Machines $M_1, M_2 \ldots \ldots M_{mj}$.

Step 3: The new processing times so obtained can now be used for solving n-job, two machines equivalent sequencing problem with the prescribed ordering HG in the same way as

$t_{2j} + t_{3j} + \ldots \ldots + t_{m-1j} = k$ (constant)

for all j = 1, 2, 3, ... m - 1, then the optimal sequence can be obtained for n-jobs and two machines M_1 and M_m in the order $M_1 M_m$ as usual.

2. If $t_{1j} = t_{mj}$ and $t_{Gj} = t_{Hj}$, for all j = 1, 2, 3, ...n, then total number of optimal sequences will be n and total minimum elapsed time in these cases would also be the same.

3. The method described above for solving n-jobs and m-machines sequencing problem is not a general method. It is applicable only to certain problems where the minimum cost (or time) of processing the jobs through first and/or last machine is more than or equal to the cost (or time) of processing the jobs through remaining machines.

EXAMPLES

1. Find an optimal sequence for the following sequencing problems of four jobs and five machines when passing is not allowed of which processing time (in hours) is given below:

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
A	7	5	2	3	9
B	6	6	4	5	10
C	5	4	5	6	8
D	8	3	3	2	6

Also find the total elapsed time.

Solution: Here,

Min $(t_{M1, j}) = 5 = t_{M1, C}$

Min $(t_{M5, j}) = 6 = t_{M5, D}$

and Max $\{ t_{M2, j}, t_{M3, j}, t_{M4, j} \} = \{6, 5, 6\}$ respectively.

Since the condition of Min $(t_{M5, j}) \geq$ Max $\{ t_{M2, j}, t_{M3, j}, t_{M4, j} \}$ is satisfied, therefore the given problem can be converted into a four jobs and two machines problem as G and H. The processing times of four jobs denoted by t_{Gj} and t_{Hj} on G and H, respectively are as follows:

Job	A	B	C	D
Machine G	17	21	20	16
Machine H	19	25	23	14

where $t_{Gj} = \sum_{i=1}^{m-1} tij$ and $t_{Hj} = \sum_{i=2}^{m} tij$.

Now using the optimal sequence algorithm, the following optimal sequence can be obtained.

A	C	B	D

The total elapsed time corresponding to the optimal sequence can be calculated as shown in Table 1, using the individual processing times given in the original problem.

Table 1 shows that the minimum total elapsed time is 51 hours. The idle time for machines M_1, M_2, M_3, M_4 and M_5 is 25, 33, 37 and 18 hrs respectively.

<center>Table 1   Minimum Elapsed Time</center>

Job Sequence	Machine				
	M_1	M_2	M_3	M_4	M_5
A	0 - 7	7 - 12	12 - 14	14 - 17	16 - 26
B	7 - 12	12 - 18	16- 21	21 - 27	27 - 35

C	12 - 18	18 - 24	24 - 28	28 - 33	35 - 45
D	18 - 26	26 - 29	29 - 32	33 - 35	45 - 51

2. Solve the following sequencing problem giving an optimal solution when passing is not allowed.

Machine	Job				
	A	B	C	D	E
M_1	11	13	9	16	17
M_2	4	3	5	2	6
M_3	6	7	5	8	4
M_4	15	8	13	9	11

Solution: From the data of the problem it is observed that

Min $(t_{M1, j}) = 9 = t_{M1, C}$

Min $(t_{M4, j}) = 8 = t_{M4, B}$

and Max $\{ t_{M2, j}\} = 6 = $, $t_{M2, E;}$ Max $\{ t_{M3, j}\} = 8 = $, $t_{M2, D}$.

Since both the conditions

Min $(t_{M1, j}) \geq$ Max $\{ t_{M2, j;} t_{M3, j}\}$; $j = 1, 2, ..., 5$

are satisfied, therefore given problem can be converted into a 5-jobs and 2-machine problem as G and H.

Further, it may be noted that, $t_{M2, j} + t_{M3, j} = 10$ (a fixed constant) for all j (j = 1, 2, ..., 5).

Thus the given problem is reduced to a problem of solving 5-jobs through 2-machines M_1 and M_4 in the order M_1 M_4. This means machines M_2 and M_4 will have no effect on the optimality of the sequences.

The processing times of 5 jobs on machine M_1 and M_4 is as follows:

Job	A	B	C	D	E
Machine M_1	11	13	9	16	17
Machine M_4	15	8	13	9	11

Now using the algorithm described earlier, the optimal sequence so obtained as follows:

C	A	E	D	B

The total elapsed time corresponding to the optimal sequence is 83 hours as shown in table 1, using the individual processing times given in the original problem:

Table 1 Minimum Total Elapsed Time

Job Sequence	Machine				
	M_1	M_2	M_3	M_4	
C	0 - 9	9 - 14	14 - 19	19 - 32	
A	9 - 20	20 - 24	24 - 30	32 - 45	
E	29 - 36	36 - 42	42 - 46	46 - 57	
D	36 - 52	52 - 54	54 - 62	62 - 71	
B	52 - 65	65 - 68	68 - 75	75 - 83	

PROCESSING TWO JOBS THROUGH m MACHINES

Let there be two jobs A and B each of which is to processed on m machines say M_1, M_2, ..., M_m in two different orders. The technological ordering of each of the two jobs through m machines is known in advance. Such ordering may not be same for both the jobs. The exact or expected processing times on the given machines are known. Each machine can perform

only one job at a time. The objective is to determine an optimal sequence of processing the jobs so as to minimise total elapsed time.

The optimal sequence in this case can be obtained by using graph. The procedure can be illustrated by taking examples.

Example 1: Use the graphical method to minimise the time needed to process the following jobs on the machines shown, i.e. each machine find the job which should be done first. Also calculate the total elapsed time to complete both jobs.

Machine

Job 1 {Sequence:	A	B	C	D	E
Time (hrs)	3	4	2	6	2

Machine

Job 2 {Sequence:	A	B	C	D	E
Time (hrs)	5	4	3	2	6

Solution

The solution procedure for solving the above problem can be summarised in the following steps:

1. Draw the set of axes at right angle to each other where x-axis represents the processing time of job 1 on different machines while job 2 remains idle and y-axis represents processing time of job 2 while job 2 remain idle.

2. Mark the processing times for jobs 1 and 2 on x-axis and y-axis, respectively according to the given order of machines as shown in the figure:
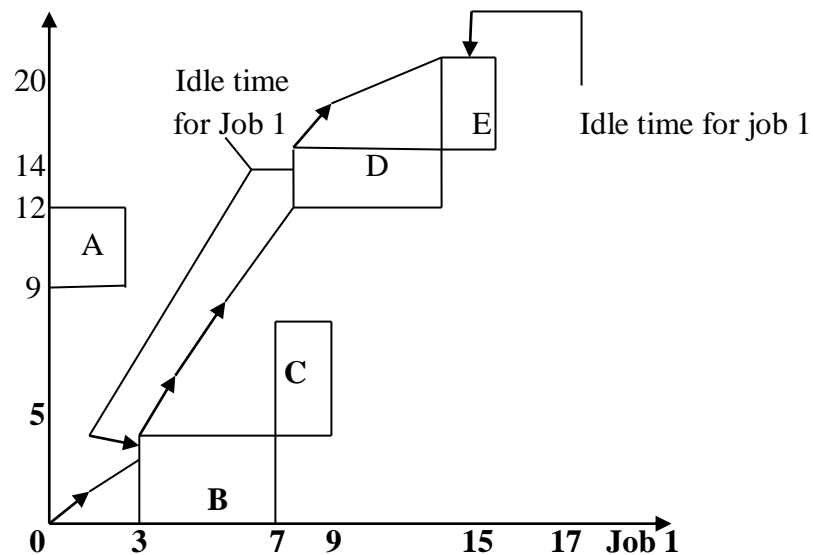
Graphical solution of 2-Jobs and m-Machines Sequencing Problem

For example, machine A takes 3 hours for job 1 and 3 hours for job 2. Construct the rectangle for machine A as shown in above Figure. Similarly, construct other rectangles for machines B, C, D and E.

3. Construct various blocks starting from the origin by pairing the same machine until a point marked 'finished' is obtained.

4. Draw a line starting from origin to the point marked 'finish' by moving horizontally, vertically and diagonally along a line which makes an angle of 45° with the horizontal axis. Moving horizontally along this line indicates that first job is under process while second job is idle. Similarly, moving vertically along this line indicates that the second job is under process while first job is idle. The diagonal movement along this line shows that both the jobs are under process simultaneously.

Since simultaneous processing of both jobs on a machine is not possible, therefore, diagonal movement is not allowed. In other words, diagonal movement through rectangle areas is not allowed.

5. An optimal path is one that minimises the idle time for both the jobs. Thus, we must choose the path on which diagonal movement is maximum as shown in the above figure.

6. Total elapsed time is obtained by adding the idle time for either job to the processing time for that job. In this example, the idle time for the chosen path is found to be 5 hrs and 2 hrs for jobs 1 and 2, respectively. The total elapsed time is calculated as follows:

Elapsed time, Job 1 = Processing time of Job 1 + Idle time for Job 1

$$= 17 + (2 + 3) = 22 \text{ hrs.}$$

Elapsed time, Job 2 = Processing time of Job 2 + Idle time for Job 2

$$= 22 + (17-15) = 24 \text{ hours.}$$

EXERCISES

1. Explain the four elements that characterise a sequencing problem.

2. Explain the principal assumptions made while dealing with sequencing problems.

3. Give Johnson's procedure for determining an optimal sequence for processing n items on two machines. Give justification of the rule used in the procedure.

4. What is no passing rule in a sequencing algorithm? Explain the principal assumptions made while dealing with sequencing problems.

5. Give three different examples of sequencing problems from your daily life.

6. We have five jobs, each of which must be processed on the two machines A and B in the order AB. Processing times in hours are given in the table below:

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

7. We have 5 jobs each of which must go through the machines A, B and C in the order ABC. Processing times (in hours) is as follows:

Job	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

8. What do you understand by the following terms in the context of sequence of jobs:

1. Job arrival pattern   2. Number of machines   3. The flow pattern in the shop

4. the criteria of evaluating the performance of a schedule.

9. Find an optimal sequence for the following sequencing problem of four jobs and five machines (when passing is not allowed) of which processing time (in hrs) is as follows:

Job	1	2	3	4
Machine M_1	6	5	4	7
Machine M_2	4	5	3	2
Machine M_3	1	3	4	2
Machine M_4	2	4	5	1
Machine M_5	8	9	7	5

Also find the total elapsed time.

10. Two jobs are to be processed on four machines A, B, C and D. The technological order for these jobs on machines is as follows:

$$\text{Job 1: A} \qquad \text{B} \qquad \text{C} \qquad \text{D}$$
$$\text{Job 2: D} \qquad \text{B} \qquad \text{A} \qquad \text{C}$$

Processing times are given in the following table:

Machines

	A	B	C	D
Job 1	4	6	7	3
Job 2	4	7	5	8

Find the optimal sequence of jobs on each of the machines.

Theory of Games or Competitive Stratagies

INTRODUCTION

In previous chapters like Linear Programming, Waiting line model, Sequencing problem and Replacement model etc., we have seen the problems related to individual industrial concern and problems are solved to find out the decision variables which satisfy the objective of the industrial unit. But there are certain problems where two or more industrial units are involved in decision making under *conflict situation*. This means that decision-making is done to maximize the benefits and minimize the losses. The decision-making much depends on the decision made or decision variables chosen by the opponent business organization. Such situations are known as **competitive strategies**. Competitive strategies are a type of **business games**. When we here the word game, we get to our mind like pleasure giving games like Foot ball, Badminton, Chess, etc., In these games we have two parties or groups playing the game with definite well defined rules and regulations. The out come of the game as decided decides winning of a group earlier. In our discussion in Theory of Games, we are not concerned with pleasure giving games but we are concerned with **business games**. What is a business game?

Every business manager is interested in capturing the larger share in the market. To do this they have to use different strategies (course of action) to motivate the consumers to prefer their product. For example you might have seen in newspapers certain company is advertising for its product by giving a number of (say 10) eyes and names of 10 cine stars and identify the eyes of the stars and match the name with the eyes. After doing this the reader has to write why he likes the product of the company. For right entry they get a prize. This way they motivate the readers to prefer the product of the company. When the opponent company sees this, they also use similar strategy to motivate the potential market to prefer the product of their company. Like this the companies advertise in series and measure the growth in their market share. This type of game is known as **business game**. Managers competing for share of the market, army chief planning or execution of war, union leaders and management involved in collective bargaining uses different strategies to fulfill their objective or to win over the opponent. All these are known as business games or **competitive situation**. In business, competitive situations arise in advertising and marketing campaigns by competing business firms.

Hence, **Game theory is a body of knowledge that deals with making decisions when two or more intelligent and rational opponents are involved under conditions of conflict or competition. The competitors in the game are called *players*.**

The beginning of **theory of games** goes back to 20 th century. But **John Von Neumann and Morgenstern** have mathematically dealt the theory and published a well-known paper **"theory of Games and Economic Behavior"** in 1944. The mathematical approach of Von Neumann utilizes the

Minimax principle, which involves the fundamental idea of **minimization of the maximum losses**. Many of the competitive problems can be handled by the game theory but not all the competitive problems can be analyzed with the game theory. Before we go to game theory, it is better for us to discuss briefly about decision-making.

` DECISION MAKING

Making decision is an integral and continuous aspect of human life. For child or adult, man or woman, government official or business executive, worker or supervisor, participation in the process of decision-making is a common feature of everyday life. What does this process of decision making involve? What is a decision? How can we analyze and systematize the solving of certain types of decision problems? Answers of all such question are the subject matter of **decision theory**. Decision-making involves listing the various alternatives and evaluating them economically and select best among them. Two important stages in decision-making is: (*i*) making the decision and (*ii*) Implementation of the decision.

Analytical approach to decision making classifies decisions according to the amount and nature of the available information, which is to be fed as input data for a particular decision problems. Since future implementations are integral part of decision-making, available information is classified according to the degree of certainty or uncertainty expected in a particular future situation. With this criterion in mind, three types of decisions can be identified. First one is that these decisions are made when **future can be predicted with certainty.** In this case the decision maker assumes that there is only one possible future in conjunction with a particular course of action. The second one is that decision making under **conditions of risk**. In this case, the future can bring more than one state of affairs in conjunction with a specific course of action. The third one is decision making under **uncertainty**. In this case a particular course of action may face different possible futures, but the probability of such occurrence cannot be estimated objectively.

The Game theory models differ from **decision-making under certainty (DMUC)** and **decision-making under risk (DMUR)** models in two respects. First the opponent the decision maker in a game theory model is an active and rational opponent in DMUC and DMUR models the opponent is the passive state of nature. Second point of importance is decision criterion in game model is the **maximin** or the **minimax** criterion. In DMUC and DMUR models the criterion is the maximization or minimization of some measure of effectiveness such as profit or cost.

10.3. DESCRIPTION OF A GAME

In our day-to-day life we see many games like Chess, Poker, Football, Baseball etc. All these games are pleasure-giving games, which have the character of a competition and are played according to well-structured rules and regulations and end in a **victory** of one or the other team or group or a player. But we refer to the word **game** in this chapter the competition between two business organizations, which has more earning competitive situations. In this chapter game is described as:

A competitive situation is called a game if it has the following characteristics (Assumption made to define a game):

1. There is finite number of competitors called **Players**. This is to say that the game is played by two or more number of business houses. The game may be for creating new market, or to increase the market share or to increase the competitiveness of the product.

2. A list of finite or infinite number of possible **courses of action is available** to each player. The list need not be the same for each player. Such a game is said to be in **normal form**. To explain this we can consider two business houses A and B. Suppose the player A has three strategies, as strategy I is to offer a car for the customer who is selected through advertising campaign. Strategy II may be a house at Ooty for the winning customer, and strategy III may a cash prize of Rs. 10,00,000 for the winning customer. This means to say that the competitor A has three strategies or courses of action. Similarly, the player B may have two strategies, for example strategy I is A pleasure trip to America for 10 days and strategy II may be offer to spend with a cricket star for two days. In this game A has three courses of action and B has two courses of actions. The game can be represented by mans of a matrix as shown below:

$$
\begin{array}{cc}
& \text{B} \\
& \begin{array}{cc} \text{I} & \quad\quad \text{II} \end{array} \\
\text{A} \quad \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \left[\begin{array}{cc} & \\ & \\ & \end{array} \right.
\end{array}
$$

3. A play is played when each player chooses one of his courses o action. The choices are made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action. But in real world, a player makes the choices after the opponent has announced his course of action.

Every play *i.e.* combination of courses of action is associated with an out come, known as ***pay off*** - (generally money or some other quantitative measure for the satisfaction) which determines a set of gains, one to each player. Here **a loss is considered to be negative gain**. Thus after each playoff the game, one player pays to other an amount determined by the courses of action chosen. For example consider the following matrix:

$$
\begin{array}{cc}
& \text{B} \\
& \begin{array}{ccc} \text{I} & \quad \text{II} & \quad \text{III} \end{array} \\
\text{A} \quad \begin{array}{c} \text{I} \\ \\ \text{II} \end{array} & \left[\begin{array}{ccc} 2 & 4 & -3 \\ \\ -1 & 2 & 2 \end{array} \right.
\end{array}
$$

In the given matrix, we have two players. Among these the player who is named on the left side matrix is known as winner, *i.e.* here A is the winner and the matrix given is the matrix of the winner. The player named above is known as the loser. The loser's matrix is the negative version of the given matrix. In the above matrix, which is the matrix of A, a winner, we can describe as follows. If A selects first strategy, and *B* selects the second strategy, the out come is +4 *i.e. A* will get 4 units of money and *B* loses 4 units of money. *i.e. B* has to give 4 units of money to *A*. Suppose *A* selects second strategy and *B* selects first strategy A's out come is –1, *i.e. A* loses one unit of money and he has to give that to *B*, it means *B* wins one unit of money.

4. All players act rationally and intelligently.
5. Each player is interested in **maximizing his gains or minimizing his losses**. The winner, *i.e.* the player on the left side of the matrix always tries to maximize his gains and is known as **Maximin player**. He is interested in maximizing his minimum gains. Similarly, the player B, who is at the top of the matrix, a loser always tries to minimize his losses and is known as **Minimax player** - *i.e.* who tries to minimize his maximum losses.
6. Each player makes individual decisions without direct communication between the players. By principle we assume that the player play a strategy individually, without knowing opponent's strategy. But in real world situations, the player play strategy after knowing the opponent's choice to maximin or minimax his returns.
7. It is assumed that each player knows complete relevant information.

Game theory models can be classified in a number of ways, depending on such factors as the:
   (*i*)   Number of players,
   (*ii*) Algebraic sum of gains and losses
   (*iii*) Number of strategies of each player, which decides the size of matrix.

Number of players: If number of players is two it is known as **Two-person game**. If the number of players is is '*n*' (where $n \geq 3$) it is known as ***n- person game***. In real world two person games are more popular. If the number of players is '*n*', it has to be reduced to two person game by two constant collations, and then we have to solve the game, this is because, the method of solving n- person games are not yet fully developed.

Algebraic sum of gains and losses: A game in which the gains of one player are the losses of other player or the algebraic sum of gains of both players is equal to zero, the game is known as **Zero sum game (ZSG)**. In a zero sum game the algebraic sum of the gains of all players after play is bound to be zero. i.e. If g_i as the pay of to a player in a n-person game, then the game will be a zero sum game if sum of all g_i is equal to zero.

In game theory, the resulting gains can easily be represented in the form of a matrix called **pay - off matrix** or **gain** matrix as discussed in S.No 3 above. A pay - off matrix is a table, which shows how payments should be made at end of a play or the game. Zero sum game is also known as **constant sum game**. Conversely, if the sum of gains and losses does not equal to zero, the game is a **nonzero -sum game.** A game where two persons are playing the game and the sum of gains and losses is equal to zero, the game is known as **Two-Person Zero-Sum Game (TPZSG).** A good example of two-person game is the game of chess. A good example of n- person game is the situation when several companies are engaged in an intensive advertising campaign to capture a larger share of the market.

BASIC ELEMENTS OF GAME THEROY

Let us consider a game by name **Two-finger morra,** where two players (persons) namely *A* and *B* play the game. *A* is the winner and *B* is the loser. The matrix shown below is the matrix of *A*, the winner. The elements of the matrix show the gains of *A*. Any positive element in the matrix shows the gain of *A* and the negative element in the matrix show the loss (negative gain) of *A*.

		(One finger)		(Two finger)
		I	B	II
(One finger)	I	2		–2
A				
(Two fingers)	II	–2		2

The game is as follows: Both the players *A* and *B* sit at a table and simultaneously raise their hand with **one** or **two** fingers open. In case the fingers shown by both the players is same, then *A* will gain Rs.2/-. In case the number of fingers shown is different (*i.e*. A shows one finger and *B* shows two fingers or vice versa) then *A* has to give *B* Rs. 2/- *i.e*. *A* is losing Rs.2/-. In the above matrix, strategy I refer to finger one and strategy II refers to two fingers. The above given matrix is the pay of matrix of A. **The negative entries in the matrix denote the payments from *A* to *B*.** The pay of matrix of *B* is the negative version of *A*'s pay of matrix; because in two person zeros sum game the gains of one player are the losses of the other player. Always we have to write the matrix of the winner, who is represented on the left side of the matrix. The winner is **the maximizing player, who wants to maximize his minimum gains. The loser is the minimizing player, who wants to minimize his maximum losses.**

Note the following and remember

1. **The numbers within the payoff matrix represent the *outcome* or the *payoffs* of the different *plays* or *strategies* of the game. The payoffs are stated in terms of a measure of effectiveness such as money, percent of market share or utility.**

    By convention, in a 2-person, zero-sum game, the positive numbers denote a gain to the row or maximizing player or winner, and loss to the *column* or *minimizing player or loser*. It is assumed that both players know the payoff matrix.

2. **A strategy is a course of action or a complete plan. It is assumed that a strategy cannot be upset by competitors or nature (chance). Each player may have any number of strategies. There is no pressure that both players must have same number of strategies.**

3. ***Rules of game* describe the framework within which player choose their strategies. An assumption made here that *player must choose their strategies simultaneously and that the game is repetitive*.**

4. **A strategy is said to be *dominant* if each payoff in the strategy is *superior* to each *corresponding* pay off of alternative strategy. For example, let us consider *A* (winner) has three strategies. The payoffs of first strategy are 2, 1, 6 and that of second strategy are -1, -2 and 3. The second strategy's outcomes are inferior to that of first strategy. Hence first strategy dominates or superior to that of second strategy. Similarly let us assume B (loser) has two strategies. The outcomes of first strategy 2, -1 and that of second strategy is 1 and -2. The payoffs of second strategy is better than that of first strategy, hence second strategy is superior and dominates the first strategy. The rule of dominance is used to reduce the size of the given matrix.**

5. The *rule of game* refers to the expected outcome per play when both players follow their best or optimal strategies. A game is known as fair game if its value is zero, and unfair if its value is nonzero.

6. An *Optimal strategy* refers to the course of action, or complete plan, that leaves a player in the most preferred position regardless of the actions of his competitors. The meaning of the *most preferred position* is that any deviation from the optimal strategy, or plan, would result in decreased payoff.

7. The purpose of the game model is to identify the optimal strategy for each player. The conditions said in serial number 1 to 3 above, the practical value of game theory is rather limited. However the idea of decision-making under conditions of conflict (or cooperation) is at the core of managerial decision. Hence the concepts involved in game theory are very important for the following reasons.
   * It develops a framework for analyzing decision making in competitive (and sometimes in cooperative) situations. Such a framework is not available through any other analytical technique.
   * It describes a systematic quantitative method (in two-person zero-sum games) that enables the competitors to select rational strategies for the attainment of their goals.
   * It describes and explains various phenomena in conflicting situations, such as bargaining and the formation of coalitions.

THE TWO-PERSON, ZERO-SUM GAME: (Pure Strategy and Mixed Strategy games)

In our discussion, we discuss two types of Two-person, Zero-sum games. In one of the most preferred position for each player is achieved by adopting a **single strategy**. Hence this game is known as **pure-strategy game.** The second type requires the adoption by both players of **a mixture or a combination of** different strategies as opposed to a single strategy. Therefore this is termed as **mixed strategy game.**

In pure strategy game one knows, in advance of all plays that he will always choose only one particular course of action. **Thus pure strategy is a decision rule always to select the same course of action.** Every course of action is pure strategy.

A mixed strategy is that in which a player decides, in advance to choose on of his course of action in accordance with some fixed probability distribution. This in case of mixed strategy we associate probability to each course of action (each pure strategy). The pure strategies, which are used in mixed strategy game with non-zero probabilities, are termed as **supporting strategies**. Mathematically, a mixed strategy to any player is an ordered set of 'm' non-negative real numbers, which add to a sum unity (m is the number of pure strategies available to a player).

It is said above that in pure strategy game a player selects same strategy always, hence the opponent will know in advance the choice. But the superiority of mixed strategy game over pure strategy games is that the player is always kept guessing about the opponent's choice as innumerable combination of pure strategies one can adopt.

The purpose of the game theory is to determine the **best strategies** for each player on the basis of **maximin and minimax criterion of optimality. In this criterion a player lists his worst possible**

outcomes and then he chooses that strategy which corresponds to the best of those worst outcomes. The **value of the game** is the maxim guaranteed gain to player. The value is denoted by **'v'**. The game whose value $v = 0$ is known as zero sum **game** or **fair game**. Solving the game mean to find the best strategies for both the players and find the value of the game.

The game theory does not insist on how a game should he played, but only tells the procedure and principles by which the action should be selected. Hence, **the game theory is a decision theory useful in competitive situations.** *The fundamental theorem assures that there exists a solution and the value of a rectangular game in terms of mixed strategies.*

CHARACTERISTICS OR PROPERTIES OF A GAME

To classify the games, we must know the properties of the game. They are:

Number of persons or groups who are involved in playing the game

Number of strategies or courses of action each player or group have (they may be finite or infinite).

Type of course of action or strategy.

How much information about the past activities of other player is available to the players. It may be complete or partly or may be no information available.

The pay off may be such that the gains of some players may or may not be the direct losses of other players.

The players are independent in decision-making and they make the decision rationally.

THE MAXIMIN AND MINIMAX PRINCIPLES

To understand, the principles of **minimax** and **maximin** let us consider a pay of matrix of two players - Player *A*, the winner and Player *B*, the looser.

		B	
		I	II
	I	−3	−4
A	II	−2	3
	III	2	4

From the matrix above if *A* plays his first strategy, his worst outcome is –4, if he plays second strategy, his worst out come is -2 and if he plays his third strategy, his worst outcome (minimum gain) is +2. Out of all these strategy, for *A* the best strategy is third strategy. He can select the third strategy. But this outcome of +2 is possible when *B* selects his first strategy. But where is the guarantee that *B* will select first strategy. He may select his second strategy because where he has an out come of 4 (negative of A's outcome). Similarly, for *B* the worst outcome (minimum loss) if he selects his first strategy is 2 (i.e. negative version is –2, a loss of two units of money) and his worst out come if selects his second strategy is 4. Hence he selects the best among the two is first strategy. By doing so, *A* is sure of getting +2 units of money when he selects first strategy and *B* is sure of losing 2 units of money, which is minimum. Hence *A* will always selects his first strategy which will guarantee him

minimum gain of -2 units of money and B will always selects his first strategy as he looses minimum amount i.e. 2 units of money. Here, **A is known as Maximin player, as he is maximizing his minimum gains. And B is known as Minimax player as he is minimizing his maximum losses.**

Mathematically, if we denote the pay off matrix of a game by $[a_{ij}]_{m \times n}$ then minimax for A and **maximin for B**

Max [min a_{ij}] and min [max a_{ij}] respectively
  i   j   j   i

Let Max [min a_{ij}] $= a_{pq}$ and Min [max a_{ij}] $= a_{rs}$
     i   j   j   i

Then a_{pq} is the minimum element in the p th row, therefore, a_{pq} is $\leq a_{ps}$ another element in the p th row. Similarly, a_{rs} is the maximum element in s th column. Hence, $a_{ps} \leq a_{rs}$

Combining these two we get, $a_{pq} \leq a_{rs}$ or

Max [min a_{ij}] $\leq$ Min [max a_{ij}]
  i   j   j   i

Maximin of a_{ij} is called the lower value of the game and is denoted by $\underline{v}$ and the minimax of a_{ij} is called the upper value of the game and is denoted by $\bar{v}$. The value of the game is always between $\underline{v}$ and $\bar{v}$ and

Satisfies the inequality **Max min for A $\leq$ Min max for B OR** $\underline{v} \leq v \leq \bar{v}$
                              i   j   j   i

If row minimum is equals to column maximum, then the element at the intersection of that row and column is known as the **saddle point** and is the value of the game. In the matrix given above, the saddle point is

$A_3 B_1$ and A's third strategy and B's first strategy are pure strategies. The element at the intersection of $A_3 B_1$ is the saddle point and the value of the game is 2. The answer for the problem is represented as shown below

Optimal strategies: A (0,0,1) and B (1,0) and the value of the game $v = 2$.

Please remember that, the strategies used by both players are the pure strategies. This is the simplest type of game. Here the solution is stable in the minimax sense because in this case neither of the players can increase his gains after deviating from their optimal strategies. In case any player deviates from his optimal strategy, he will loose and his gains are reduced.

A game for which maximin of winner is equals to minimax of loser, is known as **game with saddle point**

The game with optimal pure strategies is sometimes called as **strictly determined.** Here, $\underline{v} = v = \bar{v}$.

METHOD OF SOLVING THE GAME OR STEPS IN SOLVING THE GAME

Step 1. Find the saddle point. If the game has saddle point, the game is solved. Write the optimal strategies and the value of the game.

Step 2. If no saddle point, try to reduce the size of the matrix given ($m \times n$) to:
  (*a*) 2×2 matrix, which has formula for optimal strategies and the value of the game. Use the formula to get the answer.
  (*b*) 3×2 or 2×3 matrix and use Sub game method to get the answer. (The sub games are once again 2×2 games).
  (*c*) To m $\times$ 2 or 2 $\times$ n matrix and use graphical method to get solution. Graphical solution will give us way to 2×2 matrix.
Step 3. Use algebraic method to get the solution.
Step 4. Use Linear-programming approach to get the solution. Use simplex method to get solution (Duality principle in Linear Programming is used).
Step 5. Use Iteration method or approximate method to get the solution.
  All these methods are explained by using numerical examples in the following discussion.

Saddle Point Method

Maxi $\min_{i} \min_{j} a_{ij} = \min_{j} \max_{i} a_{ij}$ is called **a game with saddle point**. This makes us to understand that the players in the game always use pure strategies. The element at the intersection of their pure strategies is known as **saddle point**. The element at the saddle point is the value of the game. As the players uses the pure optimal strategies, the game is known as **strictly determined game**. A point to remember is that the saddle point is the **smallest element in the row and the greatest element in the column. Not all the rectangular games will have saddle point, but if the game has the saddle point, then the pure strategies corresponding to the saddle point are the best strategies and the number at the point of intersection of pure strategies is the value of the game.** Once the game has the saddle point the game is solved. The rules for finding the saddle point are:
  1. *Select the minimums of each row and encircle them.*
  2. *Select the maximums of each column and square them.*
  3. *A point where both circle and square appears in the matrix at the same point is the saddle point.*
  Another name given to saddle point is equilibrium point of the game and the corresponding strategies form the equilibrium pair of strategies.

Problem

Solve the game given below:

Player B

		I	II	III
Player A	I	1	9	2
	II	8	5	4

Solution

		Player B			
		I	II	III	Row minimum
Player A	I	1	9	2	1
	II	8	5	④	**4**
Column Maximum		8	9	4	

In the matrix given, row minimums and column maximums are indicted. The element of A's second strategy and B's third strategy *i.e.* a $_{32}$ is both row minimum and column maximum. Hence **4** is the saddle point and pure strategy for A is second strategy and pure strategy for *B* is third strategy. Hence answer is:

A (0.1), B (0, 0, 1) and the value of the game is $v = +4$. This means A will gain 4 units of money *B* will loose 4 units of money and the sum of outcomes is zero.

Problem

Solve the game whose pay of matrix is:

		B		
		I	II	III
	I	–3	–2	6
A	II	2	⓪	4
	III	5	–2	–4

Solution

		B			
		I	II	III	Row minimum
	I	–3	–2	6	–3
A	II	2	⓪	4	**0**
	III	5	–2	–4	–4
Column Maximum		5	**0**	6	

Element at A(II) and B(II) is both column maximum and row minimum. Hence the element **0** is the saddle point. The answer is: A (0, 1, 0) and B (0, 1, 0) and the value $v = 0$.

Problem

The matrix given below illustrates a game, where competitors A and B are assumed to be equal in ability and intelligence. A has a choice of strategy 1 or strategy 2, while B can select strategy 3 or strategy 4. Find the value of the game.

