MAR GREGORIOS COLLEGE
OF ARTS & SCIENCE

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DEPARTMENT OF MATHS

SUBJECT NAME: ALLIED MATHEMATICS - I

SUBJECT CODE: SM3AA

SEMESTER: I

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SYLLABUS

UNIVERSITY OF MADRAS

ALLIED MATHEMATICS-I

UNIT - I

Algebra and Numerical Methods

Algebra:

Summation of series – Simple problems.

Numerical Methods:

Operators E, Δ, ∇ , difference tables, Newton-Raphson Method – Newton's forward and backward interpolation formulae for equal intervals – Lagrange's interpolation formula.

UNIT - II

Matrices

Symmetric, Skew-symmetric, Orthogonal Hermitian skew-Hermitian and Unitary matrices – Eigen values and Eigen vectors – Cayley-Hamilton theorem (without proof) – verification – computation of inverse of matrix using Cayley-Hamilton theorem.

UNIT - III

Theory of Equations

Polynomial equations with real coefficients, irrational roots, complex roots, symmetric functions of roots, transformation of equation by increasing or decreasing roots by a constant, reciprocal equation Newton's method to find a root approximately - Simple problems.

UNIT - IV

Trigonometry

Expansions of $\sin n\theta$ and $\cos n\theta$ in a series of powers of $\sin\theta$ and $\cos\theta$ – Expansions of $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$ in a series of sines and cosines, tangents of multiples of 'θ' – Expansions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in a series of power of 'θ' – Hyperbolic and inverse hyperbolic functions – Logarithmic of complex numbers.

UNIT - V

Differential calculus

Successive differentiation – n^{th} derivatives – Leibnitz theorem (without proof) and applications – Jacobians, Curvature and radius of curvature in Cartesian co-ordinates – maxima and minima of functions of two variables – Lagrange's multiplier simple problems.

UNIT-1

Exponential and Logarithmic Series

1. In the following chapter we are about to obtain an expansion in powers of x for the expression a^x, where both a and x are real, and also to obtain an expansion for $\log_e (1 + x)$, where x is real and less than unity, and e stands for a quantity to be defined.

2. *To find the value of the quantity* $\left(1+\frac{1}{n}\right)^n$, *when n becomes infinitely great and is real.*

Since $\frac{1}{n}<1,$ we have, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+...$$

$$=1+1+\frac{1-\frac{1}{n}}{1\cdot 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\underline{|4}}+... \quad ...(1)$$

This series is true for all values of n, however great. Make then n infinite and the right-hand side

$$=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\text{... ad inf.} \quad ...(2)$$

Hence the limiting value, when n is infinite, of $\left(1+\frac{1}{n}\right)^n$ is the sum of the series.

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}\text{ ... ad inf.}$$

The sum of this series is always denoted by the quantity e.

Hence we have

$$\mathbf{\underset{n=\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e,}$$

where $\underset{n=\infty}{\text{Lt}}$ stands for "the limit when $n = \infty$."

Cor. By putting $n = \frac{1}{m}$, it follows (since m is zero when n is infinity) that

$$\underset{m=0}{\text{Lt}}\,(1+m)^{1/m} = \underset{n=\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e.$$

3. This quantity e is finite.

For since $\quad \frac{1}{\lfloor 3} < \frac{1}{2\cdot 2} < \frac{1}{2^2},$

$$\frac{1}{\lfloor 4} < \frac{1}{2\cdot 2\cdot 2} < \frac{1}{2^3},$$

. . . .

we have

$$e < 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} \text{ ...ad inf.}$$

$$< 1+\frac{1}{1-\frac{1}{2}}$$

$< 1 + 2$ *i.e.* < 3.

Also clearly $\quad e > 2$.

Hence it lies between 2 and 3.

By taking a sufficient number of terms in the series, it can be shown that

$e = 2.7182818285...$

4. *The quantity e is incommensurable.*

For, if possible, suppose it to be equal to a fraction $\frac{p}{q}$, where p and q are whole numbers.

We have then

$$\frac{p}{q} = 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+...+\frac{1}{\lfloor q}+\frac{1}{\lfloor q+1}+\frac{1}{\lfloor q+2}+... \qquad ...(1)$$

Multiply this equation by $\lfloor q$, so that all the terms of the series (1) become integers except those commencing with $\frac{\lfloor q}{\lfloor q+1}$. Hence we have

$$p\lfloor q-1 = \text{whole number} + \frac{\lfloor q}{\lfloor q+1}+\frac{\lfloor q}{\lfloor q+2}+\frac{\lfloor q}{\lfloor q+3}+...,$$

i.e. an integer $= \frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+... \qquad ...(2)$

But the right-hand side of this equation is $>\dfrac{1}{q+1}$, and

$$<\frac{1}{q+1}+\frac{1}{(q+1)^2}+\frac{1}{(q+1)^3}+\ldots,$$

i.e. $$<\frac{1}{q+1}\div\left(1-\frac{1}{q+1}\right),$$

i.e. $$<\frac{1}{q}.$$

Hence the right-hand side of (2) lies between $\dfrac{1}{q+1}$ and $\dfrac{1}{q}$, and therefore a fraction and so cannot be equal to the left-hand side.

Hence our supposition that e was commensurable is incorrect and it therefore must be incommensurable.

5. Exponential Series: *When x is real, to prove that*

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots ad\ inf.,$$

and that

$$a^x = 1 + x\log_e a + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \ldots\ ad\,inf.$$

When n is greater than unity, we have

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx}$$

$$= 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{1\cdot 2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{1\cdot 2\cdot 3}\frac{1}{n^3} + \ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1\cdot 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\cdot 2\cdot 3} + \ldots.$$

In this expression make n infinitely great. The left-hand becomes, as in Art. 2, e^x. The right-hand becomes

$$1 + x + \frac{x^3}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

Hence we have

$$\mathbf{e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ \text{ad inf.}} \qquad \ldots(1)$$

Let $a = e^c$, so that $c = \log_e a$.

$$\therefore \qquad a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots \text{ ad inf.}$$

by substituting cx for x in the series (1).

$$\therefore \qquad \mathbf{a^x = 1 + x\log_e a + \frac{x^2}{\lfloor 2}(\log_e a)^2 + \frac{x^3}{\lfloor 3}(\log_e a)^3 +} \ldots \text{ ad inf.} \qquad \ldots(2)$$

6. It can be shown (as in C. Smith's *Algebra*, Art. 278) that the series (1), and therefore (2), of the last article is convergent for all real values of x.

7. Example 1. *Prove that* $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \ldots$ *ad inf.*

By equation (1) of Art. 5 we have, by putting x in succession equal to 1 and -1,

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots \textit{ ad inf.}$$

and
$$e^{-1} = 1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots \textit{ ad inf.}$$

Hence, by subtraction,

$$e - e^{-1} = 2\left(1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \ldots\right),$$

i.e.
$$\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \ldots \textit{ ad inf.}$$

Example 2. *Find the sum of the series*

$$1 + \frac{1+2}{\lfloor 2} + \frac{1+2+3}{\lfloor 3} + \frac{1+2+3+4}{\lfloor 4} + \ldots \textit{ ad inf.}$$

The nth term $= \dfrac{1+2+3+\ldots+n}{\lfloor n} = \dfrac{\frac{1}{2}n(n+1)}{\lfloor n}$

$$= \frac{1}{2}\frac{n+1}{\lfloor n-1} = \frac{1}{2}\left[\frac{(n-1)+2}{\lfloor n-1}\right] = \frac{1}{2}\left[\frac{1}{\lfloor n-2} + \frac{2}{\lfloor n-1}\right],$$

provided that $n > 2$.

Similarly,

$$\text{the } (n-1)\text{th term} = \frac{1}{2}\left[\frac{1}{\lfloor n-3} + \frac{2}{\lfloor n-2}\right],$$

. . . .

$$\text{the 4th term} = \frac{1}{2}\left[\frac{1}{\lfloor 2} + \frac{2}{\lfloor 3}\right],$$

$$\text{the 3rd term} = \frac{1}{2}\left[\frac{1}{\lfloor 1} + \frac{2}{\lfloor 2}\right].$$

Also $$\text{the 2nd term} = \frac{1}{2}\left[1 + \frac{2}{\lfloor 1}\right].$$

and $$\text{the 1st term} = \frac{1}{2}\left[\frac{2}{1}\right].$$

Hence, by addition, the whole series

$$= \frac{1}{2}\left[1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \text{ ad inf.}\right]$$

$$+ \frac{1}{2} \cdot 2\left[1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \text{ ad inf.}\right]$$

$$= \frac{1}{2} \cdot e + e = \frac{3e}{2}.$$

8. Logarithmic Series: *To prove that, when y is real and numerically* < 1, *then*

$$\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots \textit{ ad inf.}$$

In the equation (2) of Art. 5, put

$a = 1 + y$,

and we have

$$(1+y)^x = 1 + x\log_e(1+y) + \frac{x^2}{\lfloor 2}\{\log_e(1+y)\}^2 + \ldots \qquad \ldots(1)$$

But, since y is real and numerically < unity, we have

$$(1+y)^x = 1 + x \cdot y + \frac{x(x-1)}{1 \cdot 2}y^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}y^3 + \ldots \qquad \ldots(2)$$

The series on the right-hand side of (1) and (2) are equal to one another and both convergent, when y is numerically < 1. Also it could be shown that the series on the right-hand side of (2) is convergent when it is arranged in powers of x. Hence we may equate like powers of x.

Thus we have

$$\log_e(1+y) = y - \frac{y^2}{1 \cdot 2} + \frac{(-1)(-2)}{1 \cdot 2 \cdot 3}y^3 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3 \cdot 4}y^4 + \ldots \text{ad inf.},$$

i.e. $$\mathbf{\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 +} \text{ ad inf.} \qquad \ldots(3)$$

9. If $y = 1$, the series (3) of the previous article is equal to

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{... ad inf.}$$

which is known to be convergent.

If $y = -1$, it equals $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}$ ... ad inf. which is known to be divergent.

In addition therefore to being true for all values of y between -1 and $+1$, it is true for the value $y = 1$; it is not however true for the value $y = -1$.

10. Calculation of logarithms to base e.

In the logarithmic series, if we put $y = 1$, we have

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{... ad inf.} \qquad \text{...(1)}$$

If we put $$y=\frac{1}{2},$$

we have

$$\log_e 3-\log_e 2=\log_e\frac{3}{2}=\log_e\left(1+\frac{1}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\ldots \qquad \text{...(2)}$$

If we put $$y=\frac{1}{3}.$$

we have

$$\log_e 4-\log_e 3=\log_e\left(1+\frac{1}{3}\right)=\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\frac{1}{4}\cdot\frac{1}{3^4}+\ldots \qquad \text{...(3)}$$

From these equations we could, by taking a sufficient number of terms, calculate $\log_e 2$, $\log_e 3$ and $\log_e 4$.

It would be found that a large number of terms would have to be taken to give the values of these logarithms to the required degree of accuracy. We shall therefore obtain more convenient series.

11. By Art. 8 we have

$$\log_e(1+y)\,y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\ldots \qquad \text{...(1)}$$

and, by changing the sign of y,

$$\log_e(1-y)=-y-\frac{1}{2}y^2-\frac{1}{3}y^3-\frac{1}{4}y^4+\ldots \qquad \text{...(2)}$$

In order that both these series may be true y must be numerically less than unity.

By subtraction, we have

$$\log_e(1+y)-\log_e(1-y)=\log_e\frac{1+y}{1-y}=2\left[y+\frac{1}{3}y^3+\frac{1}{5}y^5+\ldots\right] \qquad \ldots(3)$$

Let $$y=\frac{m-n}{m+n},$$

where m and n are positive integers and $m > n$, so that

$$\frac{1+y}{1-y}=\frac{m}{n}.$$

The equation (3) becomes

$$\log_e\frac{m}{n}=2\left[\left(\frac{m-n}{m+n}\right)+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\right] \qquad \ldots(4)$$

Put $m = 2$, $n = 1$ in (4) and we get $\log_e 2$.

Put $m = 3$, $n = 2$ and we get $\log_e 3 - \log_e 2$, and therefore $\log_e 3$.

By proceeding in this way we get the value of the logarithm of any number to base e.

12. Logarithms to base 10. The logarithms of the previous article, to base e, are called Napierian or natural logarithms.

We can convert these logarithms into logarithms to base 10.

For, by Art. 147 (Part I.), we have, if N be any number,

$$\log_e N = \log_{10} N \times \log_e 10.$$

$$\therefore \log_{10} N = \log_e N \times \frac{1}{\log_e 10}.$$

Now, $\log_e 10$ can be found as in the last article and then $\frac{1}{\log_e 10}$ is found to be 0.4342944819...,

Hence, $$\log_{10} N = \log_e N \times 0.43429448...,$$

so that the logarithm of any number to base 10 is found by multiplying its logarithm to base e by the quantity 0.43429448... This quantity is called the Modulus.

EXAMPLES I

Prove that

1. $\frac{1}{2}(e+e^{-1})=1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots$

2. $\left(1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right)\left(1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\ldots\right)=1.$

3. $\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots\right)^2=1+\left(1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\ldots\right)^2 \ldots$

4. $1+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 5}+\frac{4}{\lfloor 7}+\ldots=\frac{e}{2}.$

5. $\frac{2}{\lfloor 3}+\frac{4}{\lfloor 5}+\frac{6}{\lfloor 7}+\ldots=e^{-1}.$

6. $\dfrac{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots}{1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\ldots}=\frac{e-1}{e+1}.$

7. $1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\ldots=5e.$

Find the sum of the series

8. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ ad inf.

9. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\ldots$ ad inf.

Prove that

10. $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots=\log_e a=-\log_e b.$

11. $\log_e\frac{1+x}{1-x}=2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\text{ad inf.}\right).$

12. $\log_e\frac{x+1}{x-1}=2\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\ldots\text{ad inf.}\right)$, if $x>1$.

13. $\log_e(1+3x+2x^2)=3x-\frac{5x^2}{2}+\frac{9x^3}{3}-\frac{17x^4}{4}+\ldots+(-1)^{n-1}\frac{2^n+1}{n}x^n+\ldots,$

provided that $2x$ be not >1.

14. $2\log_e x-\log_e(x+1)-\log_e(x-1)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots,$ if $x>1$.

15. $\log_e 2=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\ldots$ ... ad inf.

16. $\log_e 2-\frac{1}{2}=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\ldots$ ... ad inf

17. $\tan\theta+\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+\ldots=\frac{1}{2}\log\dfrac{\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}$, if $\theta<\frac{\pi}{4}$.

18. If θ be $>\frac{\pi}{2}$ and $<\pi$, prove that

$$(1)\sin\theta+\frac{1}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta+\ldots\ldots\text{ ad inf.}$$

$$=2\left[\cot\frac{\theta}{2}+\frac{1}{3}\cot^3\frac{\theta}{2}+\frac{1}{5}\cot^5\frac{\theta}{2}+\ldots\text{ad inf.}\right],$$

NUMERICAL METHODS

If $y_0, y_1, y_2, ..., y_n$ denotes a set of values of any function y=f(x) for $x=x_0, x_1, x_2, . . . x_n$, the different values of the independent variable x is called the arguments and their corresponding y values are called the entries.

Forward Differences:

$y_1 - y_0, y_2 - y_1, y_3 - y_2, ..., y_n - y_{n-1}$ are called the first differences of the function *y*. The first differences of y_n values are denoted by

$$\Delta y_n = y_{n+1} - y_n, \quad n = 0,1,2,...$$

Here Δ acts as an operator called forward difference operator.

$$\therefore \quad \Delta y_0 = y_1 - y_0, \ \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, ... \ \Delta y_n = y_{n+1} - y_n$$

The second differences are,

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

Similarly $\Delta^2 y_1 = y_3 - 2y_2 + y_1$ and so on.

In general $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$ defines n^{th} difference where *k* and *n* are integers.

Here each difference proves to be combination of *y* values. For eg,

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$$

$$= [(y_3 - y_2) - (y_2 - y_1)] - [(y_2 - y_1) - (y_1 - y_0)]$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\vdots$$
$$\Delta^n y_0 = y_n - nC_1 y_{n-1} + nC_2 y_{n-2} - nC_3 y_{n-3} + ... + (-1)^n y_0$$

The differences are called forward differences and these differences are usually represented as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

The difference operator Δ may also be defined as $\Delta f(x) = f(x+h) - f(x)$ where h is the interval of spacing.

Backward Differences:

If $y_1, y_2, \ldots, y_n$ denote a set of values of any function $y=f(x)$ then y_1-y_0, y_2-y_1, y_3-y_2,.... y_n-y_{n-1} are the differences. Here we use another operator called the backward difference operator ∇ and is defined by

$$\nabla y_n = y_n - y_n \quad \text{for } n = 0,1,2,\ldots$$

$$\nabla y_0 = y_0 - y_{-1}$$

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1 \quad \text{and so on}$$

The second backward differences is $\nabla^2 y_n = \nabla(\nabla y_n)$

$$= \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$

$$= y_n - 2y_{n-1} + y_{n-2}$$

Similarly $\nabla^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}$ and so on.

The backward difference can also be defined in the following way, where h being the length of the interval.

$$\nabla f(x) = f(x) - f(x-h) \text{ where h is the length of the interval.}$$

The backward difference table is,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Shifting Operators E and E^{-1}

The operator E is defined such that its operation on the y value at x yields the y value at x+h, Thus

$$Ef(x)=f(x+h)$$

$$E(y_0)=Ef(x_0)=f(x_0+h)=y$$

$$E(y_1)=y_2,\ E(y_2)=y_3,\ldots\ E(y_{n-1})=y_n$$

Also $\quad E^2(y_0)=E(Ey_0)=E(y_1)=y_2$

$$E^3(y_0)=y_3,\ E^4(y_0)=y_4 \quad \text{and so on}$$

The operator E^{-1} is defined to be the inverse of E, so that from $Ef(x)=f(x+h)$, we get

$$E^{-1}f(x+h)=f(x) \Rightarrow E^{-1}f(x)=f(x-h)$$

Relation between Δ, ∇, E and E^{-1}

1) Prove that $\Delta = E-1$

Proof

$$\Delta f(x)=f(x+h)-f(x)$$

$$=Ef(x)-f(x)$$

$$=[E-1]f(x)$$

$$\Rightarrow \Delta = E-1$$

2) Prove that $\nabla = 1-E^{-1}$

Proof

$$\nabla f(x)=f(x)-f(x-h)$$

$$= f(x) - E^{-1}f(x)$$
$$= \left(1 - E^{-1}\right)f(x)$$
$$\Rightarrow \nabla = 1 - E^{-1}$$

3) Prove that $E\nabla = \nabla E = \Delta$

Proof:

$$E\nabla = E\left(1 - E^{-1}\right)$$
$$= E - EE^{-1} = E - 1 = \Delta$$

$$\nabla E = \left(1 - E^{-1}\right)E$$
$$= E - 1 = \Delta$$

Hence Proved.

4) Evaluate $\Delta\left(\cos\left(ax+b\right)\right) \& \Delta\left(\log f\left(x\right)\right)$

(i) $\Delta\cos\left(ax+b\right) = \cos\left(ax+ah+b\right) - \cos\left(x+b\right)$

$$= -2\sin\left(ax+b+\frac{ah}{2}\right)\sin\frac{ah}{2}$$

$$= 2\sin\frac{ah}{2}\cos\left(\frac{\pi}{2}+ax+b+\frac{ah}{2}\right)$$

$$= 2\sin\frac{ah}{2}\cos\left(ax+b+\frac{\pi+ah}{2}\right)$$

(ii) $\Delta\log f\left(x\right) = \log f\left(x+h\right) - \log f\left(x\right)$

$$= \log\frac{f\left(x+h\right)}{f\left(x\right)}$$

$$= \log\left[\frac{f\left(x\right)+f\left(x+h\right)-f\left(x\right)}{f\left(x\right)}\right]$$

$$= \log\left[1+\frac{\Delta f\left(x\right)}{f\left(x\right)}\right]$$

INTERPOLATION:

Consider the table

x	x_0	x_1	x_2	....	x_n
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	....	$f(x_n)$

If the value of f(y) is to be found at some point y in the interval $[x , x_n]$ and y is not one of the tabulated points, then the value of f(y) is estimated by using the known values of f(x) at the surround points. This process of computing the value of a function inside the given range is called interpolation. If the point y lies outside the domain $[x_0, x_n]$ then the estimation of f(y) is called extrapolation.

Newton's forward interpolation formula:

Let the x and y values be $x_0, x_1, ..., x_n$ and $y_0, y_1, ..., y_n$ where the x values are in the increasing order and are equally spaced. Then $x_1 - x_0 = x_2 - x_1 = ... = x_n - x_{n-1}$=h (a +ve quantity).

The Newton's forward interpolation formula is,

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

Where

$$u = \frac{x - x_0}{h}$$

This formula is used to interpolate or extrapolate the value of y corresponding to an x value close to x_0. The formula involves u and the elements in the first row of the forward difference table.

Newton's backward interpolation formula:

For the same data given above the Newton;s backward interpolation formula is,

$$y = f(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + ...$$

Where

$$v = \frac{x - x_n}{h}$$

This formula is used to interpolate or extrapolate the value of y corresponding to an x value close t x_n. This formula involves u and the elements in the last row of the backward difference table.

LAGRANGE'S FORMULA:

For the same data given above, if $x_0, x_1, x_2, ..., x_n$ are not equally spaced, we use the formula,

$$y=f(x)=y_0\frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}+y_1\frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}+...$$

$$+y_n\frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})}$$

Which is known as Lagrange's formula for interpolation. The calculation can be made easier by using the following table.

$$\begin{array}{ccccccc} - & x & x_0 & x_1 & \cdots & x_n & \text{Product} \\ x & 0 & x-x_0 & x-x_1 & \cdots & x-x_n & P \\ x_0 & x_0-x & 0 & x_0-x_1 & \cdots & x_0-x_n & P_0 \\ x_1 & x_1-x & x_1-x_0 & 0 & \cdots & x_1-x_n & P_1 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ x_n & x_n-x & x_n-x_0 & x_n-x_i & \cdots & 0 & P_n \end{array}$$

$$y=f(x)=-P\left[\frac{y_0}{P_0}+\frac{y_1}{P_1}+\frac{y_2}{P_2}+...+\frac{y_n}{P_n}\right]$$

In the table, the elements in the diagonal are zero, above the diagonal are negatives of the elements below the diagonal.

Here $x_0, x_1, ..., x_n$ can be equally spaced. Regarding increasing or decreasing nature, $x_0, x_1, ..., x_n$ can be in any jumbled order.

Differences of a Constant Function:

If f(x)=C, a constant, then

$$\Delta f(x)=f(x+h)-f(x)=C-C=0$$

Differences of a Polynomial

If f(x) is a Polynomial in x of degree n, then it can be shown that, the n^{th} order difference $\Delta^n f(x)$ is a constant and the $(n+1)^{th}$ order difference $\Delta^{n+1}f(x)=0$.

Example:

1) Using Newton's formula, find y when x=142 **Nov'15**

x	**140**	**150**	**160**	**170**	**180**
y	**3.685**	**4.854**	**6.302**	**8.076**	**10.225**

Solution:

Now the x value 142 lies in the first interval (140,150), so we shall use Newton's forward formula.

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

Forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

From the table y_0=3.685, Δy_0=1.169, $\Delta^2 y_0$=0.279, $\Delta^3 y_0$=0.047, $\Delta^4 y_0$=0.002.

$$u = \frac{x - x_0}{h} \qquad x=142,\ x_0 = 140, \qquad h=10$$

$$u = \frac{142-140}{10} = 0.2$$

$$y = f(142) = 3.685 + \frac{0.2}{1!}(1.169) + \frac{(0.2)(-0.8)(0.279)}{2!} + \frac{(0.2)(-0.8)(-1.8)(0.047)}{3!} + \frac{(0.2)(-0.8)(-1.8)(-2.8)(0.002)}{4!}$$

y = 3.685 + 0.2338 – 0.02232 + 0.002256 – 0.0000672

=3.8986688

y =3.899 (app)

2) Using Newton's formula find y corresponding to x=45

x	**40**	**50**	**60**	**70**	**80**
y	**3.7**	**4.9**	**6.3**	**8**	**10.2**

Solution:

Since x=45 lies in the first half of the arguments, we use Newton's forward interpolation formula to find the value of y.

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

Forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	3.7				
		1.2			
50	4.9		0.2		
		1.4		0.1	
60	6.3		0.3		0.1
		1.7		0.2	
70	8		0.5		
		2.2			
80	10.2				

Here $y_0=3.7$, $\Delta y_0=1.2$, $\Delta^2 y_0=0.2$, $\Delta^3 y_0=0.1$, $\Delta^4 y_0=0.1$

$$u = \frac{x - x_0}{h} \qquad x=45, \quad x_0 = 40, \qquad h=10$$

$$u = \frac{45-40}{10} = 0.5$$

$$y = 3.7 + \frac{0.5(1.2)}{1!} + \frac{(0.5)(-0.5)(0.2)}{2!} + \frac{(0.5)(-0.5)(-1.5)(0.1)}{3!} + \frac{(0.5)(-0.5)(-1.5)(-2.5)(0.1)}{4!}$$

$$y = 3.7 + 0.6 - 0.025 + 0.00625 - 0.0039$$

$$y = 4.27735$$

3) If y(75) = 246, y(80)=202, y(85)-118, y(90)=40 find y(79).

Solution:

Given that

x	75	80	85	90
y	246	202	118	40

Since x=79 lies in the first half of x values, we use Newton's forward interpolation formula.