B

	3	4
1	+4	+6
2	+3	+5

A

Solution

B

	3	4	Row minimum
1	(+4)	**+6**	+4
2	+3	+5	+3
Column Maximum:	**+4**	+6	

A

The element a_{11} is the row minimum and column maximum. Hence the element $a_{11} = 4$ is the saddle point and the answer is **A (1, 0) and B (1, 0) and value of the game = v = 4.**

Problem

In a certain game player has three possible courses of action L, M and N, while B has two possible choices P and Q. Payments to be made according to the choice made.

Choices	Payments.
L,P	A pays B Rs.3
L,Q	B pays A Rs. 3
M,P	A pays B Rs.2
M,Q	B pays A Rs.4
N,P	B pays A Rs.2
N,Q	B pays A Rs.3

What are the best strategies for players A and B in this game? What is the value of the game for A and B?

Solution

The pay of matrix for the given problem is:

		B		
		P	Q	Row minimum
	L	−3	+3	−3
A	M	−2	+4	−2
	N	⊕**2**	+3	**+2**
Column Maximum:		**+2**	+4	

Optimal strategies for *A* and *B* are: *A* **(0, 0, 1)** and *B* **(1, 0)** and the value of the game is *v* = **+2**

Problem

Consider the game *G* with the following payoff.

		B	
		I	II
	I	2	6
A			
	II	−2	p

(*a*) Show that *G* is strictly determinable, whatever the value of *p* may be.
(*b*) Determine the value of *p*

Solution

(*a*) Ignoring whatever the value of *p* may be, the given payoff matrix represents:

		B		
		I	II	Row minimum
	I	②	6	**2**
A				
	II	−2	p	−2
Column Maximum:		**2**	6	

Maximin value = 2 and Minimax value = 2. Therefore, the game is strictly determinable as the saddle pointy is $a_{11} = 2$.

(*b*) The value of the game is $v = +2$. And optimal strategies of players are **A (1, 0) and B (1, 0)**.

Problem

For what value of q, the game with the following payoff matrix is strictly determinable?

		B		
		I	II	III
	I	q	6	2
A	II	−1	q	−7
	III	−2	4	q

Solution

Ignoring whatever the value of q may be, the given payoff matrix represents:

		B			Row minimum
		I	II	III	
	I	q	6	2	**2**
A	II	−1	q	−7	−7
	III	−2	4	q	−2
Column maximum:		**−1**	6	2	

Maximin value = 2 and Minimax value = −1. So the value of the game lies between −1 and 2. *i.e.* $-1 \le v \le 2$.

For strictly determinable game since maximin value = minimax value, we must have $-1 \le q \le 2$.

Problem

Find the ranges of values of p and q, which will render the entry (2,2) a saddle point for the game.

		B		
		I	II	III
	I	2	4	5
A	II	10	7	q
	III	4	p	6

Solution

Let us ignore the values of p and q and find the row minimum and column maximum.

		B			
		I	II	III	Row minimum
	I	2	4	5	2
A	II	10	**7**	q	**7**
	III	4	p	6	4
Column maximum:		10	**7**	6	

Maximin value $= 7 =$ Minimax value. This means that $p \leq 7$ *i.e.* column maximum and $q \geq 7$ *i.e.* row minimum. Hence the range of p and q will be $p \leq 7$ and $q \geq 7$.

Problem

Find the solution of the game whose payoff matrix is given below:

		B				
		I	II	III	IV	V
	I	−4	−2	−2	3	1
A	II	1	0	−1	0	0
	III	−6	−5	−2	−4	4
	IV	3	1	−6	0	−8

Solution

		B					
		I	II	III	IV	V	Row Minimum
	I	−4	−2	−2	3	1	4
A	II	1	0	$\boxed{-1}$	0	0	**−1**
	III	−6	−5	−2	-4	4	−6
	IV	3	1	−6	0	−8	−8
Column Maximum:		3	1	**−1**	3	4	

Optimal strategies for $A = A$ **(0, 1, 0, 0)** and for $B = B$ **(0, 0, 1, 0, 0)** and the value of the game $v = -1$. This means that B always wins 1 unit of money.

Problem

Find the range of values of p and q which will render the entry (2,2) a saddle point in the game with the following payoff matrix.

	B		
	1	2	3
A 1	1	q	3
A 2	p	5	10
A 3	6	2	3

Solution

	B			Row minimum
	1	2	3	
A 1	1	q	3	1
A 2	p	⑤	10	**5**
A 3	6	2	3	2
Column maximum	6	**5**	10	

In order to have element (2,2) as the saddle point *i.e.* 5 as the saddle point, q should be less than or equal to 5 and p should be greater than or equals to 5. Hence range for p and q are $p \geq 5$ and $q \leq 5$ or $q \leq 5 \leq p$.

Principle of Dominance in Games

In case there is no saddle point the given game matrix $(m \times n)$ may be reduced to $m \times 2$ or $2 \times n$ or 2×2 matrix, which will help us to proceed further to solve the game. The ultimate way is we have to reduce the given matrix to 2×2 to solve mathematically.

To discuss the principle of dominance, let us consider the matrix given below:

	B				Row minimum
	I	II	III	IV	
A I	2	–4	–3	4	–4
A II	4	–3	–4	2	–4
Column Maximum	4	–3	–3	4	

The row minimums and column maximums show that the problem is not having saddle point. Hence we have to use method of dominance to reduce the size of the matrix.

(*i*) Consider the first and second strategies of B. If B plays the first strategy, he looses 2 units of money when A plays first strategy and 4 units of money when A plays second strategy. Similarly, let us consider B's second strategy, B gains 4 units of money when A plays his first strategy and gains 3 units of money when A plays second strategy. Irrespective of A's choice, B will gain money. **Hence for B his second strategy is superior to his first**

strategy. In other words, *B*'s second strategy dominates *B*'s first strategy. Or *B*'first strategy is dominated by B's second strategy. Hence we can remove the first strategy of B from the game. The reduced matrix is:

$$
\begin{array}{c}
 & & \multicolumn{3}{c}{B} \\
 & & \text{II} & \text{III} & \text{!V} \\
\hline
 & \text{I} & -4 & -3 & 4 \\
A & & & & \\
 & \text{II} & -3 & -4 & 2 \\
\end{array}
$$

(*ii*) Consider *B*'s III and IV strategy. When *B* plays IV strategy, he loose 4 units of money when A plays his first strategy and 2 units of money when A plays his second strategy. Where as, when *B* plays his III strategy, he gains 3 units of money and 4 units of money, when A plays his I and II strategy respectively. Hence B's IV strategy (pure strategy) is dominating the third strategy. Hence we can remove the same from the game. The reduced matrix is:

$$
\begin{array}{c}
 & & \multicolumn{2}{c}{B} \\
 & & \text{II} & \text{III} \\
\hline
 & \text{I} & -4 & -3 \\
A & & & \\
 & \text{II} & -3 & -4 \\
\end{array}
$$

In the above example, if we keenly observe, we see that the elements of second column are smaller or less than the elements of column 4, similarly elements of III column also smaller or less than the elements of I and IV column and I. Hence, we can write the dominance rule for columns as **When elements of a column, say *i*th are less than or equals to the corresponding elements of *j*th column, then *j*th column is dominated by *i*th column or *i*th column dominates *j*th column.**

Consider the matrix given below

$$
\begin{array}{c}
 & & \multicolumn{2}{c}{B} \\
 & & \text{I} & \text{II} \\
\hline
 & \text{I} & -2 & -4 \\
A & & & \\
 & \text{II} & 1 & 2 \\
\end{array}
$$

Let *A* play his first strategy, then he looses 2 units of money and looses 4 units of money when *B* plays his second strategy. But when *A* plays his second strategy, he gains 1 unit of money for *B*'s first strategy and gains 2 units of money, for *B*'s second strategy. Hence, **A's second strategy (pure strategy) is superior to A's first strategy or A's second strategy dominates A's first strategy or**

***A's* first strategy is dominated by *A's* second strategy.** We can closely examine and find that elements of *A's* second strategy are greater than the elements of first strategy. Hence we can formulate general rule of dominance for rows. **When the elements of *r*th row are greater than or equals to elements of *s*th row, then *r*th row dominates *s*th row or *s*th row is dominated by *r*th row.**

The general rules of dominance can be formulated as below

1. **If all the elements of a column (say ith column) are greater than or equal to the corresponding elements of any other column (say *j*th column), then *i*th column is dominated by *j*th column.**
2. **If all the elements of *r*th row are less than or equal to the corresponding elements of any other row, say *s*th row, then *r*th row is dominated by *s*th row.**
3. **A pure strategy of a player may also be dominated if it is inferior to some convex combinations of two or more pure strategies, as a particular case, inferior to the averages of two or more pure strategies.**

Note: At every reduction of the matrix, check for the existence of saddle point. If saddle point found, the game is solved. Otherwise continue to reduce the matrix by method of dominance.

Solutions to 2 x 2 games without saddle point: (Mixed strategies)

In rectangular games, when we have saddle point, the best strategies were the pure strategies. Now let us consider the games, which do not have saddle points. In such cases, the best strategies are the **mixed strategies**. While dealing with mixed strategies, we have to determine the probabilities with which each action should be selected. Let us consider a 2×2 game and get the formulae for finding the probabilities with which each strategy to be selected and the value of the game.

Points to be remembered in mixed strategy games are

(*a*) If one of the players adheres to his optimal mixed strategy and the other player deviates from his optimal strategy, then the deviating player can only decrease his yield and cannot increase in any case (at most may be equal).

(*b*) If one of the players adheres to is optimal strategy, then the value of the game does not alter if the opponent uses his supporting strategies only either singly or in any combination.

(*c*) If we add (or subtract) a fixed number say 1,to (from) each elements of the payoff matrix, then the optimal strategies remain unchanged while the value of the game increases (or decreases) by 1.

Consider the 2×2 game given below:

$$B$$

		y_1 I	y_2 II
x_1	I	a_{11}	a_{12}
x_2	II	a_{21}	a_{22}

A

Let x_1 and x_2 be the probability with which A plays his first and second strategies respectively. Similarly B plays his first and second strategies with probability of y_1 and y_2 respectively. Now

$x_1 + x_2 = 1$, and $y_1 + y_2 = 1$. Let us work out expected gains of A and B when they play the game with probabilities of x_1, x_2 and y_1 and y_2.

A's expected gains when:

B plays his first strategy $= a_{11} x_1 + a_{21} x_2$

When B plays his second strategy $= a_{12} x_1 + a_{22} x_2$

Similarly B's gains when:

A plays his first strategy $= a_{11} y_1 + a_{12} y_2$

When A plays his second strategy $= a_{21} y_1 + a_{22} y_2$

Now let us assume that the v is the value of the game. As A is the miximin player, he wants to see that his gains are $\geq v$. As B is the minimax player, he wants to see that his gains must be always $\leq v$.

Therefore, we have:

$$a_{11} x_1 + a_{21} x_2 \geq v$$
$$a_{12} x_1 + a_{22} x_2 \geq v \text{ and}$$
$$a_{11} y_1 + a_{12} y_2 \leq v$$
$$a_{21} y_1 + a_{22} y_2 \leq v$$

To find the value of x_1, x_2 and y_1, y_2 we have to solve the above given inequalities. For convenience, let us consider them to be equations to find the values of x_1, x_2 and y_1, y_2. Therefore, we have:

$$a_{11} x_1 + a_{21} x_2 = v$$
$$a_{12} x_1 + a_{22} x_2 = v \text{ and}$$
$$a_{11} y_1 + a_{12} y_2 = v$$
$$a_{21} y_1 + a_{22} y_2 = v$$

Always we workout a solution of a 2×2 game by considering the above inequalities as strict equalities. Now we can write above as:

$a_{11} x_1 + a_{21} x_2 = v = a_{12} x_1 + a_{22} x_2$ or this can be written as $x_1 (a_{11} - a_{12}) = x_2 (a_{22} - a_{21})$ or
$(x_1 / x_2) = (a_{22} - a_{21}) / (a_{11} - a_{12})$, Similarly we can write:
$(y_1 / y_2) = (a_{22} - a_{12}) / (a_{11} - a_{12})$, by simplifying, we get:
$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x^2$
$x_2 = (a_{11} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x^1$
$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y^2$

$$y_2 = (a_{11} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) \text{ or } = 1 - y^1, \text{ and the value}$$

of the game is $v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$

Hints to remember formula:

The matrix is

			B	
			y_1	y_2
			I	II
x_1		I	a_{11}	a_{12}
A				
x_2		II	a_{21}	a_{22}

> **For x_1 Numerator $= a_{22} - a_{21}$ i.e. x_1 is in the first row, for numerator we have to take the difference of second row elements from right to left.**
>
> **For x_2, which comes in second row, we have to take difference of the first row elements from left to right.**
>
> **For y_1 which comes in the first column, we have to take the difference of second column elements from bottom to top.**
>
> **For y_2, which comes in second column, we have to take the difference of the elements of first column from top to bottom.**
>
> **As for the denominator is concerned, it is common for all formulae. It is given by sum of diagonal elements from right hand top corner to left-hand bottom corner minus the sum of the elements diagonally from left-hand top corner to right hand bottom corner.**
>
> **For value of the game, the numerator is given by products of the elements in denominator in the first bracket minus the product of the elements in the second bracket.**

When the game does not have saddle point, the two largest elements of its payoff matrix must constitute one of the diagonals.

Now, let us consider the 2×2 matrix we got by reducing the given matrix in the article 10.8.2 and get the answer by applying the formula.

The reduced matrix is:

		B		
		II	III	Row minimum.
A	1	−4	−3	−4
	II	−3	−4	−4
Column maximum:		−3	−3	

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x^2$

$x_1 = (-4 - [-3]) / (-4 + [-4]) - (-3 + [-3]) = (-4 + 3) / (-4 - 4) - (-3 - 3) = -1 / (-8) - (-6) =$
$-1 / -8 + 6 = -1 / -2 = 1/2 = 0.5$.

$x_2 = 1 - x_1 = 1 - 0.5 = 0.5$.

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = [-4 + (-3)] / [-4 + (-3)] - [-3 + (-3)] =$
$(-4 + 3) / (-4 - 3) - (-3 - 3) = -1 / (-7 + 6) = 1$ (*i.e.* pure strategy).

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$[12 - 12) / [`-4 - 3] - [-3 - 3] = 0$

Problem

Solve the game whose payoff matrix is:

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	1	6

Solution

		B			
		I	II	III	Row minimum.
A	I	1	7	2	1
	II	6	2	7	2
	III	5	1	6	1
Column Maximum.		6	7	7	

No saddle point. Hence reduce the matrix by method of dominance.

B's third strategy gives him 2,7,6 units of money when A plays his I, II, and III strategies. When we compare this with the *B's* first strategy, it clearly shows that the payoffs of first strategy are superior or better to that of third strategy. Hence B's third strategy is dominated by the B's first strategy. Hence we remove the third of B strategy from the game.

The reduced matrix is

		B		
		II	III	Row minimum
A	I	1	7	1
	II	6	2	2
	III	5	1	1
Column maximum:		6	7	

No Saddle point. Reduce the matrix by method of dominance. Consider A's II strategy. The payoffs are 6 and 2 units of money when *B* plays his II and III strategy. When we compare this with A's III strategy, which fetches only 5 and 1 units of money, which is inferior to payoffs of II strategy. Hence we can remove A's third strategy form the game. The reduced matrix is:

B

		II	III	Row minimum
	I	1	7	1
A				
	II	6	2	2
Column maximum:		6	7	

No saddle point. Hence apply the formula.

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (2 - 6) / (1 + 2) - (6 + 7) = -4 / -10 = (2/5)$ or 0.4

Hence $x_2 (1 - 2/5) = 3 / 5$ or 0.6.

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (2 - 7) / (1 + 2) - (6 + 7) = (-5 / -10) = (1/2) = 0.5$

$y_2 = 1 - y_1 = 1 - (1/2) = 1 / 2 = 0.5$

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$= (1 \times 2) - (6 \times 7) / (1 + 2) - 6 + 7) = -40 / -10 = 4$

Solution to the game is: A (2/5, 3/5, 0) and B (0, ½, ½) and value of the game is $v = 4$ *i.e.* A allays win 4 units of money.

Problem

Use the concept of dominance to solve the game.

B

		I	II	III	IV	Row minimum
	I	3	2	4	0	0
	II	3	4	2	4	2
A	III	4	2	4	0	0
	IV	0	4	0	8	0
Column maximum		4	4	4	8	

No saddle point. Let us reduce the matrix by method of dominance.

Compare *A*'s I strategy and III strategy, we find that third strategy is superior to first strategy as the elements of III row are greater than or equal to that of elements of first row. Hence, A's III strategy dominates *A*'s I strategy. Hence *A*'s first strategy can be removed from the game. The reduced matrix is:

		B				
		I	II	III	IV	Row minimum
	II	3	4	2	4	2
A	III	4	2	4	0	0
	IV	0	4	0	8	0
Column maximum		4	4	4	8	

No saddle point, try to reduce the matrix by dominance method. Compare *B's* first strategy and III strategy. As the elements of III strategy are less than or equal to that of first strategy, the III strategy dominates the first strategy. Hence, *B's* first strategy is removed from the game. The reduced matrix is:

		B			
		II	III	IV	Row minimum
	II	4	2	4	2
A	III	2	4	0	0
	IV	4	0	8	0
Column Maximum		4	4	8	

No saddle point and there is no dominance among pure strategies. Hence let us take the averages of two or more pure strategies and compare with other strategies, to know whether there is dominance or not. Let take *B's* III and IV strategy and take the average and compare with elements of first strategy.

Average of elements of *B's* III and IV strategy are: $(2 + 4 = 6/2 = 3)$, $(4 + 0 = 4/2 = 2)$ and $(0 + 8 = 8/2 = 4)$.

Hence the reduced matrix is:

		B			
		II	Avg. of III & IV	Row minimum	
	II	4	3	3	
A	III	2	2	2	(do not consider saddle point)
	IV	4	4	4	
Column maximum		4	4		

As all the elements of *B's* second strategy are greater than or equal to that of averages of III and IV strategies, *B's* second strategy is inferior to that of III and IV strategies. Hence the matrix is:

		B		
		III	IV	Row minimum
A	II	2	4	2
	III	4	0	0
	IV	0	8	0
Column maximum		4	8	

No saddle point. Hence, let us try the dominance by comparing the averages of two A's strategies with elements of other strategy. Averages of A's II and IV pure strategies is:

(4 + 2 = 6 / 2 = 3) and (0 + 8 = 8 / 2 = 4). The matrix is:

		B	
		III	IV
A	II	2	4
	Avg. of III & IV	3	4

As the elements of A's II strategy are inferior to averages of III and IV strategy, II strategy is removed from the matrix. The reduced matrix is:

		B		
		III	IV	Row minimum
A	III	4	0	0
	IV	0	8	0
Column maximum		4	8	

No saddle point. By applying the formulae:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (8 - 0) / (4 + 8) - (0 + 0) = 8 / 12 = $ **2 / 3.** Hence $x_2 = 1 - (2/3) = $ **1/3.**

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (8 - 0) / (4 + 8) - (0 + 0) = 8 / 12 = $ **2/3.** Hence $y_2 = 1 - (2/3) = $ **1/3.**

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =$
(32 – 0) / (4 + 8) – (0 + 0) = 32 / 12 = **8/3.**

Hence the solution is A **(0, 0, 2/3, 1/3), B (0, 0, 2/3, 1/3) and v = 8/3**

A will always win 8/3 units of money.

Problem

Two players P and Q play the game. Each of them has to choose one of the three colours: White (W), Black (B) and Red (R) independently of the other. Thereafter the colours are compared. If both P and Q has chosen white (W, W), neither wins anything If player P selects white and Player Q black (W, B), player P loses Rs.2/- or player Q wins the same amount and so on. The complete payoff table is shown below. Find the optimum strategies for P and Q and the value of the game.

		Q		
		W	B	R
	W	0	−2	7
P	B	2	5	6
	R	3	−3	8

Solution

The payoff matrix is:

		Q			
		W	B	R	Row minimum
	W	0	−2	7	−2
P	B	2	5	6	2
	R	3	−3	8	−3
Column maximum:		3	5	8	

No saddle point. Reduce the matrix by method of dominance. Comparing the elements of B's strategy R, the elements of strategy R are greater than the elements of other strategies; hence it can be removed from the matrix as it is dominated by strategies W and B. Reduced matrix is:

		Q		
		W	B	Row minimum
	W	0	−2	−2
P	B	2	5	2
	R	3	−3	−3
Column maximum:		3	5	

There is no saddle point. Comparing P's strategies, W and B, we see that the elements of W strategy are less than the elements of strategy B. Hence Strategy B dominates strategy W and is removed from the matrix. The reduced matrix is:

		W	B	Row minimum
P	B	2	5	2
	R	3	–3	–3
Column maximum:		3	5	

There is no saddle point. By applying the formulae:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x^2 = (-3 - 3) / [2 + (-3)] - [3 + 5] = -6 / [(-1) - (8) = -6 / -9 = 6 / 9 = 2 / 3$. Hence $x_2 = (1 - 2/3) = $ **1/3**

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y^2 = (-3 - 5) / (-9) = -8 / -9 = $ **(8 / 9)**. Hence $y_2 = 1 / (8 / 9) = $ **(1 / 9).**

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = (-6 - 15) / -9 = -21 / -9 = $ **(21 /9). The solution is: P (0, 2/3, 1/3), Q (8/9, 1/9, 0) and v = 21 /9.**

Problem

Solve the game whose payoff matrix is:

		B				
		1	2	3	4	5
	1	1	3	2	7	4
A	2	3	4	1	5	6
	3	6	5	7	6	5
	4	2	0	6	3	1

Solution

		B					Row minimum
		1	2	3	4	5	
	1	1	3	2	7	4	1
A	2	3	4	1	5	6	1
	3	6	5	7	6	5	5
	4	2	0	6	3	1	0
Column maximum:		6	5	7	7	6	

The game has the saddle point (3, 2). Hence the value of the game is $v = $ **5** and the optimal strategies of *A* and *B* are: **A (0, 0, 1, 0) , B (0, 1, 0, 0, 0)**

Problem

Solve the following game whose payoff matrix is:

		B					
		I	II	III	IV	V	VI
	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
A	III	4	3	7	–5	1	2
	IV	4	3	4	–1	2	2
	V	4	3	3	–2	2	2

Solution

		B						Row minimum
		I	II	III	IV	V	VI	
	I	4	2	0	2	1	1	0
	II	4	3	1	3	2	2	1
A	III	4	3	7	–5	1	2	–5
	IV	4	3	4	–1	2	2	–1
	V	4	3	3	–2	2	2	–2
Column maximum:		4	3	7	3	2	2	

The game has no saddle point. Let us reduce the size of the matrix by method of dominance.

Compare *A*'s I and II strategies, I strategy is dominated by II strategy. Similarly, compare *A*'s IV and V strategies, elements of IV strategy are greater than that of V strategy; hence V strategy is dominated by IV strategy. Hence *A*'s I ad V strategies can be eliminated and the reduced matrix is:\

		B						Row minimum
		I	II	III	IV	V	VI	
	II	4	3	1	3	2	2	1
A	III	4	3	7	-5	1	2	-5
	IV	4	3	4	-1	2	2	-1
Column maximum:		4	3	7	3	2	2	

The matrix has no dominance. Compare B's I and II strategies are dominated by B's V and VI strategy, as the elements of I and II columns are greater than that of V and VI columns. Hence, B's I and II strategies can be eliminated. Similarly, elements of VI column are greater than that of V column, hence V strategy dominates VI strategy, and hence VI strategy is eliminated. Reduced matrix is:

		III	IV	V	Row minimum
	II	1	3	2	1
A	III	7	–5	1	–5
	IV	4	–1	2	–1
Column maximum:		7	3	2	

(Header above III IV V is: B)

As the pure strategies do not have dominance, let us take the average of B's III and IV strategies and compare with V strategy. The averages are: $(1 + 3)/2 = 2$, $(7 – 5)/2 = 1$, and $(4 – 1)/2 = 3/2$. These average when compared with the elements of V strategy, they are smaller, hence, III and IV strategies dominates V strategy of B. The reduced matrix is:

		III	IV	Row minimum.
	II	1	3	1
A	III	7	–5	–5
	IV	4	–1	–1
Column maximum.		7	3	

(Header above III IV is: B)

No saddle point. Let us take average of A's II and III strategy and compare with IV strategy.

Average is: $(1 + 7)/2 = 4$, and $(3 – 5)/2 = –1$. II and III strategies of A dominate IV strategy. Hence is eliminated from the matrix. The reduced matrix is:

		III	IV	Row minimum
	II	1	3	1
A				
	III	7	–5	–5
Column maximum.		7	3	

(Header above III IV is: B)

No dominance. Hence applying the formulae:

$x_1 = (a_{22} – a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ **or** $= 1 – x_2 =$ $(–5 – 7) / ([1 + (– 5)] – (3 + 7) = –12 /(–4 – 10) = 12/14 = $ **6 / 7**, hence $x_2 = 1 – (6 / 7) =$ **(1 / 7)**

$y_1 = (a_{22} – a_{12}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ **or** $= 1 – y_2 = (–5 – 3) / – 14 = 8 / 14 =$ **(2 / 7)** hence, $y_2 = 1 – 2 / 7 =$ **5 / 7.**

Value of the game = $v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = (-5 - 21) / -14 = 26 / 14$ = **13 / 7**

The answer is: A (0, 6 / 7, 1 / 7, 0, 0), B (0, 0, 2 / 7, 7 / 7, 0, 0) and v = 13 / 7.

Problem

A and B play a game in which each has three coins, *a* 5 paise, 10 paise and 20 paise coins. Each player selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B's coins. If the sum is even, B wins A's coins. Find the optimal strategies for the players and the value of the game.

Solution

The pay of matrix for the given game is: Assume 5 paise as the I strategy, **10 paise as the** II strategy and the 20 paise as the III strategy.

			B 5 I	10 II	20 III
	5	I	–10	15	25
A	10	II	15	–20	–30
	20	III	25	–30	–40

In the problem it is given when the sum is odd, A wins B's coins and when the sum is even, B will win A's coins. Hence the actual pay of matrix is:

			B 5 I	10 II	20 III	Row minimum
	5	I	–5	10	20	–5
A	10	II	5	–10	–10	–10
	20	III	5	–20	–20	–20
Column maximum.			5	10	20	

The problem has no saddle point. Column I and II are dominating the column III. Hence it is removed from the game. The reduced matrix is:

			B 5 I	10 II	Row minimum
	5	I	–5	10	–5
A	10	II	5	–10	–10
	20	III	5	–20	–20
Column maximum.			5	10	

The problem has no saddle point. Considering *A*, row III is dominated by row II, hence row III is eliminated from the matrix. The reduced matrix is:

		B		Row minimum
		I	II	
A	I	–5	10	–5
	II	5	–10	–10
Column maximum.		5	10	

No saddle point. By application of formulae:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (-10 - 5) / [-5 + (-10)] - (10 - 5)$
$= -15 / (-15 - 5) = (-15 / -20) = (15 / 20) = $ **3 / 4,** hence $x_2 = 1 - (3 / 4) = $ **1 / 4**

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (-10 - 10) / -20 = 20 / 20 = $ **1** and $y_2 = 0$

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = (50 - 50) / - 20 = $ **0**

Answer is *A* (3/ 4, 1 / 4, 0), B (1, 0, 0) , *v* = 0.

Method of Oddments (for 2 × 2 games)

Once the game matrix is reduced to 2×2 the players has to resort to mixed strategies. We have already seen how using formulae can an algebraic method to find optimal strategies and value of the game. There is one more method available for the same that is the method of oddments. Steps involved in method of dominance are:

1. Subtract the two digits in column 1 and write them under column 2, ignoring sign.
2. Subtract the two digits in column 2 and write them under column 1 ignoring sign.
3. Similarly proceed for the two rows.

These values are called **oddments.** They are the frequencies with which the players must use their courses of action in their optimum strategies. Let us take a simple example and get the answer for a game.

Problem

In a game of matching coins, player A wins Rs.2/-, if there are two heads, wins nothing if there are two tails and loses Re.1/- when there are one head and one tail. Determine the pay off matrix and best strategies and value of the game.

Solution: (by using method of oddments)

The payoff matrix is:

		B		Row minimum
		I H	II T	
	I H	2	–1	–1
A	II T	–1	0	–1
Column maximum.		2	0	

There is no saddle point. Let us apply method of oddments.

The given matrix is:

		B		Oddments of A	Probability
		H	T		
	H	2	−1	(−1− 0) = 1	1 / (1 + 3) = 1/4
A					
	T	−1	0	(2 − (−1) = 3	3 / (1 = 3) = 3/4

Oddments of *B*   (−1–0) = 1   (2 − (−1)) = 3
Probability: 1/ (1 + 3) = 1 /4   3 / (3 + 1) = 3/4

Hence optimal strategies for A and B are: A (1/4, 3/4), B (1/4, 3/4),

Value of the game can be written if the sum of oddments of both the players is equal. Other wise we have to apply the formula of 2×2 games and get the value. In this case the sum of oddments of both players is 4. Hence, we can find the value of the game.

Using *A*'s oddments:

When *B* lays his I strategy (*H*), $v = [1 \times 2 + (3 \times -1)] / (3 + 1) = (2 - 3) / 4 = - (1 / 4)$,

When *B* plays his II strategy, (*T*) $= v = (1 \times -1 + 3 \times 0) / (3 + 1) = - (1 / 4)$

Using *B*'s Oddments:

When *A* plays his I strategy (*H*): $v = (1 \times 2 + 3 \times -1) / (3 + 1) = - (1 / 4)$,

When *A* plays his II strategy (*T*) $v = (-1 \times 1 + 3 \times 0) / (3 + 1) = - (1 / 4)$.

Value of the game is $- 1 / 4$.

Problem

By using the oddments of *A* and *B* solve the game.

		B	
		I	II
	I	4	0
A			
	II	0	8

There is no saddle point. The oddments are:

		B		Oddments of A	Probability
		I	II		
	I	4	0	(0 − 8) = 8 = 2	(2/3)
A					
	II	0	8	(4 − 0) = 4 = 1	(1/3)

Oddments of B (0 – 8)   = 8 (4 – 0) = 4

                         =2   = 1

Probability   = 2 / 3   1 / 3.

Value of the game:(The sums of two oddments is same) v = For B playing I strategy:

$v = 2 \times 4 + 1 \times 0 / (2 + 1) = (8 / 3)$

Answer is A (2 / 3, 1 / 3,), B (2 / 3, 1 / 3) and $v = 8 / 3$

Problem

Two armies are at war. Army A has two air bases, one of which is thrice as valuable as the other. Army B can destroy an undefended air base, but it can destroy only one of them. Army A can also defend only one of them. Find the strategy for A to minimize the losses.

Solution

                              B (attacker)

	I (attack smaller)	II (attack larger)	Row min.
(Defend smaller) I	Both survive: 0	The larger destroyed: –3	–3
Defender: A			
(Defend larger) II	Smaller destroyed: –1	Both survive: 0	–1
Column Max:	0	0	

No saddle point.   By method of oddments:

                              B

		I	II	Oddments of A	Probability
	I	0	– 3	(–1 – 0) = 1	1 / 4
A					
	II	– 1	0	(0 – (–3) = 3	3 / 4

Oddments of B (0 – (–3) = 3   (–1–0) = 1

Probabilities: 3 / 4   1 / 4

Value of the game is: (Note the sum of oddments is same: Taking the oddments of A, When B plays his I strategy, A's expected winning for army A =

(3/4) [(0) × (1 / 4) + (–1) × (3 / 4)] + (1 / 4) [(–3) × (1 / 4) + (0) × (3 / 4)] = –(9 / 16) – (3 / 16) = – (12 / 16) = –(3 / 4)

Solutions to 2 × *n* or *m* × 2 games

When we can reduce the given payoff matrix to 2 × 3 or 3 × 2 we can get the solution by method of **sub games**. If we can reduce the given matrix to 2 × *n* or *m* × 2 sizes, then we can get the solution by **graphical method**. A game in which one of the players has two strategies and other player has

number of strategies is known as $2 \times n$ or $m \times 2$ games. If the game has saddle point it is solved. If no saddle point, if it can be reduced to 2×2 by method of dominance, it can be solved. When no more reduction by dominance is possible, we can go for Method of Sub games or Graphical method. We have to identify 2×2 sub games within $2 \times n$ or $m \times 2$ games and solve the game.

Problem

Solve the game whose payoff matrix is:

		B		
		I	II	III
A	I	−4	3	−1
	II	6	−4	−2

Solution

Given pay of matrix is 2×3 matrix.

		B			Row minimum
		I	II	III	
A	I	−4	3	−1	−4
	II	6	−4	−1	−4
Column maximum.		6	3	−1	

No saddle point.
The sub games are:
Sub game I:

		B		Row minimum
		I	II	
A	I	−4	3	−4
	II	6	−4	−4
Column Maximum.		6	3	

No saddle point. First let us find the value of the sub games by applying the formula. Then compare the values of the sub games; which ever is favorable for the candidate, that sub game is to be selected. Now here as *A* has only two strategies and *B* has three strategies, the game, which is favorable to *B*, is to be selected.

Value of the game $= v_1 = (a_{11} a_{22} − a_{12} a_{21}) / (a_{11} + a_{22}) − (a_{12} + a_{21})$
$= (−4 \times −4) − (3 \times 6) / [(−4 + −4] − (3 + 6) = 2 / 17$

Sub game II:

B

		I	III	Row minimum
A	I	−4	−1	−4
	II	6	−2	−2
Column maximum.		6	−1	

No saddle point, hence value of the game = $v_2 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
= [(−4) × (−2)] − [(−1) × 6] / [(−4) + (−2) − (6 − 1) = − (14 / 11)

Sub game III:

B

		II	III	Row minimum
A	I	3	**−1**	**−1**
	II	−4	−2	−4
Column Maximum.		3	**−1**	

The game has saddle point (1,3), the element is (−1). Hence the value of the game v_3 =-1.

Comparing the two values v_1 and v_2, v_2, and v_3, both v_2 and v_3 have negative values, which are favorable to player *B*. But v_2 is more preferred by *B* as it gives him good returns. Hence *B* prefers to play strategies I and III. Hence sub game II is selected. For this game we have to find the probabilities of strategies. For sub game II the probabilities of strategies are:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** = $1 - x_2$ = [(−2) − 6] / (− 11) = (8 / 11), hence $x_2 = 1 - (8 / 11) = 3 / 11$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** = $1 - y_2$ = [(−2) − (−1)] / −11 = (1 / 11), Hence $y_2 = 1 / (1 / 11) = (10 / 11)$.

Hence optimal strategies for the players are:

A (8/11, 3 / 11), B (1 / 11, 0, 10 / 11) and the value of the game is – (14 / 11).

Problem

Solve the following 2 × n sub game:

B

		I	II
	I	1	8
A	II	3	5
	III	11	2

Solution

The given game is $m \times 2$ game.

		B		
		I	II	Row minimum
	I	1	8	1
A	II	3	5	3
	III	11	2	2
Column maximum.		11	8	

No saddle point. Hence A's Sub games are:

A's sub game No.1.

		B		
		I	II	Row minimum
	I	1	8	1
A				
	II	**3**	5	**3**
Column Maximum.		**3**	8	

The game has saddle point and hence value of the game is $v_1 = 3$

A's sub game No.2.

		B		
		I	II	Row minimum
	I	1	8	1
A				
	III	11	2	2
Column maximum.		11	8	

No saddle point. Hence the value of the game $v_2 = (a_{11} \, a_{22} - a_{12} \, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
$= [(1) \times (2) - (8) \times (11)] / (3) - (19) = (3 - 88) / (- 16) = (85 / 16)$

A's Sub game No. 3:

		B		
		I	II	Row minimum
	II	3	5	3
A	III	11	2	2
Column maximum.		11	5	

No saddle point. Hence the value of the game $v_3 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
$(3 \times 2) - (11 \times 5) / (3 + 2) - (5 + 11) = (6 - 55) / (5 - 16) = -(49 / 11)$

Now $v_1 = 3$, $v_2 = 85 / 16 = 5.31$, and $v_3 = 49 / 11 = 4.45$. Comparing the values, as far as A is concerned, v_2 gives him good returns. Hence A prefers to play the sub game No. 2. For this game we have to find out the probabilities of playing the strategies. For sub game No.2:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (2 - 11) / (-16) = (9 / 16)$,
$x_2 = 1 - (9 / 16) = (7/16)$
$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (2 - 8) / (-16) = (6 / 16)$
$y_2 = 1 - (6 / 16) = (10 / 16)$.

Therefore optimal strategies for A and B are:
A (9 / 16, 0, 7 / 16), B (6 / 16, 10 / 16) and value of the game v = (85 / 16) = 5.31.

Problem

Solve the game by method of sub games whose payoff matrix is:

		B	
		I	II
A	I	6	5
	II	3	6
	III	8	4

Solution

The given payoff matrix is

		B		
		I	II	Row minimum
A	I	6	5	5
	II	3	6	3
	III	8	4	4
Column maximum:		8	6	

No saddle point. Let us form sub games of A and find the optimal strategies.
Sub game No. 1.

		B		
		I	II	Row minimum
A	I	6	5	5
	II	3	6	3
Column maximum.		6	6	

No saddle point. The value of the game $= v_1 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =$
$(5 \times 6) - (5 \times 3) / (6 + 6) - (5 + 3) = (36 - 15) / (12 - 8) = 21 / 4$.
Sub game No. 2:

		B		Row minimum.
		I	II	
A	I	6	5	5
	III	8	4	4
Column maximum.		8	5	

The game has saddle point (1,2) and the element is 5. Hence the value of the game is $v_2 = 5$.
Sub game No. 3.

		B		Row minimum.
		I	II	
A	II	3	6	3
	III	8	4	4
Column maximum.		8	6	

No saddle point. Let us find the value of the game.
Value of the game $= v_3 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =$
$(12 - 48) / (7 - 14) = (36 / 7)$.

Now $v_1 = (21 / 4) = 5.25$, $v_2 = 5$, and $v_3 = (36 / 7) = 5.14$. Among all the three $v_1 = 5.25$ is good return to A. Hence he selects sub game No.1. Let us find the probabilities of strategies for this game.

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x_2 = (6 - 3) / 4 = 3 / 4$. Therefore $x_2 = 1/4$.
$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y_2 = (6 - 5) / 4 = 1/4$, Therefore, $y_2 = 3 / 4$.
Answer: A (3 / 4, 1 / 4, 0), B (1 / 4, 3 / 4) and the value of the game $= v = 36 / 7 = 5.25$.

Problem

Solve the game whose payoff matrix is given below by method of sub games.

		B		
		I	II	III
A	I	−5	5	0
	II	8	−4	−1

Solution

Given matrix is

	B			
	I	II	III	Row minimum
A I	–5	5	0	–5
A II	8	–4	–1	–4
Column maximum.	8	5	0	

No saddle point.

Sub game No. 1 of B:

	B		
	I	II	Row minimum
A I	–5	5	–5
A II	8	–4	-4
Column maximum	8	5	

No saddle point. Hence, Value of the game = $v_1 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
$[(-5 \times -4) - (5 \times 8) / [(-5) + (-4)] - [5 + 8] = (20 - 40) / (-9 - 13) = -20 / -21 = (20 / 21)$

Sub game 2 of B:

	B		
	I	III	Row minimum
A I	–5	0	–5
A II	8	–1	–1
Column maximum.	8	0	

No saddle point. Hence, value of the game = $v_2 = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
$= [(-5 \times -1) - (0)] / [(-5 + (-1)] - (0 + 8) = (5 - 0) / (-6 - 8) = - (5 / 14).$

Sub game No.3 of B:

	B		
	II	II	Row minimum.
A I	5	**0**	**0**
A II	–4	–1	–4
Column maximum.	5	0	

The game has saddle point. $V_3 = 0$

Comparing all the three values, $v_1 = (20 / 21)$, $v_2 = - (5 / 14)$ and $v_3 = 0$. The sub game 2 will give good returns to B. Hence, B prefers to play the sub game 2. Now let us find the probabilities of the strategies.

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (-1 - 8) / (- 14) = -9 / -14 = 9/14$
$x_2 = 1 - (9 / 14) = 5 / 14,$
$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (-1 - 0) / (-14) = (-1 / -15) = 1/15$
$y_2 = 1 - (1 / 15) = 14 / 15.$
Answer: A (9 / 14, 5 / 14), B (1 / 15, 0, 14 / 15) and the value of the game $v = - (5 / 14)$.

Graphical Method

When a $m \times n$ pay of matrix can be reduced to $m \times 2$ or $n \times 2$ pay off matrix, we can apply the sub game method. But too many sub games will be there it is time consuming. Hence, it is better to go for **Graphical method** to solve the game when we have $m \times 2$ or $n \times 2$ matrixes.

Problem

Solve the game whose pay of matrix is:

		B			
		I	II	III	IV
A	I	1	4	-2	-3
	II	2	1	4	5

Solution

Given payoff matrix is:
Solve the game whose pay of matrix is:

			B				
			I	II	III	IV	Row minimum
A	x	I	1	4	-2	-3	-3
	(1-x)	II	2	1	4	5	1
Column maximum			2	4	4	5	

No saddle point. If sub game method is to be followed, there will be many sub games. Hence, graphical method is used.

Let A play his first strategy with a probability of x, and then he has to play his second strategy with a probability of $(1 - x)$. Let us find the payoffs of A when B plays his various strategies.

Step 1

Find the payoffs of A when B plays his various strategies and A plays his first strategy with a probability x and second strategy with a probability $(1 - x)$. Let pay off be represented by P. Then A's payoffs, when

B plays his first strategy: $P_1 = 1 \, x \, (x) + 2 \, (1 - x) = 1x + 2 - 2x = 2 - x.$
B plays his second strategy: $P_2 = 4x + 1 \, (1 - x) = 4x + 1 - x = 1 + 3x.$
B plays his third strategy: $P_3 = -2x + 4 \, (1 - x) = -2x + 4 - 4x = 4 - 6x.$
B Plays his fourth strategy: $P_4 = -3x + 5 \, (1 - x) = -3x + 5 - 5x = 5 - 8x.$

Step 2

All the above payoff equations are in the form of $y = mx + c$. Hence we can draw straight lines by giving various values to x. To do this let us write two **vertical lines,** keeping the distance between lines at least four centimeters. Then write a horizontal line to represent the probabilities. Let the left side vertical line represents, A's first strategy and the probability of $x = 1$ and right side vertical line represents A's second strategy and the probability of $1 - x$. Mark points 1, 2, 3 etc on vertical lines above the horizontal line and $-1, -2, -3$ etc, below the horizontal lines, to show the payoffs.

Step 3

By substituting $x = 0$ and $x = 1$ in payoff equations, mark the points on the lines drawn in step 2 above and joining the points to get the payoff lines.

Step 4

These lines intersect and form open polygon. These are known as upper bound above the horizontal line drawn and the open polygon below horizontal line is known as lower bound. The upper bound (open polygon above the horizontal line is used to find the decision of player B and the open polygon below the line is used to find the decision of player A. This we can illustrate by solving the numerical example given above.

Step 5

Remember that the objective of graphical method is also to reduce the given matrix to 2×2 matrix, so that we can apply the formula directly to get the optimal strategies of the players.

For $P_1 = 2 - x$, when $\times = 0$, $P_1 = 2$ and when $\times = 1$, $P_1 = 1>$ Mark these points on the graph and join the points to get the line P_1. Similarly, we can write other profit lines.

$P_2 = 1 + 3x$, when $x = 0$, $P_2 = 1$, $x = 1$, $P_2 = 4$.
$P_3 = 4 - 6x$. When $x = 0$, $P_3 = 4$ and When $x = 1$, $P_3 = -2$.
$P_4 = 5 - 8x$, When $x = 0$, $P_4 = 5$ and When $x = 1$, $P_4 = -3$.
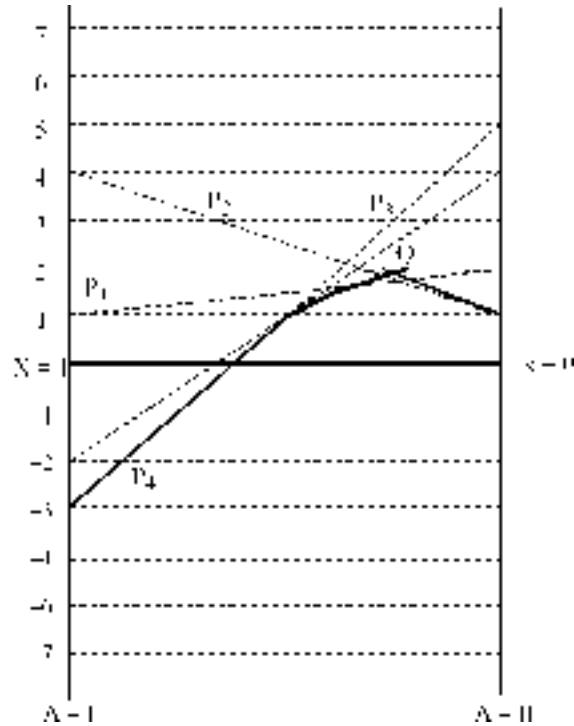
After drawing the graph, the lower bound is marked, and the highest point of the lower bound is point Q, lies on the lines P_1 and P_2. Hence B plays the strategies II, and I so that he can minimize his losses. Now the game is reduced to 2×2 matrix. For this payoff matrix, we have to find optimal strategies of A and B. The reduced game is:

		B		
		I	II	Row minimum
	I	1	4	1
A				
	II	2	1	1
Column Maximum:		2	4	

No saddle point. Hence we have to apply formula to get optimal strategies.

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x_2 =$

$x_1 = (1 - 2) / (1 + 1) - (4 + 2) = -1 / (2 - 6) = (-1 / -4) = 1 / 4.$ and $x_2 = 1 - (1 / 4) = 3 / 4$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y_2$

$y_1 = (1 - 4) / (- 4) = (3 / 4), y_2 = 1 - (3 / 4) = (1 / 4)$

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =$
$(1 \times 1) - (4 \times 2) / -4 = (1 - 8) / -4 = -4 / -4 = (7 / 4)$
Answer: **A (1 / 4, 3 / 4), B (3 / 4, 1 / 4, 0, 0) , v = 7 /4. A always wins 7/4 units of money.**

Problem

Solve the given payoff matrix by Graphical method and state optimal strategies of players A and B.

		B				
		1	2	3	4	5
A	1	−5	5	0	−1	8
	2	8	−4	−1	6	−5

Solution

Given Payoff Matrix is

			B					Row minimum
			1	2	3	4	5	
x	A	1	−5	5	0	−1	8	−5
(1 − x)		2	8	−4	−1	6	−5	−5
Column Max:			8	5	0	6	8	

No saddle point. Reduce the given matrix by using graphical method. Let us write the payoff equations of B when he plays different strategies. A has only two strategies to use. Let us assume that A plays his first strategy with a probability x and his second strategy with a probability $(1 - x)$. The B's payoffs are:

P_1 for B's first strategy $= -5x + 8 (1 - x)$, i.e. $P_1 = -5x + 8 - 8x = 8 - 13x$. When, $x = 0$, $P_1 = 8$, $x = 1$,

$P_1 = -5$.

P_2 for B's second Strategy $= 5x - 4 (1 - x)$, i.e. $P_2 = 5x - 4 + 4x = 9x - 4$, When $x = 0$, $P_2 = -4$. When $x = 1$, $P_2 = 5$.

P_3 for B's third strategy $= 0x - 1 (1 - x) = x - 1$, When $x = 0$, $P_3 = -1$, and When $x = 1$, $P_3 = 0$.

P_4 for B's fourth strategy $= -1x + 6 (1 - x) = x + 6 - 6x = 6 - 5x$. When $x = 0$, $P_4 = 6$ and when $x = 1$, $P_4 = 1$

P_5 for B's fifth strategy $= 8 x - 5 (1 - x) = 8x - 5 + 5x = 13x - 5$. When $x = 0$, $P_5 = -5$, when $x = 1$, $P_5 = 8$.

If we plot the above payoffs on the graph:
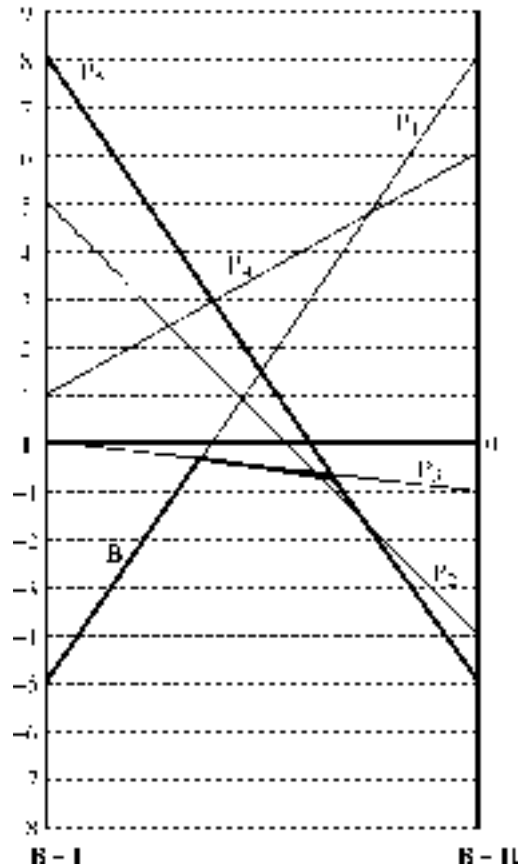
Now, player *B* has to select the strategies, as player *A* has only two strategies. To make A to get his minimum gains, *B* has to select the point *B* in the lower bound, which lies on both the strategies *B* –1 and *B* – 3. Hence now the 2 × 2 game is:

		B		
		1	3	Row minimum
	1	–5	0	–5
A				
	2	8	–1	–1
Column maximum.		8	0	

No saddle point. Hence apply the formula to get the optimal strategies.

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 =$

$x_1 = [-1 - 8]/(-5 - 1) - (0 + 8) = (-9)/(-6 - 8) = (-9/-14) = (9/14)$ and $x_2 = [1 - (9/14) = (5/14)$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y_2$

$y_1 = [-1 - 0] / - (14) = - (-1 / -14) = (1 / 14)$ and $y_2 = 1 - (1 / 14) = (13 / 14)$

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = [(-1 \times -5) - (0 \times 8)] / (-14) = (-5 / -14) = (5 / 14)$.

Answer: $v = (5 / 14)$, A $(9 / 14, 5 / 14)$, B $(1/14, 0, 13 / 14, 0, 0)$. A always wins a sum of 5/14.

> **Note: While calculating the profits to draw graph, it is shown that first to write the equation and then substituting the values of 0 and 1 to x we can get the profits for each strategy. Students as well can directly write the profit points, without writing the equation. For example, in the given problem, we know that A plays his first strategy with x and then the second strategy with $(1- x)$ probability. When $x = 0$, the value is 8, *i.e.* the element a_{21} in the matrix. Similarly, when $x = 1$, the values is –5 *i.e.* the element a_{11}. We can write other values similarly. But it is advised it is not a healthy practice to write the values directly. At least show one equation and calculate the values and then write the other values directly. This is only a measure for emergency and not for regular practice.**

Problem

Solve the game graphically, whose pay off matrix is:

		B	
		I	II
	1	−6	7
	2	4	−5
A	3	−1	−2
	4	−2	5
	5	7	−6

Solution

The given pay off matrix is:

		B		Row minimum
		I	II	
	1	−6	7	−6
	2	4	−5	−5
A	3	−1	−2	−2
	4	−2	5	−2
	5	7	−6	−6
Column maximum:		7	7	

No saddle point. For graphical method, let us assume that *B* plays his first strategy with a probability of '*y*' and the second strategy with a probability of $(1 - y)$. Then the equations of various pay offs when *A* plays his different strategies are:

A's pay off when he plays his strategy 1 is $P_1 = -6y + 7(1 - y) = -6y + 7 - 7y = 7 - 13y$, When $y = = 0$,

$P_1 = 7$ and when $y = 1$, $P_1 = -6$.

A's pay off when he plays his strategy 2 is $P_2 = 4y - 5(1 - y) = 4y - 5 + 5y = -5 + 9y$, When $y = 0$, $P_2 = -5$,

When $y = 1$, $P_2 = 4$.

A's pay off when he plays his strategy 3 is $P_3 = -1y - 2(1 - y) = -1y - 2 + 2y = y - 2$, When $y = 0$, $P_3 = -2$,

When $y = 1$, $P_3 = -1$.

A's pay off when he plays his strategy 4 is $P_4 = -2y + 5(1 - y) = -2y + 5 - 5y = 5 - 7y$, When $y = 0$, $P_4 = 5$

When $y = 1$, $P_4 = -2$

A's pay off when he plays his strategy 5 is $P_5 = 7y - 6(1 - y) = 7y - 6 + 6y = 13y - 6$, When $y = 0$, $P_5 = -6$

When $y = 1$, $P_5 = 7$.
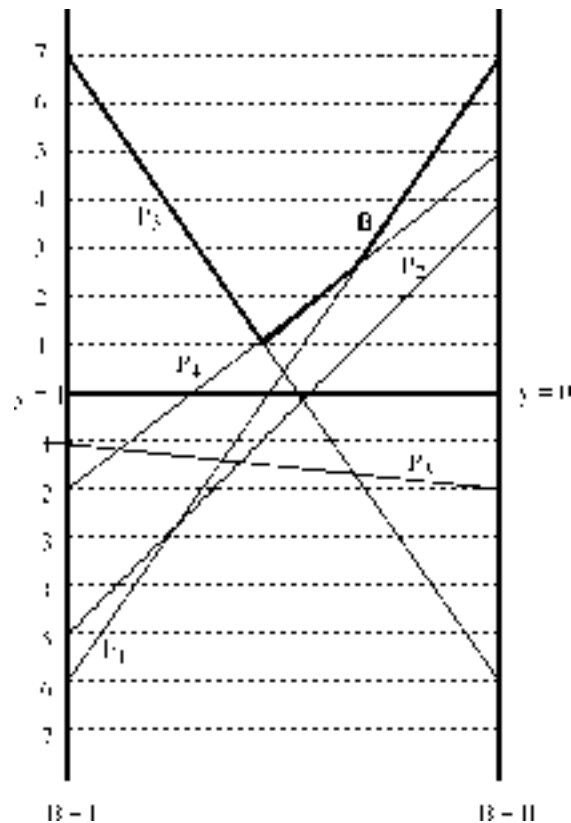
In the figure, point B lies on the strategies 4 and 5 of player A. Now B has to select lowest point in the lower bound (Thick lines) *i.e.* point B. The reduced 2×2 game is

B

	I	II	Row minimum.
4	–2	5	–2
5	7	– 6	–6

(A is the row player label)

Column maximum   7   5

No saddle point. To find the optimal strategies of A and B and the value of the game the formula is used.

$x_1 = (a_{22} - a_{21})/(a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 =$

$x_1 = (-6 - 7)/(-2 - 6) - (5 + 7) = (-13)/(-8) - (12) = 13/20$, hence $x_2 = 7/20$.

$y_1 = (a_{22} - a_{12})/(a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2$

$y_1 = (-6 - 5)/(-20) = 11/20$, $y_2 = 9/20$.

Value of the game $= v = (a_{11} a_{22} - a_{12} a_{21})/(a_{11} + a_{22}) - (a_{12} + a_{21}) = (-2 \times -6 - 5 \times 7)/(-20)$

$= (12 - 35)/(-20) = 23/20. = 1.15$.

Answer: A (0, 0, 0, 13 / 20, 7 / 20), B (11 / 20, 9 / 20), and $v = 1.15$.

Problem

Solve the game whose pay off matrix is:

B

	I	II	III	IV
I	3	4	10	12
II	8	4	3	2

(A is the row player label)

Solution

The pay of matrix is:

B

	I	II	III	IV	Row minimum
I	3	4	10	12	3
II	8	4	3	2	2
Column maximum.	8	4	10	12	

(A is the row player label)

No Saddle point. Hence reduce the matrix by graphical method. Let *A* play his I strategy with a probability of *x* and the II strategy with a probability of $(1 - x)$. The pay offs of *B* when he plays different strategies is as follows:

B's Strategy	Pay off Equation (P_i)	When x = 0 P_i =	When x = 1 P_i =
I	$3x + 8 (1 - x) = 3x + 8 - 8x = 8 - 5x.$	8	3
II	$4x + 4 (1 - x) = 4x + 4 - 4x = 4$	4	4
III	$10x + 3 (1 - x) = 10x + 3 - 3x = 3 + 7x$	3	10
IV	$12x + 2 (1 - x) = 12x + 2 - 2x = 2 + 10x$	2	12
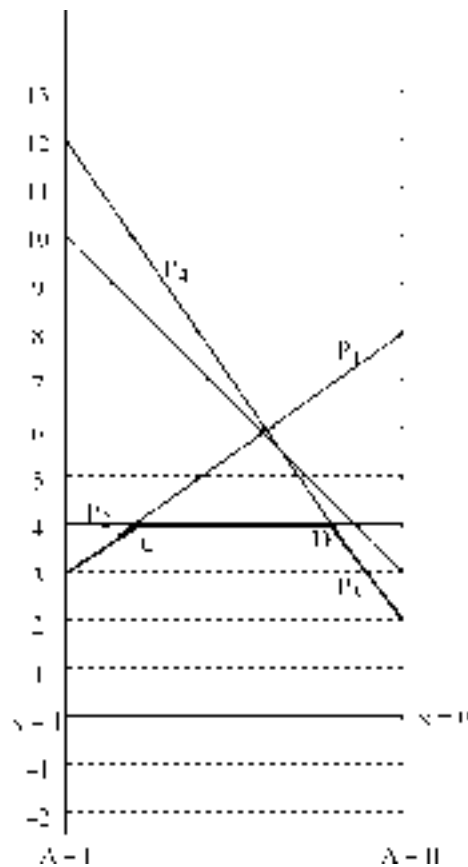
In the figure points *C* and *D* lies on a horizontal line and point *C* lies on P_2 and P_1, similarly, point *D* lies on P_2 and P_3. Hence we have two 2 × 2 games. If we solve by applying the formula, the pay offs or the value of the game is same for both games.

The games are:

		B		Row minimum				B		Row minimum
		I	II					II	IV	
	I	3	4	3			I	④	12	**4**
A						A				
	II	8	④	**4**			II	4	2	2
Column maximum		8	**4**	Column maximum				**4**	12	

Both the games have saddle point and the value of the game in both cases is $v = 4$.

Optimal strategies are:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - x_2 = (4 - 8) / (7 - 11) = 1$, Then $x_2 = 0$.

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or** $= 1 - y_2 = (0) / -4 = 0$, then $y_2 = 1$. (This means they are playing pure strategy game.

Answer: A (1, 0), B (0, 1, 0, 0) and $v = 4$.

Algebraic Method

When the given pay off matrix of a game cannot be reduced to 2×2, or 2×3 or $m \times 2$ or $2 \times n$, then, we can solve the game by using the algebraic method. This is a straightforward and lengthy and time-consuming method. Here we have to write a system of inequalities and consider them as equations and solve the simultaneous equations as usual. To illustrate the method, let us take a numerical example and solve by using Algebraic method.

Problem

Solve the game whose pay off matrix is as given:

		B		
		I	II	III
	I	−1	2	1
A	II	1	−2	2
	III	3	4	−3

Solution

Given matrix is:

		B			Row minimum
		I	II	III	
	I	−1	2	1	−1
A	II	1	−2	2	−2
	III	3	4	−3	−3
Column maximum			3	4	2

No saddle point. Let us solve the game by Algebraic method. Let A play his strategies with a probability of x_1, x_2 and x_3 and B play his strategies with a probability of y_1, y_2 and y_3. Now we know $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$. Let us write the inequalities, which show the pay offs of both players. A is a maximizing (maximin) player and he expects his pay off should be $\geq v$ and B is the minimizing player (minimax) he expects his pay off must be $\leq v$. The inequalities are:

$$-1x_1 + 1x_2 + 3x_3 \geq v$$
$$2x_1 - 2x_2 + 4x_3 \geq v$$
$$1x_1 + 2x_2 - 3x_3 \geq v$$

$$-1y_1 + 2y_2 + 1y_3 \leq v$$
$$1y_1 - 2y_2 + 2y_3 \leq v$$
$$3y_1 + 4y_2 - 3y_3 \leq v$$
$$x_1 + x_2 + x_3 = 1$$
$$y_1 + y_2 + y_3 = 1 \text{ and } x_1, x_2, x_3, y_1, y_2, y_3 \text{ all } \geq 0$$

Now let us consider all the above inequalities into equations so that we can go ahead to solve the equations. The equations are:

$$-1x_1 + 1x_2 + 3x_3 = v \qquad \qquad ...1$$
$$2x_1 - 2x_2 + 4x_3 = v \qquad \qquad ...2$$
$$1x_1 + 2x_2 - 3x_3 = v \qquad \qquad ...3$$
$$-1y_1 + 2y_2 + 1y_3 = v \qquad \qquad ...4$$
$$1y_1 - 2y_2 + 2y_3 = v \qquad \qquad ...5$$
$$3y_1 + 4y_2 - 3y_3 = v \qquad \qquad ...6$$
$$x_1 + x_2 + x_3 = 1 \qquad \qquad ...7$$
$$y_1 + y_2 + y_3 = 1 \qquad \qquad ...8$$
$$\text{And } x_1, x_2, x_3, y_1, y_2, y_3 \text{ all } \geq 0 \qquad \qquad ...9$$

Add equations 1 and 3:

$$-1x_1 + 1x_2 + 3x_3 = v \qquad \qquad ...1$$
$$\underline{1x_1 + 2x_2 - 3x_3 = v} \qquad \qquad ...3$$
$$3x_2 = 2v \text{ or } x_2 = (2 / 3) v$$

Add two times of equation 1 to equation 2:

$$-2x_1 + 2x_2 + 6x_3 = 2v \qquad \qquad ...1$$
$$\underline{2x_1 - 2x_2 + 4x_3 = v} \qquad \qquad ...2$$
$$10x_3 = 3v \text{ or } x_3 = (3 / 10) v$$

Substituting the values of x_2 and x_3 in equations 1 to 3 we get $x_1 = (17 / 30) v$. Substituting the values of x_1, x_2 and x_3 in equation number 7 we get $v = (15 / 23)$. Substituting the value of v we get $x_1 = (17 / 46)$,

$x_2 = (10 / 23)$ and $x_3 = (9 / 46)$.

As we know the value of v, we can substitute this value in equations 4, 5 and 6 and solving for y_1, y_2, y_3 we get the values as: $y_1 = (7 / 23)$, $y_2 = (6 / 23)$ and $y_3 = (10 / 23)$.

Answer: A (17 / 46, 10 / 23, 9 / 46), B (7 / 23, 6 / 23, 10 / 23) and v = 15 / 23.

Method of oddments for solving the games

Problem

Solve the given game by method of oddments:

		B		
		I	II	III
A	I	3	−1	−3
	II	−3	3	−1
	III	−4	−3	3

Solution

The matrix given is:

		B			
		I	II	III	Row minimum
A	I	3	−1	−3	−3
	II	−3	3	−1	−3
	III	−4	−3	3	−4
Column maximum		3	3	3	

The game has no saddle point. Let us apply method of oddments.

*Step.*1: Subtract each row from the row above it. That is subtract second row from first row and third row from second row etc. Write the difference of these rows in the form of two-succssie rows below the rows of the matrix.

Step 2: Subtract each column from the column to its left i.e. subtract second column from the first and the third column from the second and so on and write the difference in the form of two successive columns to the right of the given matrix. This is shown below:

		B				
		I	II	III		
A	I	3	−1	−3	4	2
	II	−3	3	−1	−6	4
	III	−4	−3	3	−1	−6
		6	−4	−2		
		1	6	−4		

Step 3: Calculate the oddments for *A*'s I, II, and III strategies and *B*'s I, II, and III strategies.
Oddment for *A*'s first strategy = Determinant

$$= \begin{vmatrix} -6 & 4 \\ -1 & -6 \end{vmatrix} = (-6 \times -6) - (4 \times -1) = 40$$

Oddment for A's Second strategy = Determinant

$$= \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} = (4 \times -6) - (2 \times -1) = -24 + 2 = -22$$

Oddment for A's third strategy = Determinant

$$= \begin{vmatrix} 4 & 2 \\ -6 & 4 \end{vmatrix} = (4 \times 4) - (2 \times -6) = 16 + 12 = 28$$

Oddment for B's First strategy = Determinant

$$= \begin{vmatrix} -4 & -2 \\ 6 & -4 \end{vmatrix} = (-4 \times -4) - (-2 \times 6) = 16 + 12 = 28$$

Oddment for B's second strategy = Determinant

$$= \begin{vmatrix} 6 & -2 \\ 1 & -4 \end{vmatrix} = (6 \times -4) - (-2 \times 1) = -24 + 2 = -22$$

Oddment for B's third strategy = Determinant

$$= \begin{vmatrix} 6 & -4 \\ 1 & 6 \end{vmatrix} = (6 \times 6) - (-4 \times 1) = 36 + 4 = 40.$$

Step 4: Write these oddments, neglecting the signs as shown below:

		I	II	III	
		B			
	I	3	−1	−3	40
A	II	−3	3	−1	22
	III	−4	−3	3	28
		28	22	40	Col total = 90
					Row total = 90

Step 5: Now verify the sums of oddments of *A* and *B*. They must be same to solve the game by matrix method. In this example both sums are equal to 90. This means that both players use their pure strategies and hence the game is conformable for matrix method. **The necessary condition for solving the game is the sums of two oddments must be same. In case the sums of oddments are different, then both the players do not use their all-pure strategies and hence matrix method fails.**

Step.6: Divide the oddments by the sum of the oddments to get the optimal strategies of players.
A (40 / 90, 22 / 90, 28. 90), B (28 / 90, 22 / 90, 40 / 90) OR A (20 / 45, 11 / 45, 14 / 45),
B (14 / 45, 11 / 45, 20 / 45).