Forward difference table is,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	$\boxed{246}$			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

Here $y_0=246, \Delta y_0=-44, \Delta^2 y_0=-40, \Delta^3 y_0=46$.

$$u=\frac{x-x_0}{h} \qquad x=79,\quad x_0=75,\quad h=5$$

$$u=\frac{79-75}{5}=0.8$$

$$y=246+\frac{(0.8)(-44)}{1!}+\frac{(0.8)(-0.2)(-40)}{2!}+\frac{(0.8)(-0.2)(-1.2)(46)}{3!}$$

$$=246-35.2+3.2+1.472=215.472$$

4) Find the value of θ when x=84 using suitable Newton's Formula. **Nov'16**

x:	**40**	**50**	**60**	**70**	**80**	**90**
θ:	**184**	**204**	**226**	**250**	**276**	**304**

Solution:

Since x=84 lies in the second half of the x values to find θ we use Newton's backward formula,

$$y=f(x)=y_n+\frac{v}{1!}\nabla y_n+\frac{v(v+1)}{2!}\nabla^2 y_n+\frac{v(v+1)(v+2)}{3!}\nabla^3 y_n+\ldots$$

Backward difference table is

x	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$	$\nabla^5\theta$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

Here $y_n=304$, $\nabla y_n=28$, $\nabla^2 y_n=2$, $\nabla^3 y_n=\nabla^4 y_n=\nabla^5 y_n=0$.

$$v=\frac{x-x_n}{h} \qquad x=84, \quad x_n=90, \quad h=10$$

$$v=\frac{84-90}{10}=-0.6$$

$$y=304+\frac{(-0.6)(28)}{1!}+\frac{(-0.6)(0.4)(2)}{2!}=286.96$$

5) Using Newton's formula, find y when x=27 **Apr' 17**

x:	**10**	**15**	**20**	**25**	**30**
y:	**35.4**	**32.2**	**29.1**	**26.0**	**23.1**

Solution:

The x-value x=27 lies between (25, 30), the second half, we use Newton's backward interpolation formula to find corresponding y value.

$$f(x)=y=y_n+\frac{v\nabla y_n}{1!}+\frac{v(v+1)}{2!}\nabla^2 y_n+\frac{v(v+1)(v+2)}{3!}\nabla^3 y_n+\dots$$

$$v=\frac{x-x_n}{h} \qquad x=27, \quad x_n=30, \quad h=5$$

$$v=\frac{27-30}{5}=-0.6$$

Backward difference table is

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
10	35.4				
		–3.2			
15	32.2		0.1		
		–3.1		–0.1	
20	29.1		0.0		0.3
		–3.1		0.2	
25	26.0		0.2		
		–2.9			
30	23.1				

Here $y_n=23.1$, $\nabla y_n=-2.9$, $\nabla^2 y_n=0.2$, $\nabla^3 y_n=0.2$, $\nabla^4 y_n=0.3$.

$$y=23.1+\frac{(-0.6)(-2.9)}{1!}+\frac{(-0.6)(0.4)(0.2)}{2!}+\frac{(-0.6)(0.4)(1.4)(0.2)}{3!}$$
$$+\frac{(-0.6)(0.4)(1.4)(2.4)(0.3)}{4!}$$

y = 23.1 + 1.74 – 0.024 – 0.0112 – 0.01008

y = 24.79472

6) Given the table

x:	**0**	**0.1**	**0.2**	**0.3**	**0.4**
e^x:	**1**	**1.1052**	**1.2214**	**1.3499**	**1.4918**

Find the value of y when x=0.38

Solution:

Since x=0.38 lies in the second half of arguments, we use Newton's backward formula.

$$y = f(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + ...$$

$$v = \frac{x - x_n}{h} \qquad x=0.38, \quad x_n=0.4, \quad h=0.1$$

$$v = \frac{0.38 - 0.4}{0.1} = -0.2$$

Backward difference table is,

x	$y=e^x$	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0.0	1				
		0.1052			
0.1	1.1052		0.0110		
		0.1162		0.0013	
0.2	1.2214		0.0123		–0.0002
		0.1285		0.0011	
0.3	1.3499		0.0134		
		0.1419			
0.4	1.4918				

$$y = 1.4918 + \frac{(-0.2)(0.1419)}{1!} + \frac{(-0.2)(0.8)(0.0134)}{2!} + \frac{(-0.2)(0.8)(1.8)(0.0011)}{3!} + \frac{(-0.2)(0.8)(1.8)(2.8)(-0.0002)}{4!}$$

y = 1.4918 – 0.02838 – 0.001072 – 0.0000528 + 0.000001

y=1.4623

7) The values of x and y are given by

x:	**5**	**6**	**9**	**11**
y:	**12**	**13**	**14**	**16**

Using Lagrange's interpolation formula, find y when x=10

Solution:

Lagrange's interpolation formula is,

$$y = \frac{(x-x_1)(x-x_2)\dots}{(x_0-x_1)(x_0-x_2)\dots}y_0 + \frac{(x-x_0)(x-x_2)\dots}{(x_1-x_0)(x_1-x_2)\dots}y_1 + \dots$$

Here $x_0 = 5$, $x_1=6$, $x_2=9$, $x_3=11$, $x=10$

and $y_0 = 12$, $y_1=13$, $y_2=14$, $y_3=16$.

$$\therefore y = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}13$$
$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}16$$

$$= \frac{(4)(1)(-1)(12)}{(-1)(-4)(-6)} + \frac{(5)(1)(-1)(13)}{(1)(-3)(-5)} + \frac{(5)(4)(-1)(14)}{(4)(3)(-2)} + \frac{(5)(4)(1)(16)}{(6)(5)(2)}$$

$$= 2 - \frac{13}{3} + \frac{70}{6} + \frac{16}{3} = 2 - 4.333 + 11.667 + 5.333$$

$y = 14.66733.$

8) Using suitable formula, find $\log_{10}301$ from the following table. **Nov'17**

x:	**300**	**304**	**305**	**307**
$\log_{10}x$	**2.4771**	**2.4829**	**2.4843**	**2.4871**

Solution:

The interval of x values (arguments) is unequal. Hence we use Lagrange's interpolation formula to find y.

$$y = \frac{(x-x_1)(x-x_2)\dots}{(x_0-x_1)(x_0-x_2)\dots}y_0 + \frac{(x-x_0)(x-x_2)\dots}{(x_1-x_0)(x_1-x_2)\dots}y_1 + \dots$$

Here $x = 301$, $x_0 = 300$, $x_1=304$, $x_2=305$, $x_3=307$

$y_0 = 2.4771$, $y_1=2.4829$, $y_3=2.4843$, $y_4=2.4871$.

$$y=\frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}(2.4771)$$
$$+\frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}(2.4829)$$
$$+\frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)(2.4843)}$$
$$+\frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)}(2.4871)$$

$$y=\frac{(-3)(-4)(-6)(2.4771)}{(-4)(-5)(-7)}+\frac{(1)(-4)(-6)(2.4829)}{(4)(-1)(-3)}$$
$$+\frac{(1)(-3)(-6)(2.4843)}{(5)(1)(-2)}+\frac{(1)(-3)(-4)(2.4871)}{(7)(3)(2)}$$

$$=\frac{-178.3512}{-140}+\frac{59.5896}{12}-\frac{44.7174}{10}+\frac{29.8452}{42}$$
$$=1.2739371429+4.9658-4.47174+0.7106$$
$$=6.9503371429-4.47174$$

$y = 2.4785971429$

9) Using Lagrange's interpolation formula, find a polynomial to the data,

x:	**0**	**1**	**3**	**4**
y:	**–12**	**0**	**6**	**12**

Solution:

$x_0 = 0$, $x_1=1$, $x_2=3$, $x_3=4$, $x=x$

$y_0 = -12$, $y_1=0$, $y_2=6$, $y_3=12$.

$$\therefore y=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12)+\frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0)$$
$$+\frac{(x-4)(x-0)(x-1)}{(3-4)(3-0)(3-1)}(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$= (x-1)(x-3)(x-4) - x(x-1)(x-4) + (x)(x-1)(x-3)$

$= x^3-8x^2-13x-12-x^3+5x^2-4x+x^3-4x^2+3x$

$y = x^3-7x^2-14x-12$ is the required polynomial.

10) Apply Newton's backward interpolation formula find a polynomial of degree 3, using the given table.

x:	**3**	**4**	**5**	**6**
y:	**6**	**24**	**60**	**120**

Solution:

Newton's backward formula is,

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \ldots$$

$$v = \frac{x - x_n}{h} \qquad x_n=6, \quad h=1$$

$v=x-6$

Backward difference table is,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	6			
		18		
4	24		18	
		36		6
5	60		24	
		60		
6	120			

$y_n=120, \quad \nabla y_n=60, \quad \nabla^2 y_n=24, \quad \nabla^3 y_n=6$

$$\therefore y = 120 + \frac{(x-6)}{1!}60 + \frac{(x-6)(-5)}{2!}(24) + \frac{(x-6)(x-5)(x-4)}{3!}(6)$$

$$y = 120 + 60(x-6) + 12(x^2 - 11x + 30) + (x-4)(x^2 - 11x + 30)$$

$$= 120 + 60x - 360 + 12x^2 - 132x + 360 + x^3 - 15x^2 + 74x - 120$$

$$y = x^3 - 3x^2 + 2x$$

10) Find the missing term from the following data.

x:	**0**	**5**	**10**	**15**	**20**	**25**
y:	**7**	**11**	**14**	**–**	**24**	**32**

Solution:

Since five values of y are given, we can find a polynomial of degree 4, and $\Delta^5 y=0$. The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
			–1			
5	11			K–16		
		3			–4K+71	
			K–17			
10	14					10K–180
				–3K+55		
		K–14				
			–2K+38		6K–109	
15	K(say)					
				3K–54		
		24–K				
20	24		K–16			
		8				
25	32					

$\Delta^5 y=0 \Rightarrow \quad 10K-180=0$

$$\boxed{K=18}$$

Hence the missing y value is 18.

UNIT -2
MATRICES

A matrix is defined to be a rectangular array of numbers arranged into rows and columns. It is written as follows:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots & a_{3n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{bmatrix}$$

Special Types of Matrices:

(i) A **row matrix** is a matrix with only one row. E.g., [2 1 3].

(ii) A **column matrix** is a matrix with only one column. E.g., $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(iii) **Square matrix** is one in which the number of rows is equal to the number of columns.

If A is the square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots & a_{3n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{bmatrix}$$

then the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots & a_{3n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{vmatrix}$$

is called the determinant of the matrix A and it is denoted by |A| or detA.

(iv) **Scalar matrix** is a diagonal matrix in which all the elements along the main diagonal are equal.

E.g., $\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$

(v) **Unit matrix** is a scalar matrix in which all the elements along the main diagonal are unity.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vi) **Null or Zero matrix**. If all the elements in a matrix are zeros, it is called a null or zero matrix and is denoted by 0.

(vii) **Transpose matrix**. If the rows and columns are interchanged in matrix A, we obtain a second matrix that is called the transpose of the original matrix and is denoted by A^t.

(viii) **Addition of matrices.** Matrices are added, by adding together corresponding elements of the matrices. Hence only matrices of the same order may be added together. The result of addition of two matrices is a matrix of the same order whose elements are the sum of the same elements of the corresponding positions in the original matrices.

E.g., $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{bmatrix}$

Problem:

Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$; compute 3A-4B

Solution :

$$3A-4B=3\begin{bmatrix}1 & 0 & 2\\ 3 & 1 & 4\\ 5 & 0 & 6\end{bmatrix}-4\begin{bmatrix}2 & 1 & -1\\ 3 & 0 & -2\\ 0 & 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0 & 6\\ 9 & 3 & 12\\ 15 & 0 & 18\end{bmatrix}-\begin{bmatrix}8 & 4 & -4\\ 12 & 0 & -8\\ 0 & 4 & 4\end{bmatrix}$$

$$=\begin{bmatrix}-5 & -4 & 10\\ -3 & 3 & 20\\ 15 & -4 & 14\end{bmatrix}$$

Problem: Find values of $x, y, z\ and\ \omega$ that satisfy the matrix relationship

$$\begin{bmatrix}x+3 & 2y+5\\ z+4 & 4x+5\\ \omega-2 & 3\omega+1\end{bmatrix}=\begin{bmatrix}1 & -5\\ -4 & 2x+1\\ 2\omega+5 & -20\end{bmatrix}$$

Solution :

From the equality of these two matices we get the equations

$x+3=1 \qquad 4x+5=2x+1$

$2y+5=-5 \qquad \omega-2=2\omega+5$

$z+4=-4 \qquad 3\omega+1=-20$

Solving these equations we get

$x=-2, y=-5,\ z=-8,\ \omega=-7$

Multiplication of Matrices.

If A is a m × n matrix with rows $A_1, A_2, \ldots, A_m$ and B is a n × p matrix with columns $B_1, B_2, \ldots, B_p$, then the prodduct AB is a m × p matrix C whose elements are given by the formula $C_{ij} = A_i \,.\, B_j$.

$$\text{Hence C = AB} = \begin{bmatrix}A_1.B_1 & A_1.B_2 & \cdots & A_1.B_p\\ A_2.B_1 & A_2.B_2 & \cdots & A_2.B_p\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ A_m.B_1 & A_m.B_2 & \cdots & A_m.B_p\end{bmatrix}$$

Inverse of a Matrix

Problem: Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution:

$$\det\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = 2\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix}$$

= 2(1 + 3) – 1(-6) – 1(-2)

= 8 + 6 + 2

= 16.

Form the matrix of minor determinants:

$$\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -6 & -2 \\ 0 & 4 & -4 \\ 4 & 6 & 2 \end{pmatrix}.$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} 4 & 6 & -2 \\ 0 & 4 & 4 \\ 4 & -6 & 2 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{16}\begin{pmatrix} 4 & 0 & 4 \\ 6 & 4 & -6 \\ -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$.

Problem: Show that A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$. Hence determine its inverse.

Solution: $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $A^2 - 4A - 5I = 0$.

Multiplying by A^{-1}, we have

$A^{-1}A^2 - 4\,A^{-1}A - 5\,A^{-1}I = 0$

i.e., $A - 4I - 5\,A^{-1} = 0$

Therefore $5\,A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$.

Rank of a Matrix

A sub-matrix of a given matrix A is defined to be either A itself or an array remaining after certain rows and columns are deleted from A.

The determinants of the square sub-matrices are called the minors of A.

The rank of an $m \times n$ matrix A is r iff every minor in A of order r + 1 vanishes while there is at least one minor of order r which does not vanish.

Problem: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$.

Solution:

Minor of third order = $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{vmatrix}$

= 0.

The minors of order 2 are obtained by deleting any one row and any one column.

One of the minors of orders 2 is $\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}$

Its value is 8.

Hence the rank of the given matrix is 2.

Rank of a Matrix by Elementary Transformations:

Problem: Find the rank of the matrix A = $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

Solution: The given matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -6 \\ 0 & -1 & -8 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & -1 & -8 \end{bmatrix} R_2 \to R_2(-1)$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 5C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} C_3 \to C_3 - 6C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{R_3}{-2}$$

Hence A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is a unit matrix of order 3.

Hence the rank of the given matrix is 3.

Procedure for finding the solutions of a system of equations:

Let the given system of linear equations be

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n = b_2$

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n = b_m$

Step 1: Construct the coefficient matrix which is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

Step 2: Construct the augmented matrix which is denoted by [A, B]

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & \ldots\ldots & a_{3n} & b_3 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

Step 3: Find the ranks of both the coefficient matrix and augmented matrix which are denoted by R(A) and R(A, B).

Step 4: Compare the ranks of R (A) and R(A, B) we have the following results.

(a) If R(A) = R(A, B) = n (number of unknowns) then the given system of equations are consistent and have unique solutions.

(b) If R(A) = R(A, B) < n (number of unknowns) then the given system of equations are consistent and have infinite number of solutions.

(c) If R(A) ≠ R(A, B) then the given system of equations are inconsistent (that is the given system of equations have no solution).

Problem: Test for consistency and hence solve $x-2y+3z=2, 2x-3z=3, x+y+z=0.$

Solution: The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -9 & -1 \\ 0 & -3 & -2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-5}{19} \end{bmatrix} R_3 \to \frac{4R_3}{19}$$

Here rank of coefficient matrix is 3.

Rank of augmented matrix is 3.

Hence the given system of equations are consistent and have unique solution.

Problem: Test the consistency of the following system of equations and if consistent solve

$$2x - y - z = 2, x + 2y + z = 2, 4x - 7y - 5z = 2.$$

Solution:

The coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix} R_1 \sim R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 5 & 3 & 2 \end{bmatrix} R_3 \to R_3 + R_2$$

Here rank of coefficient matrix is R(A) = 2.

Rank of augmented matrix is R(A, B) = 2.

i.e., R(A) = R(A, B) < 3 (the number of unknowns)

Hence the given system of equations are consistent but have infinite number of solutions.

Here the reduced system is

5y +3z = 2

x + 2y + z = 2

i.e., y = $\frac{2-3z}{5}$

x = 2 – z – 2 ($\frac{2-3z}{5}$)

= $\frac{6+z}{5}$

i.e., x = $\frac{6+k}{5}$, y = $\frac{2-3z}{5}$, z = k where z = k is the parameter.

Solution of Simultaneous Equations

Problem: Solve the system of equations $\begin{aligned} 2x+y+z&=6 \\ x+2y+3z&=6.5 \\ 4x-2y-5z&=2 \end{aligned}$

Solution:

It can be represented as: $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix}$.

To see whether a solution exists we need to find $\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

This determinant is $2\begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}$ = 2(-4) – (-17) + (-10) = -1

Therefore we know that the equations do have a unique solution.

To find the solution we need to find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

Find the determinant: we have already found that this is -1.

Form the matrix of minor determinants (which, for a particular entry in the matrix, is the determinant of the 2 by 2 matrix that is left when the row and column containing the entry are deleted):

$$\begin{pmatrix} -4 & -17 & -10 \\ -3 & -14 & -8 \\ 1 & 5 & 3 \end{pmatrix}$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} -4 & 17 & -10 \\ 3 & -14 & 8 \\ 1 & -5 & 3 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{-1}\begin{pmatrix} -4 & 3 & 1 \\ 17 & -14 & -5 \\ -10 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$.

Hence the solutions to the equations are found by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1 \\ 2 \end{pmatrix}.$$

Therefore x = 2.5, y = -1 and z = 2.

Cayley – Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Problem: *Verify Cayley – Hamilton theorem for the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ *and hence find the inverse of A.*

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(1+4+6) + \lambda(-1-3+0) - [1(-1)-2(-3)+3(-2)] = 0$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$, which is the characteristic equation.

By Cayley – Hamilton theorem , we have to prove

$A^3 - 11A^2 - 4A + 1 = 0$

$$A^2 = A \times .A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix}$$

$$A^3 - 11A^2 - 4A + I = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix} - 11 \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence the theorem is verified.

To find A^{-1}

We have $A^3 - 11A^2 - 4A + I = 0$

$I = -A^3 + 11A^2 + 4A$

$A^{-1} = -A^2 - 11A + 4I$

$$= - \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 11 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{A^{-1}} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

Problem: Find all the eigen values and eigen vectors of the matrix A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+1+1) + \lambda(-3+1+1) - [2(-3)-1(-1)-1(-1)] = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$, which is the characteristic equation.

1	1	-4	-1	4
	0	1	-3	-4
	1	-3	-4	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4 , -1$

Hence **λ = 1 , 4 , -4.**

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \text{................(1)}$$

When $\lambda = 1$, equation (1) becomes

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = 0$

Take first and second equation,

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2+0} = \frac{-x_2}{-2+1} = \frac{x_3}{0-1}$$

$$\Rightarrow \quad \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore \mathbf{x_1} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

When λ = -1 , equation (1) becomes

$3x_1+x_2-x_3 = 0$

$x_1+2x_2-2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2+2} = \frac{-x_2}{-6+1} = \frac{x_3}{6-1}$$

$$\Rightarrow \quad \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore \mathbf{x_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

When λ = 4 , equation (1) becomes

$-2x_1+x_2-x_3 = 0$

$x_1-3x_2-2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2-3} = \frac{-x_2}{4+1} = \frac{x_3}{6-1}$$

$$\Rightarrow \quad \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore \mathbf{x_3} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Hence **Eigen vector =** $\begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

Problem: Find all the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+3+2) + \lambda(4+3+4) - [2(4)-2(1)+1(-1)] = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, which is the characteristic equation.

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \lambda = 1, 5$

Hence λ = 1 , 1 , 5.

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + x_3 &= 0 \\ x_1 + (3-\lambda)x_2 + x_3 &= 0 \\ x_1 + 2x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

Here all the equations are same.

Put $x_3 = 0$, we get $x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$$

$$\therefore \mathbf{x_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 1$, put $x_2 = 0$,we get

$x_1+x_3 = 0$

$x_1= -x_3$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$$

$$\therefore \mathbf{x_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

When $\lambda = 5$, equation(1) becomes

$-3x_1+2x_2+x_3 = 0$

$x_1-2x_2+x_3 = 0$ (taking first and second equation)

$$\Rightarrow \frac{x_1}{2+2} = \frac{-x_2}{-3-1} = \frac{x_3}{6-2}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\therefore$ $\mathbf{x_3}= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. **Hence Eigen vector =** $\begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Problem: ***Find the eigen values and eigen vectors of*** $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Solution : The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(1+2-1) + \lambda(-3-1+3) - [1(-3) - 1(1) - 2(-1)] = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

2	1	-2	-1	2
	0	2	0	-2
	1	0	-1	0

λ = 2 is a root.
The other roots are

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda = 1, -1$$

Hence **λ = 2 , 1 , -1**

The eigen vectors of matrix A is given by

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (-1-\lambda)x_3 &= 0 \end{aligned}\right\} \text{.............................(1)}$$

When λ = 1 ,Equation (1) becomes

$$0x_1 + x_2 - 2x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$\Rightarrow \frac{x_1}{1+2} = \frac{-x_2}{0-2} = \frac{x_3}{0+1}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

When λ = -1 ,Equation (1) becomes

$x_2=0$

$2x_1-2x_3 = 0$

$x_1 = x_3$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

When λ = 2,

Equation (1) becomes

$-x_1+x_2-2x_3=0$

$-x_1+0x_2+x_3=0$(taking first and second equation)

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{-1-2} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Hence **Eigen vector =** $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

Unit – 3
Theory of Equations

Let us consider

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

This a polynomial in 'x' of degree 'n' provided $a_0 \neq 0$.

The equation is obtained by putting f(x) = 0 is called an **algebraic equation** of degree n.

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF EQUATIONS

Let the given equation be $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$

Let α_1, α_2,...... , α_n be its roots.

$\sum \alpha_1$ = sum of the roots taken one at a time = $-\frac{a_1}{a_0}$

$\sum \alpha_1 \alpha_2$ = sum of the product of the roots taken two at a time = $\frac{a_2}{a_0}$

$\sum \alpha_1 \alpha_2 \alpha_3$ = sum of the product of the roots taken three at a time = $-\frac{a_3}{a_0}$

finally we get $\alpha_1 . \alpha_2 ...... \alpha_n = (-1)^n \frac{a_n}{a_0}$.

Problem:

If α and β are the roots of $2x^2 + 3x + 5 = 0$, find α + β, αβ.

Solution:

Here $a_0 = 2, a_1 = 3, a_3 = 5.$

$\sum \alpha$ = α + β = $-\frac{a_1}{a_0} = -\frac{3}{2}$

αβ = $\frac{a_2}{a_0} = \frac{5}{2}$.

Problem:

Solve the equation $x^3 + 6x + 20 = 0$, one root being 1 + 3i.

Solution:

Given equation is cubic. Hence we have 3 roots. One root is (1+3i) = α (say) complex roots occur in pairs.

∴ β = 1 – 3i is another root.

To find third root γ (say)

Sum of the roots taken one at a time

$$\alpha + \beta + \gamma = \frac{0}{1} = 0.$$

i.e., $1 + 3i + 1 - 3i + \gamma = 0$

$$\gamma = -2$$

∴ The roots of the given equation are $1 + 3i$, $1 - 3i$, -2.

Problem:

Solve the equation $3x^3 - 23x^2 + 72x - 20 = 0$ having given that $3 + \sqrt{-5}$ is a root.

Solution:

Given equation is cubic. Hence we have three roots.

One root is $3 + i\sqrt{5} = \alpha$

Since complex roots occur in pairs, $3 - i\sqrt{5} = \beta$ is another root.

Sum of the roots is $\alpha + \beta + \gamma = \frac{23}{3}$

i.e., $3 + i\sqrt{5} + 3 - i\sqrt{5} + \gamma = \frac{23}{3}$

$$6 + \gamma = \frac{23}{3}$$

$$\gamma = \frac{23}{3} - 6$$

$$\gamma = \frac{5}{3}$$

Hence the roots of the given equation are $3 + i\sqrt{5}$, $3 - i\sqrt{5}$, $\frac{5}{3}$.

Problem:

Solve the equation $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$ which has a root $2 + \sqrt{3}$.

Solution:

Given $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$. ----- (1)

This equation is biquadratic, i.e., fourth degree equation.

∴ It has 4 roots. Given $2 + \sqrt{3}$ is a root which is clearly irrational. Since irrational roots occur in pairs, $2 - \sqrt{3}$ is also a root of the given equation.

∴ $[x - (2 + \sqrt{3})]\,[x - (2 - \sqrt{3})]$ is a factor of (1)

i.e., $x^2 - 4x + 1 = 0$ is a factor.

Dividing (1) by $x^2 - 4x + 1 = 0$, we get

$$
\begin{array}{r|l}
 & x^2 + 6x + 7 \\
\hline
x^2 - 4x + 1 & x^4 + 2x^3 - 16x^2 - 22x + 7 \\
 & x^4 - 4x^3 + x^2 \\
\hline
 & (-)\quad 6x^3 - 17x^2 - 22x + 7 \\
 & \qquad 6x^3 - 24x^2 + 6x \\
\hline
 & (-)\qquad 7x^2 - 28x + 7 \\
 & \qquad\quad 7x^2 - 28x + 7 \\
\hline
 & \qquad\qquad 0 \\
\hline
\end{array}
$$

Hence the quotient is $x^2 + 6x + 7 = 0$. Solving this quadratic equation, we get $= -3 \pm \sqrt{2}$.

Hence the roots of the given equation are $2 + \sqrt{3}$, $2 - \sqrt{3}$, $-3 + \sqrt{2}$, $-3 - \sqrt{2}$.

Problem:
Form the equation, with rational coefficients one root of whose roots is $\sqrt{2} + \sqrt{3}$.
Solution:

One root is $\sqrt{2} + \sqrt{3}$

i.e., $x = \sqrt{2} + \sqrt{3}$

i.e., $x - \sqrt{2} = \sqrt{3}$

Squaring on both sides we get

$$(x - \sqrt{2})^2 = 3$$

$$x^2 - 2\sqrt{2}x + 2 = 3$$

$$x^2 - 1 = 2\sqrt{2}x$$

Again squaring, we get

$$(x^2 - 1)^2 = \left(2\sqrt{2}x\right)^2$$

$$x^4 - 2x^2 + 1 = 4.2.x^2$$

$x^4 - 10x^2 + 1 = 0$, which is the required equation.

Problem:

Form the equation with rational coefficients having 1 + $\sqrt{5}$ and 1 + $\sqrt{-5}$ as two of its roots.

Solution:

Given x = 1 + $\sqrt{5}$ and x = 1 + $i\sqrt{5}$

i.e., [x − (1 + $\sqrt{5}$)] [x − (1 + $i\sqrt{5}$)] are the factors of the required equation.

Since complex and irrational roots occur in pairs, we have x = 1 − $\sqrt{5}$, x = 1 − $i\sqrt{5}$ are also the roots of the required equation.

i.e., x − (1 − $\sqrt{5}$) and x − (1 − $i\sqrt{5}$) are also factors of the required equation.

Hence the required equation is,

[x − (1 + $\sqrt{5}$)] [x − (1 + $i\sqrt{5}$)] [x − (1 − $\sqrt{5}$)] [x − (1 − $i\sqrt{5}$)] = 0

i.e., $[(x-1)^2 - 5][(x-1)^2 + 5]$ = 0

$$(x^2 - 2x - 4)(x^2 - 2x + 6) = 0$$

Simplifying we get

$x^4 - 4x^3 + 6x^2 - 4x - 24 = 0$ which is the required equation.

Problem:

Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$ whose roots are in A.P.