Value of the game is given by: $v = [40 \times 3 + 22 \times (-3) + 28 \times (-4)] / (40 + 22 + 28) = (-58 / 90)$ = (−29 / 45) OR

$v = [40 \times (-1) + 22 \times 3 + 28 \times (-3)] / 90 = -58 / 90$ OR
$v = [40 \times (-3) + 22 \times (-1) + 28 \times 3] / 90 = -58 / 90$ OR
$v = [28 \times 3 + 22 \times (-1) + 40 \times (-3)] / 90 = -58 / 90$ OR
$v = [28 \times (-3) + 22 \times 3 + 40 \times (-1)] / 90 = -58 / 90$ OR
$v = [28 \times (-4) + 22 \times (-3) + 40 \times 3] / 90 = -58 / 90$

The value can be found by any one of the above.

Problem

Solve the given game by method of matrices:

		B		
		I	II	III
	I	1	−1	−1
A	II	−1	−1	3
	III	−1	2	−1

Solution

The given pay of matrix is:

		B			Row minimum
		I	II	III	
	I	1	−1	−1	−1
A	II	−1	−1	3	−1
	III	−1	2	−1	−1
Column Maximum:		1	2	3	

No Saddle point. To solve by method of oddments:

		B				
		I	II	III		
	I	1	−1	−1	2	0
A	II	−1	−1	3	0	−4
	III	−1	2	−1	−3	3
		2	0	4		
		0	−3	4		

Oddment for AI = Determinant $= \begin{vmatrix} 0 & -4 \\ -3 & 3 \end{vmatrix} = 0 - 12 = -12$

Oddment for A II = Determinant $= \begin{vmatrix} 2 & 0 \\ -3 & 3 \end{vmatrix} = 0 - 6 = -6$

Oddment for A III = Determinant $= \begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix} = -8 - 0 = -8$

Oddment for B I = Determinant $= \begin{vmatrix} 0 & -4 \\ -3 & -4 \end{vmatrix} = 0 - 12 = -12$

Oddment for B II = Determinant $= \begin{vmatrix} 2 & -4 \\ 0 & -4 \end{vmatrix} = 0 - 8 = -8$

Oddment for B III = Determinant $= \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6 - 0 = -6$

Now neglecting the signs of oddments, we write as below:

		B					
		I	II	III			
	I	1	−1	−1	12	12 / 23	6 / 13
A	II	−1	−1	3	6	6 / 26	3 / 13
	III	−1	2	−1	8	8 / 26	4 / 13
		12	8	6	26 = Total of oddments.		
		12 / 26	8 / 26	6 / 26			
		6 / 13	4 / 13	3 / 13			

Value of the game $v = [12 \times 1 - 6 \times 1 - 8 \times 1] / (12 + 6 + 8) = 2 / 26 = 1 / 13$.

Answer = A (6 / 13, 3 / 13, 4 / 13), B (6 / 13, 4 / 13, 3 / 13) and $v = 1 / 13$.

Problem

Solve the game by using oddments:

$$
\begin{array}{c|cc}
 & \multicolumn{2}{c}{B} \\
 & 1 & 2 \\
\hline
A\quad 1 & 5 & 1 \\
A\quad 2 & 3 & 4 \\
\end{array}
$$

Solution

Given matrix is

	B 1	2	Row minimum
A 1	5	1	1
A 2	3	4	3
Column maximum:	5	4	

No saddle point. To find the oddment it is easy as it is 2×2 matrix.

$$
\begin{array}{c|cc|c}
 & \multicolumn{2}{c}{B} & \\
 & 1 & 2 & \\
\hline
A\quad 1 & 5 & 1 & 1 \\
A\quad 2 & 3 & 4 & 4 \\
\hline
 & 3 & 2 & 5 \\
\end{array}
$$

Sum of both oddments is 5. Hence optimal strategies for $A = (1 / 5, 4 / 5)$ and that for $B = (3 / 5, 2 / 5)$

And the value of the game is $v = (1 \times 1 + 4 \times 4) / 5 = 17 / 5$.

Answer: *A* (1/5, 4/5), *B* (3 / 5, 2 / 5) and *v* = 17 / 5.

Method of Linear Programming

When the given pay of matrix cannot be reduced into lesser degree, (in case it does not have pure strategy for players), the mixed strategy game can easily be solved by applying the principles of linear programming. If the problem of maximizing player is primal one, the problem of minimizing player will be the dual of the primal. Hence by solving either primal or dual, we can get the answer of the problem. As the linear programming problem insists on non-negativity constraint, we must take care to see that all the elements in the given matrix are positive elements. In case, there are negative elements in the

given matrix, we can add a suitable, large and positive number to the matrix, so that all the elements in the matrix will become positive elements. Or by writing the row minimums and column maximums, we can know that the range of the value of 'v' and to keep the v as positive, a positive, sufficiently large number is added to the all elements of the matrix, so that we can satisfy the non - negativity constraint of the linear programming inequalities.

Problem

Two oil companies; Indian oil Company and Caltex Company operating in a city are trying to increase their market at the expense of the other. The Indian Oil Company is considering possibilities of decreasing price; giving free soft drinks on Rs. 40/- purchases of oil or giving away a drinking glass with each 40-liter purchase. Obviously, Caltex cannot ignore this and comes out with its own programme to increase its share in the market. The payoff matrix from the viewpoints of increasing or decreasing market shares is given in the matrix below:

		Caltex Oil Company		
		Decrease Price.	Free soft drink on Rs. 40/- purchase.(II)	Free drinking glass On 40 liters or more.(III)
	Decrease Price. (I)	4%	1%	–3%
Indian Oil Co.,	Free soft Drink on Rs.40-/ Purchase. (II)	3	1	6
	Free drinking Glass on 40 Lts or more. (III)	–3	4	–2

Find the optimal strategies and the value of the game.

Solution

Given pay off matrix is:

		Caltex Company			
		I	II	III	Row minimum
	I	4	1	–3	–3
Indian Oil Company.	II	3	1	6	1
	III	–3	4	–2	–3
Column maximum		4	4	6	

The game has no saddle point. Moreover, the value of the game lies between 1 (best out of the worst or maximum of minimum gains for (*A*) and 4 *i.e.* minimum of maximum losses for *B* (minimax). As both are positive numbers, we can proceed further for linear programming method. If, in any way the value of the game lies between a negative element and a positive element, then we have to add suitable positive element to the matrix to see that the value lies between two positive elements. (Some times to make the things easy, we can make all the elements of matrix positive by adding a suitable

positive element). Let us assume that A plays his strategies with probabilities of x_1, x_2, and x_3. Similarly B plays his strategies with probabilities of y_1, y_2 and y_3. The inequalities are:

For B:

$$1y_1 + 1y_2 - 1y_3 = 1$$
$$4y_1 + 1y_2 - 3y_3 \leq v$$
$$3y_1 + 1y_2 + 6y_3 \leq v$$
$$-3y_1 + 4y_2 - 2y_3 \leq v$$

For A:

$$1x_1 + 1x_2 + 1x_3 = 1$$
$$4x_1 + 3x_2 - 3x_3 \geq v$$
$$1x_1 + 1x_2 + 4x_3 \geq v$$
$$-3x_1 + 6x_2 - 2x_3 \geq v$$

Now divide all the inequalities and equations by 'v' and keep $(x_i / v) = X_i$ and $(y_j / v) = Y_j$ and write the inequalities and equations.

> **(Note: Dividing the above relations by 'v' is valid only if $v > 0$. If however, V < 0, the direction of inequality constraints must be reversed and if $v = 0$, division is meaning less). However, both these cases can be easily solved by adding a positive constant 'L' (Where $L \geq$ the negative game value) to all the elements of the matrix, thus ensuring that the game value for the *revised matrix* is greater than zero. After obtaining the optimal solution, the true value of the game can be obtained by subtracting L from the game value so obtained. In general, if the maxmin value of the game is non-negative, then the value of the game is greater than zero, provided the game does not have a saddle point.**

If we keenly observe the above inequalities, and recollect the knowledge of linear programming, particularly the duality in linear programming, we understand that the B's constraints are primal then the A's constraints are that of dual. After dividing by 'v' the inequalities and equations are:

$$(y_1 / v) + (y_2 / v) + (y_3 / v) = 1$$
$$(4y_1 / v) + (1y_2 / v) - (3y_3 / v) \leq 1$$
$$(3y_1 / v) + (1y_2 / v) + (6y_3 / v) \leq 1$$
$$(-3y_1 / v) + (4y_2 / v) - (2y_3 / v) \leq 1$$

Similarly we can write for A also. For A the inequalities are:

$$(x_1 / v) + (x_2 / v) + (x_3 / v) = 1$$
$$(4x_1 / v) + (3x_2 / v) - (3x_3 / v) \geq 1$$
$$(1x_1 / v) + (1x_2 / v) + (4x_3 / v) \geq 1$$
$$(-3x_1 / v) + (6x_2 / v) - (2x_3 / v) \geq 1$$

Now putting $x_i / v = X_i$ and $y_j / v = Y_j$ the above inequalities will become:

$$X_1 + X_2 + X_3 = 1/v$$
$$4X_1 + 3X_2 - 3X_3 \geq 1$$

$$1X_1 + 1\,X_2 + 4X_3 \geq 1$$
$$-3\,X_1 + 6X_2 - 2X_3 \geq 1$$

For B

$$Y_1 + Y_2 + Y_3 = 1/\,v$$
$$4Y_1 + 1Y_2 - 3Y_3 \leq 1$$
$$3Y_1 + 1Y_2 + 6Y_3 \leq 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 \leq 1$$

If B want to minimize 'v' he has to maximize $(1/v)$, This means he has to maximize $Y_1 + Y_2 + Y_3$ which is the objective function for B.

Maximize $\qquad Z = Y_1 + Y_2 + Y_3 = 1/\,v$ S.T.
$$4Y_1 + 1Y_2 - 3Y_3 \leq 1$$
$$3Y_1 + 1Y_2 + 6Y_3 \leq 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 \leq 1 \text{ and all } Y_1, Y_2 \text{ and } Y_3 \text{ are } \geq 0.$$

Writing this in Simplex format:

Maximize $Z = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$ S.T.
$$4Y_1 + 1Y_2 - 3Y_3 + 1S_1 + 0S_2 + 0S_3 = 1$$
$$3Y_1 + 1Y_2 + 6Y_3 + 0S_1 + 1S_2 + 0S_3 = 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 + 0S_1 + 0S_2 + 1S_3 = 1 \text{ and all } Y_1, Y_2 \text{ and } Y_3 \text{ are } \geq 0.$$

Initial basic feasible solution is:

Table I $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $S_1 = 1$, $S_2 = 1$, $S_3 = 1$ and Z = Rs. 0.

Problem	Profit in	Capacity	1	1	1	0	0	0	Replacement
Variable	Rs.		Y_1	Y_2	Y_3	S_1	S_2	S_3	Ratio.
S_1	0	1	**4**	1	−3	1	0	0	1 / 4 ←
S_2	0	1	3	1	6	0	1	0	1 / 3
S_3	0	1	−3	4	−2	0	0	1	1 / −3
		Net Ev.	1	1	1	0	0	0	

Table II : $Y_1 = 1/4$, $Y_2 = 0$. $Y_3 = 0$, $S_1 = 0$, $S_2 = ¼$, $S_3 = 7/4$ and Z = Rs. 1 × (1 / 4)

Problem	Profit in	Capacity	1	1	1	0	0	0	Replacement
Variable	Rs.		Y_1	Y_2	Y_3	S_1	S_2	S_3	Ratio.
Y_1	1	1 / 4	1	1 / 4	− 3 / 4	1 / 4	0	0	− 1 / 3
S_2	0	1 / 4	0	1 / 4	**33 / 4**	−3 / 4	1	0	1 / 33 ←
S_3	0	7 / 4	0	19 / 4	− 17 / 4	3 / 4	0	1	−7 / 17
		N.E	0	3 / 4	7 / 4	− 1 / 4	0	0	

Table III: $Y_1 = 3 / 11$, $Y_2 = 0$. $Y_3 = 1 / 33$, $S_1 = 0$, $S_2 = 0$, $S_3 = 66 / 33$ and $Z = $ Rs. $1 \times (3 / 11) + 1 \times (1 / 33)$

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	0 S_1	0 S_2	0 S_3	Replacement Ratio.
Y_1	1	3 / 11	1	3 / 11	0	2 / 11	1 / 11	0	1
Y_3	1	1 / 33	0	1 / 33	1	−1 / 11	4 / 33	0	1
S_3	0	62 / 33	0	**161 / 33**	0	4 / 11	17 / 33	1	62 / 161←
		N.E.	0	23 / 33	0	− 1 / 11	− 7 / 33	0	

↑

Table III: $Y_1 = 27 / 161$, $Y_2 = 62 / 161$. $Y_3 = 3 / 161$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ and $Z = $ Rs. 4 / 7

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	0 S_1	0 S_2	0 S_3	Replacement Ratio.
Y_1	1	27 / 161	1	0	0	26 / 161	10 / 161	−9 / 161	
Y_3	1	3 / 161	0	0	1	− 15 / 161	19 / 161	−1 / 161	
Y_2	1	62 / 161	0	1	0	12 / 161	17 / 161	33 / 161	
		N.E.	0	0	0	−23 / 161	−46 / 161	−23 / 161	

Now the value of $(1 / v) = (4 / 7)$

As, $$y_j / v = Y_j$$
$$y_1 = Y_1 \times v = (27 / 161) \times (7 / 4) = (27 / 92)$$
$$y_2 = Y_2 \times v = (62 / 161) \times ((7 / 4) = (62 / 92)$$
$$y_3 = Y_3 \times v = (3 / 161) \times (7 / 4) = (3 / 92).$$

A's best strategies we can get from net evaluation row and the elements under slack variables column. We have:

$$X_1 = (23 / 161), X_2 = (46 / 161), \text{ and } X_3 = 23 / 161.$$
$$x_1 = X_1 \times v = (23 / 161) \times (7 / 4) = (1 / 4)$$
$$x_2 = X_2 \times v = (46 / 161) \times (7 / 4) = (1 / 2)$$
$$x_3 = X_3 \times v = (23 / 161) \times (7 / 4) = (1 / 4)$$

Therefore the optimal strategies of A and B are:

Indian Oil Company = A (1/4, 1/2 , 1/4)

Caltex Company   = B (27 / 92, 62 / 92, 3 / 92) and value of the game is (7 / 4) for A.

Problem

Solve the game given in the pay off matrix below:

```
                        B
                1       2       3
         1      3      −4       2
A        2      1      −3      −7
         3     −2       4       7
```

Solution

The given matrix is:

		B 1	2	3	Row minimum
	1	3	−4	2	−4
A	2	1	−3	−7	−7
	3	−2	4	7	**−2**
Column maximum:		3	4	7	

The game has no saddle point. The value lies between –2 and 3. Hence a constant $L = \geq 2$ (*i.e.* $L = 3$) is added to all the elements of the matrix, so that the value of the game will be positive. Let x_1, x_2 and x_3 be the probabilities with which A plays his strategies and y_1, y_2 and y_3 be the probabilities with which B plays his strategies. As the player B is the minimizing player, let us write his inequalities and solve by Linear Programming method. A's strategies can be found from the net evaluation row of the final table of B. Modified matrix is:

			B y_1 1	y_2 2	y_3 3
	x_1	1	6	−1	5
A	x_2	2	4	0	−4
	x_3	3	1	7	10

The inequalities of B when A plays his different strategies are:

$$6y_1 - 1y_2 + 5y_3 \leq v$$
$$4y_1 + 0y_2 - 4y_3 \leq v$$
$$1y_1 + 7y_2 + 10y_3 \leq v \text{ and } y_1 + y_2 + y_3 = 1.$$

Dividing by v and keeping $y_i / v = Y_i$ we get,

$$6Y_1 - 1Y_2 + 5Y_3 \leq 1$$
$$4Y_1 + 0Y_2 - 4Y_3 \leq 1$$
$$1Y_1 + 7Y_2 + 10Y_3 \leq 1 \text{ and } Y_1 + Y_2 + Y_3 = (1 / v) \text{ and all } Y_1, Y_2, Y_3 \text{ all } \geq 0.$$

Writing the same in Simplex format, we have:

As B has to minimize v, he has to maximize $(1 / v)$. Therefore,

Maximize $\qquad\qquad Z = 1 / v = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$ S.t.

$$6Y_1 - 1Y_2 + 5Y_3 + 1S_1 + 0S_2 + 0S_3 = 1$$
$$4Y_1 + 0Y_2 - 4Y_3 + 0S_1 + 1S_2 + 0S_3 = 1$$
$$1Y_1 + 7Y_2 + 10Y_3 + 0S_1 + 0S_2 + 1S_3 = 1 \text{ and } Y_1, Y_2, \text{ and } Y_3 \text{ all } \geq 0$$

Table I: $S_1 = 1$, $S_2 = 1$, $S_3 = 1$ and $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$ and Z = Rs.0/-

Problem	Profit in	Capacity	1	1	1	0	0	0	Replacement
Variable	Rs.		Y_1	Y_2	Y_3	S_1	S_2	S_3	Ratio.
S_1	0	1	**6**	−1	5	1	0	0	1 / 6 ←
S_2	0	1	4	0	−4	0	1	0	1 / 4
S_3	0	1	1	7	10	0	0	1	1
		N.E	1	1	1	0	0	0	

Table II: $S_1 = 0$, $S_2 = 1/3$, $S_3 = 5/43$ and $Y_1 = 1/6$, $Y_2 = 0$, $Y_3 = 0$ and $Z = $ Rs.1 × (1 / 6)

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	0 S_1	0 S_2	0 S_3	Replacement Ratio.
Y_1	1	1/6	1	–1/6	5/6	1/6	0	0	–1
S_2	0	1/3	0	2/3	–22/3	–2/3	1	0	1/2
S_3	0	5/43	0	**43/6**	55/6	–1/6	0	1	5/43 ←
		N.E.	0	7/6	1/6	–1/6	0	0	

↑

Table III: $S_1 = 0$, $S_2 = 11/43$, $S_3 = 0$, and $Y_1 = 8/43$, $Y_2 = 5/43$, $Y_3 = 0$ and $Z = $ Rs.1 × (8 / 43) + 1 × (5 / 43) = 13 / 43.

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	0 S_1	0 S_2	0 S_3	Replacement Ratio.
Y_1	1	8/43	1	0	45/43	7/43	0	1/43	
S_2	0	11/43	0	0	–352/43	–28/43	1	–4/43	
Y_2	1	5/43	0	1	55/43	–1/43	0	6/43	
		N.E.	0	0	–57/43	–6/43	0	–7/43	

$Y_1 = (8/43)$, $Y_2 = (5/43)$, $Y_3 = 0$ and $v = (13/43)$.

Now we know that $\quad\quad\quad Y_j = (y_j / v)$, therefore,

$y_1 = Y_1 \times v = (8/43) \times 43/13) = (8/13)$

$y_2 = Y_2 \times v = (5/43) \times (43/13) = (5/13)$ and $y_3 = 0$

$X_1 = 6/43$, $X_2 = 0$, $X_3 = 7/43$.

$x_1 = (6/43) \times (43/13) = (6/13)$

$x_2 = 0$ and $x_3 = (7/43) \times (43/13) = (7/13)$.

Optimal strategies for $\quad\quad\quad$ A = A (6/13, 0, 7/13),

For B = (8/13, 5/13, 0) and value of the game = $v = (43/13) - 3 = 4/13$. (The element 3 was added to get the value of the v as positive).

Problem

Solve the game by L.P.P. method whose pay off matrix is:

		B I	II	III	IV
	I	3	2	4	0
A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Solution: The given matrix is:

		B I	II	III	IV	Row minimum
	I	3	2	4	0	0
A	II	3	4	2	4	2
	III	4	2	4	0	0
	IV	0	4	0	8	0
Column Maximum		4	4	4	8	

The game has no saddle point and the value of the game falls between 2 and 4. Hence we can write the inequalities directly, without adding any positive number to the matrix. As the player is minimizing player let us write his inequalities and apply Linear Programming approach.

Let y_j, where $j = 1, 2, 3,$ and 4 be the probabilities with which player B plays his strategies and x_1, where, $i = 1, 2, 3,$ and 4 with which player A plays his strategies. Then the inequalities of B are:

$$3y_1 + 2y_2 + 4y_3 + 0y_4 \leq 1$$
$$3y_1 + 4y_2 + 2y_3 + 4y_4 \leq 1$$
$$4y_1 + 2y_2 + 4y_3 + 0y_4 \leq 1$$
$$0y_1 + 4y_2 + 0y_3 + 8y_4 \leq 1 \text{ and}$$
$$y_1 + y_2 + y_3 + y_4 = 1$$

Dividing all the above inequalities by v and keeping $y_j / v = Y_j$ and writing entire thing in simplex model, we get:

Maximize $Z = (I / v) = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3 + 0S_4$ s.t.

$$3Y_1 + 2Y_2 + 4Y_3 + 0Y_4 + 1S_1 + 0S_2 + 0S_3 + 0S_4 = 1$$
$$3Y_1 + 4Y_2 + 2Y_3 + 4Y_4 + 0S_1 + 1S_2 + 0S_3 + 0S_4 = 1$$
$$4y_1 + 2y_2 + 4y_3 + 0y_4 + 0S_1 + 0S_2 + 1S_3 + 0S_4 = 1$$
$$0y_1 + 4y_2 + 0y_3 + 8y_4 + 0S_1 + 0S_2 + 0S_3 + 1S_4 = 1$$

And all Y_j are ≥ 0 where $j = 1, 2, 3$ and 4.

Table I: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $Y_4 = 0$, $S_1 = 1$, $S_2 = 1$, $S_3 = 1$, $S_4 = 1$ and Z = Rs. 0

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	1 Y_4	0 S_1	0 S_2	0 S_3	0 S_4	Replacement Ratio.
S_1	0	1	3	2	4	0	1	0	0	0	∞
S_2	0	1	3	4	2	4	0	1	0	0	1 / 4
S_3	0	1	4	2	4	0	0	0	1	0	∞
S_4	0	1	0	4	0	**8**	0	0	0	1	1 / 8 ←
		N.E.	1	1	1	1	0	0	0	0	

Table II: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $Y_4 = 1 / 8$, $S_1 = 1$, $S_2 = 1 / 2$ $S_3 = 1$, $S_4 = 0$ and Z = Rs. 1 x (1 / 8)

Problem Variable	Profit in Rs.	Capacity in Units	1 Y_1	1 Y_2	1 Y_3	1 Y_4	0 S_1	0 S_2	0 S_3	0 S_4	Replacement Ratio.
S_1	0	1	3	2	4	0	1	0	0	0	1 / 4
S_2	0	1 / 2	3	2	2	0	0	1	0	–1 / 2	1 / 4
S_3	0	1	4	2	**4**	0	0	0	1	0	1 / 4 ←
Y_4	1	1 / 8	0	1/2	0	1	0	0	0	1 / 8	1
		N.E.	1	1/2	1	0	0	0	0	– 1 / 8	

Table III: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 1 / 4$, $Y_4 = 1 / 8$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ and

Z = Rs. 1 × (1 / 8) + 1 × (1 / 4) = 3 / 8

Problem Variable	Profit in Rs.	Capacity	1 Y_1	1 Y_2	1 Y_3	1 Y_4	0 S_1	0 S_2	0 S_3	0 S_4	Replacement Ratio.
S_1	0	0	– 1	0	0	0	1	0	– 1	0	
S_2	0	0	1	1	0	0	0	1	–1 / 2	–1 / 2	
Y_3	1	1 / 4	1	1 / 2	1	0	0	0	1 / 4	0	
Y_4	1	1 / 8	0	1 / 2	0	1	0	0	0	1 / 8	
		N.E.	0	0	0	0	0	0	–1 / 4	–1 / 8	

$1 / v$ = (3 / 8), Hence the value of the game = v = (8 / 3). As $y_j = Y_j \times v$

$y_1 = 0 \times (8 / 3) = 0$, $y_2 = 0 \times (8 / 3) = 0$, $y_3 = (1 / 4) \times (8 / 3) = (2 / 3)$, $y_4 = (1 / 8) \times (8 / 3) =$
(1 / 3). Therefore, B's optimal policy = (0, 0, 2 / 3, 1 / 3)

From simplex table, $X_1 = 0$, $X_2 = 0$, $X_3 = (1/4)$, $X_4 = (1/8)$ and $x_i = X_i \times v$

$X_1 = 0 \times (8/3) = 0$, $x_2 = 0 \times (8/3) = 0$, $x_3 = (1/4) \times (8/3) = (2/3)$, and $x_4 = (1/8) \times (8/3) = (1/3)$

Answer: A (0, 0, 2 / 3, 1 / 3), B (0, 0, 2 / 3, 1 / 3) and $v = 8 / 3$

Iterative Method for Approximate Solution

Many a times the operations research manager is satisfied with an approximate answer, nearer to the optimal answer for making quick decisions. It is well known fact that, to get an accurate solution for the problem on hand by using well programmed methods will take time and by the time one get the correct answer, the situation or variables of the may change due to change in the conditions of market or environment. Hence, the manager, who has to take quick decision to solve the problem on hand, will be more satisfied with an approximate answer rather than the correct answer. The same applies to game theory also. One of the methods of determining the approximate solution is the method of iteration. The principle of the approximate method is:

The two players are supposed to play the plays of the game iteratively and at each play the players choose a strategy which is best to himself or say worst to opponent, in view of what the opponent has done up to the iteration.

In the given game, one of the players starts the game by selecting one of his strategies. The second player looking to the out comes of the strategy selected by the first player, the second player selects his strategy, which is best to him or worst to the opponent. Then the pay offs are written as shown. The first player selects the strategy best to him in the same way and writes his outcomes. Every time the out comes are added and written as shown. More number of iterations they play, they get answer very close to the optimal answer. But it is time consuming. Hence, the students **can write up to 10 iterations while dealing with the problem.** Generally the approximate method or method iteration is selected, when the game has no saddle point, cannot be reduced due to domination, and when we need to get the solution quickly. But students are advised not to use this method directly, without trying - Saddle point, Domination, Sub games, Graphical method etc. Let us try to understand by taking a numerical example.

Problem

Solve the following game by using method of iteration.

		B		
		I	II	III
A	I	−1	2	1
	II	1	−2	2
	III	3	4	−3

Solution

Given pay off matrix is:

		B													
		I	II	III	1	2	3	4	5	6	7	8	9	10	
A	I	−1	2	1	2	[3]	[4]	[5]	[6]	[5]	4	5	6	7	(5)
	II	1	−2	2	−2	−4	−2	0	2	4	5	[7]	[9]	[11]	(3)
	III	3	4	−3	4	1	−2	−5	−8	4	[7]	4	1	2	(2)
	1	1	[−2]	2											
	2	4	8	[−1]											
	3	3	10	[0]											
	4	2	12	[1]											
	5	4	14	[2]											
	6	[3]	16	3											
	7	[2]	18	4											
	8	5	22	[1]											
	9	6	20	[3]											
	10	7	18	[5]											
		(2)	(1)	(7)											

1. Let A select his second strategy. Write this below the last row of the matrix. Now, B looks at the out comes and selects the best strategy to him *i.e.* second strategy, because he gets 2 units of money. Write the outcomes B's second strategy beside the third column of the matrix.

2. Now, A examines his out come for various strategies and selects the strategy which is most advantageous to him, that is third strategy, which gives him four units of money. The selected strategy is generally encircled or the element is written in thick letter.

3. Now, the outcomes of A's third strategy is added to the elements written in the first row below the given matrix and the out come is written against S.No. 2, below the matrix. Once again in this B selects the best one for him, *i.e.* (−1) for third strategy. The elements of third strategy is added and added to the first column written right to the matrix and written against S.No. 2.

4. Continue the procedure until all the 10 plays are completed.

5. Look at the 6 th play of B (below the matrix), we have two elements having the same numerical value *i.e.* 3. As B is already played third strategy many times, we can give chance to his first strategy. *i.e.* Game is to be played judiciously.

6. To find the optimal strategies, let us see how many times each player has played each of his strategies.Now take A. A has played his first strategy 5 times, second strategy 3 times and his third strategy 2 times. Hence, the optimal strategy is A **(5 / 10, 3 / 10, 2 / 10)**. Similarly, for B the optimal strategy is B **(2 / 10, 1/ 10, 7 / 10).**

7. To find the values of the game, we can fix the higher and lower limits of value. Take the highest element in the last column i.e. 11 and lowest element, in the last row, *i.e.* 5. Divide these two by 10, *i.e.* the number of times the game is played. Then the **value of the game lies between**

$$v = 5 / 10 \le v \le 11 / 10.$$

Problem

Solve the game given below by method of iteration.

B

		1	2	3	4
	1	2	3	−1	0
A	2	5	4	2	−2
	3	1	3	8	2

Solution

Given matrix is:

B

		1	2	3	4	1	2	3	4	5	6	7	8	9
	1	2	3	−1	0	−1	−1	1	3	5	5	5	5	5
A	2	5	4	2	−2	2	0	5	10	(15)	13	11	9	7
	3	1	3	8	2	(8)	(10)	(11)	(12)	13	(15)	(17)	(19	(21)

Total

	1	2	3	4			
1	5	4	(2)	3		10	
2	6	7	10	(5)		5	0
3	(7)	10	18	(7)		5	1
4	(8)	13	26	9		(23)	9
5	(9)	16	34	11			
6	14	20	36	(9)			
7	15	23	44	(11)			
8	16	26	52	(13)			
9	17	29	60	(15)			
10	18	32	68	(17)			

Total   3   0   0   7

Optimal strategies: **A (0, 1 / 10, 9 / 10), B (3 / 10, 0, 0, 7 / 10). Value of the game lies between: 0 $\le v \le$ 23 / 10.**

Typical Problems

So far we have solved the problems, where the pay off matrix is given. Sometimes, we have to construct the payoff matrix, which is a very difficult job. Once the pay off matrix is written, solving

can be done by any one of the suitable methods discussed so far. In the following problems, the pay off matrix is constructed and the students have to solve the game to get the optimal strategies of the players and the value of the game.

Problem

A and B each take out one or two matches and guess how many matches the opponent has taken. If one of the players guesses correctly then the opponent has to pay him as many rupees as the sum of the numbers of matches had by both the players, otherwise the pay-out is zero. Write down the pay off matrix and obtain the optimal strategies for both the players.

Solution

Let A be the guessing fellow or the winner and the player B is the loser. They have two strategies; one to take one matches and the second is to take two matches. If both have taken one matches and winning player guesses correctly, then the opponent has to pay him the sum of rupees equal to the sum of matches. If he guesses wrongly the pay out is zero. The pay off matrix is as shown.

		B	
		1 matches	2 matches
A	1 matches	2	0
	2 matches	0	4

By solving with the formulae of 2×2 game, we get, $v = 4 / 3$, A (2 / 3, 1 / 3), B (2 / 3, 1 / 3)

Problem

Two players A and B, without showing each other put on a table a coin of Re.1/- with head or tail up. If the coin shows the same side (both head or both tail), the player A takes both the coins, otherwise B get them Construct the game and solve it.

Solution

If both are heads or tails, A will win the game and he gets Re.1/- + Re.1/- = Rs.2 /-. If the one is head and the other is tail, then B will get Rs.2/-. Therefore, the pay of matrix is:

		B	
		Head	Tail
A	Head	2	–2
	Tail	–2	2

As the pay of matrix of A is written, A's outcomes are positive and B's outcomes are negative, because B is winning and A is losing.

Optimal strategies are: A (1 / 2, 1 / 2), B (1 / 2, 1 / 2) and value of the game is $= 0$

Problem

In a game of matching coins with two players, suppose *A* wins one unit of vale when there are two heads; wins nothing when there are two tails and loses 1 / 2, unit of value when there is one head and one tail. Determine the pay off matrix, and the optimal strategies for the players.

Solution

A wins one unit of money when there are two heads (*H*, *H*), Wins nothing when there are two tails (*T*, *T*), loses ½ unit of money when there are two tails (*H*, *T*). The pay of matrix is:

		B	
		H	T
A	H	1	−1 / 2
	T	−1 / 2	0

The optimal strategies are: *A* (1 / 4, 3 / 4), *B* (1 / 4, 3 / 4), Value of the game is – (1 / 8). *B* gets always 1 / 8 units of money.

Problem

Consider a modified form of "matching coins" game. The matching player is paid Rs. 8/- if the two coins are both heads, and Re.1/- if both are tails. The non-matching player is paid Rs.3 /- when the coins do not match. Given the choice of being the matching or non-matching player both, which would you choose and what would be your strategy.

Solution

Let *A* be the matching player. He gets Rs.8/- when both are heads (*H*, *H*), he gets Re. 1 /- when both are tails (*T*, *T*). The non-matching player gets Rs. 3/ - when there is (*H*, *T*) or (*T*, *H*). Hence the pay off matrix is:

		B	
		H	T
A	H	8	-3
	T	−3	1

The optimal strategies are: *A* (4 / 15, 11 / 15), *B* (4 / 15, 11 / 15), and value of the game is – (1 / 15). Because the non - matching player is getting the money, it is better to be a non-matching player.

Problem

Consider the two person zero sum game in which each player selects independently an integer from the set of integers: 1, 2, and 3. The player with the smaller number wins one point unless his

number is less than his opponents by one unit. When the numbers are equal, there is no score. Find the optimal strategies of the players.

Solution

		B		
		1	2	3
	1	0	1	1
A	2	−1	0	1
	3	1	−1	0

Value of the game is $v = 0$, Optimal strategies of A and B are A (1/3, 1/3, 1/3), B (1/3, 1/3, 1/3). This type of games is known as symmetric games.

Problem

Two children play the following game, named as **'Scissors, Paper,** and **Stone'** **(S, P, St.).** Both players simultaneously call one of the three: Scissors, Paper or Stone. Scissors beat paper as paper can be cut by scissor, Paper beats stone as stone can be wrapped in paper, and stone beats scissors as stone can blunt the scissors. If both players name the same item, then there is a tie. If there is one point for win, zero for the tie and –1 for the loss. Form the pay of matrix and write the optimal strategies.

Solution

If both call same (S, S) or (P, P) or (St, St) the pay out is zero. There is one point for winning (S, P), (P, St) and (St, S) and –1 point for (S, St), (P, S) and (St., P). the pay off matrix is:

		B		
		S	P	St
	S	0	1	−1
A	P	−1	0	1
	St	1	−1	0

This is also symmetric game. Hence Value $= v = 0$. Strategies are: A (1/3, 1/3, 1/3), B (1 / 3, 1 / 3, 1/ 3).

Problem

Party A attacks an object party B defends it. A has two airplanes B has three anti-aircraft guns. To attack the object, it is enough that one airplane of A breaks through B's defenses. The planes of A can choose any one of the three regions I, II, and III of space for approaching the object as shown in figure. The party B can place his guns to defend in any of the regions of space. A gun can only defend one particular region and is incapable of engaging a plane approaching the object form a different region of approach. Each gun is capable of shooting down only one plane with a probability 1.

Party A does not know where the guns are placed, party B also does not know how the airplanes approach the object. The problem of A is to attack. The problem of party B is to defend this. Construct the game and determine best strategies for both the parties.
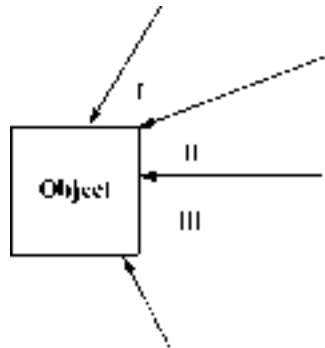
Solution

Strategies of party A are: A_1: Send airplanes in two different regions and A_2: Send both airplanes in one region.

Strategies of B are: B_1 To place three guns in three different regions.

B_2 To place one gun in one region, two guns in other region, and keep one region undefended.

B_3 To place all the three guns in only one region.

To analyze the situation: The pay of matrix is of the order 2×3 as A has only two strategies and B has three strategies. Now:

$A_1 B_1 = a_{11} = 0$, because the three regions are defended by one gun and hence, probability of hitting the object $= 0$.

$A_2 B_1 = a_{21}$ since one of the two airplanes, which attack the same region is shot down and the other hits the object, hence the probability of attacking the object $= 1$.

$A_1 B_2 = a_{12} =$ Probability of attacking the object is probability of the airplanes selecting the undefended region is equals to $(1 / 3) + (1 / 3) = (2 / 3)$, as in an undefended region, the airplane definitely hit the object.

$A_2 B_2 = a_{22} =$ Probability of sending the airplanes to undefended region or the region defended by one gun, is $(1 / 3) + (1 / 3) = (2 / 3)$.

$A_1 B_3 = a_{13} =$ As all the guns are placed in one region and the two airplanes are sent in two different regions, at least one of the planes will definitely go to one of the undefended regions. Hence the probability of striking the object is 1.

$A_2 B_3 = a_{23} =$ Probability of sending both the planes to any one of the two undefended regions is $(1 / 3) + (1 / 3) = (2 / 3)$. Hence the required pay off matrix is given by:

		B		
		B_1	B_2	B_3
A	A_1	0	2/3	1
	A_2	1	2/3	2/3

B_2 is dominating B_3 hence the game is reduced to 2×2 game.

		B		
		B_1	B_2	Row minimum
A	A_1	0	2 / 3	0
	A_2	1	**2 / 3**	2 / 3
Column Maximum		1	**2 / 3**	

Optimal strategies are A (0, 1), B (0, 1, 0) and the value of the game is (2 / 3).

Problem

A has two ammunition stores, one of which is twice as valuable as the other. *B* is an attacker, who can destroy an undefended store, but he can only attack one of them. *A* knows that *B* is about to attack one of the stores but does not know which one? What should he do? Note that A can successfully defend only one store at a time.

Solution

Let us assume the value of the smaller store is 1, and then the value of the bigger store is 2. By analysis, If both stores survive, A loses nothing *i.e.* 0 (a_{11}). If the smaller survives, *i.e.* larger is destroyed, then *A* loses 2 (a_{12}). If the larger store survives and the smaller is destroyed, A losses 1. Therefore the pay off matrix of *A* is

	B	
	B_1 attack smaller Store	B_2 Attack larger store
A_1 defend small Store.	0	–2
A_2 defend larger Store.	–1	0

(A on the left spanning both rows)

Solving the game by formulae, we get A (1 / 3, 2 / 3), $v = - (2 / 3)$

Problem

The game known as **'The prisoner's dilemma'.** The district authority has two prisoners in different cells and knows that both are guilty. To provide the sufficient evidence to convict them, he plays a game. He offers them a chance of confessing and declares that if one confesses and the other refuses to confess, the penalty will be great particularly for the one who denies the charge say 10 years, whilst the one who confesses will go free for giving the testimony against the other. Both prisoners know that if neither confesses they will both receive at most a minor sentence say 1 year for a technical offence. Also if both confess, they will get 8 years. What the prisoners do?

Solution

Each prisoner's sentence (in years) may be represented by the matrix given below:

II Prisoner

	Denies	Confesses
Denies	1	10
Confesses	0 go free	8

I Prisoner

The game is not a zero sum game as the II prisoner's matrix is not the negative of the above matrix as usual the case may be. The following matrix may also represent this sentenced of the two prisoners in order.

II Prisoner

	Denies	confesses
Denies	$(1, 1)$	$(10, 0)$
Confesses	$(0, 10)$	$(8, 8)$

I Prisoner

We can analyze the game as follows: No doubt when both will study the situation both will decide to play the first strategy (I, I). But however, with some reflection first prisoner may give reasons as follows: If second prisoner plays his first strategy, then he should play second because he can go free. But when the first prisoner plays his second strategy, another prisoner also decided to play his second strategy. If both prisoners play their second strategy, both get a sentence of 8 years. The pair of strategies (II, II) forms an equilibrium point, because departing from this, neither, without the other doing so, can do better for himself. Therefore, both play the game (II, II)

Bidding Problems

Bidding problems are of two types. They are ***Open or Auction bids*** in which two or more bidders bid on an item of certain value until none is willing to increase the bid. The last bid is then the winner of the bid.

The second one is ***Closed bids*** in which each bidder submits his bid in a closed envelop and the envelopes are opened all at one time and the highest (or lowest) bid is accepted. In this case none knows his opponent's bid.

Problem

Two items of worth Rs. 100/- and Rs. 150/- are to be auctioned at a public sale. There are only two bidders *A* and *B*. Bidder A has Rs. 125/- and the bidder *B* has Rs. 155/- with him. If each bidder wants to maximize his own return, what should be his strategy?

Solution

Let each bidder increase the bid successively by λ. At any bid, each player has the option to increase the bid or to leave the opponent's bid stand. If *B* bids Rs. x on the first item (Rs. 100/- value), then *A* has the following options.

If A lets B win the first item, for Rs. x/-, then B will be left with Rs. $(155 - x)$ only for bidding the second item i.e. he cannot make a bid more than Rs. $(155 - x)$ for the second item. Thus A will be positively able to win the second item for Rs. $(155 - x + \lambda)$. Therefore, A's gain by allowing B to win the first item for Rs. x/- will be Rs. $[150 - (155 - x + \lambda)] = (x - \lambda - 5)$.

On the other hand, if A bids Rs. $(x + \lambda)$, for the first item and B lets him to win the bid, then A's gain will be Rs. $([100 - (x + \lambda)] = (100 - x - \lambda)$.

Now since A wants to maximize his return, he should bid Rs. $(x + \lambda)$ for the first item provided $(100 - x - \lambda) \geq (x - \lambda - 5)$ or $x \leq$ Rs. 52.50

Thus A should bid for first item until $x \leq$ Rs. 52.50. In case $x >$ Rs. 5.50, he should allow B to win the first item.

Similarly, B's gains in the two alternatives are: Rs. $[150 - (125 - y) - \lambda]$ and Rs. $(100 - y - \lambda)$, where y denote A's bid for the firs item. Thus B should bid Rs. $(y + \lambda)$ for the first item provided:

$(100 - y - \lambda) \geq [150 - (125 - y) - \lambda]$ or $y \leq$ Rs. 37.50

Obviously, A will win the fist item for Rs. 37.50 because he can increase his bid without any loss up to Rs. 52.50, and B will get the second item for Rs. 125 – Rs. 37.50 = Rs. 87.50 because a, after winning the first item in Rs. 37.50 cannot increase his bid for the second item beyond Rs. 87.50. Thus B will get the second item for Rs. 87.50. Therefore A's gain is Rs. 100 – Rs.37.50 = Rs. 62.50 and B's gain is Rs. 250 – Rs. 87.50 = Rs. 62.50.

Problem

Two items of values Rs.100/- and Rs. 120 respectively are to be bid simultaneously by two bidders A and B. Both players intend to devote a lot of sum of Rs. 130 to the two bids. If each uses a minimax criterion, find the resulting bids.

Solution

Here the bids are closed since they are to be made simultaneously. Let A_1 and A_2 are the A's optimum bids for the first and second items respectively. Obviously, A's optimum bids are the ones that fetch the same profit to A on both the items. If p denote the profit earned by the successful bid, then,

$2p = (100 - A_1) + (120 - A_2)$ or $2p = 220 - A_1 - A_2$

Since both A and B intend to devote only Rs. 130/- for both the bids, $A_1 + A_2 =$ Rs. 130/-

Therefore, $2p =$ Rs.$(220 - 130) =$ Rs. 90/- or $p =$ Rs. 45/-

Now $p = 100 - A_1$ or $A_1 = 100 - p = 100 - 45 =$ Rs. 55/-

Also, $p = 120 - A_2$ or $A_2 = 120 - p = 120 - 45 =$ RS. 75/-

Thus optimum bids for A are Rs. 55/- and Rs. 75/- for the first and second items respectively. Likewise, optimum bids for B can be determined and will be Rs 55/- and Rs. 75/- respectively for the two items.

n- Person Zero sum games

Whenever more than two persons are involved in the game, they are treated as if two coalitions are formed by n - persons involved. The properties of such games are values of the various games between every possible pair of coalitions. For example, for a player A, B, C, and D the following coalitions can be formed:

A against B, C, D;
B against A, C, D;
C against A, B, D;
D against A, B, C;
A, B against C, D;
A, C against B, D;
A, D against B, C.

If the value of the games for B, C, D coalition is V, then the value of the game for A is $-V$, since it is zero sum game. Thus in a four person zero sum game, there will be seven values or characteristics for the game, which are obtained from the seven different coalitions.

Problem

Find the value of the three person zero sum game in which player A has two choices, X_1 and X_2; player B has two choices, Y_1, and Y_2 and player C has two choices, Z_1 and Z_2. They pay offs are as shown below:

Choices			Pay offs.		
A	B	C	A	B	C
X_1	Y_1	Z_1	3	2	–2
X_1	Y_1	Z_2	0	2	1
X_1	Y_2	Z_1	0	–1	4
X_1	Y_2	Z_2	1	3	–1
X_2	Y_1	Z_1	4	–1	0
X_2	Y_1	Z_2	–1	1	3
X_2	Y_2	Z_1	1	0	2
X_2	Y_2	Z_2	0	2	1

Solution

There are three possible coalitions:

(1). A against B and C; (2). B against A and C and (3). C against A and B.

Now we shall solve the resulting games.

1. *A* against *B* and *C*:

<center>B and C</center>

		$Y_1 Z_1$	$Y_1 Z_2$	$Y_2 Z_1$	$Y_2 Z_2$	Row minimum
	X_1	3	**⓪**	0	1	**0**
A						
	X_2	4	−1	1	0	−1
Column maximum:		4	**0**	1	1	

The game has saddle point. Hence *A*'s best strategy is X_1 and *B* and *C*'s best combination is $Y_1 Z_2$. Value of the game for $A = 0$, and that of *B* and *C* also equals to zero.

2. *B* against *A* and *C*:

<center>A and C</center>

		$X_1 Z_1$	$X_1 Z_2$	$X_2 Z_1$	$X_2 Z_2$	Row minimum
	Y_1	2	2	−1	1	−1
B						
	Y_2	−1	3	0	2	−1
Column maximum:		2	3	0	2	

Game has no saddle point. First and third columns dominate second and fourth columns respectively hence dominated columns are cancelled. The reduced matrix is:

<center>A and C</center>

		$X_1 Z_1$	$X_2 Z_1$	Oddments.	Probabilities.
	Y_1	2	−1	1	1 / 4
B					
	Y_2	−1	0	3	3 / 4
Oddments:		1	3		
Probability		1 / 4	3 / 4		

By solving the method of oddments, *B*'s best strategy is Y_1 with a probability of 1 / 4 and choice Y_2 with a probability of 3 / 4. Now *A* and *C* has to play X_1 and Z_1 with a probability of 1 / 4 and X_2 and Z_1 with a probability of 3 / 4.

Value of game for $B = [(2 / 4) – (3 / 4)] / [(1 / 4 + (3 / 4)] = – (1 / 4)$

Value of game for A and C is $(1 / 4)$

3. C against A and B. Pay of matrix is:

		A and B				
		$X_1 Y_1$	$X_1 Y_2$	$X_2 Y_1$	$X_2 Y_2$	Row minimum
C	Z_1	–2	4	0	2	–2
	Z_2	1	–1	3	1	–1
Column Maximum:		1	4	3	2	

The game has no saddle point. First column dominates third and fourth column. Hence the reduced matrix is:

		A and B			
		$X_1 Y_1$	$X_1 Y_2$	oddment	Probability
C	Z_1	–2	4	2	2 / 8
	Z_2	1	–1	6	6 / 8
Oddment		5	3		
Probability.		5 / 8	3 / 8		

C's best strategy is to play $(Z_1, Z_2) = [(2 / 8), (6 / 8)]$

For A and $B = (X_1 Y_1, X_1 Y_2) = [(5 / 8), (3 / 8)]$

Value of the game for $C = [–(10 / 8) + (12 / 8)] / [(5 / 8) + (3 / 8)] = (2 / 8) / 1 = (1 / 4)$

Value of the game for A and B is $– (1 / 4)$ as this is a zero sum game.

Therefore, the characteristics of the game are:

$V (A) = 0$, $V (B) = – (1 / 4)$, $V (C) = (1 / 4)$ and $V (B, C) = 0$, $V (A, C) = (1 / 4)$. $V (A, B) = – (1 / 4)$

QUESTIONS

1. What are competitive situations? Explain with the help of an example.
2. What is a business game? Enlist the properties of the game. What assumptions are made in game theory?
3. Explain Maximin and Minimax principle with respect to game theory.
4. By means of an example, explain what do you mean by Two Person Zero Sum game.
5. Solve the following game, whose pay of matrix is:

		B		
		I	II	III
	I	–2	15	–2
A	II	–5	–6	–4
	III	–5	20	–8

6. Solve the game whose pay of matrix is:

		B			
		I	II	III	IV
	I	3	2	4	0
A	II	2	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

7. Explain the theory of Dominance in solving a given game.

8. Explain the graphical method of solving a game.

9. Solve the following game:

		B				
		1	2	3	4	5
	1	2	–1	5	–2	6
A						
	2	–2	4	–3	1	0

10. In a small town, there are two discount stores *ABC* and *XYZ*. They are the only stores that handle Sunday goods. The total number of customers is equally divided between the two; because the price and quality are equal. Both stores have good reputations in the community, and they render equally good customer service. Assume that a gain customer by *ABC* is a loss to *XYZ* and vice versa. Both stores plan to run annual pre Diwali sales during the first week of the month in which Diwali falls. Sales are advertised through local newspaper, radio and television media. With aid of an advertising firm *ABC* Store constructed the game matrix given below, which gives the gain and loss to each customer. Find the optimal strategies of the stores.

		XYZ		
		Newspaper	Radio	television.
	Newspaper	30	40	–80
ABC	Radio	0	15	–20
	Television.	90	20	50

11. Players *A* and *B* play the following game. *A* has a bag containing three coins, one worth 4 units, one 6 units and the rest 9 units of money. *A* takes one coin from the bag and before exposure *B* guesses. If *B* is right he takes the coin and if wrong he pays to *A* the same worth money to *A*. Find the optima strategies of *A* and *B* and the value of the game.

Game Theory: MULTIPLE CHOICE QUESTIONS

1. If the value of the game is zero, then the game is known as:
   (*a*) Fair strategy   (*b*) Pure strategy
   (*c*) Pure game   (*d*) Mixed strategy.   ( )
2. The games with saddle points are:
   (*a*) Probabilistic in nature,   (*b*) Normative in nature
   (*c*) Stochastic in nature,   (*d*) Deterministic in nature.   ( )
3. When Minimax and Maximin criteria matches, then
   (*a*) Fair game is exists.   (*b*) Unfair game is exists,
   (*c*) Mixed strategy exists   (*d*) Saddle point exists.   ( )
4. When the game is played on a predetermined course of action, which does not change throughout game, then the game is said to be
   (*a*) Pure strategy game,   (*b*) Fair strategy game
   (*c*) Mixed strategy game   (*d*) Unsteady game.   ( )
5. If the losses of player *A* are the gins of the player *B*, then the game is known as:
   (*a*) Fair game   (*b*) Unfair game
   (*c*) Non- zero sum game   (*d*) Zero sum game.   ( )
6. Identify the wrong statement:
   (*a*) Game without saddle point is probabilistic
   (*b*) Game with saddle point will have pure strategies
   (*c*) Game with saddle point cannot be solved by dominance rule.
   (*d*) Game without saddle point uses mixed strategies,   ( )
7. In a two person zero sum game, the following does not hold correct:
   (*a*) Row player is always a loser;   (*b*) Column Player is always a winner.
   (*c*) Column player always minimizes losses
   (*d*) If one loses, the other gains.   ( )
8. If a two person zero sum game is converted to a Linear Programming Problem,
   (*a*) Number of variables must be two only,
   (*b*) There will be no objective function,
   (*c*) If row player represents Primal problem, Column player represent Dual problem,
   (*d*) Number of constraints is two only.   ( )

9. In case, there is no saddle point in a game then the game is
   (*a*) Deterministic game,   (*b*) Fair game,
   (*c*) Mixed strategy game,   (*d*) Multi player game.   (   )
10. When there is dominance in a game then
   (*a*) Least of the row ≥ highest of another row
   (*b*) Least of the row ≤ highest of another row
   (*c*) Every element of a row ≥ corresponding element of another row.
   (*d*) Every element of the row ≤ corresponding element of another row.   (   )
11. When the game is not having a saddle point, then the following method is used to solve the game:
   (*a*) Linear Programming method,   (*b*) Minimax and maximin criteria
   (*c*) Algebraic method   (*d*) Graphical method.   (   )
12. Consider the matrix given, which is a pay off matrix of a game. Identify the dominance in it.

		B		
		X	Y	Z
	P	1	7	3
A	Q	5	6	4
	R	7	2	0

   (*a*) P dominates Q   (*b*) Y dominates Z
   (*c*) Q dominates R   (*d*) Z dominates Y   (   )
13. Identify the unfair game:

	C	D		C	D
(*a*) A	0	0	(*b*) A	1	−1
B	0	0	B	−1	1

	C	D		C	D
(*c*) A	−5	+5	(*d*) A	1	0
B	+ 10	− 10	B	0	1

(   )

14. If there are more than two persons in a game then the game is known as:
   (*a*) Non zero sum game   (*b*) Open game
   (*c*) Multiplayer game   (*d*) Big game   (   )
15. For the pay of matrix the player *A* always uses:

		B	
		I	II
	I	−5	−2
A			
	II	10	5

   (*a*) First strategy   (*b*) Mixed strategy of both II and I
   (*c*) Does not play game   (*d*) Second strategy.   (   )

16. For the pay off matrix the player prefers to play

$$
\begin{array}{c@{\qquad}c}
 & \text{B} \\
\end{array}
$$

	I	II
A I	–7	6
II	–10	8

(*a*) Second strategy   (*b*) First strategy

(*c*) Keep quite   (*c*) Mixed strategy. ( )

17. For the game given the value is:

B

	I	II
A 1	2	3
II	–5	5

(*a*) 3,   (*b*) –5

(*c*) 5   (*d*) 2 ( )

19. In the game given the saddle point is:

B

	I	II	III
A I	2	–4	6
II	0	–3	–2
III	3	–5	4

(*a*) –2   (*b*) 0

(*c*) –3   (*d*) 2 ( )

20. A competitive situation is known as:

(*a*) Competition   (*b*) Marketing

(*c*) Game   (*d*) None of the above. ( )

21. One of the assumptions in the game theory is:

(*a*) All players act rationally and intelligently,

(*b*) Winner alone acts rationally

(*c*) Loser acts intelligently,

(*d*) Both the players believe luck ( )

22. A play is played when:
    (*a*) The manager gives green signal
    (*b*) Each player chooses one of his courses of action simultaneously
    (*c*) The player who comes to the place first says that he will start the game
    (*d*) When the latecomer says that he starts the game.   ( )
23. The list of courses of action with each player ……………
    (*a*) Is finite
    (*b*) Number of strategies with each player must be same
    (*c*) Number of strategies with each player need not be same
    (*d*) None of the above.   ( )
24. A game involving '*n*' persons is known as:
    (*a*) Multi member game   (*b*) Multi player game
    (*c*) *n* - person game   (*d*) not a game.   ( )
25. Theory of games and economic behavior is published by:
(*a*) John Von Neumann and Morgenstern   (*b*) John Flood
(*c*) Bellman and Neumann   (*d*) Mr. Erlang,   ( )
26. In the matrix of a game given below the negative entries are:

<center>B</center>

		I	II
	I	1	−1
A	II	−1	1

    (*a*) Payments from *A* to *B*   (*b*) Payments from *B* to *A*
    (*c*) Payment from players to organizers   (*d*) Payment to players from organizers. (   )

ANSWERS

1. (*c*)	2. (*d*)	3. (*d*)	4. (*a*)
5. (*d*)	6. (*c*)	7. (*a*)	8. (*c*)
9. (*c*)	10. (*d*)	11. (*b*)	12. (*d*)
13. (*d*)	14. (*c*)	15. (*d*)	16. (*b*)
17. (*d*)	18. (*c*)	19. (*c*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*c*)	24. (*c*)
25. (*a*)	26. (*a*)		

UNIT - 5

Programme Evaluation and Review Technique and Critical Path Method (PERT and CPM)

15.1. INTRODUCTION

Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques that are widely used in planning and scheduling the large projects. A project is a combination of various activities. For example, Construction of a house can be considered as a project. Similarly, conducting a public meeting may also be considered as a project. In the above examples, construction of a house includes various activities such as searching for a suitable site, arranging the finance, purchase of materials, digging the foundation, construction of superstructure etc. Conducting a meeting includes, printing of invitation cards, distribution of cards, arrangement of platform, chairs for audience etc. In planning and scheduling the activities of large sized projects, the two network techniques — PERT and CPM — are used conveniently to estimate and evaluate the project completion time and control the resources to see that the project is completed within the stipulated time and at minimum possible cost. Many managers, who use the PERT and CPM techniques, have claimed that these techniques drastically reduce the project completion time. But it is wrong to think that network analysis is a solution to all bad management problems. In the present chapter, let us discuss how PERT and CPM are used to schedule the projects.

Initially, projects were represented by **milestone chart** and **bar chart**. But they had little use in controlling the project activities. **Bar chart** simply represents each activity by bars of length equal to the time taken on a common time scale as shown in figure 15.1. This chart does not show interrelationship between activities. It is very difficult to show the progress of work in these charts. An improvement in bar charts is **milestone chart**. In milestone chart, key events of activities are identified and each activity is connected to its preceding and succeeding activities to show the logical relationship between activities. Here each key event is represented by a node (a circle) and arrows instead of bars represent activities, as shown in figure 15.2. The extension of milestone chart is PERT and CPM network methods.
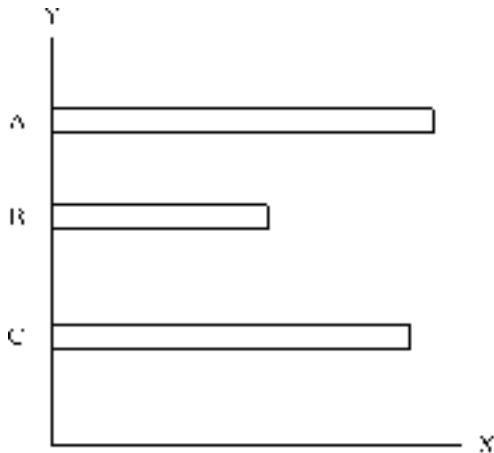
Figure 15.I. Bar chart.
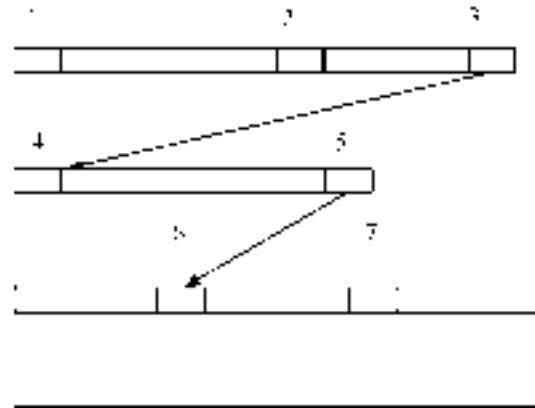
Figure 15.2. Milestone chart.

15.2. PERT AND CPM

In PERT and CPM the milestones are represented as *events*. Event or node is either starting of an activity or ending of an activity. Activity is represented by means of an arrow, which is resource consuming. Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a *network*, which is the basic document in network-based management. The basic steps for writing a network are:

(*a*) List out all the activities involved in a project. Say, for example, in building construction, the activities are:

   (*i*) Site selection,
   (*ii*) Arrangement of Finance,
  (*iii*) Preparation of building plan,
  (*iv*) Approval of plan by municipal authorities,
   (*v*) Purchase of materials,
  (*vi*) Digging of foundation,
 (*vii*) Filling up of foundation,
(*viii*) Building superstructure,
  (*ix*) Fixing up of doorframes and window frames,
   (*x*) Roofing,
  (*xi*) Plastering,
  (*xii*) Flooring,
 (*xiii*) Electricity and water fittings,
 (*xiv*) Finishing.

(*b*) Once the activities are listed, they are arranged in sequential manner and in logical order. For example, foundation digging should come before foundation filling and so on.

(*c*) After arranging the activities in a logical sequence, their time is estimated and written against each activity. For example: Foundation digging: 10 days, or 1½ weeks.

(*d*)   Some of the activities do not have any logical relationship, in such cases; we can start those activities simultaneously. For example, foundation digging and purchase of materials do not have any logical relationship. Hence both of them can be started simultaneously. Suppose foundation digging takes 10 days and purchase of materials takes 7 days, both of them can be finished in 10 days. And the successive activity, say foundation filling, which has logical relationship with both of the above, can be started after 10 days. Otherwise, foundation digging and purchase of materials are done one after the other; filling of foundation should be started after 17 days.

(*e*)   Activities are added to the network, depending upon the logical relationship to complete the project network.

Some of the points to be remembered while drawing the network are

(*a*) There must be only one beginning and one end for the network, as shown in figure 15.3.
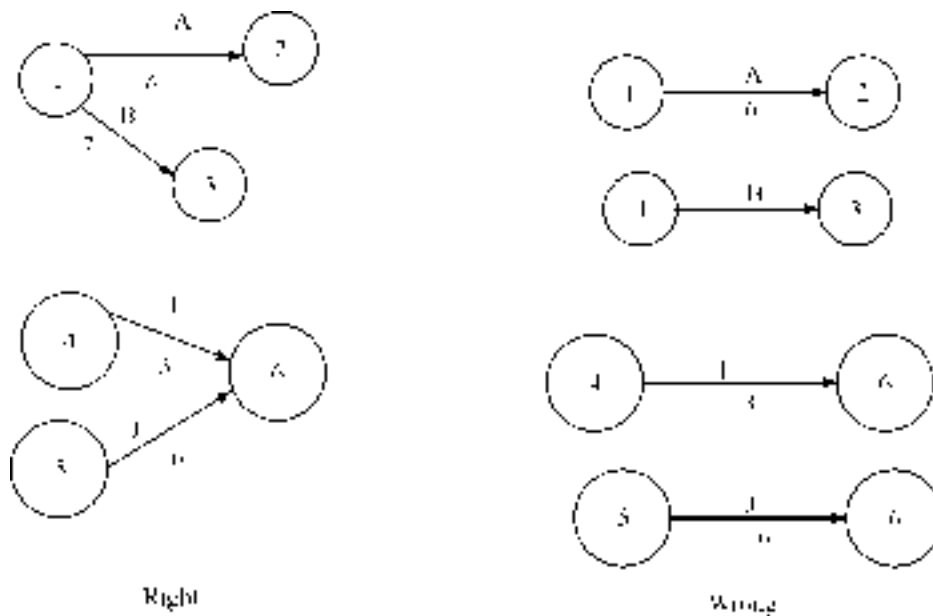
Figure 15. 3. Writing the network.

(*b*) Event number should be written inside the circle or node (or triangle/square/rectangle etc). Activity name should be capital alphabetical letters and would be written above the arrow. The time required for the activity should be written below the arrow as in figure 15. 4
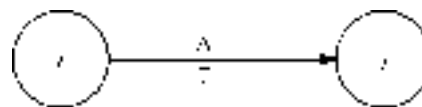
Figure 15.4. Numbering and naming the activities.

(*c*) While writing network, see that activities should not cross each other. And arcs or loops as in figure 15.5 should not join Activities.
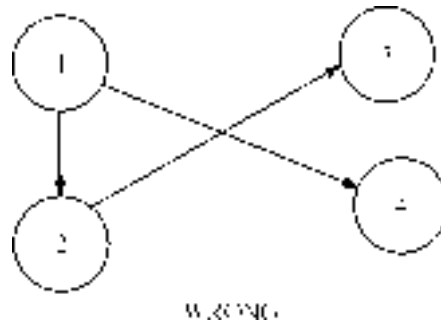
Figure 15.5. Crossing of activities not allowed.

(*d*) While writing network, looping should be avoided. This is to say that the network arrows should move in one direction, *i.e.* starting from the beginning should move towards the end, as in figure 15.6.
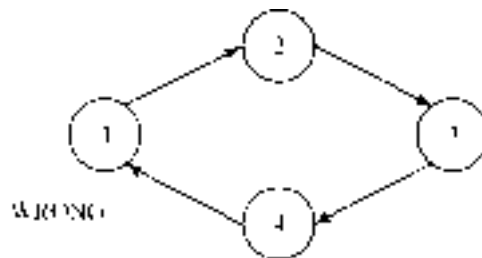
Figure 15. 6. Looping is not allowed.

(*e*) When two activities start at the same event and end at the same event, they should be shown by means of a **dummy activity** as in figure 15.7. Dummy activity is an activity, which simply shows the logical relationship and does not consume any resource. It should be represented by a dotted line as shown. In the figure, activities *C* and *D* start at the event 3 and end at event 4. *C* and *D* are shown in full lines, whereas the dummy activity is shown in dotted line.
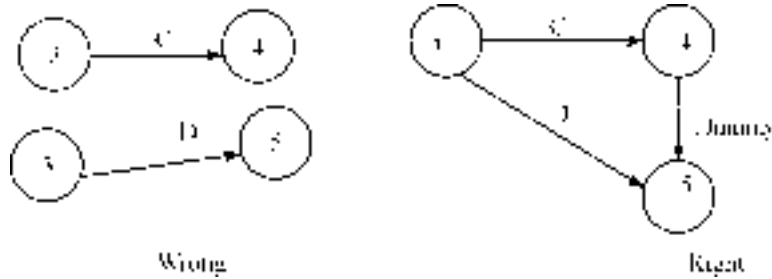
Figure 15.7. Use of Dummy activity.