Solution:

Let the roots be α – d, α, α + d.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{48}{32}$

$3\alpha = \frac{48}{32}$

$\alpha = \frac{1}{2}$

$\therefore \frac{1}{2}$ is a root of the given equation. By division we have,

$\frac{1}{2}$	32	-48	22	-3
	0	16	-16	3
	32	-32	6	0

The reduced equation is $32x^2 - 32x + 6 = 0$

Solving this quadratic equation we get the remaining two roots $\frac{1}{4}, \frac{3}{4}$.

Hence the roots of the given equation are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Problem:

Find the value of k for which the roots of the equation $2x^3 + 6x^2 + 5x + k = 0$ are in A.P.

Solution:

Given $2x^3 + 6x^2 + 5x + k = 0$ --------- (1)

Let the roots be $\alpha - d, \alpha, \alpha + d$.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{-6}{2}$

$3\alpha = -3$

i.e., $\alpha = -1$

i.e., $\alpha = -1$ is a root of (1).

$\therefore$ put x = - 1 in (1), we get k = 1.

Problem:

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.

Solution:

Given $27x^3 + 42x^2 - 28x - 8 = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time is $\frac{\alpha}{r} . \alpha . \alpha r = \frac{8}{27}$

i.e., $\alpha^3 = \frac{8}{27}$

i.e., $\alpha = \frac{2}{3}$.

i.e., $\alpha = \frac{2}{3}$ is a root of the given equation (1)

i.e., $x = \frac{2}{3}$ is a root of the given equation (1)

i.e., $(x - \frac{2}{3})$ is a factor of (1).

$$\begin{array}{r|l}
 & 27x^2 + 60x + 12 \\
\hline
x - \frac{2}{3} & 27x^3 + 42x^2 - 28x - 8 \\
 & 27x^3 - 18x^2 \\
\hline
 & (-) \quad 60x^2 - 28x - 8 \\
 & \quad\quad 60x^2 - 40x \\
\hline
 & (-) \quad\quad 12x - 8 \\
 & \quad\quad\quad 12x - 8 \\
\hline
 & \quad\quad\quad\quad 0
\end{array}$$

Hence the quotient is $27x^2 + 60x + 12 = 0$

i.e., $9x^2 + 20x + 4 = 0$

Solving this quadratic equation we get x = $-2 \; or \; -\frac{2}{9}$

Hence the roots of the given equation are $-2, -\frac{2}{9}, \frac{2}{3}$.

Problem:

Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in G.P.

Solution:

Given $x^3 - px^2 + qx - r = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time $\frac{\alpha}{r}$. α . αr = r

i.e., $\alpha^3 = r$ -------- (2)

But α is a root of the equation (1). Put x = α in (1), we get,

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0 \quad \text{-------- (3)}$$

Substituting (2) in (3) we get

$$r - p\alpha^2 + q\alpha - \mathrm{r} = 0$$

$$p\alpha^2 - q\alpha = 0$$

$$\alpha(\mathrm{p}\alpha - \mathrm{q}) = 0$$

$\alpha \neq 0$ $\therefore$ pα − q = 0

i.e., $\mathrm{p}\alpha = \mathrm{q}$

i.e., $\alpha = \frac{q}{p}$

$$\therefore \alpha^3 = \frac{\alpha^3}{\beta^3}$$

$$\mathrm{r} = \frac{q^3}{p^3}$$

Hence the required condition is $p^3r = q^3$.

Transformation of Equations:

Problem:

If the roots of $x^3 - 12x^2 + 23x + 36 = 0$ are -1, 4, 9, find the equation whose roots are

1, -4, -9.

Solution:

Given $x^3 - 12x^2 + 23x + 36 = 0$ -------- (1)

The roots are -1, 4, 9.

Now we find an equation whose roots are 1, -4, -9 ie., to find an equation whose roots are the roots of (1) but the signs are changed. Hence in (1) we have to change the sign of odd powers of x.

Hence the required equation is

$$-x^3 - 12x^2 - 23x + 36 = 0$$

i.e., $$x^3 + 12x^2 + 23x - 36 = 0$$

This gives the required equation.

Problem:

Multiply the roots of the equation $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ by $\frac{1}{2}$.

Solution:

Given $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ -------- (1)

To multiply the roots of (1) by $\frac{1}{2}$, we have to multiply the successive coefficients beginning with the second by $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4$

i.e., $$x^4 + \frac{1}{2}\,2x^3 + \left(\frac{1}{2}\right)^2 4x^2 + \left(\frac{1}{2}\right)^3 6x + \left(\frac{1}{2}\right)^4 8 = 0$$

$$x^4 + x^3 + x^2 + \frac{3}{4}x + \frac{1}{2} = 0$$

i.e., $$4x^4 + 4x^3 + 4x^2 + 3x + 2 = 0$$

which is the required equation.

Problem:

Remove the fractional coefficients from the equation $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$.

Solution:

Given $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$ --------- (1)

Multiply by the roots of (1) by m, we get

$$x^3 - \frac{m}{4}x^2 + \frac{m^2}{3}x - m^3 = 0 \quad \text{--------- (2)}$$

If m = 12 (L.C.M. of 4 and 3), the fractions will be removed. Put m = 12 in (2), we get

i.e., $x^3 - 3x^2 + 48x - 1728 = 0$.

Problem:

Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ given that its roots are in H.P.

Solution:

Given $6x^3 - 11x^2 - 3x + 2 = 0$ ---------- (1)

Its roots are in H.P. x to $\frac{1}{x}$ in (1), we get

$$6\left(\frac{1}{x}\right)^3 - 11\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 2 = 0$$

$$\Rightarrow 2x^3 - 3x^2 - 11x + 6 = 0 \quad ----- (2)$$

Now the roots of (2) are in A.P. (Since H.P. is a reciprocal of A.P.). Let the roots of (2) be $\alpha - d,\ \alpha,\ \alpha + d$.

Sum of the roots

$$\alpha - d + \alpha + \alpha + d = \frac{3}{2}$$

$$\Rightarrow 3\alpha = \frac{3}{2}$$

$$\alpha = \frac{1}{2} \quad ----- (3)$$

Product of the roots taken 3 at the time is $(\alpha - d) \times \alpha \times (\alpha + d) = \frac{-11}{2}$

d = $\pm\frac{5}{2}$.

Case(i) :

$When\ \ d = \frac{5}{2}\ and\ \ \alpha = \frac{1}{2}, the\ roots\ of\ (\ )\ are\ \frac{1}{2} - \frac{5}{2}, \frac{1}{2}, \frac{1}{2} + \frac{5}{2}$

$i.e., -2, \frac{1}{2}, 3.$

$\therefore The\ roots\ of\ the\ given\ equation\ are\ the\ reciprocal\ of\ the\ roots\ of\ (\ )$

$i.e., -\frac{1}{2}, 2, \frac{1}{3}.\ are\ roots\ of\ (\ )$

Case (ii) :

$When\ \ d = \frac{-5}{2}\ and\ \ \alpha = \frac{1}{2}, the\ roots\ of\ (\ )\ are\ \frac{1}{2} + \frac{5}{2}, \frac{1}{2}, \frac{1}{2} - \frac{5}{2}$

$i.e., 3, \frac{1}{2}, -2.$

$\therefore The\ roots\ of\ the\ given\ equation\ are\ the\ reciprocal\ of\ the\ roots\ of\ (\ )$

$i.e., \frac{1}{3}, 2, -\frac{1}{2}.\ are\ roots\ of\ (\ )$

Problem:
Diminish the roots of $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0\ by\ 2$ and find the transformed equation.
Solution :
Diminishing the roots by 2, we get

2	1	-5	7	-4	5	
	0	2	-6	2	-4	
2	1	-3	1	-2	1	(constant term of the transformed equation)
	0	2	-2	-2		
2	1	-1	-1	-4		(coefficient of x)
	0	2	2			
2	1	1	1			(coefficient of x^2)
	0	2				
2	1	3				(coefficient of x^3)
	0					
	1					(coefficient of x^4 in the transformed equation)

The transformed equation whose roots are less by 2 of the given equation is
$x^4 + 3x^3 + x^2 - 4x + 1 = 0$

Problem:

Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$ and find the transformed equation.

Solution :

Increasing by 7 the roots of the given equation is the same as diminishing the roots by -7.

-7	3	7	-15	1	-2	
	0	-21	98	-581	4060	
-7	3	-14	83	-580	4058	(constant term of the transformed equation)
	0	-21	245	-2296		
-7	3	-35	328	-2876	(coefficient of x)	
	0	-21	392			
-7	3	-56	720	(coefficient of x^2)		
	0	-21				
-7	3	-77	(coefficient of x^3)			
	0					
	3	(coefficient of x^4 in the transformed equation)				

The transformed equation is $3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$.

Problem:

Find the equation whose roots are the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ increased by 2.

Solution :

-2	1	-1	-10	4	24	
	0	-2	6	8	24	
-2	1	-3	-4	12	0	(constant term of the transformed equation)
	0	-2	10	-12		
-2	1	-5	6	0	(coefficient of x)	
	0	-2	14			
-2	1	-7	20	(coefficient of x^2)		
	0	-2				
-2	1	-9	(coefficient of x^3)			
	0					
	1	(coefficient of x^4 in the transformed equation)				

The transformed equation is $x^4 - 9x^3 + 20x = 0$.

Problem:

If α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$, find an equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.

Solution :

2	1	-6	12	-8
	0	2	-8	8
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	2		
2	1	0		
	0			
	1			

The transformed equation is $x^3 = 0$.

i.e., the roots are = 0, 0, 0.

i.e., $\alpha - 2 = 0,\ \beta - 2 = 0,\ \gamma - 2 = 0$

i.e., $\alpha = 2,\ \beta = 2,\ \gamma = 2$.

Problem:

Find the transformed equation with sign changed $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$.

Solution:

Given that $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$

Given sign =	+	+	+	-	+	-
	+	-	+	-	+	-
	+	-	+	+	+	+

Now the transformed equation $x^5 - 6x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$ which is the required equation.

Special Cases

If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha+\beta = \frac{-b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

If α and β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then $\alpha+\beta+\gamma = \frac{-b}{a}$,

and $\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$ *and* $\alpha\beta\gamma = \frac{-d}{a}$.

Illustrative Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a = \frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow \left(-\frac{p}{3}\right)^3 + p.\left(-\frac{p}{3}\right)^2 + q.\left(-\frac{p}{3}\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\frac{a}{r}, a, ar$

Then , $\frac{a}{r}.a.ar = a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$

Since $a = \frac{2}{3}$ is a root, $\left(x - \frac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\frac{-60 \pm \sqrt{60^2 - 4\times 27\times 12}}{2\times 27} = \frac{-2}{9}$ *or* -2

Hence the roots of the given polynomial eqution are $\frac{-2}{9}$, -2, $\frac{2}{3}$.

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $^1/_a$, $^1/_b$, $^1/_c$ are in arithmetic progression and hence $b = \frac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{9}{15}$ ……….. (1)

$$\alpha\beta\gamma = \frac{1}{15} \quad \text{……… (2)}$$

Since α, β, γ are in harmonic progression, $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \frac{.9}{15} \Rightarrow 3\alpha\gamma = \frac{9}{15}$

$$\Rightarrow \alpha\gamma = \frac{3}{15}.$$

Substitute in (2), we obtain $\frac{3}{15}\beta = \frac{1}{15}$

$\Rightarrow \beta = \frac{1}{3}$ is a root of the given polynomial.

Proceeding as in the above problem, we find that the roots are $\frac{1}{3}$, 1, $\frac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\frac{k}{r}, \ k, \ kr$

Then $\frac{k}{r}.k.kr = \frac{-d}{a}$

i.e., $k^3 = \frac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$$ak^3 + bk^2 + ck + d = 0$$

$$a\left(\frac{-d}{a}\right) + bk^2 + ck + d = 0$$

$$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow \; bk^2 = -ck$$

$$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \; b^3k^6 = -c^3k^3$$

$$\Rightarrow b^3 \frac{d^2}{a^2} = -c^3\left(\frac{-d}{a}\right)$$

$$\Rightarrow \quad \frac{b^3 d}{a} = c^3 \quad \Rightarrow \quad b^3 d = c^3 a.$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio 3: 2.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $\quad 3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$

$$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$$

i.e., $\; 6\alpha^2 + 5\alpha\beta = 14 \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$

and $\quad 3\alpha.2\alpha.\beta = 6\alpha^2\beta = -24$

$$\Rightarrow \alpha^2\beta = -4 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

i.e., $19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\alpha = 2 \; or \; \frac{7}{19}$.

When $\alpha = \frac{7}{19}$, from (1), we get $\beta = \frac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.

1.3. Symmetric Functions of the Roots

Consider the expressions like $\alpha^2 + \beta^2 + \gamma^2, (\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$, $(\beta + \gamma)(\gamma + \alpha)(\alpha - \beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions**.

Generally, a function $f(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ is said to be a symmetric function of $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ if it remains unchanged by interchanging any two of $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

Remark:

The expressions $S_1, S_2, \ldots.., S_n$ where S_r is the sum of the products of $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ taken r at a time, are symmetric functions. These are called **elementary symmetric functions**.

Now we discuss some results about the sums of powers of the roots of a given polynomial equation.

1.3.1. Theorem

The sum of the r^{th} powers of the roots of the equation f(x) = 0 is the coefficient of x^{-r} in the expansion of $\frac{xf'(x)}{f(x)}$ in descending powers of x.

Proof:

Let f(x) = 0 be the given n^{th} degree equation and let its roots be $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ then, f(x) = $a_0(x-\alpha_1)(x-\alpha_2)\ldots.(x-\alpha_n)$ where a_0 is some constant.