(*f*)   When the event is written at the tail end of an arrow, it is known as **tail event**. If event is written on the head side of the arrow it is known as **head event**. A tail event may have any number of arrows (activities) emerging from it. This is to say that an event may be a tail event to any number of activities. Similarly, a head event may be a head event for any number of activities. This is to say that many activities may conclude at one event. This is shown in figure 15.8.

Figure 15.8. Tail event and Head event.

The academic differences between PERT network and CPM network are:

(*i*)   PERT is event oriented and CPM is activity oriented. This is to say that while discussing about PERT network, we say that Activity 1-2, Activity 2-3 and so on. Or event 2 occurs after event 1 and event 5 occurs after event 3 and so on. While discussing CPM network, we say that Activity *A* follows activity *B* and activity *C* follows activity *B* and so on. Referring to the network shown in figure 9, we can discuss as under.

PERT way: Event 1 is the predecessor to event 2 or event 2 is the successor to event 1. Events 3 and 4 are successors to event 2 or event 2 is the predecessor to events 3 and 4.

CPM way: Activity 1-2 is the predecessor to Activities 2-3 and 2-4 or Activities 2-3 and 2-4 are the successors to activity 1-2.

(*ii*) PERT activities are probabilistic in nature. The time required to complete the PERT activity cannot be specified correctly. Because of uncertainties in carrying out the activity, the time cannot be specified correctly. Say, for example, if you ask a contractor how much time it takes to construct the house, he may answer you that it may take 5 to 6 months. This is because of his expectation of uncertainty in carrying out each one of the activities in the construction of the house. Another example is if somebody asks you how much time you require to reach railway station from your house, you may say that it may take 1 to 1½ hours. This is because you may think that you may not get a transport facility in time. Or on the way to station, you may come across certain work, which may cause delay in your journey from house to station. Hence PERT network is used when the activity times are probabilistic.
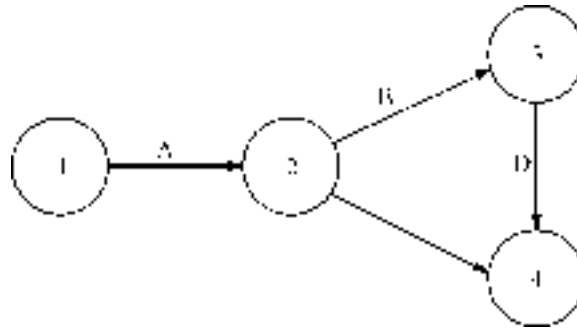
Figure 15.9. Logical relationship in PERT and CPM.
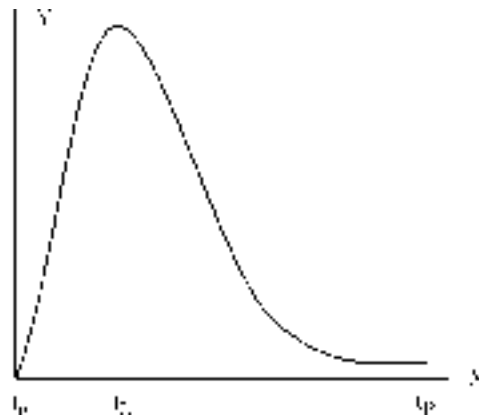
Figure 15.10. Three Time estimates.

There are three time estimates in PERT, they are:

(*a*) **OPTIMISTIC TIME:** Optimistic time is represented by t_O. Here the estimator thinks that everything goes on well and he will not come across any sort of uncertainties and estimates lowest time as far as possible. He is optimistic in his thinking.

(*b*) **PESSIMISTIC TIME:** This is represented by t_P. Here estimator thinks that everything goes wrong and expects all sorts of uncertainties and estimates highest possible time. He is pessimistic in his thinking.

(*c*) **LIKELY TIME:** This is represented by t_L. This time is in between optimistic and pessimistic times. Here the estimator expects he may come across some sort of uncertainties and many a time the things will go right.

So while estimating the time for a PERT activity, the estimator will give the three time estimates. When these three estimates are plotted on a graph, the probability distribution that we get is closely associated with **Beta Distribution curve**. For a Beta distribution curve as shown in figure 6.10, the characteristics are:

$$\text{Standard deviation} = (t_P - t_O)/_6 \ = \ \sigma, t_P - t_O \text{ is known as range.}$$

$$\text{Variance} = \{(t_P - t_O)/_6\}^2 \ = \ \sigma^2$$

Expected Time or Average Time $= t_E = (t_O + 4t_L + t_P) / 6$

These equations are very important in the calculation of PERT times. Hence the student has to remember these formulae.

Now let us see how to deal with the PERT problems.

(*g*) **Numbering of events:** Once the network is drawn the events are to be numbered. In PERT network, as the activities are given in terms of events, we may not experience difficulty. Best in case of CPM network, as the activities are specified by their name, is we have to number the events. For numbering of events, we use D.R. Fulkerson's rule. As per this rule:

An initial event is an event, which has only outgoing arrows from it and no arrow enters it. Number that event as 1.

Delete all arrows coming from event 1. This will create at least one more initial event. Number these initial events as 2, 3 etc.

Delete all the outgoing arrows from the numbered element and which will create some more initial events. Number these events as discussed above.

Continue this until you reach the last event, which has only incoming arrows and no outgoing arrows.

While numbering, one should not use negative numbers and the initial event should not be assigned 'zero'. When the project is considerably large, at the time of execution of the project, the project manager may come to know that some of the activities have been forgotten and they are to be shown in the current network. In such cases, if we use **skip numbering**, it will be helpful. Skip numbering means, skipping of some numbers and these numbers may be made use to represent the events forgotten. We can skip off numbers like 5, 10, 15 etc. or 10, 20 and 30 or 2, 12, 22 etc. Another way of numbering the network is to start with 10 and the second event is 20 and so on. This is a better way of numbering the events.

Let now see how to write network and find the project completion time by solving some typical problems.

Problem 15.1.

A project consists of 9 activities and the three time estimates are given below. Find the project completion time (T_E).

1. Write the network for the given project and find the project completion time?

Activities		Days		
i	j	T_O	T_L	T_P
10	20	5	12	17
10	30	8	10	13
10	40	9	11	12
20	30	5	8	9
20	50	9	11	13
40	60	14	18	22
30	70	21	25	30
60	70	8	13	17
60	80	14	17	21
70	80	6	9	12

Solution

In PERT network, it is easy to write network diagram, because the successor and predecessor event relationships can easily be identified. While calculating the project completion time, we have to calculate t_e i.e. expected completion time for each activity from the given three-time estimates. In case, we calculate project completion time by using t_O or t_L or t_P separately, we will have three completion times. Hence it is advisable to calculate t_E expected completion time for each activity and then the project completion time. Now let us work out expected project completion time.

Predecessor event	Successor event	Time in days			$T_E =$ $(t_O + 4t_L + t_P)/6$	Range $t_P - t_O$	S.D (σ) $(t_P - t_O)$ /6	Variance σ^2
10	20	5	12	17	9.66 (10)	12	2	4
10	30	8	10	13	10.17 (10)	5	0.83	0.69
10	40	9	11	12	10.83 (11)	3	0.5	0.25
20	30	5	8	9	7.67 (8)	4	0.66	0.44
20	50	9	11	13	11.00 (11)	4	0.66	0.44
40	60	14	18	22	18.00 (18)	8	1.33	1.78
30	70	21	25	30	25.18 (25)	9	1.5	2.25
60	70	8	13	17	12.83 (13)	9	1.5	2.25
50	80	14	17	21	17.17 (17)	7	1.16	1.36
70	80	6	9	12	9.00 (9)	6	1.0	1.0

For the purpose of convenience the t_E got by calculation may be rounded off to nearest whole number (the same should be clearly mentioned in the table). The round off time is shown in sbrackets. In this book, in the problems, the decimal, will be rounded off to nearest whole number.

To write the network program, start from the beginning i.e. we have 10 – 20, 10 – 30 and 10 – 40. Therefore from the node 10, three arrows emerge. They are 10 – 20, 10 – 30 and 10 – 40. Next from the node 20, two arrows emerge and they are 20 – 30 and 20 – 50. Likewise the network is constructed. The following convention is used in writing network in this book.
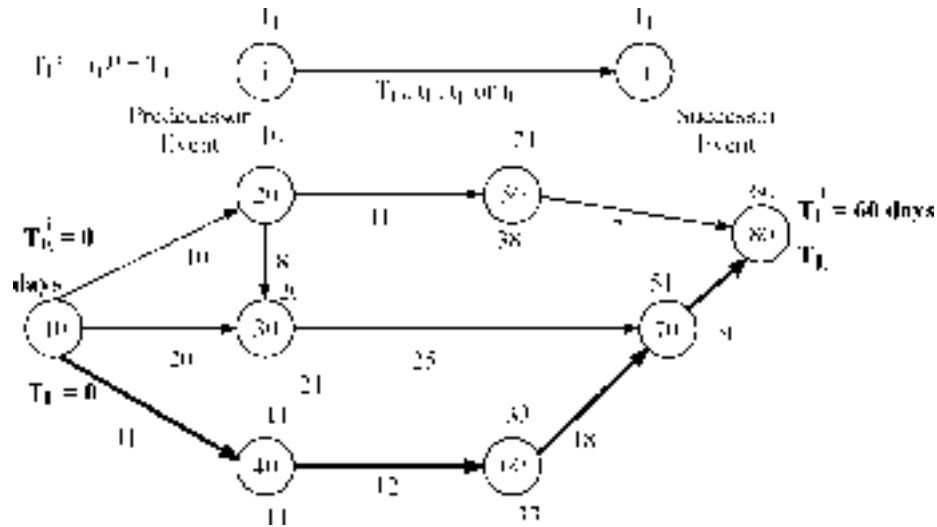
Figure 15.11. Network for Problem 15.1

Let us start the event 10 at 0th time *i.e.* expected time $T_E = 0$. Here T_E represents the occurrence time of the event, whereas t_E is the duration taken by the activities. T_E belongs to event, and t_E belongs to activity.

$$T_E^{10} = 0$$
$$T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 10 = 10 \text{ days}$$
$$T_E^{30} = T_E^{10} + t_E^{10-30} = 0 + 10 = 10 \text{ days}$$
$$T_E^{30} = T_E^{20} + t_E^{20-30} = 10 + 8 = 18 \text{ days}$$

The event 30 will occur only after completion of activities 20–30 and 10–30. There are two routes to event 30. In the **forward pass** *i.e.* when we start calculation from 1st event and proceed through last event, we have to workout the times for all routes and select the **highest one** and the **reverse** is the case of the **backward pass** *i.e.* we start from the last event and work back to the first event to find out the occurrence time.

$$T_E^{40} = T_E^{10} + t_E^{10-40} = 0 + 11 = 11 \text{ days}$$
$$T_E^{50} = T_E^{20} + t_E^{20-30} = 10 + 11 = 21 \text{ days}$$
$$T_E^{60} = T_E^{40} + t_E^{40-60} = 11 + 18 = 29 \text{ days}$$
$$T_E^{70} = T_E^{30} + t_E^{30-70} = 18 + 25 = 43 \text{ days}$$
$$T_E^{70} = T_E^{60} + t_E^{60-70} = 29 + 13 = 42 \text{ days}$$
$$T_E^{80} = T_E^{70} + t_E^{70-80} = 43 + 9 = 52 \text{ days}$$
$$T_E^{80} = T_E^{50} + t_E^{50-80} = 21 + 17 = 38 \text{ days}$$

$T_E^{80} = 52$ days. Hence the project completion time is 52 days. The path that gives us 52 days is known as

Critical path. Hence 10–20–30–70–80 is the critical path. Critical path may be represented by double line ($\Longrightarrow$) or thick line ($\longrightarrow$) or hatched line ($/\!\!/\!\!\!\rightarrow$). In this book thick line is used. All other parts *i.e.* 10–40–60–70–80, 10–20–50–80 and 10–30–70–80 are known as **non-critical paths**. All activities on critical path are **critical activities**.

The significance of critical path is delay in completion of critical activities which will increase the project completion time.

Now in the above project, the project completion time is 52 days. In case everything goes correctly the project will be completed in 52 days. Suppose the manager may want to completed the project in 50 days, then what is the probability of completing the project in 50 days? To find the answer for this, let us recollect what is discussed earlier.

Activity $i - j$ is given three time estimates *i.e.* t_O, t_L, and t_P and assumed that the distribution of these time estimates follows ß distribution curve. The approximate mean time for each activity is given by

$$t_E = (t_O + 4t_L + t_P)/6$$

The meaning of this expected time is that there is a fifty-fifty chance of completing the activities in a time duration t_E as shown in the curve.
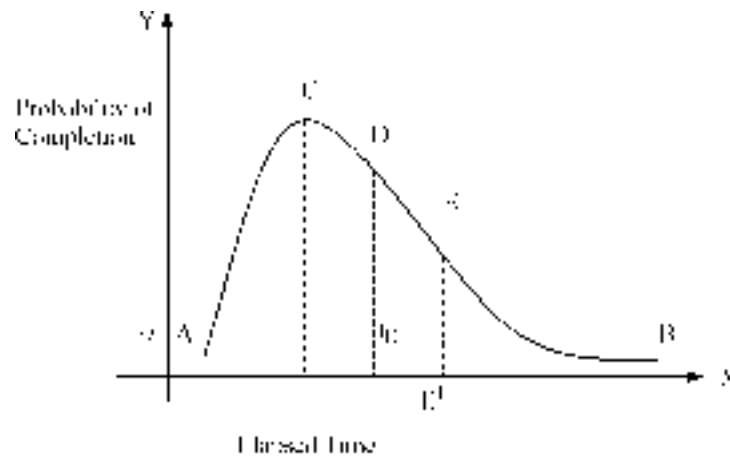
Figure 15.12

The vertical line at D represents t_E and the chance of completing activities at t_E is ½ Suppose we want to find out probabilities of completing the activities at EE ?,

The probability = Area under ACE / Area under ACB

While calculating the probability of completing the project (having number of activities), the following procedure is applied (here, we apply central limit theorem).

Step 1: Identify critical path and critical activities

Step 2: Find variance (σ^2) for critical activities.

$$\sigma_{ij}^2 = [(t_p^{ij} - t_o^{ij})/6]^2$$

Step 3: List out critical activities and their σ^2

Step 4: Find the sum of variance of critical activities *i.e.* $\Sigma\sigma^2$

Step 5: Find the square root of sum of variance *i.e.* $\sqrt{\Sigma\sigma^2}$

Step 6: Find the difference between the contractual time (T_L) *i.e.*, time by which the project is to be completed and project completion time T_E, i.e. $T_L - T_E$

Depending on the value of T_L, this may be +ve or 0 or –ve number. That is

If $T_L = T_E$ then $T_L - T_E = 0$

If $T_L > T_E$ then $T_L - T_E$ = Positive Number

If $T_L < T_E$ then $T_L - T_E$ = Negative Number

Step 7: Find the ratio $(T_L - T_E)/\sqrt{\Sigma\sigma^2} = Z$, this is the length of ordinate at T_L on the curve.

Step 8: Refer to Table 15.1, which gives the height of Z and the probability of completing the project.

If $T_L = T_E$ the probability is ½.

If $T_L > T_E$ then Z is Positive, the probability of completing the project is higher than 0.5

If $T_L < T_E$ then Z is Negative, the probability of completing the project is lower than 0.5

Table: 15.I
Standard Normal Distribution Function

Z (+)	Probability P_r (%)	Z (–)	Probability (P_r) (%)
0	50.0	0	50.0
+0.1	53.98	–0.1	46.02
+0.2	57.95	–0.2	42.07
+0.3	61.79	–0.3	38.21
+0.4	65.54	–0.4	34.46
+0.5	69.15	–0.5	30.85
+0.6	72.57	–0.6	27.43
+0.7	75.80	–0.7	24.20
+0.8	78.81	–0.8	21.19
+0.9	81.59	–0.9	18.41
+1.0	84.13	–1.0	15.87
+1.1	86.43	–1.1	13.57
+1.2	88.49	–1.2	11.51
+1.3	90.32	–1.3	9.68
+1.4	91.92	–1.4	8.08
+1.5	93.32	–1.5	6.68
+1.6	94.52	–1.6	5.48
+1.7	95.54	–1.7	4.46
+1.8	96.41	–1.8	3.59
+1.9	97.13	–1.9	2.87
+2.1	98.21	–2.1	1.79
+2.2	96.61	–2.2	1.39
+2.3	98.93	–2.3	1.07
+2.4	99.19	–2.4	0.82
+2.5	99.38	–2.5	0.62
+2.6	99.53	–2.6	0.47
+2.7	99.65	–2.7	0.35
+2.8	99.74	–2.8	0.26
+2.9	99.81	–2.9	0.19
+3.0	99.87	–3.0	0.13

Now coming to the problem Number 15.1, given that $T_L = 52$ days.

Critical activities		σ^2
I	J	
10	20	4.00
20	30	0.44
30	70	2.25
70	80	1.00
$\Sigma\sigma^2$		7.69

$$\sqrt{\Sigma\sigma^2} = \sqrt{7.69} = 2.77$$

$$T_L - T_E = 50 - 52 = -2$$

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = -2 / 2.77 = -0.722 = Z = \text{Normal deviate.}$$

–0.722 falls at between probability 22.7. The probability is very low. Hence the manager should not accept to complete the project in 50 days.

Say for example given that $T_L = 58$ days then $58 - 55 = +3$

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = 3 / 2.77 = 1.08 = Z = \text{Normal deviate.}$$

1.08 falls at 85% probability. The probability of completing the project is high. The manager can accept the offer.

Let us continue further discussion on problem no. 15.1
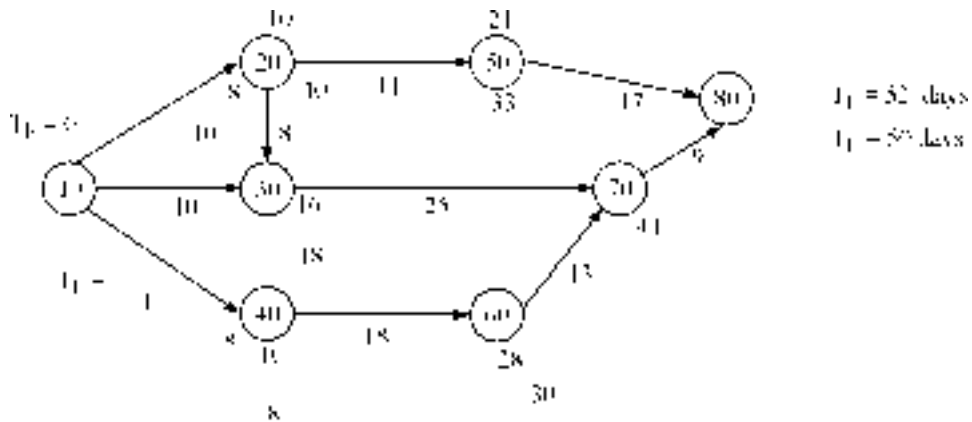
Figure 15.13.

Let us assume that the contractual time = 50 days.

This is written at the end event. Now let us work back to find out when the project should be started if the delivery time is 52 days.

$$T_L{}^{80} = 50 \text{ days}$$

$$T_L{}^{70} = T_L{}^{80} - t_E{}^{70-80} = 50 - 9 = 41 \text{ days this we write below the node}$$

$$T_L{}^{50} = T_L{}^{80} - t_E{}^{50-80} = 50 - 17 = 33 \text{ days}$$

$$T_L{}^{60} = T_L{}^{70} - t_E{}^{60-70} = 41 - 13 = 28 \text{ days}$$
$$T_L{}^{40} = T_L{}^{60} - t_E{}^{40-60} = 28 - 22 = 6 \text{ days}$$
$$T_L{}^{20} = T_L{}^{50} - t_E{}^{20-50} = 33 - 11 = 22 \text{ days}$$
$$T_L{}^{30} = T_L{}^{70} - t_E{}^{30-70} = 41 - 25 = 16 \text{ days}$$
$$T_L{}^{20} = T_L{}^{30} - t_E{}^{20-30} = 16 - 8 = 8 \text{ days}$$

$T_L{}^{20}$ has two values *i.e.* 22 days and 8 days. Here as we are going back to find out when the project is to be started, take lowest of the two *i.e.* $T_L{}^{20}$ = 8 days

$$T_L{}^{10} = T_L{}^{20} - t_E{}^{10-20} = 8 - 12 = -4 \text{ days}$$
$$T_L{}^{10} = T_L{}^{30} - t_E{}^{10-30} = 16 - 10 = 6 \text{ days}$$
$$T_L{}^{10} = T_L{}^{40} - t_E{}^{10-40} = 10 - 11 = -1 \text{ days}$$

Take $T_L = -1$ days which is lowest. Hence the project is to be started 1 day before the scheduled starting time.

Now at the critical events calculate ($T_L - T_E$). For all critical events it is –1 day.

This ($T_L - T_E$) is known as slack and is represented by Greek letter 'τ'. On the critical path τ remains to be same. In fact slack is the breathing time for the contractor. If ($T_L - T_E$), slack for all critical events is zero. If ($T_L > T_E$) it is a positive number and if ($T_L < T_E$) it will be a negative number for all critical events. For non-critical activities this difference between T_L and T_E *i.e.* ($T_L - T_E$) shows the breathing time available to the manager at that activity. For example take the event 50

For this event $T_L = 33$days and $T_E = 21$ days *i.e.* 33 – 21 = 12 days of time available for this manager. In case of any inconvenience he can start the activity 50-80 any day between 21st day and 33rd day. Now let us work out some more examples.

Problem 15. 2.

Steps involved in executing an order for a large engine generator set are given below in a jumbled manner. Arrange them in a logical sequence, draw a PERT network and find the expected execution time period.

Activities (not in logical order)	Time in weeks		
	t_O	t_L	t_P
Order and receive engine	1	2	3
Prepare assembly drawings	1	1	1
Receive and study order	1	2	3
Apply and receive import license for generator	3	5	7
Order and receive generator	2	3	5
Study enquiry for engine generator set	1	2	3
Fabricate switch board	2	3	5
Import engine	1	1	1
Assemble engine generator	1	2	3
Submit quotation with drawing and full	1	2	3
Prepare base and completing	2	3	4
Import generator	1	1	1
Order and receive meters, switch gears for switch board	2	3	4
Test assembly	1	1	1

Solution

As the activities given in the problem are not in logical order, first we have to arrange them in a logical manner.

S.No.	Activities	Time in weeks		
		t_o	t_L	t_P
A	Study enquiry for engine generator set	1	2	3
B	Prepare assembly drawings	1	1	1
C	Submit quotation with drawing and full	1	2	3
D	Receive and study order	1	2	3
E	Apply and receive import license for generator	3	5	7
F	Order and receive engine	1	2	3
G	Order and receive generator	2	3	5
H	Inspect engine	1	1	1
I	Order and receive meters, switch gears for switch board	2	3	4
J	Prepare base	2	3	4
K	Complete assemble engine generator	1	2	3
L	Fabricate switch board	2	3	5
M	Test assembly	1	1	1

Solution
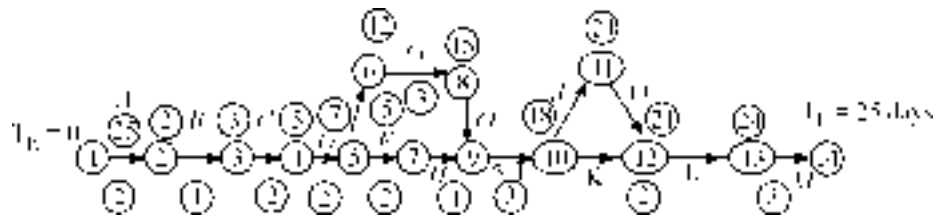
Figure 15.14.

The second step is to write network and number the events

Activities	Predecessor event	Successor Event	Weeks			$t_E = \dfrac{t_O + 4t_L + t_P}{6}$	$\sigma = (t_P - t_O)/6$	σ^2
			t_O	t_L	t_P			
A	1	2	1	2	3	2	1/3 = 0.33	0.102
B	2	3	1	1	1	1	0	0
C	3	4	1	2	3	2	0.33	0.102
D	4	5	1	2	3	2	0.33	0.102
E	4	6	3	5	7	5	0.66	0.44
F	4	7	1	2	3	2	0.33	0.102
G	6	9	2	3	5	3	0.5	0.25
H	7	10	1	1	1	1	0	0
I	5	8	2	3	4	3	0.33	0.102
J	10	11	2	3	4	3	0.33	0.102
K	11	12	1	2	3	2	0.33	0.102
L	8	12	2	3	5	3	0.5	0.25
M	12	13	1	1	1	1	0	0

CRITICAL PATH = 1 – 2 –3 – 4 – 6 – 8 – 9 – 10 – 11 – 12 – 13

T_E = 25 Weeks

Problem 15.3.

A small project is composed of 7 activities whose time estimates are listed below. Activities are being identified by their beginning (i) and ending (j) node numbers.

Activities		Time in weeks		
i	j	t_o	t_l	t_p
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

1. Draw the network
2. Calculate the expected variances for each
3. Find the expected project completed time
4. Calculate the probability that the project will be completed at least 3 weeks than expected
5. If the project due date is 18 weeks, what is the probability of not meeting the due date?

Solution

Activities		Weeks			$t_E = t_E + 4t_L + t_P/6$	t_E	$\sigma = (t_p - t_o)/6$	σ^2
i	j	t_O	t_L	t_P				
1	2	1	1	7	2	6	1	1
1	3	1	4	7	6	6	1	**1**
1	4	2	2	8	3	6	1	1
2	5	1	1	1	1	0	0	0
3	5	2	5	14	6	12	2	**4**
4	6	2	5	8	5	6	1	1
5	6	3	6	15	7	12	2	**4**
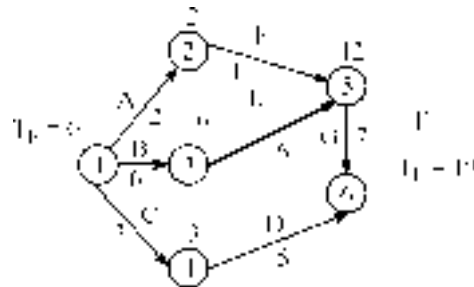
Figure 15.15.

Critical activities	Variance
1 – 3	1
3 - 5	4
5 - 6	4
$\Sigma\sigma^2$	9

$$\sqrt{\Sigma\sigma^2} = \sqrt{9} = 3$$

4. Probability of completing the project at least 3 weeks earlier i.e. 16 in weeks

$$T_L = 16 \text{ weeks}, \ T_E = 19 \text{ weeks}.$$

$$T_L - T_E = -3 \text{ weeks}$$

$$Z = (T_L - T_E) / \sqrt{\Sigma\sigma^2} = -3 / 3 = -1$$

From table the probability of completing the project = 15.9%

5. if $T_L = 18$ weeks. Probability of completing in 11 weeks is $(18 - 19) / 3 = -1/3$

From table the probability = 38.2%

Probability of not meeting due date = 100 – 38.2 = 61.8%

i.e. 61.8% of the time the manager cannot complete the project by due date.

Example 15.4

There are seven activities in a project and the time estimates are as follows

Activities	Time in weeks		
	t_O	t_L	t_P
A	2	6	10
B	4	6	12
C	2	3	4
D	2	4	6
E	3	6	9
F	6	10	14
G	1	3	5

The logical of activities are:

1. Activities *A* and *B* start at the beginning of the project.
2. When *A* is completed *C* and *D* start.
3. *E* can start when *B* and *D* are finished.
4. *F* can start when *B*, *C* and *D* are completed and is the final activity.
5. *G* can start when *F* is finished and is final activity the.

(*a*) What is the expected time of the duration of the project?

(*b*) What is the probability that project will be completed in 22 weeks?

Solution

First we use to establish predecessor and successor relationship and then find standard deviation σ, variance σ^2 and expected time of completing activities, t_E.

| Activities | Predecessor Event | Weeks | | | $t_E =$ $t_O + 4t_L + t_P / 6$ | $\sigma =$ $(t_P - t_O)/6$ | σ^2 |
		t_O	t_L	t_P			
A	–	2	6	10	6	8/6 = 1.33	1.77
B	–	4	6	12	10	8/6 = 1.33	1.77
C	A	2	3	4	3	2/6 = 0.33	0.11
D	A	2	4	6	4	4/6 = 0.66	0.44
E	B, D	3	6	9	5		1.
F	B, C, D	6	10	14	10	8/6 = 1.33	1.77
G	F	1	3	5	3	4/6=0.66	0.44

Now to write network the logical (predecessor) relationship is considered.
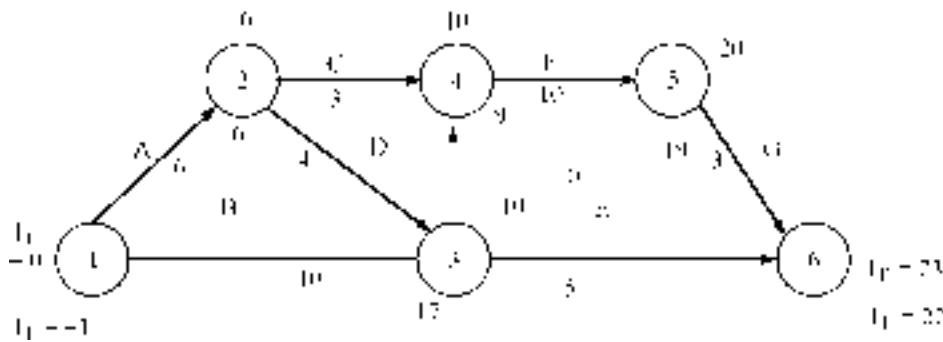
Figure. 15.16.

After writing the network, numbering of events and t_E is entered on the network. Next the project completion time is worked out. The project completion time $T_E = 23$ weeks. This project has two critical paths i.e. $A - D - F - G$ and $B - F - G$.

(1)

Critical path	Variance σ^2
A	1.77
D	0.44
F	1.77
G	0.44
$\Sigma\sigma^2$	4.42

$$\sqrt{\Sigma\sigma^2} = \sqrt{4.42} = 2.10$$

(2)

Critical path	Variance σ^2
B	1.77
F	1.77
G	0.44
$\Sigma\sigma^2$	3.98

$$\sqrt{\Sigma\sigma^2} = \sqrt{3.98} = 1.99$$

In the problem T_L is given as 22 weeks. Therefore $T_L - T_E = 22 - 23 = -1$

Therefore probability of completing the project in 22 weeks

$$-1 / 2.10 = -0.476 \text{ OR } -1 / 1.99 = 0.502$$

The probability of completing the project is approximately 49%.

15.3. CRITICAL PATH METHOD (CPM) FOR CALCULATING PROJECT COMPLETION TIME

In critical path method, the time duration of activity is deterministic in nature *i.e.* there will be a single time, rather than three time estimates as in PERT networks. The network is activity oriented. The three ways in which the CPM type of networks differ from PERT networks are

CPM		PERT	
(*a*)	Network is constructed on the basis of jobs or activities (activity oriented).	(*a*)	Network is constructed basing on the events (event oriented)
(*b*)	CPM does not take uncertainties involved in the estimation of times. The time required is deterministic and hence only one time is considered.	(*b*)	PERT network deals with uncertainties and hence three time estimations are considered (Optimistic Time, Most Likely Time and Pessimistic Time)
(*c*)	CPM times are related to cost. That is can be by decreasing the activity duration direct costs increased (crashing of activity duration is possible)	(*c*)	As there is no certainty of time, activity duration cannot be reduced. Hence cost cannot be expressed correctly. We can say expected cost of completion of activity (crashing of activity duration is not possible)

15.3.1. Writing the CPM Network

First, one has to establish the logical relationship between activities. That is predecessor and successor relationship, which activity is to be started after a certain activity. By means of problems let us see how to deal with CPM network and the calculations needed.

Problem 15.5.

A company manufacturing plant and equipment for chemical processing is in the process of quoting tender called by public sector undertaking. Help the manager to find the project completion time to participate in the tender.

S.No.	*Activities*		*Days*
1	A	–	3
2	B	–	4
3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3
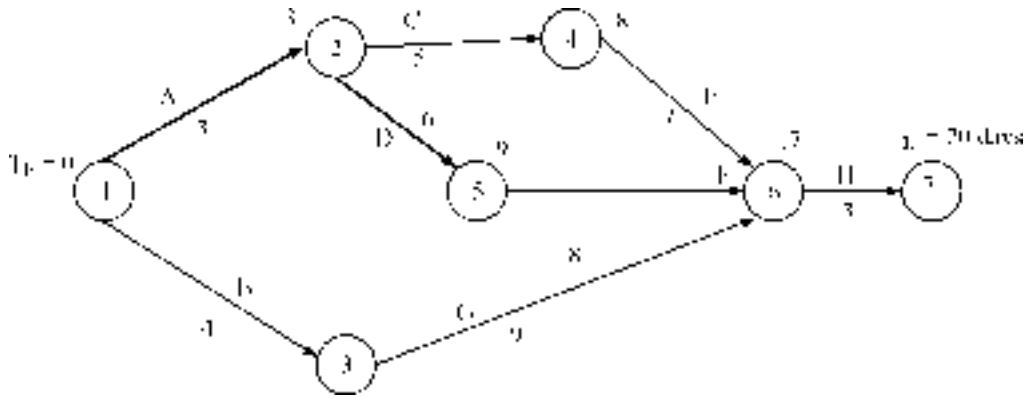
Figure 15.17.

(1) Write the network referring to the data

(2) Number the events as discussed earlier.

(3) Calculate T_E as done in PERT network $T_E^j = (T_E^i + T_E^{ij})]$

(4) Identify the critical path

Project completion time = 20 weeks and the critical path $= A - D - F - H.$

Problem 15. 6.

A small project has 7 activities and the time in days for each activity is given below:

Activity	Duration in days
A	6
B	8
C	3
D	4
E	6
F	10
G	3

Given that activities A and B can start at the beginning of the project. When A is completed C and D can start. E can start only when B and D are finished. F can start when B, C and D are completed and is the final activity. G can start when E is finished and is the final activity. Draw the network and find the project completion time.

Activity	Immediate predecessor	Time in days
A	–	6
B	–	8
C	A	3
D	A	4
E	B , D	6
F	B, C or D	10
G	E	3

Draw the network and enter the times and find T_E .

Solution
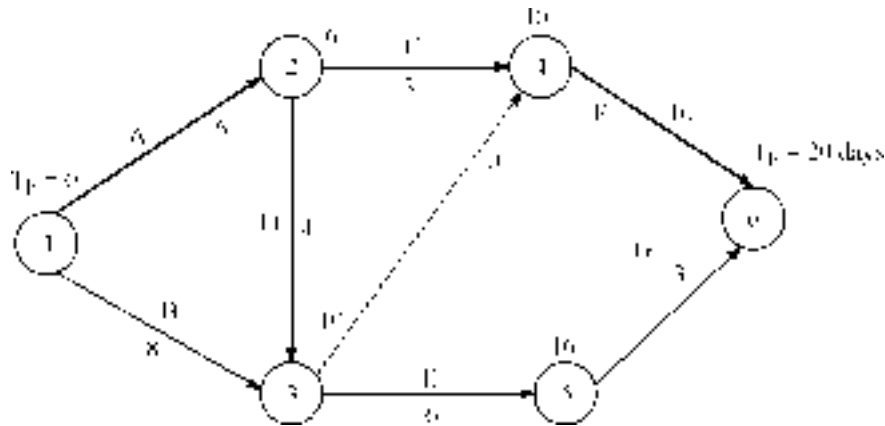
Figure 15.18

Project completion time = 20 days and critical path is $A - D - F$.

15.3.2. Time Estimation in CPM

Once the network is drawn the nextwork is to number the events and enter the time duration of each activity and then to calculate the project completion time. As we know, the CPM activities have single time estimates, and no uncertainties are concerned, the system is deterministic in nature. While dealing with CPM networks, we came across the following times.

(a) **Earliest Event Time:** We have defined event as either starting or ending of an activity. Earliest event time means what is the earliest time by which that event occurs. Let us consider a small example to understand this. Consider the figure 6.20.

Figure 15.19

In figure, the network has two activities A and B. Activity A i.e. 'ij' is predecessor to activity B i.e. activity jk. The time taken by activity A is t_E^{ij} and that of B is t_E^{jk}. If the event 'i' occurs at time 0, then event 'j' occurs at the earliest at $0 + t_E^{ij}$ i.e. $t_E^i + t_E^{ij} = t_E^j$ and the earliest time by which event 'k' occurs is $T_E^K = t_E^j + t_E^{jk}$. But when various lines as shown in the figure 15.23 connect a node, the procedure is as follows.
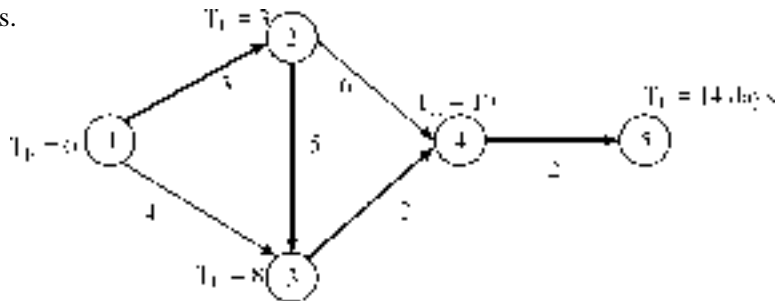
Figure 15.20

Event 3 is having two routes $1 - 2 - 3$ and $1 - 3$

$$T_E^1 = 0$$
$$T_E^2 = T_E^1 + t_E^{12} = 0 + 3 = 3$$
$$T_E^3 = T_E^1 + t_E^{13} = 0 + 4 = 4 \text{ also } T_E^3 = T_E^2 + t_E^{23} = 3 + 5 = 8$$

As the rule says that event 3 occurs only after the completion of activities 1-4 and 2-3. Activity 1-3 ends on 4th day and event 2-3 ends on 8th day. Hence event 3 occurs on 8th day. This means the formula for finding T_E is

$$T_E^j = (T_E^i + t_E^{ij})_{\max}$$

When the event has more routes, we have to calculate T_E for all routes and take the maximum of all the routes.

Problem 15. 7.

Find the slack of each event
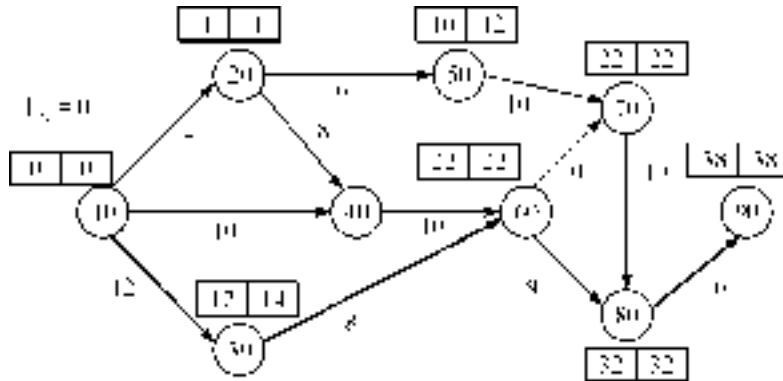
Figure 15.21

$T_E^{90} = 38$ days, $T_L^{90} = 38$ days

Critical path $= 10 – 20 – 40 – 60 – 70 – 80 – 90$

Event 'i'	Event 'j'	Duration T^{ij} days	$t_E^j = t_E^i + t^{ij}$ ↑	T_E^j	T_L^i ↓	T_L^j	Slack $= T = T_L - T_E$
90	80	6	**38**	38	**32**	38	0
80	70	10	**32**	32	**22**	32	0
80	60	8	30	32	24	32	0
70	60	0	**22**	22	**22**	22	0
70	50	10	20	22	**12**	22	0
60	40	10	**22**	22	**12**	22	0
60	30	8	20	22	14	22	0
50	20	6	**10**	10	6	12	+2
40	20	8	**12**	12	4	12	0
40	10	10	10	12	2	12	0
30	10	12	12	12	2	14	+2
20	10	4	4	**4**	**0**	4	0

Thick numbers   Thin numbers
are maximums   are minimums

15.3.3. Latest Allowable Occurrence Time

The next one is the Latest Allowable Occurrence time represented by $T_L{}^i$. This is illustrated by a simple example.
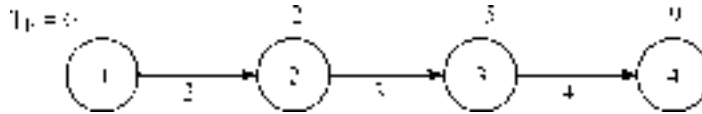
Figure 15.22

Earliest occurrence time of event 4 = 9 days. As the activities 3 - 4 take 4 days, the latest time by which activity starts is $T_L{}^4 - t_E{}^{3-4} = 9 - 4 = $ 5th day, which is also the Earliest Occurrence time of event 3. Similarly, Latest Time by which event 2 occurs in $T_L{}^3 - t_E{}^{2-3} = 5 - 3 = 2$ and so on. If a node is connected by number of paths then we have to find Latest Allowable Occurrence time as discussed below.

Consider the figure given below:
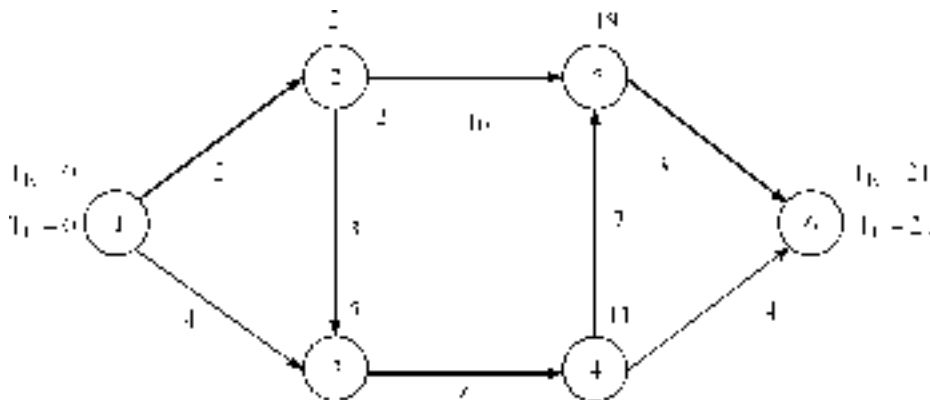
Figure 15.23

The earliest occurrence time of event 6 is 21 days. As activities 4 – 6 take 4 days the earliest occurrence time is

$T_L{}^6 - t_E{}^{46} = 21 - 4 = 17$ days. But there is another route 6 – 5 – 4. If we consider this route $T_L{}^5 = T_L{}^6 - t_E{}^{56} = 21 - 3 = $ 18th day, $T_E{}^4 = T_E{}^5 - t_E{}^{54} = 18 - 7 = 11$ days. As the latest allowable occurrence time for event 4 is 17th day and 11th day, the event 4 will not occur until activities 6 – 4 and 6 –5 are completed. As 11th day is the smallest, the event 4 occurs on 11th day. Hence the formula for $T_L{}^i = (T_L{}^i - t_E{}^{ij})_{\text{minimum}}$.

15.3.4. Slack time

Slack time is difference between latest event and the earliest event time *i.e.* $T_L{}^i - t_E{}^i = \tau^i$

Float: Now let us define the times for activities $i - j$.

(*i*) Earliest start time: - This is the earliest occurrence time for the event from which the activity arrow originates and is represented by $T_E{}^i$

(*ii*) Earliest finish time: - This is the earliest occurrence time of the event from which the activity arrow originates plus the duration of the activity. $T_E{}^i + t_E{}^{ij}$

(*iii*) Latest start time:- This is the latest occurrence time for the node at which the activity arrow terminates minus the duration of activity *i.e.* $T_L{}^j - t_E{}^{ij}$

(*iv*) Latest finish time:- This is the latest occurrence time for the node at which the activity arrow terminates, represented by $T_L{}^j$

(*v*) Maximum time available for activity is $T_L{}^j - T_E{}^i$

(*vi*) Total float: - If the job $i - j$ requires time t^{ij} units, the actual float for jobs $i - j$ is the difference between the maximum time available for the job and the actual time.

$$\text{Total float for } i - j \quad = \quad (T_L{}^j - t_E{}^i) - t^{ij}$$
$$= \quad (T_L{}^j - t^{ij}) - T_E{}^i$$

This is the latest time for the activity minus the earliest start time.

(*vii*) Free float: - Free float for an activity is based on the possibility that all events occur at their earliest times that means all activities start as early as possible. If you have two activities $i - j$ and $j - k$ *i.e.*, activity $j - k$ is a successor activity to activity $i - j$

Let $T_E{}^i$ = Earliest Occurrence Time for event '*i*'.

$T_E{}^j$ = Earliest Occurrence Time for event '*j*'.

This means that the earliest possible start time for activity $i - j$ is $T_E{}^i$ and for the activity $j - k$ is $T_E{}^j$. Let the activity duration be $t_E{}^{ij}$. In case $T_E{}^j$ is greater than $T_E{}^i + t_E{}^{ij}$ activity $j - k$ cannot start until $T_E{}^j$. The difference between $T_E{}^j - (T_E{}^i + t_E{}^{ij})$ is known as Free Float. Therefore, Free Float for activity $i - j = T_E{}^j - (T_E{}^i + t_E{}^{ij})$. But $(T_E{}^i + t_E{}^{ij})$ is earliest finish time for activity $i - j$. Hence free float $= T_E{}^j - $ Earliest Finish Time of $i - j$. Free float for activity $i - j$ is the difference between its Earliest Finish Time and Earliest Start time of its successor activity.
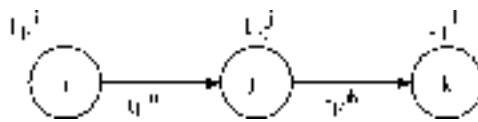
Figure 15.24

(*viii*) Another type of float is "Independent Float". Referring to figure 15.25. Consider the activity i–j. Activity $h - i$ is predecessor to $i - j$ and activity $j - k$ is a successor to activity $i - j$ and $T_L{}^i$ is to latest finish time of activity $h - i$.
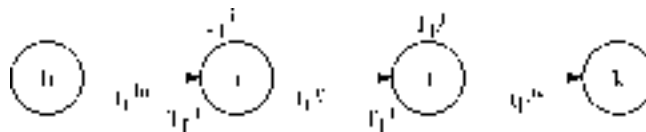
Figure 15.25

And activity *j-k* starts at the earliest possible moment i.e. $T_E{}^j$. This means activity $i - j$ can take time duration between $T_E{}^i$ to $T_E{}^j - T_L{}^i$, without affecting the networks. The difference between the $T_E{}^j - T_L{}^i$ and $t_E{}^{ij}$ is known as Independent Float.

Independent float for $i - j = (T_E{}^j - T_L{}^i) - t_E{}^{ij}$

(*ix*) Another type of float is Interference Float. Interference Float is the difference between Total Float and the Free Float. In fact it is head event slack.

Interference float = Total Float – Free Float

$$F_{IT} = F_T - F_F$$
$$F_T = (T_L{}^j - T_E{}^j) - t_E{}^{ij}$$
$$F_F = (T_E{}^j - T_E{}^i) - t_E{}^{ij}$$
$$F_{IT} = (T_L{}^j - T_E{}^i - t_E{}^{ij}) - (T_E{}^j - T_E{}^i - t_E{}^{ij})$$
$$F_{IT} = (T_L{}^j - T_E{}^j) = \text{Head event slack.}$$

Summary of float

S.No.	Type of float		Formulae
1.	Total float (FT)	Excess of maximum available time over the activity time.	$F_T = (T_L{}^j - T_E{}^j) - t_E{}^{ij}$
2.	Free float (FF)	Excess of available time over the activity time when all jobs start as early as possible.	$F_F = (T_E{}^j - T_E{}^i) - t_E{}^{ij}$
3.	Independent float F_{ID}	Excess of maximum available time over the activity time.	$F_{ID} = (T_E{}^j - T_L{}^i) - t_E{}^{ij}$
4.	Interfering float (F_{IT})	Difference between total float and free float	$F_{iT} = F_T - F_F$

15.4. PROJECT COST ANALYSIS

So far we have dealt with how to find project completion time in PERT and CPM networks. In CPM network, when the time required by an activity is deterministic in nature, we may come across a situation that we may have to reduce the activity duration. This is not possible in PERT activity; because activity duration is probabilistic in nature and we have three time estimates. Which time (either t_O, t_L or t_P) is to be reduced is a question. Hence activity time crashing is possible in critical path network only.

Before crashing the activity duration, we must understand the costs associated with an activity.

15.4.1. Direct Cost

Direct costs are the costs that can be identified with activity. For example, labour costs, material cost etc. When an activity whose duration is to be reduced (crashed), we have to supply extra resources, specially manpower. Let us say an activity takes 7 days with 2 men. If 4 men works it can be done in 4 days. The cost of 2 workmen increases. As we go on reducing the activity time, cost goes on increasing as shown in figure 15.26.
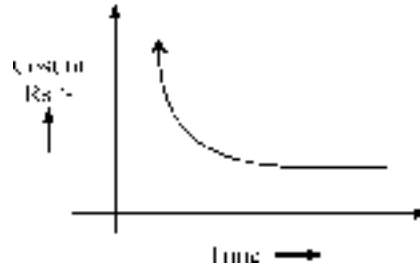
Figure 15.26 Direct Cost.

15.4.2. Indirect Cost

These are the costs, which cannot be identified with the activity. Say the salary of a manager, who is in-charge of many projects. Exact amount of his salary that should be charged to a particular project cannot be estimated correctly as it is very difficult to say how much time he has spent on each project. We can express all indirect costs put together in terms of an amount per time period, for example say Rs. 100/- per day, as the indirect costs are expressed as so much of amount per time period, as the duration of project goes on reducing the indirect cost also goes on decreasing as in figure 15.27.
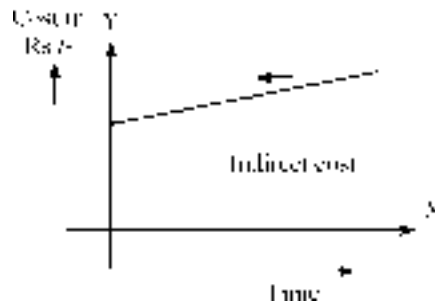
Figure 15.27 Indirect Cost.

15.4.3. Total cost

The total cost which is the sum of direct cost and indirect cost is shown in figure 15.28. As the project duration goes on reducing the total cost reduces from *B* to *C* and if duration is still crashed the total cost increases to *A*. Hence our problem here is to find out the optimal duration of the project and optimal cost.
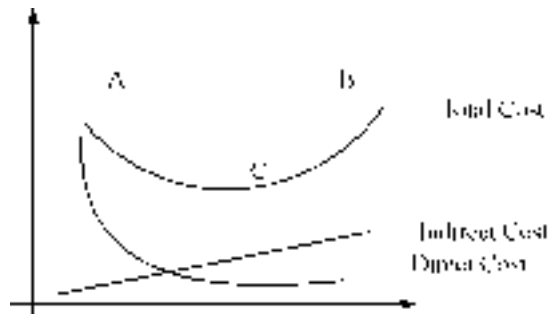
Figure15.28 Total cost curve.

15.4.4. Cost Slope

Consider a small portion of total cost curve and enlarge it. It appears like a straight line as shown in figure 15.29.
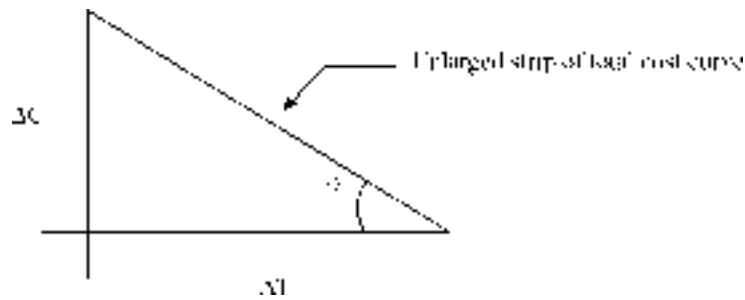
Figure 15.29 Cost Slope.

If θ is the inclination then tan $\theta = \Delta C / \Delta t$. This indicates how much cost increases for crashing a unit of time period.

In other words cost slope is the slope of the direct cost curve, approximated as a straight line. It is given by

$$\text{Cost Slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{\Delta C}{\Delta t}$$

Where ΔC = increase in cost, Δt = is decrease in time.

Problem 15.8.

A project consists of 4 activities. Their logical relationship and time taken is given along with crash time and cost details. If the indirect cost is Rs. 2000/- per week, find the optimal duration and optimal cost.

Activity	Predecessor	Normal		Crash	
		Time in days	Cost in Rs/-	Time in days	Cost in Rs/-
A	-	4	4,000	2	12,000
B	A	5	3,000	2	7,500
C	A	7	3,600	5	6,000
D	B	4	5,000	2	10,000
		TOTAL	15,600		35,500

Solution

Slopes

(1) Find ΔC = Crash cost – Normal cost
(2) Find Δt = Normal time – Crash time
(3) Find $\Delta C / \Delta t$ = cost slope.
(4) Identify the critical path and underline the cost slopes of the critical activities.

(5) As the direct cost increases and indirect cost reduces, crash such activities whose cost slopes are less than the indirect cost given.

(6) Select the lowest cost slope and crash it first, then next highest and so on.

(7) Do not crash activities on non-critical path until they become critical activities in the process of crashing.

(8) In case any non-critical activity becomes critical activity at the time of crashing consider the cost slopes of both the critical activities, which have same time span and the costs slopes of both activities.

(9) Crashing should be continued until the cost slope becomes greater than the indirect cost.

(10) Do not crash such activities whose cost slope is greater than the indirect cost.

(11) Crashing is done on a graph sheet with squared network drawn to scale.

Activity	Predecessor	Normal		Crash		ΔC	Δt	$\dfrac{\Delta C}{\Delta t}$
		Time in days	Cost in Rs./-	Time in days	Cost in Rs./-			
A	-	4	4,000	2	12,000	8,000	2	**4,000**
B	A	5	3,000	2	7,500	4,500	3	**1,500**
C	A	7	3,600	5	6,000	2,400	2	1,200
D	B	4	5,000	2	10,000	5,000	2	**2,500**
		TOTAL	15,600		35,500			
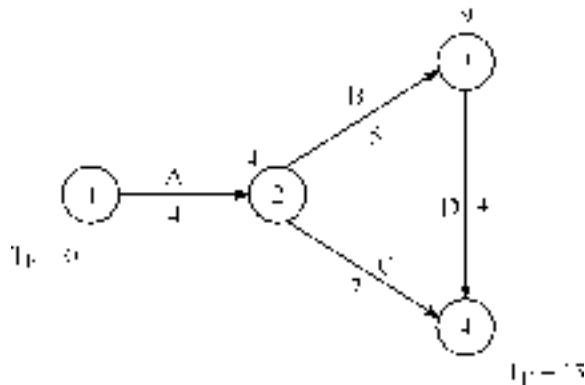
Figure 15.30

Now activities *A, B* and *D* are critical activities. Activity *B* is the only activity whose cost slope is less than indirect cost. Hence we can crash only activity *B*. For crashing we have to write the squared network. While writing squared network critical activities are shown on a horizontal line and non-critical activities are shown as in the figure *i.e.* above and / or below the critical path as the case may be. That is non-critical paths above critical path are shown above vice versa.

Though the activity *B* can be crashed by 3 days, only 2 days are crashed because after crashing 2 days at 11th day, activity 2-4 (C) also becomes critical activity. At this stage if we want to crash one more day we have to crash activity 2–4 *i.e.* C also along with 2 – 3. Now the cost slopes of activities *B* and *C* are to be considered which will be greater than indirect cost. Hence no crashing can be done. 11 days is the optimal period and optimal cost is Rs. 39, 100/-.
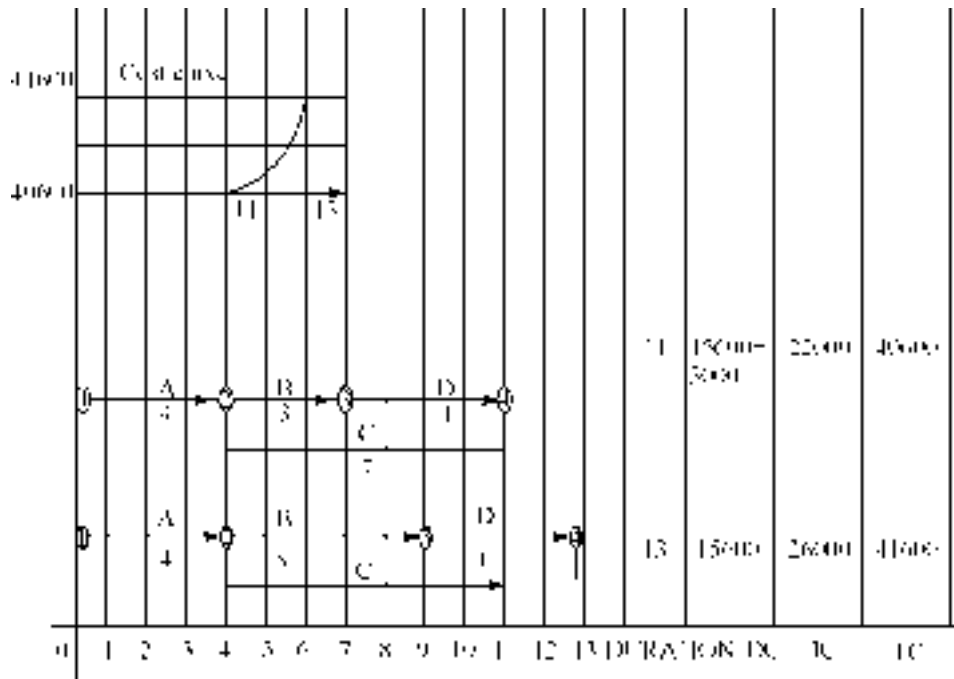
Figure 15.31

Problem 15.9.

(*a*) A maintenance project has following estimates of times in hours and cost in rupees for jobs. Assuming that jobs can be done either at normal or at fast pace, but not any pace in between. Plot the relationship between project completion time and minimum project cost.

(*b*) Assuring a relationship between job duration and job cost and with overhead cost of Rs. 25/- per hour, plot the cost – time relationship.

Jobs	Predecessor	Normal		Crash	
		Time in hrs	Cost in Rs/-	Time in hrs	Cost in Rs/-
A	-	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B, C, D	6	50	6	50
		TOTAL	300		530

Solution
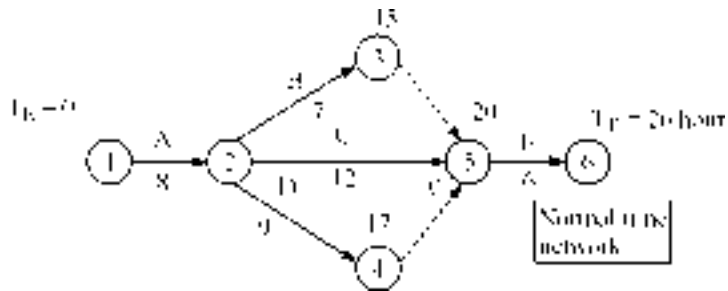
Figure 15.32.
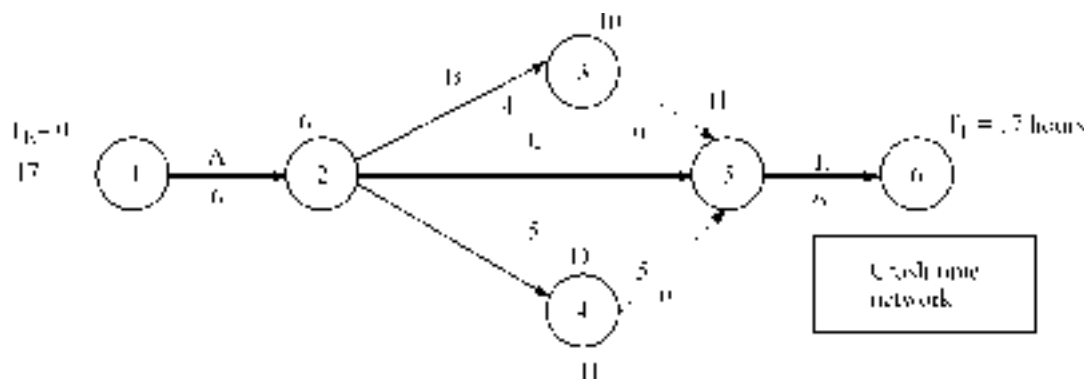
Figure 15.33.
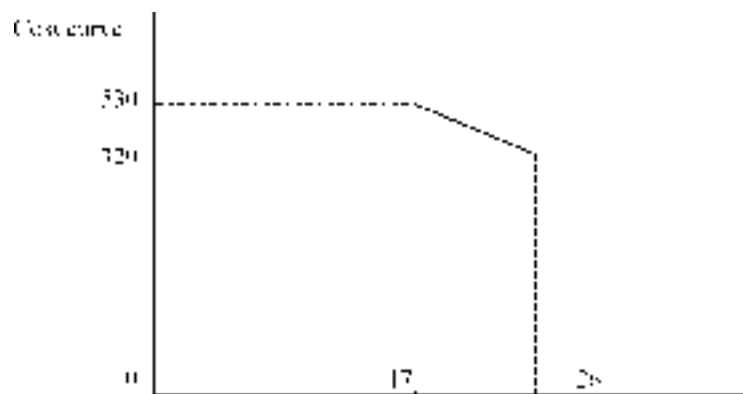
Figure 15.34.

Indirect cost = Rs. 25/- per hour

Jobs	Predecessor	Normal		Crash		ΔC	Δt	$\dfrac{\Delta C}{\Delta t}$
		Time in hrs	Cost in Rs./-	Time in hrs	Cost in Rs./-			
A	-	8	80	6	100	2	20	10
B	A	7	40	4	94	3	54	18
C	A	12	100	5	184	7	84	12
D	A	9	70	5	102	4	32	8
E	B, C, D	6	50	6	50	-	-	-
		Total	300		530			
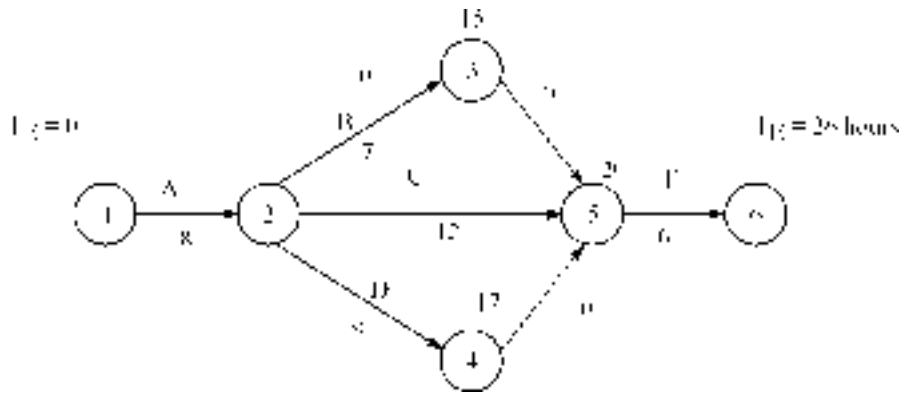
Figure 15.35
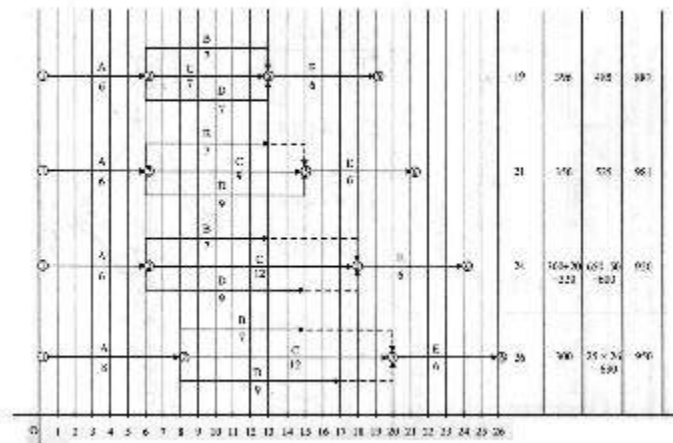
Figure 15.36 (a) Squared Network for Problem 6.8

(*a*) Figure 15.36 (*a*) shows the squared network.

(*b*) As critical activity A has got cost slope of Rs. 10/-, which is less than the indirect cost it is crashed by 2 days.

Hence duration is 24 hrs.

Direct Cost   = Rs. 300 + 2 × 10 = Rs. 320

Indirect Cost   = Rs. 650 – 2 × 25 = Rs. 600

   Total Cost   = Rs. 920

(*c*) Next, critical activity *C* has got a cost slope 12, which is less than 25. This is crashed by 3 days, though it can be crashed 7 days. This is because, if we crash further, activity *D* becomes critical activity, hence its cost slope also to be considered.

Duration is 21 hrs.

Direct Cost   = Rs. 320 + 3 × 12 = Rs. 356

Indirect Cost   = Rs. 600 – 3 × 25 = Rs. 525

   Total Cost   = Rs. 881 (Rs. 356 + Rs. 525)

(*d*) Now cost slope of activities *C* and *D* put together = Rs.12 + 8 = Rs. 20 which is less than Rs. 25/-, indirect cost, both are crashed by 2 hrs

Duration is 19 hrs.