Taking logarithm, we obtain

$$\log\ f(x) = \log a_0 + \log(x-\alpha_1) + \ldots. + \log(x-\alpha_n)$$

Differentiating w.r.t. x, we have:

$$\frac{f^1(x)}{f(x)} = \frac{1}{x-\alpha_1} + \ldots\ldots. + \frac{1}{x-\alpha_n}$$

Multiplying by x,

$$\frac{x f^1(x)}{f(x)} = \frac{x}{x-\alpha_1} + \ldots\ldots. + \frac{x}{x-\alpha_n}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1} + \ldots. + \left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= \left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^{\,2}}{x^2}+\ldots.\right) + \ldots\ldots.. + \left(1+\frac{\alpha_n}{x}+\frac{\alpha_n^{\,2}}{x^2}+\ldots.\right)$$

$$= n + \left(\sum\alpha_i\right)x^{-1} + \left(\sum\alpha_i^{\,2}\right)x^{-2} + \ldots\ldots + \ldots\ldots$$

Therefore $\sum\alpha_i^r$ is the coefficient of x^{-r} in the expansion of $\frac{xf'(x)}{f(x)}$ in descending powers of x.

1.3.2. Theorem (Newton's Theorem on the Sum of the Powers of the Roots)

If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are the roots of the equation $x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots. + P_n = 0$,

and $S_r = \alpha_1^r + \ldots. + \alpha_n^r$. Then, $S_r + S_{r-1}P_1 + \ldots.. + S_1 P_{r-1} + rP_r = 0, \quad if \quad r \le n.$

and $S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots + S_{r-n}P_n = 0$ if $r > n$.

Proof:

We have $x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots. + P_n = (x-\alpha_1)(x-\alpha_2)\ldots..(x-\alpha_n)$

Put $x = \dfrac{1}{y}$

$$\Rightarrow \quad \frac{1}{y^n} + \frac{P_1}{y^{n-1}} + \frac{P_2}{y^{n-2}} + \ldots. + P_n = (\frac{1}{y} - \alpha_1)(\frac{1}{y} - \alpha_2)\ldots..(\frac{1}{y} - \alpha_n),$$

and then multiplying by y^n, we obtain:

$$1 + P_1 y + P_2 y^2 + \ldots. + P_n y^n = (1-\alpha_1 y)(1-\alpha_2 y)\ldots..(1-\alpha_n y)$$

Taking logarithm and differentiating w.r.t y, we get

$$\frac{P_1 + 2P_2 y + 3P_3 y^2 + \ldots\ldots + nP_n y^{n-1}}{1 + P_1 y + P_2 y^2 + \ldots\ldots + P_n y^n} = \frac{-\alpha_1}{1-\alpha_1 y} + \frac{-\alpha_2}{1-\alpha_2 y} + \ldots. + \frac{-\alpha_n}{1-\alpha_n y}$$

$$= -\alpha_1(1-\alpha_1 y)^{-1} - \alpha_2(1-\alpha_2 y)^{-1} - \ldots\ldots - \alpha_n(1-\alpha_n y)^{-1}$$

$$= -\alpha_1(1+\alpha_1 y + \alpha^2 y^2 + \ldots.) - \alpha_2(1+\alpha_2 y + \alpha_2^2 y^2 + \ldots.) - \ldots\ldots - \alpha_n(1+\alpha_n y + \alpha_n^2 y^2 + \ldots.)$$

$$= -S_1 - S_2 y - S_3 y^2 - \ldots.. - S_{r+1} y^r - \ldots$$

Cross – multiplying, we get

$$P_1 + 2P_2 y + 3P_3 y^2 + \ldots\ldots + nP_n y^{n-1} = -(1 + P_1 y + P_2 y^2 + \ldots. + P_n y^n)$$

$$[S_1 + S_2 y + \ldots.. + S_{r+1} y^r + \ldots..]$$

Equating coefficients of like powers of y, we see that

$P_1 = -S_1 \quad \Rightarrow \quad S_1 + 1.P_1 = 0$

$2P_2 = -S_2 - S_1 P_1 \quad \Rightarrow \quad S_2 + S_1 P_1 + 2P_2 = 0$

$3p_3 = -S_3 - S_2 P_1 - S_1 P_2 \quad \Rightarrow \quad S_3 + S_2 P_1 + S_1 P_2 + 3P_3 = 0$, and so on .

If $r \le n$, equating coefficients of y^{r-1} on both sides,

$$rP_r = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - \ldots\ldots - S_1 P_{r-1}$$

$$\Rightarrow \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_1P_{r-1} + rP_r = 0$$

If $r > n$, then $r-1 > n-1$.

Equating coefficients of y^{r-1} on both sides,

$$0 = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - \ldots\ldots - S_{r-n}P_n$$

i.e., $S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_{r-n}P_n = 0$

Remark:

To find the sum of the negative powers of the roots of $f(x) = 0$, put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the new equation.

Illustrative Examples

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of the following in terms of the coefficients.

$$\text{(i)}\, \Sigma\frac{1}{\beta\gamma} \text{(ii)}\, \Sigma\frac{1}{\alpha}\ \text{(iii)}\, \Sigma\alpha^2\beta$$

Solution:

Here $\alpha + \beta + \gamma = -p,\ \alpha\beta + \beta\gamma + \alpha\gamma = q,\ \alpha\beta\gamma = -r$

(i) $$\sum\frac{1}{\beta\gamma} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

(ii) $$\sum\frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$$

(iii) $$\Sigma\alpha^2\beta = \alpha^2\beta + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\gamma + \beta^2\gamma$$

$$= (\alpha\beta + \beta\gamma + \alpha\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = (q \,.\, -p) - 3(-r) = 3r - pq\ .$$

2. If α is an imaginary root of the equation $x^7 - 1 = 0$ form the equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4$.

Solution:

Let $a = \alpha + \alpha^6 \quad b = \alpha^2 + \alpha^5 \quad c = \alpha^3 + \alpha^4$

The required equation is $(x - a)(x - b)(x - c) = 0$

i.e., $x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc = 0$ …………… (1)

$$a + b + c = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = \frac{\alpha(\alpha^6 - 1)}{\alpha - 1} = \frac{\alpha^7 - \alpha}{\alpha - 1} = \frac{1 - \alpha}{\alpha - 1} = -1$$

(Since α is a root of $x^7 - 1 = 0$, we have $\alpha^7 = 1$)

Similarly we can find that $ab + bc + ac = -2$, $abc = 1$.

Thus from (1), the required equation is

$x^3 + x^2 - 2x - 1 = 0$

3. If α, β, γ are the roots of $x^3 + 3x^2 + 2x + 1 = 0$, find $\sum \alpha^3$ and $\sum \alpha^{-2}$.

Solution:

Here $\alpha + \beta + \gamma = -3$, $\alpha\beta + \beta\gamma + \alpha\gamma = 2$, $\alpha\beta\gamma = -1$

Using the identity $a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ac)$, we find that

$$\sum \alpha^3 = (\alpha + \beta + \gamma)\left[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= (\alpha + \beta + \gamma)\left[[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)] - (\alpha\beta + \beta\gamma + \alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= -3[(9 - 4) - 2] - 3$$

$$= -9 - 3 = -12$$

Also, $\sum \alpha^{-2} = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$

$$= \frac{(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\sum \alpha^2\beta\gamma}{\alpha^2\beta^2\gamma^2} \quad \text{……………..} \ (1)$$

We have:

$\sum \alpha^2\beta\gamma = (\alpha + \beta + \gamma)\alpha\beta\gamma = -3 . -1 = 3$

$$(1) \Rightarrow \sum \alpha^{-2} = \frac{4 - 2.3}{1} = -2$$

4. Find the sum of the 4th powers of the roots of the equation $x^4 - 5x^3 + x - 1 = 0$.

Solution:

Let $f(x) = x^4 - 5x^3 + x - 1 = 0$

Then $f^1(x) = 4x^3 - 15x^2 + 1$

can be evaluated as follows :

Now, $\dfrac{xf^1(x)}{f(x)}$

$$\begin{array}{r l}
 & \underline{4+5+25+122} + 609 + \ldots\ldots \\
1-5+0+1-1) & 4-15+0+1+0 \\
 & \underline{4-20+0+4-4} \\
 & \quad 5+0-3+4 \\
 & \quad \underline{5-25+0+5-5} \\
 & \qquad 25-3-1+5 \\
 & \qquad \underline{25-125+0+25-25} \\
 & \qquad\quad 122-1-20+25 \\
 & \qquad\quad \underline{122-610+0+122-122} \\
 & \qquad\qquad 609-20-97+122 \\
 & \qquad\qquad \underline{609-3045+0+609-609} \\
 & \qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

Therefore,

$$\frac{xf^{|}(x)}{f(x)} = 4 + \frac{5}{x} + \frac{25}{x^2} + \frac{122}{x^3} + \frac{609}{x^4} + \ldots\ldots$$

Sum of the fourth powers of the roots = coefficient of x^{-4}.

= 609.

5. If $\alpha+\beta+\gamma = 1$, $\alpha^2+\beta^2+\gamma^2 = 2$, $\alpha^3+\beta^3+\gamma^3 = 3$. Find $\alpha^4+\beta^4+\gamma^4$.

Solution:

Let $x^3 + P_1x^2 + P_2x + P_3 = 0$ be the equation whose roots are α, β, γ, then

$$\alpha+\beta+\gamma = -P_1 \Rightarrow P_1 = -1$$

By Newton's theorem,

$$S_2 + S_1P_1 + 2P_2 = 0$$

i.e., $2 + 1.(-1) + 2P_2 = 0 \Rightarrow P_2 = {}^{-1}/_2$

Again, by Newton's theorem

$$S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$$

i.e., $3 + 2.-1 + 1.{}^{-1}/_2 + 3.P_3 = 0$

$\Rightarrow P_3 = {}^{-1}/_6$

Also $S_4 + S_3P_1 + S_2P_2 + S_1P_3 = 0$ (By Newton's theorem for the case $r < n$)

Substituting and simplifying, we obtain $S_4 = {}^{25}/_6$

Thus $\alpha^4+\beta^4+\gamma^4 = \dfrac{25}{6}$

6. Calculate the sum of the cubes of the roots of $x^4 + 2x + 3 = 0$

Solution:

Let the given equation be

$$x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$$

Here $P_1 = P_2 = 0$, $P_3 = 2$ and $P_4 = 3$

By Newton's theorem, $S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$

i.e., $S_3 + 0 + 0 + 3 . 2 = 0$

$\Rightarrow S_3 = -6$

i.e., sum of the cubes of the roots of $x^4 + 2x + 3 = 0$, is -6.

1.4. Transformations of Equations

Let $f(x) = 0$ be a polynomial equation. Without explicitly knowing the roots of $f(x) = 0$, we can often transform the given equation into another equation whose roots are related to the roots of the first equation in some way. Now we discuss some important such transformations.

1. To form an equation whose roots are k-times the roots of a given equation.

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n$ ----------- (1)

Suppose that $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of $f(x) = 0$

Then $f(x) = a_o(x-\alpha_1)(x-\alpha_2)........(x-\alpha_n)$ …………….. (2)

Put $y = kx$ in (2), we obtain:

$$f\left(\frac{y}{k}\right) = a_o\left(\frac{y}{k} - \alpha_1\right)\left(\frac{y}{k} - \alpha_2\right)......\left(\frac{y}{k} - \alpha_n\right)$$

Thus the roots of $f(y/k) = 0, \ are \ k\alpha_1, ..., k\alpha_n$

Therefore the required equation is

$$f\left(\frac{y}{k}\right) = a_o\left(\frac{y}{k}\right)^n + a_1\left(\frac{y}{k}\right)^{n-1} + ... + a_n = 0$$

$$\text{i.e.,} \quad a_o y^n + k a_1 y^{n-1} + k^2 a_2 y^{n-2} + ... + k^n a_n = 0$$

Thus; to obtain the equation whose roots are k times the roots of a given equation, we have to multiply the coefficients of $x^n, x^{n-1}, ..., x$ and the constant term by 1, k, k^2,……k^{n-1} and k^n respectively.

Remark:

To form an equation whose roots are the negatives of the roots of a given equation of degree n, multiply the coefficients of x^n, x^{n-1}, ... by 1, -1, 1, -1, … respectively.

2. To form an equation whose roots are the reciprocals of the roots of a given equation.

Consider, $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ ………….. (1)

Let $\alpha_1, \alpha_1, ..., \alpha_n$ be the roots of the equation. Then,

$$f(x) = a_o (x - \alpha_1)(x - \alpha_2)......(x - \alpha_n) \quad \text{………….. (2)}$$

In (1), put $y = \frac{1}{x}$ *i.e.,* $x = \frac{1}{y}$

Then $f\left(\frac{1}{y}\right) = a_o \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)........\left(\frac{1}{y} - \alpha_n\right)$

The roots of this equation are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ...\frac{1}{\alpha_n}$

But from (1) , $f\left(\frac{1}{y}\right) = a_o \left(\frac{1}{y}\right)^n + a_1 \left(\frac{1}{y}\right)^{n-1} + ... + a_n = 0$

i.e., $a_o + a_1 y + a_o y^2 + ... + a_n y^n = 0$

Therefore, the required equation is $a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y + a_o = 0$

3. To form an equation whose roots are less by 'h' then the roots of a given equation. (i.e., Diminishing the roots by h)

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ …………. (1)

Suppose that $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of f(x) = 0

Therefore, $f(x) = a_o (x - \alpha_1)(x - \alpha_2).......(x - \alpha_n)$ …………. (2)

Put y = x – h so that x = y + h

From (2), $f(y + h) = a_o (y + h - \alpha_1)(y + h - \alpha_2).......(y + h - \alpha_n)$

$$= a_o (y - (\alpha_1 - h))(y - (\alpha_2 - h))......(y - (\alpha_n - h))$$

The roots of f (y + h) = 0 are $\alpha_1 - h, ..., \alpha_n - h$.