Direct Cost   = Rs. 356 + 2 × 20 = Rs. 396

Indirect Cost   = Rs. 525 – 2 × 25 = Rs. 475

   Total Cost   = Rs. 871 (Rs. 396 + 475)

As we see from the network, no further crashing can be done. Optimal time = 19 hrs and optimal cost = Rs. 871/-
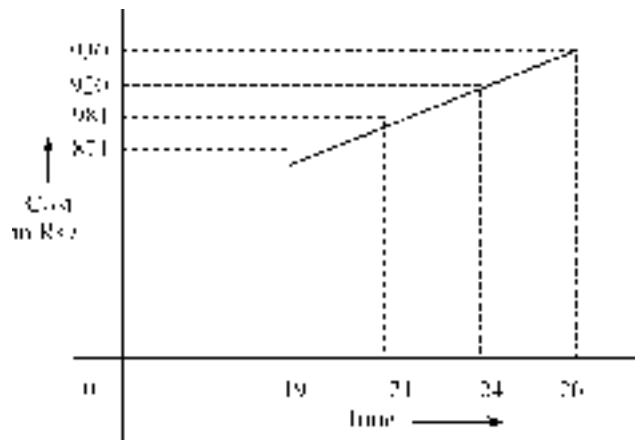
Figure 15.37

Problem 15.9.

The following details pertain to a job, which is to be scheduled to optimal cost.

| Jobs | Predecessor | Normal | | Crash | | ΔC | Δt | $\dfrac{\Delta C}{\Delta t}$ |
		Timein hrs	Cost in Rs./-	Time in hrs	Cost in Rs./-			
A	–	3	1,400	2	2,100	700	1	700
B	C	6	2,150	5	2,750	600	1	600
C	–	2	1,600	1	2,400	800	1	800
D	A, B	4	1,300	3	1,800	500	1	500
E	C	2	1,700	1	2,500	800	1	800
F	D	7	1,650	4	2,850	400	3	133
G	E, F	4	2,100	3	2,900	800	1	800
H	D	3	1,100	2	1,800	500	1	500
		TOTAL	13,000		18,900			

We can enter Δt in last column.

Assume that indirect cost is Rs. 1100/- per day. Draw least cost schedule. The related network is shown below:
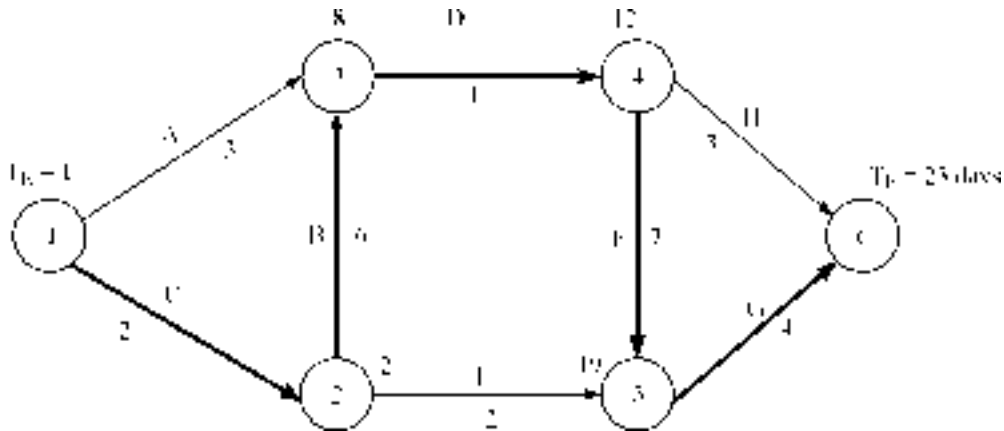
Figure 15.38

Project completion time is 23 days. Critical path is $C - B - D - F - G$. Lowest cost slope is Rs. 133 for critical activity F, this can be crashed by 3 days.

Next cost slope is Rs.500/- for activity D. This can be crashed by 1 day.

Next cost slope is Rs.600/- for activity B. This can be crashed by 1 day. Next lower cost slope is Rs. 800/- for critical activities C and G. C can be crashed by 1 day and G can be crashed by 1 day.

All critical activities have been crashed and non-critical activities have slack. Hence they are not to be crashed. Hence optimal cost is Rs. 33699 and optimal time is 16 days.
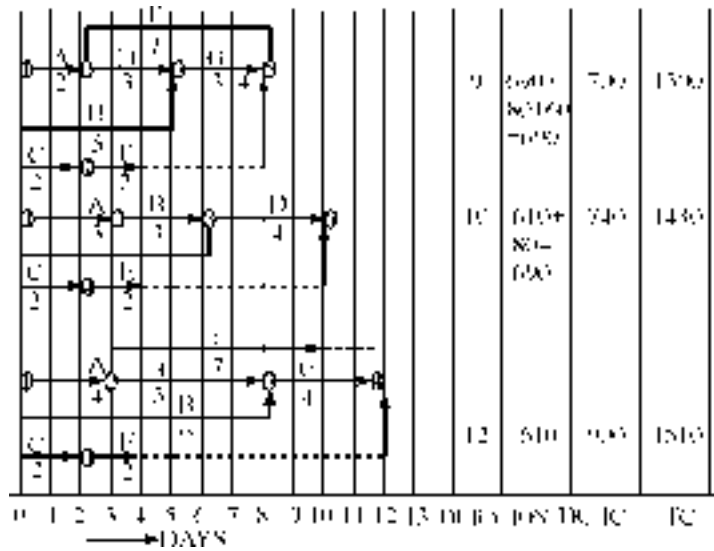
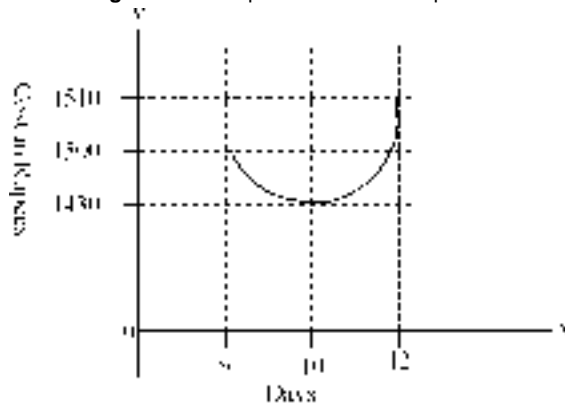**Figure 15.39** Squared network for problem 15.9

Figure 15.40

Problem 15.10.

Given below are network data and time–cost trade off data for small maintenance work.

| Jobs | Predecessor | Normal | | Crash | Cost slope |
		Time in hrs	Cost in Rs./-	Time in hrs	Rs. / day ΔC
A	–	3	50	2	**50**
B	–	6	140	4	60
C	–	2	50	1	30
D	A	5	100	2	**40**
E	C	2	55	2	–
F	A	7	115	5	30
G	B, D	4	100	2	**70**
		TOTAL	610		

Assume that the indirect cost including the cost of lost production and associated costs to be as given below:

Project duration in days	12	11	10	9	8	7
Indirect cost in Rs./-	900	820	740	700	660	620

Work out the minimum total cost for various project duration and suggest the duration for minimum total cost.
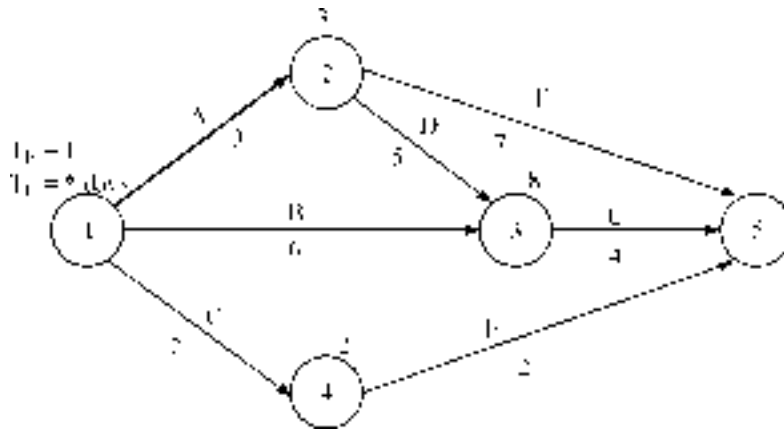
Figure 15.41.

$A - D - G$ = critical path, $T_E = 12$ days.

(1) Activity D has lowest cost slope. It can be crashed by 3 days. It is less than the cost slope for 12 days *i.e.* Rs. 980/-. By crashing activity D by 3 days, activities F and B have become critical activities.

(2) Next we can crash activity A, whose cost slope is Rs. 50/-, it can be crashed by 1 day, but we have to crash activity B also along with A.

Total Cost slope = Rs. 110/- per day. Here the total cost increases. Hence optimal duration is 10 days and optimal cost is Rs. 1430/-.
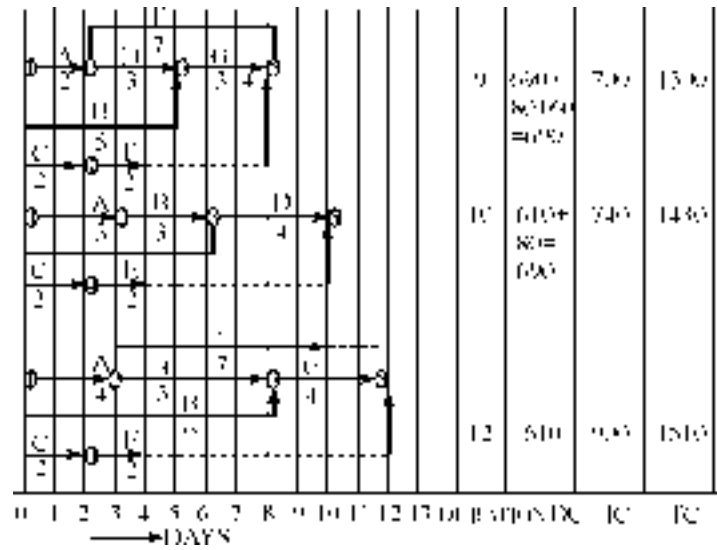
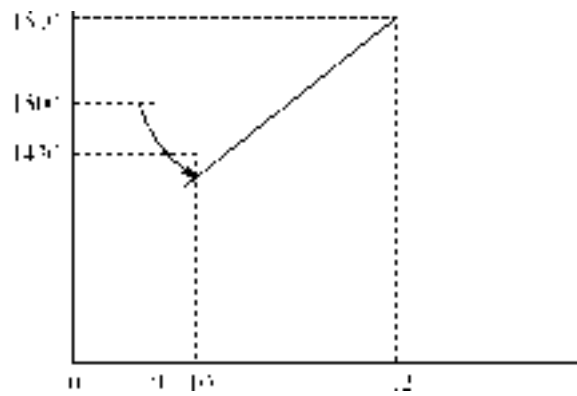**Figure 15. 42** Squared network for problem 15.10

Figure 15.43

Before concluding this chapter, it is better to introduce the students to further developments or advanced topics in network techniques.

1. **Updating the network:** In large project works, as the project progresses, we may come across situation like

   (*a*) The time estimates made before may be wrong that a particular activity may take less or more time. And we may also sense that we have forgotten certain activities. In such cases, we have to **update the project**

   Leaving the executed activities, remaining activities may have to be modified and the remaining network is redrawn. This is known as updating the network.

2. **Resource leveling and Resource smoothing:** When we have to manage project with available resources, we have two options. First one is resource leveling. Here when the resources availability is less than the maximum resources required for an activity, then delay the job having largest float and divert the resources to critical activities. When two or more jobs compete for same resource, first try to allocate to an activity, which is of short duration and next to the activity which having next highest duration. Here available resource is **a constraint**. The **project duration time may increase** during the process.

3. **Resource smoothing:** Here total project duration is **maintained to the minimum level**. By shifting the activities having floats the demand for resources are smoothened. Here main **constraint is project duration time**.

1.0   INTRODUCTION

Monte Carlo (MC) approach to analysis was developed in the 1940's, it is a computer based analytical method which employs statistical sampling techniques for obtaining a probabilistic approximation to the solution of a mathematical equation or model by utilizing sequences of random numbers as inputs into a model which yields results that are indications of the performance of the developed model.

Monte Carlo simulation was developed as part of the atomic program. Scientist at the Los Alamos National Laboratory originally used it to model the random diffusion of neutrons. The scientist who developed this simulation technique gave it the name 'Monte Carlo" after the city in Monaco and its many casinos. Monte Carlo simulation are used in a wide array of applications, including physics, finance, and system reliability

Monte Carlo analysis utilizes statistical tools to mathematically model a real life system or process and then it estimates the probability of obtaining a successful outcome. The statistical distribution of the process to be modeled must be determined first before Monte Carlo simulation can be applied.

Monte Carlo entails using random numbers as a tool to compute something that is not random. MC simulation is a versatile tool to analyze and evaluate complex measurements using a model of a system, and to experiment with the model towards drawing inferences of the system's behavior.

QUESTIONS TO ASK WHEN CONSIDERING USING MONTE CARLO
1. Consider the problem. Does it have many sources of uncertainty?
2. Is there an analytical solution? If so, use that.
3. Choose the software you will use for Monte Carlo simulation.
4. Decide on the nature and source of inputs to use
5. What distribution do the random variables have?
6. How do we generate these random variables for the simulation?
7. How do we analyze the output of the simulation?
8. How many simulation runs do we need?
9. How do we improve the efficiency of the simulation?
10. How do we interpret the generated result?

ATTRIBUTES OF MONTE CARLO SIMULATION

There is no single Monte Carlo method—the term covers a wide range of approaches to simulation. However, these approaches use a certain pattern in which:

1. A domain of possible inputs is defined;
2. Inputs are randomly generated from the domain;
3. Using the inputs, a deterministic computation is performed;
4. The results are aggregated from the individual computations to give a final result

2.0   MONTE CARLO ANALYSIS

PROCEDURE FOR APPLYING MONTE CARLO

1. Determine the pseudo-population or model that represents the true population of interest.
2. Use a sampling procedure to sample from the pseudo-population.
3. Calculate a value for the statistic of interest and store it.
4. Repeat steps 2 and 3 for N trials.
5. Use the N values found in step 4 to study the distribution of the statistic.

It is often asked that when do we simulate? and when do we approach a problem using analytical methods? The catch is this '*Calculate when you can, simulate when you can't*!"

One favorable feature of Monte Carlo is that it is possible to estimate the order of magnitude of statistical error, which is the dominant error in most Monte Carlo computations. These estimates are often called *error bars* because of the way they are indicated on plots of Monte Carlo results. Monte Carlo error bars are essentially statistical confidence intervals. At times algorithm used for MC simulations could result in wide variations in output for different computations and this necessitates developing better algorithms. The search for more accurate alternative algorithms is often called *variance reduction*.

ADVANTAGES

1. Using Monte Carlo simulation is quite straightforward.

2. It can provide statistical sampling for numerical experiments using a computer.

3. In optimization problems, Monte Carlo simulation can often reach the optimum and overcome local extremes.

4. It provides approximate solutions to many mathematical problems.

5. Monte Carlo analysis produces a narrower range of results than a "what if" analysis.

DISADVANTAGES

1. Monte Carlo simulation is not universally accepted in simulating a system that is not in equilibrium (i.e. in a transient state).

2. A large number of samples are required to reach the desired results. This can be time-consuming compared to using a spreadsheet program, such as Excel, which can generate a simple calculation fairly quickly.

3. A single sample cannot be used in simulation; to obtain results there must be many samples.

4. The results are only an approximation of the true value.

5. Simulation results can show large variance.

6. It may be very expensive to build simulation

7. It is easy to misuse simulation by stretching it beyond the limits of credibility

Simulation

13.1. INTRODUCTION

Simulation is the most important technique used in analyzing a number of complex systems where the methods discussed in previous chapters are not adequate. There are many real world problems which cannot be represented by a mathematical model due to stochastic nature of the problem, the complexity in problem formulation and many values of the variables are not known in advance and there is no easy way to find these values.

Simulation has become an important tool for tackling the complicated problem of managerial decision-making. Simulation determines the effect of a number of alternate policies without disturbing the real system. Recent advances in simulation methodologies, technical development and software availability have made simulation as one of the most widely and popularly accepted tool in Operation Research. Simulation is a quantitative technique that utilizes a computerized mathematical model in order to represent actual decision-making under conditions of uncertainty for evaluating alternative courses of action based upon facts and assumptions.

John Von Newmann and Stainslaw Ulam made first important application of simulation for determining the complicated behaviour of neutrons in a nuclear shielding problem, being too complicated for mathematical analysis. After the remarkable success of this technique on neutron problem, it has become more popular and has many applications in business and industry. The development of digital computers has further increased the rapid progress in the simulation technique.

Designers and analysts have long used the techniques of simulation by physical sciences. Simulation is the representative model of real situation. Fore example, in a city, a children's park with various signals and crossing is a simulated model of city traffic. A planetarium is a simulated model of the solar system. In laboratories we perform a number of experiments on simulated model to predict the behaviours of the real system under true environment. For training a pilot, flight simulators are used. The simulator under the control of computers gives the same readings as the pilot gets in real flight. The trainee can intervene when there is signal, like engine failure etc. Simulation is the process of generating values using random number without really conducting experiment. Whenever the experiments are costly or infeasible or time-consuming simulation is used to generate the required data.

13.2. DEFINITION

1. Simulation is a representation of reality through the use of model or other device, which will react in the same manner as reality under a given set of conditions.

2. Simulation is the use of system model that has the designed characteristic of reality in order to produce the essence of actual operation.

3. According to Donald G. Malcolm, simulation model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.

4. According to Naylor, et al. simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended period of time.

There are two types of simulation, they are:

1. Analog Simulation: Simulating the reality in physical form (*e.g.*: Children's park, planetarium, etc.) is known as analog simulation.

2. Computer Simulation: For problems of complex managerial decision-making, the analogue simulation may not be applicable. In such situation, the complex system is formulated into a mathematical model for which a computer programme is developed. Using high-speed computers then solves the problem. Such type of simulation is known as computer simulation or system simulation.

13.3. CLASSIFICATION OF SIMULATION MODELS

Simulation models are classified as:

(*a*) **Simulation of Deterministic models:**

In the case of these models, the input and output variables are not permitted to be random variables and models are described by exact functional relationship.

(*b*) **Simulation of Probabilistic models:**

In such cases method of random sampling is used. The techniques used for solving these models are termed as Monte-Carlo technique.

(*c*) **Simulation of Static Models:**

These models do not take variable time into consideration.

(*d*) **Simulation of Dynamic Models:**

These models deal with time varying interaction.

13.4. ADVANTAGES OF SIMULATION

Simulation is a widely accepted technique of operations research due to the following reasons:

* It is straightforward and flexible.
* It can be used to analyze large and complex real world situations that cannot be solved by conventional quantitative analysis models.
* It is the only method sometimes available.
* It studies the interactive effect of individual components or variables in order to determine which ones are important.
* Simulation model, once constructed, may be used over and again to analyze all kinds of different situations.

* It is the valuable and convenient method of breaking down a complicated system into subsystems and their study. Each of these subsystems works individually or jointly with others.

13.5. LIMITATIONS OF SIMULATION TECHNIQUE

* Since simulation model mostly deals with uncertainties, the results of simulation are only reliable approximations involving statistical errors, optimum results cannot be produced by simulation.

* In many situations, it is not possible to identify all the variables, which affect the behaviour of the system.

* In very large and complex problems, it is very difficult to make the computer program in view of the large number of variables and the involved inter–relationship among them.

* For problems requiring the use of computer, simulation may be comparatively costlier and time consuming in many cases.

* Each solution model is unique and its solutions and inferences are not usually transferable to other problems, which can be solved by other techniques.

13.6. MONTE-CARLO SIMULATION

The Monte-Carlo method is a simulation technique in which statistical distribution functions are created using a series of random numbers. Working on the digital computer for a few minutes we can create data for months or years. The method is generally used to solve problems which cannot be adequately represented by mathematical models or where solution of the model is not possible by analytical method. Monte-Carlo simulation yields a solution, which should be very close to the optimal, but not necessarily the exact solution. But this technique yields a solution, which converges to the optimal solution as the number of simulated trials tends to infinity. The Monte-Carlo simulation procedure can be summarized in the following steps:

Step 1: Clearly define the problem:
(*a*) Identify the objectives of the problem.
(*b*) Identify the main factors, which have the greatest effect on the objective of the problem.

Step 2: Construct an approximate model:
(*a*) Specify the variables and parameters of the mode.
(*b*) Formulate the appropriate decision rules, *i.e.* state the conditions under which the experiment is to be performed.
(*c*) Identify the type of distribution that will be used. Models use either theoretical distributions or empirical distributions to state the patterns of the occurrence associated with the variables.
(*d*) Specify the manner in which time will change.

Problem 13.1.

With the help of a single server queuing model having inter-arrival and service times constantly 1.4 minutes and 3 minutes respectively, explain discrete simulation technique taking 10 minutes as the simulation period. Find from this average waiting time and percentage of idle time of the facility of a customer. Assume that initially the system is empty and the first customer arrives at time t = 0.

Solution

Data: System is initially empty. Service starts as soon as first customer arrives. First customer arrives at t = 0.

The departure time of first customer = 0 + 3 *i.e.* arrival time + service time = 3 minutes (Dep) in the table. The second customer arrives at 1.4 minutes and third arrives at 2.8 minutes (Arr). Until the first customer leaves the system, second and third customers have to wait for service. We can calculate waiting time for second customer by taking the difference of time of departure of first customer and the time of arrival of second customer *i.e.* 3 – 1.4 = 1.6 minutes. The procedure is shown in the table below:

Time	Event Arr = arrival Dep = departure	Customer Number	Waiting time.
0.0	Arr.	1	--
1.4	Arr.	2	-
2.8	Arr.	3	--
3.0	Dep	1	3.00 – 1.40 = 1.6 min. for customer 2.
4.2	Arr.	4	
5.6	Arr.	5	
6.00	Dep	2	6.00 – 2.80 = 3.2 min. for customer 3.
7.00	Arr.	6	
8.4	Arr.	7	
9.00	Dep.	3	9.00 – 4.20 = 4.8 min. for customer 4
9.80	Arr.	8	
10.00	End of given time	-	10.00 – 5.60 = 4.4 min. for customer 5
			10.00 – 7.00 = 3.0 min. for customer 6
			10.00 – 8.4 = 1.6 min. for customer 7
			10.00 – 9.80 = 0.2 min. for customer 8.

Average waiting time per customer for those who must wait = Sum of waiting time of all customers / number of waiting times taken = (1.4 + 2.8 + 4.2 + 5.6 + 7.0 + 8.4 + 9.8) / 7 = 18.8 / 7 = 2.7 minutes.

Percentage of idle time of server = Sum of idle time of server / total time = 0%.

13.7. RANDOM NUMBERS

Random number is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

13.7.1. Pseudo-Random Numbers

Random numbers which are generated by some deterministic process but which satisfy statistical test for randomness are called **Pseudo-random numbers.**

13.7.2. Generation of Random Numbers

Using some arithmetic operation one can generate Pseudo-random numbers. These methods most commonly specify a procedure, where starting with an initial number called **seed** is generates the second number and from that a third number and so on. A number of recursive procedure are available, the most common being the **congruence method** or the **residue method.** This method is described by the expression:

$$r_{i+1} = (ar_{i+b}) \text{ (modulo } m),$$

Where a, b and m are constants, r_i and r_{i+1} are the ith and (i + 1)th random numbers.

The expression implies multiplication of a by r_i and addition of b and then division by m. Then r_{i+1} is the remainder or residue. To begin the process of random number generation, in addition to a, b and m, the value of r_0 is also required. It may be any random number and is called **seed**.

Problem 13.2

With the help of an example explain the additive multiplicative and mixed types of the congruence random number generators.

Solution

The ongruence random number generator is described by the recursive expression

$r_{i+1} = (ar_{i+b}) \text{ (modulo } m),$

Where a, b and m are constants. The selection of these constants is very important as it determines the starting of random number, which can be obtained by this method. The above expression is for a **mixed type congruential method** as it comprises both multiplication of a and r_i and addition of ar_i and b.

If $a = 1$, the expression reduces to $r_{i+1} = (r_{i+b}) \text{ (modulo } m)$. This is known as **additive type expression.**

When $b = 0$, the expression obtained is $r_{I+1} = (ar\ I) \text{ (modulo } m)$, this is known as **multiplicative method.**

To illustrate the different types of the congruence methods, let us take $a = 16$, $b = 18$ and $m = 23$ and let the starting random number or seed be $r_0 = 1$.

(*a*) **Mixed Congruential method:** $r_{i+1} = (ar_i + b) \text{ (modulo } m),$ therefore,

r_i	$r_{i+1} = (ar_i + b) \,(modulo\ m),$	$=$	*Residue*
r_1	$(16 \times 1 + 18) / 23$	34 / 23	1 residue 11
r_2	$(16 \times 11 + 18) /23$	194 / 23	8 + residue 10
r_3	$(16 \times 10 + 18) / 23$	178 / 23	7 + residue 17
r_4	$(16 \times 17 + 18) / 23$	290 / 23	12 + residue 14
r_5	$(16 \times 14 + 18) / 23$	242 / 23	10 + residue 12
r_6	$(16 \times 12 + 18) / 23$	210 / 23	9 + residue 3
r_7	$(16 \times 3 + 18) / 23$	66 / 23	2 + residue 20
r_8	$(16 \times 20 + 18) / 23$	338 / 23	14 + residue 16
r_9	$(16 \times 16 + 18) / 23$	274 / 23	11 + residue 21
r_{10}	$(16 \times 21 + 18) / 23$	354 / 23	15 + residue 9
r_{11}	$(16 \times 9 + 18) / 23$	162 / 23	7 + residue 1

The random numbers generated by this method are: 1, 11, 10, 17, 14, 12, 3, 20, 16, 21, and 9.

    (*b*) **Multiplicative Congruential Method:** $r_{i+1} = ar_i$ (modulo *m*)

r_i	$r_{i+1} = ar_i \ \ (modulo\ m)$	*Random Number*
r_1	$(16 \times 1) / 23$	0 + Residue 16
r_2	$(16 \times 16) / 23$	11 + Residue 3
r_3	$(16 \times 3) / 23$	2 + Residue 2
r_4	$(16 \times 2) / 23$	1 + Residue 9
r_5	$(16 \times 9) / 23$	6 + Residue 6
r_6	$(16 \times 6) / 23$	4 + Residue 4
r_7	$(16 \times 4) / 23$	2 + Residue 18
r_8	$(16 \times 18) / 23$	12 + Residue 12
r_9	$(16 \times 12) / 23$	8 + Residue 8
r_{10}	$(16 \times 8) / 23$	5 + Residue 13
r_{11}	$(16 \times 13) / 23$	9 + residue 1

    The string of random numbers obtained by multiplicative congruential method is 1, 16, 3, 2, 9, 6, 4, 18, 12, 8, and 13.

(*c*) **Additive Congruential Method:** $r_{i+1} = (r_{i+b})$ **(modulo** *m***).**

r_i	$r_{i+1} = (r_i + b)$ *(modulo m)*	*Random Number*
r_1	(1 + 18) / 23	0 + residue 19
r_2	(19 + 18) / 23	1 + residue 14
r_3	(14 + 18) / 23	1 + residue 9
r_4	(9 + 18) / 23	1 + residue 4
r_5	(4 + 18) / 23	0 + residue 22
r_6	(22 + 18) / 23	1 + residue 17
r_7	(17 + 18) / 23	1 + residue 12
r_8	(12 + 18) / 23	1 + residue 7
r_9	(7 + 18) / 23	1 + residue 2
r_{10}	(2 + 18) / 23	0 + residue 20
r_{11}	(20 + 18) / 23	1 + residue 15

The random numbers generated are: 1, 19, 14, 19, 4, 22, 17, 12, 7, 2, 20, and 15.

Problem 13.3.

		$T = 1$	2	3
The distribution of inter-arrival time in a single server model is		$f(T) = \frac{1}{4}$	½	¼
		$S = 1$	2	3
And the distribution of service time is		$F(S) = \frac{1}{2}$	¼	¼

Complete the following table using the two digit random numbers as 12, 40, 48, 93, 61, 17, 55, 21, 85, 68 to generate arrivals and 54, 90, 18, 38, 16, 87, 91, 41, 54, 11 to generate the corresponding service times.

Arrival Number	Random Number	Arrival Time	Time Service Begins	Random number	Time Servic ends	Waiting time in Queue

Solution

The distribution of inter-arrival times and the two-digit random numbers assigned to different values of T is as below:

T	f(T)	$\sum f(T)$	Random numbers
1	0.25	0.25	00 to 24
2	0.50	0.75	25 to 74
3	0.25	1.00	75 to 99

Inter-arrival times corresponding to random numbers 12, 40, 48, 93, 61, 17, 55, 21, 85 and 68 are Given 1, 2, 2, 3, 2, 1, 2, 1, 3, 2 respectively. Similarly, the distribution of service times and two-digit random numbers assigned to different values of S are as follows:

S	f (s)	$\sum f(s)$	Random number
1	0.50	0.50	00 to 49
2	0.25	0.75	25 to 74
3	0.25	1.00	75 to 99

The simulation is done as follows:

Arrival number	Random number	Arrival time	Time Service Begins in Mins.	Random number	Time Service Ends in Mins.	Waiting Time in Queue.
1	12	1	1	54	3	-
2	40	3	3	90	6	-
3	48	5	6	18	7	1
4	93	8	8	38	9	-
5	61	10	10	16	11	-
6	17	11	11	87	14	-
7	55	13	14	91	17	1
8	21	14	17	41	18	3
9	85	17	18	54	20	1
10	68	19	20	11	21	1

The working of the above table is as below: The simulation of the single-server system starts at zero time. First customer arrives at 1 time unit after that and the service immediately begins. Since the service time for the first customer is 2 time units, service ends at 3 time units. The second customer arrives after an inter-arrival time of 2 time units and goes to service immediately at 3 time units. The third customer who arrives at 5 time units has to wait till the service of 2nd customer ends at 6 units of time. The other entries are also filled on the same logic.

Problem 13.4.

A coffee house in a busy market operates counter service. The proprietor of the coffee house has approached you with the problem of determining the number of bearers he should employ at the counter. He wants that the average waiting time of the customer should not exceed 2 minutes. After recording the data for a number of days, the following frequency distribution of inter-arrival time of customers and the service time at the counter are established. Simulate the system for 10 arrivals of various alternative number of bearers and determine the suitable answer to the problem.

Inter-arrival time in mins.	Frequency (%)	Service time in mins.	Frequency (%)
0	5	1.0	5
0.5	35	2.0	25
1.0	25	3.0	35
1.5	15	4.0	20
2.0	10	5.0	15
2.5	7		
3.0	3		

Solution

It is queuing situation where customers arrive at counter for taking coffee. Depending upon the number of bearers, the waiting time of the customers will vary. It is like a single queue multi-channel system and waiting customer can enter any of the service channel as and when one becomes available. By taking two-digit random number interarrival and interservice times are as follows:

Random number for arrivals:

Inte-arrival time in minutes	Frequency	Cumulative frequency	Random numbers
0	5	5	00 to 04
0.5	35	40	05 to 39
1.0	25	65	40 to 64
1.5	15`	80	65 to 79
2.0	10	90	80 to 89
2.5	7	97	90 to 96
3.0`	3	100	97 to 100

Random number for Service:

Service time in minutes	Frequency	Cumulative frequency	Random number
1.0	5	5	00 to 04
2.0	25	30	05 to 29
3.0	35	65	30 to 64
4.0	20	85	65 to 84
5.0	15	100	85 to 99

Arrival Number	Random Number	Inter Arrival Time	Random Number	Service Time	Arrival Time	Bearer One Service Begins	Bearer One Service Ends	Bearer Two Service Begins	Bearer Two Service Ends	Customer Waiting Time
1	–	–	31	3	0	0.0	3.0			0
2	48	1.0	46	3	1.0			1.00	4.00	0
3	51	1.0	24	2	2.0	3.0	5.0			1.0
4	06	0.5	54	3	2.5			4.00	7.00	1.5
5	22	0.5	63	3	3.0	5.0	8.00			2.0
6	80	2.01	82	4	5.0			7.00	11.00	2.0
7	56	1.0	32	3	6.0	8.0	11.00			2.0
8	06	0.5	14	2	6.5			11.00	13.00	4.5
9	92	2.5	63	3	9.0	11.0	14.00			2.0
10	51	1.0	18	2	10.0			13.00	15.00	3.0

The customer waiting time with two servers is sometimes greater than 2 minutes. Hence let us try with one more bearers. The table below shows the waiting time of customers with three bearers.

With two bearers, total waiting time is 18 minutes. Hence average waiting time is 18 / 10 = 1.8 minutes.

Arrival Number	Server One Service Begins	Server One Service Ends	Server Two Service Begins	Server Two Service Ends	Server Three Service Begins	Service Three Service Ends	Customer Waiting Time
1	0.0	3.0					0
2			1.0	4.0			0
3					2.0	4.0	0
4	3.0	6.0					0.5
5			4.0	7.0			1.0
6					5.0	9.0	0
7	6.0	9.0					0
8			7.0	9.0			0
9					9.0	12.0	0
10	10.0	12.0					0

With three bearers, the total waiting time is 1.5 minutes. Average waiting time is 0.15 minutes. Similarly, we can also calculate the average waiting time of the bearers.