By (1), we obtain,

$$a_o (y + h)^n + a_1 (y + h)^{n-1} + ... + a_n = 0$$

Expanding using binomial theorem and combining like terms, we get an equation of the form

$$b_o y^n + b_1 y^{n-1} + ... + b_n = 0 \qquad .... (3)$$

Replacing y = x – h, we get

$$b_o(x-h)^n + b_1(x-h)^{n-1} + ... + b_n = 0 \qquad .... (4)$$

Now, equation (1) and (4) represents the same equation.

Dividing equation (4) continuously by (x – h), we obtain the remainders as $b_n, b_{n-1}, ..., b_o$

Substituting these in (3), we obtain the required equation.

Remark:

Increasing the roots by h is equivalent to decreasing the roots by –h.

4. To form an equation in which certain specified terms of the given equation are absent.

Consider the equation $a_o x^n + a_1 x^{n-1} + ... + a_n = 0 \qquad .... (1)$

Suppose it is required to remove the second term of the equation (1). Diminish the roots of the given equation by h.

For this, put $y = x - h$ i.e., x = y + h in (1), we obtain the new equation as

$$a_o(y+h)^n + a_1(y+h)^{n-1} + ... + a_n = 0$$

$$\text{ie} \quad a_o y^n + (na_o h + a_1)y^{n-1} + ... + a_n = 0$$

Now to remove the second term of the equation (1), we must have $na_o h + a_1 = 0$

i.e., we must have $h = {-a_1}/{na_o}$.

Thus to remove the second term of the equation (1), we have to diminish its roots by $h = {a_1}/{na_o}$

Remarks:

If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the polynomial equation f(x) = 0. Formation of an equation whose roots are $\phi(\alpha_1), \phi(\alpha_2), ..., \phi(\alpha_n)$ is known as a **general transformation** of the given equation.

In this case, the relation between a root x of f(x) = 0 and a root y of the transformed equation is that $y = \phi(x)$. Also, to obtain this new equation we have to eliminate x between f(x) = 0 and $y = \phi(x)$.

Solved Problems

1. Form an equation whose roots are three times those of the equation

$$x^3 - x^2 + x + 1 = 0.$$

Solution:

To obtain the required equation, we have to multiply the coefficients of x^3, x^2, x, and 1 by 1, 3, 3^2, and 3^3 respectively.

Thus $x^3 - 3x^2 + 9x + 27 = 0$ is the desired equation.

2. Form an equation whose roots are the negatives of the roots of the equation

Solution:

By multiplying the coefficients successively by 1, -1, 1, -1 we obtain the required equation as $x^3 + 6x^2 + 8x + 9 = 0$.

3. Form an equation whose roots are the reciprocals of the roots of

$$x^4 - 5x^3 + 7x^2 - 4x + 5 = 0.$$

Solution:

We obtain the required equation, by replacing the coefficients in the reverse order, as $5x^4 - 4x^3 + 7x^2 - 5x + 1 = 0$

4. Find the equation whose roots are less by 2, than the roots of the equation $x^5 - 3x^4 - 2x^3 + 15x^2 + 20x + 15 = 0$.

Solution:

To find the desired equation, divide the given equation successively by x – 2.

2	1	-3	-2	+15	+20	+15
		2	-2	-8	+14	68
	1	-1	-4	+7	+34	83
		2	+2	-4	+6	
	1	+1	-2	+3	+40	
		+2	+6	+8		
	1	+3	+4	+11		
		2	+10			
	1	+5	+14			
		+2				
	1	+7				
	1					

Thus the required equation is

$$x^5 + 7x^4 + 14x^3 + 11x^2 + 40x + 83 = 0$$

5. Solve the equation $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ by removing its second term.

Solution:

To remove the second term, we have to diminish the roots of the given equation by $h = \frac{-a_1}{na_o} = \frac{8}{4.1} = 2$.

Dividing the given equation successively by x – 2, we obtain the new equation as

$x^4 - 25x^2 + 144 = 0$

On solving, we get $x =$ - 4 , 4, - 3 , 3.

Thus the roots of the original equation are –2, 6, -1 and 5.

6. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$. Form the equation whose roots are $\alpha\beta, \beta\gamma, \gamma\alpha$.

Solution:

Note that $\alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = \frac{-c}{\gamma}$

Put $y = \frac{-c}{x} \quad \Rightarrow \quad x = \frac{-c}{y}$

Hence the given equation becomes

$$\left(\frac{-c}{y}\right)^3 + a\left(\frac{-c}{y}\right)^2 + b\left(\frac{-c}{y}\right) + c = 0$$

i.e., $y^3 - by^2 + acy - c^2 = 0$, which is the required equation.

7. If α, β, γ are the roots of $x^3 - x + 1 = 0$, show that $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = 1$

Solution:

We have to form the equation whose roots are $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$.

For this, put $y = \frac{1+x}{1-x}$ i.e., $x = \frac{y-1}{y+1}$

Therefore the required equation is $\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) + 1 = 0$

On simplifying, we obtain $y^3 - y^2 + 7y + 1 = 0$

The sum of the roots of this equation is 1. i.e., $\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}=1$

1.5. Reciprocal Equations

Let f(x) = 0 be an equation with roots $\alpha_1, \alpha_2, ..., \alpha_n$.

If $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ are also roots of the same equation, then such equations are called **reciprocal equations**.

Suppose that $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ …... (1) is a reciprocal equation with roots $\alpha_1, \alpha_2, ..., \alpha_n$.

Then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ are also roots of the same equation. The equation with roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ is : $a_n x^n + a_{n-1} x^{n-1} + ... + a_o = 0$ …… (2)

Since (1) and (2) represents the same equation, we must have $\frac{a_o}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_n}{a_o} = k$

Taking the first and last terms in the above equality, we obtain $k^2 = 1$ i.e., $k = \pm 1$

when k = 1, we have $a_o = a_n$, $a_1 = a_{n-1}$ . . . .

Such equations are called reciprocal equations of **first type**.

When k = –1, we have $a_o = -a_n$, $a_1 = -a_{n-1}$, …… These type of equations are called reciprocal equations of **second type**.

A reciprocal equation of first type and even degree is called a **standard reciprocal equation**.

Note:

1. If f(x) = 0 is a reciprocal equation of first type and odd degree, the x = –1 is always a root. If we remove the factor x + 1 corresponding to this root, we obtain a standard reciprocal equation.
2. If f(x) = 0 is a reciprocal equation of second type and odd degree, then x = 1 is always a roots. If we remove the factor x –1 corresponding to this root, we obtain a standard reciprocal equation.

3. If $f(x) = 0$ is a reciprocal equation of second type and even degree, then $x = 1$ and $x = -1$ are roots. If we remove the factor $x^2 - 1$ corresponding to these roots, we obtain a standard reciprocal equation.

Solved Problems

1. Solve the equation $60x^4 - 736x^3 + a433x^2 - 736x + 60 = 0$

Solution:

The given equation is a standard reciprocal equation. Dividing throughout by x^2, we obtain,

$$60x^2 - 736x + 1433 - \frac{736}{x} + \frac{60}{x^2} = 0$$

$$60\left(x^2 + \frac{1}{x^2}\right) - 736\left(x + \frac{1}{x}\right) + 1433 = 0$$

Putting $y = x + 1/x$ and simplifying, we obtain

$$60y^2 - 736y + 1313 = 0$$

On solving, we get $y = \frac{101}{10}$ or $\frac{13}{6}$

When $y = \frac{101}{10}, x + \frac{1}{x} = \frac{101}{10} \Rightarrow 10x^2 - 101x + 10 = 0$

i.e., $x = 10, \frac{1}{10}$

Similarly when $y = \frac{13}{6}$, we get $x = \frac{3}{2}, 2/3$

Thus the roots of the given equation are $10, \frac{1}{10}, \frac{3}{2}, \frac{2}{3}$

2. Solve :

$$x^2 - 5x^2 + 9x3 - 9x^2 + 5x - 1 = 0$$

Solution:

This is a second type reciprocal equation of odd degree. So $x = 1$ is a root. On division by the corresponding factor $x - 1$, we obtain the other factor as $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$, which is a standard reciprocal equation.

Proceeding exactly as in the above problem, we may find that

$$x = \frac{1 \pm i\sqrt{3}}{2} \text{ or } x = \frac{3 \pm \sqrt{5}}{2}$$

Hence the roots of the given equation are $1, \frac{1 \pm i\sqrt{3}}{2}, \frac{3 \pm \sqrt{5}}{2}$

3. Show that on diminishing the roots of the equation

$$6x^4 - 43x^3 + 76x^2 + 25x - 100 = 0$$

by 2, it becomes a reciprocal equation and hence solve it.

Solution:

To diminish the roots of the given equation by 2, divide it successively by (x – 2), we obtain:

2	6	-43	+76	+25	-100
		+12	-62	+28	+106
	6	-31	+14	+53	+6
		+12	-38	-48	
	6	-19	-24	+5	
		+12	-14		
	6	-7	-38		
		+12			
	6	+5			
	6				

$\Rightarrow 6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ is the required equation, which is a standard reciprocal equation.

It can be written as

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$$

Putting $x + \frac{1}{x} = y$ and solving for y, we get $y = \frac{-10}{3}$ or $\frac{5}{2}$

When $y = \frac{5}{2}$, we have $x + \frac{1}{x} = \frac{5}{2}$. On solving we get: $x = 2, \frac{1}{2}$

When $y = \frac{-10}{3}$, we have $3x^2 + 10x + 3 = 0$ or $x = -3$ or $-\frac{1}{3}$

Thus the roots of the original equation are $4, \frac{5}{2}, -1, \frac{5}{3}$ (by adding 2 to each of the above roots)

UNIT-4
TRIGNOMETRY

Expansions of Cosnθ And Sinnθ in Powers of Sinθ and Cosθ

By Demoivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

By Binomial theorem,

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \cos^n\theta + nC_1\cos^{n-1}\theta(i\sin\theta) \\ &\quad + nC_2\cos^{n-2}\theta(i\sin\theta)^2 \\ &\quad + nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \ldots\end{aligned}$$

i.e. $$\begin{aligned}\cos n\theta + i\sin n\theta &= \cos^n\theta + i\,n\cos^{n-1}\theta\sin\theta - nC_2\cos^{n-2}\theta\sin^2\theta \\ &\quad - i\,nC_3\cos^{n-3}\theta\sin^3\theta + nC_4\cos^{n-4}\theta\sin^4\theta + \ldots\end{aligned}$$

Equating real and imaginary parts,

$$\cos n\theta = \cos n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - \ldots..$$

$$\&\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta - \ldots.$$

Expansion of tannθ interms of tanθ

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + \ldots.}{\cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + \ldots}$$

Divide Nr. and Dr. by cosnθ

$$\tan n\theta = \frac{nC_1\tan\theta - nC_3\tan^3\theta + nC_5\tan^5\theta - \ldots.}{1 - nC_2\tan^2\theta + nC_4\tan^4\theta + \ldots}$$

Expansion of tan $(\theta_1+\theta_2+....+\theta_n)$

$$\tan\left(\theta_1+\theta_2+...+\theta_n\right)=\frac{S_1-S_3+S_5-S_7+...}{1-S_2+S_4-S_6}$$

Where $S_1=\sum\tan\theta_1 \quad S_2=\sum\tan\theta_1\tan\theta_2......$

Problems:

1) Prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Proof:

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - ...$$

$$\cos 4\theta = \cos^4\theta - 4C_2\cos^2\theta\sin^2\theta + 4C_4\cos^0\theta\sin^4\theta$$

$$= \cos^4\theta - \frac{4\times 3}{1\times 2}\cos^2\theta\left(1-\cos^2\theta\right) + 1\cdot\left(1-\cos^2\theta\right)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + \left[1-2\cos^2\theta+\cos^4\theta\right]$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Hence the proof.

2) Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Proof:

$$\cos 7\theta = \cos^7\theta - 7C_2\cos^5\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta - 7C_6\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta\left(1-\cos^2\theta\right) + 35\cos^3\theta\left(1-\cos^2\theta\right)^2 - 7\cos\theta\left(1-\cos^2\theta\right)^3$$

$$= \cos^7\theta - 21\cos^5\theta\left(1-\cos^2\theta\right) + 35\cos^3\theta\left[1+\cos^4\theta-2\cos^2\theta\right]$$

$$-7\cos\theta\left[1-\cos^6\theta-3\cos^2\theta+3\cos^4\theta\right]$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta + 24\cos^7\theta - 70\cos^5\theta$$

$$-7\cos\theta + 7\cos^7\theta + 21\cos^3\theta - 21\cos^5\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

3) Find cos8θ interms of sinθ **Nov'15**

Solution:

$$\text{Cos}\,8\theta = \cos^8\theta - 8C_2\cos^6\theta\sin^2\theta + 8C_4\cos^4\theta\sin^4$$

$$-8C_6\cos^2\theta\sin^6\theta + 8C_8\sin^8\theta$$

$$= \left(1-\sin^2\theta\right)^4 - 28\left(1-\sin^2\theta\right)^3\sin^2\theta + 70\left(1-\sin^2\theta\right)^2\sin^4\theta$$
$$-28\left(1-\sin^2\theta\right)\sin^6\theta + \sin^8\theta$$

$$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta - 28\sin^2\theta + 84\sin^4\theta - 84\sin^6\theta$$
$$+ 28\sin^8\theta + 70\sin^4\theta - 140\sin^6\theta + 70\sin^8\theta - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta$$

$$\mathrm{Cos}\,8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$$

4) Prove that $\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ **Apr'18**

Proof:

$$\sin 6\theta = 6C_1\cos^5\theta\sin\theta - 6C_3\cos^3\theta\sin^3\theta + 6C_5\cos\theta\sin^5\theta$$

$$\sin 6\theta = \cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$$\frac{\sin 6\theta}{\sin\theta} = \cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 20\cos^3\theta\left(1-\cos^2\theta\right) + 6\cos\theta\left(1-\cos^2\theta\right)^2$$
$$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

1) Show that $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$ **Nov'15**

Proof:

We know that,

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta$$
$$-nC_6\cos^{n-6}\theta\sin^6\theta + nC_8\cos^{n-8}\theta\sin^8\theta$$

Put n=8

$$\cos 8\theta = \cos^8\theta - 8C_2\cos^{8-2}\theta\sin^2\theta + 8C_4\cos^{8-4}\theta\sin^4\theta$$
$$-8C_6\cos^{8-6}\theta\sin^6\theta + 8C_8\cos^{8-8}\theta\sin^8\theta$$

$$= \cos^8\theta - \frac{8\times7}{1\times2}\cos^6\theta\sin^2\theta + \frac{8\times7\times6\times5}{1\times2\times3\times4}\cos^4\theta\sin^4\theta$$
$$-\frac{8\times7\times6\times5\times4\times3}{1\times2\times3\times4\times5\times6}\cos^2\theta\sin^6\theta + \frac{8\times7\times6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6\times7\times8}\cos^0\theta\sin^8\theta$$

$$= \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$$

$$= \cos^8\theta - 28\cos^6\theta\left(1-\cos^2\theta\right) + 70\cos^4\theta\left(1-\cos^2\theta\right)^2$$

$$-28\cos^2\theta\left(1-\cos^2\theta\right)^3 + \left(1-\cos^2\theta\right)^4$$

$$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta\left(1-2\cos^2\theta+\cos^4\theta\right)$$

$$-28\cos^2\theta\left(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\right)^3 + \left(1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta\right)$$

$$= \cos^8\theta\left[1+28+70+28+1\right] + \cos^6\theta\left[-28-140-84-4\right]$$

$$+\cos^4\theta\left[70+84+6\right] + \cos^2\theta\left[-28-4\right] + 1$$

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$$

$$\cos^8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$$

Hence the Proof.

4) Prove that $\dfrac{\cos 9\theta}{\cos\theta} = 256\cos^8\theta - 576\cos^6\theta + 432\cos^4\theta - 120\cos^2\theta + 9$ Apr' 17.

Proof:

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta$$

$$-nC_6\cos^{n-6}\theta\sin^6\theta + nC_8\cos^{n-8}\theta\sin^8\theta$$

Put n=9,

$$\cos 9\theta = \cos^9\theta - 9C_2\cos^{9-2}\theta\sin^2\theta + 9C_4\cos^{9-4}\theta\sin^4\theta$$

$$-9C_6\cos^{9-6}\theta\sin^6\theta + 9C_8\cos^{9-8}\theta\sin^8\theta$$

$$= \cos^9\theta - \frac{9\times 8}{1\times 2}\cos^7\theta\sin^2\theta + \frac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4}\cos^5\theta\sin^4\theta$$

$$-\frac{9\times 8\times 7\times 6\times 5\times 4}{1\times 2\times 3\times 4\times 5\times 6}\cos^3\theta\sin^6\theta + \frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}\cos\theta\sin^8\theta$$

$$\cos 9\theta = \cos^9\theta - 36\cos^7\theta\sin^2\theta + 126\cos^5\theta\sin^4\theta$$

$$-84\cos^3\theta\sin^6\theta + 9\cos\theta\sin^8\theta$$

Now,

$$\frac{\cos 9\theta}{\cos\theta} = \cos^8\theta - 36\cos^6\theta\sin^2\theta + 126\cos^4\theta\sin^4\theta$$

$$-84\cos^2\theta\sin^6\theta + 9\sin^8\theta$$

$$= \cos^8\theta - 36\cos^6\theta\left(1-\cos^2\theta\right) + 126\cos^4\theta\left(1-\cos^2\theta\right)^2$$

$$-84\cos^2\theta\left(1-\cos^2\theta\right)^3 + 9\left(1-\cos^2\theta\right)^4$$

$$= \cos^8\theta - 36\cos^6\theta + 36\cos^8\theta + 126\cos^4\theta\left(1-2\cos^2\theta+\cos^4\theta\right)$$

$$-84\cos^2\theta\left(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\right)$$

$$+9\left(1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta\right)$$

$$= \cos^8\theta\left[1+36+126+84+9\right] + \cos^6\theta\left[-36-252-252-36\right]$$

$$+\cos^4\theta\left[126+252+54\right] + \cos^2\theta\left[-84-36\right] + 9$$

$$\frac{\cos 9\theta}{\cos\theta} = 256\cos^8\theta - 576\cos^6\theta + 432\cos^4\theta - 120\cos^2\theta + 9$$

Hence the proof.

3) Express $\frac{\sin 7\theta}{\sin\theta}$ in a series of power of sinθ **Apr' 16.**

Solution:

We know that

$$\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta$$

$$+ nC_5\cos^{n-5}\theta\sin^5\theta - nC_7\cos^{n-7}\theta\sin^7\theta + ...$$

Put =n=7

$$\sin 7\theta = 7C_1\cos^{7-1}\theta\sin\theta - 7C_3\cos^{7-3}\theta\sin^3\theta$$

$$+ 7C_5\cos^{7-5}\theta\sin^5\theta - 7C_7\cos^{7-7}\theta\sin^7\theta$$

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$\cos^2\theta = 1-\sin^2\theta;\ \cos^4\theta = \left(1-\sin^2\theta\right)^2;\ \cos^6\theta = \left(1-\sin^2\theta\right)^3$$

$$\therefore \frac{\sin 7\theta}{\sin\theta} = 7\left(1-\sin^2\theta\right)^3 - 35\sin^2\theta\left(1-\sin^2\theta\right)^2 + 21\left(1-\sin^2\theta\right)\sin^4\theta - \sin^6\theta$$

$$= 7\left[1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta\right] - 35\sin^2\theta\left[1 - 2\sin^2\theta + \sin^4\theta\right]$$
$$+ 21\sin^4\theta\left(1 - \sin^2\theta\right) - \sin^6\theta$$
$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta$$
$$- 35\sin^6\theta + 21\sin^4\theta - 21\sin^6\theta - \sin 6\theta$$
$$= -64\sin^6\theta + 112\sin^4\theta - 56\sin^2\theta + 7$$

5) Prove that $\dfrac{\cos 5\theta}{\cos\theta} = 1 - 12\sin^2\theta + 16\sin^4\theta$ **Nov'16**

Proof:

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta$$
$$\cos 5\theta = \cos^5\theta - 5C_2\cos^{5-2}\theta\sin^2\theta + 5C_4\cos^{5-4}\theta\sin^4\theta$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\frac{\cos 5\theta}{\cos\theta} = \cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta$$
$$\cos^2\theta = 1 - \sin^2\theta; \qquad \cos^4\theta = \left(1 - \sin^2\theta\right)^2$$
$$= 1 - 2\sin^2\theta + \sin^4\theta$$
$$\therefore \frac{\cos 5\theta}{\cos\theta} = 1 - 2\sin^2\theta + \sin^4\theta - 10\sin^2\theta\left(1 - \sin^2\theta\right) + 5\sin^4\theta$$
$$= 1 - 2\sin^2\theta + \sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + 5\sin^4\theta$$
$$= 1 - 12\sin^2\theta + 16\sin^4\theta$$

Hence the proof.

Expansion of $\sin^n\theta$ and $\cos^n\theta$ interms of multiples of θ.

Let

$$x = \cos\theta + i\sin\theta \qquad x^n = \cos n\theta + i\sin n\theta$$
$$\frac{1}{x} = \cos\theta - i\sin\theta \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$
$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$
$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\left(x+\frac{1}{x}\right)^n=(2\cos\theta)^n=2^n\cos^n\theta$$

$$\left(x-\frac{1}{x}\right)^n=(2i\sin\theta)^n=2^n i^n\sin^n\theta$$

Problem:

1) Expand $\cos^8\theta$ in a series of cosines of multiples of θ.

Solution:

Let

$$x=\cos\theta+i\sin\theta \qquad x^n=\cos n\theta+i\sin n\theta$$

$$\frac{1}{x}=\cos\theta-i\sin\theta \qquad \frac{1}{x^n}=\cos n\theta-i\sin n\theta$$

$$x+\frac{1}{x}=2\cos\theta \qquad x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x-\frac{1}{x}=2i\sin\theta \qquad x^n-\frac{1}{x^n}=2i\sin n\theta$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

To find $\cos^8\theta$,

Consider $\left(x+\frac{1}{x}\right)^8=(2\cos\theta)^8$

i.e. $$2^8\cos^8\theta=x^8+8x^7\left(\frac{1}{x}\right)+28x^6\left(\frac{1}{x}\right)^2+56x^5\left(\frac{1}{x}\right)^3+70x^4\left(\frac{1}{x}\right)^4$$

$$+56x^3\left(\frac{1}{x}\right)^5+28x^2\left(\frac{1}{x}\right)^6+8x\left(\frac{1}{x}\right)^7+\left(\frac{1}{x}\right)^8$$

$$=x^8+8x^6+28x^4+56x^2+70+\frac{56x}{x^2}+\frac{28}{x^4}+\frac{8}{x^6}+\frac{1}{x^8}$$

$$=\left[x^8+\frac{1}{x^8}\right]+8\left[x^6+\frac{1}{x^6}\right]+28\left[x^4+\frac{1}{x^4}\right]+56\left[x^2+\frac{1}{x^2}\right]+70$$

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$$\cos^8\theta=\frac{2}{2^7}\left[\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35\right]$$

$$\left(\because x^n+\frac{1}{x^n}=2\cos n\theta\right)$$

2) Expand $\sin^5\theta$ in a series of sines of multiples of θ.

Solution:

$$x+\frac{1}{x}=2\cos\theta \qquad x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x-\frac{1}{x}=2i\sin\theta \qquad x^n-\frac{1}{x^n}=2i\sin n\theta$$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad -5 \quad 10 \quad -10 \quad 5 \quad -1 \end{array}$$

To find $\sin^5\theta$,

For sin expansion change signs alternatively in the last step

$i^2=-1$

$i^4=+1$

$i^5=i$

$$\text{consider}\left(x-\frac{1}{x}\right)^5=(2i\sin\theta)^5=2^5 i^5\sin^5\theta$$

$$\text{i.e.}\quad 2^5 i\sin^5\theta=x^5-5x^4\cdot\frac{1}{x}+10x^3\cdot\frac{1}{x^2}-10x^2.\frac{1}{x^3}+5x\frac{1}{x^4}-\frac{1}{x^5}$$

$$=\left[x^5-\frac{1}{x^5}\right]-5\left[x^3-\frac{1}{x^3}\right]+10\left[x-\frac{1}{x}\right]$$

$$2^5 i\sin^5\theta=2i\sin5\theta-5[2i\sin3\theta]+10[2i\sin\theta]$$

$$\therefore\sin^5\theta=\frac{1}{2^4}[\sin5\theta-5\sin3\theta+10\sin\theta]$$

3) Expand $\sin^3\theta\cos^5\theta$ in a series of sines of multiples of θ.

Solution:

$$x+\frac{1}{x}=2\cos\theta \qquad x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x-\frac{1}{x}=2i\sin\theta \qquad x^n-\frac{1}{x^n}=2i\sin n\theta$$

Power $\boxed{8}$

$$\begin{array}{cl} 1 & \\ 1 \quad 1 & \\ 1 \quad 2 \quad 1 & \\ 1 \quad -3 \quad 3 \quad -1 & \to 3 \\ 1 \quad -2 \quad 0 \quad 2 \quad -1 & \to 4 \\ 1 \quad -1 \quad -2 \quad 2 \quad 1 \quad -1 & \to 5 \\ 1 \quad 0 \quad -3 \quad 0 \quad 3 \quad 0 \quad -1 & \to 6 \\ 1 \quad 1 \quad -3 \quad -3 \quad 3 \quad 3 \quad -1 \quad -1 & \to 7 \\ 1 \quad 2 \quad -2 \quad -6 \quad 0 \quad 6 \quad 2 \quad -2 \quad -1 & \to 8 \end{array}$$

To find $\sin^3\theta\cos^5\theta$,

$$\text{consider}\left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^5=(2\cos\theta)^5(2\sin\theta)^3$$

$$\sin^3\theta\cos^5\theta=\frac{1}{2^5\cdot i^3\cdot 2^3}\left[\left(x-1/x\right)^3\left(x+1/x\right)^5\right]$$

Write the Pascal triangle upto 3 steps (for sine), change the sign alternatively and continue the triangle upto 5 steps (for cosine).

$$= \frac{1}{-i2^8}\left[\left[x^8 - \frac{1}{x^8}\right] - 2\left[x^6 - \frac{1}{x^6}\right] - 2\left[x^4 - \frac{1}{x^4}\right] - 6\left[x^2 - \frac{1}{x^2}\right]\right]$$

$$= \frac{1}{-i2^8}\left[2i\sin 8\theta - 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)\right]$$

$$\sin^3\theta\cos^5\theta = \frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

4) Express $\sin^4\theta\cos^2\theta$ in a series of cosines of multiples of θ.

Solution:

$$x + \frac{1}{x} = 2\cos\theta \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$x - \frac{1}{x} = 2i\sin\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

To find $\sin^4\theta\cos^2\theta$

							1							
						1		1						
					1		2		1					
	sin			1		3		3		1				
Change sign alternatively	power 4→		1		−4		6		−4		1			
	(1)-			1	−3	2		2	−3	1				
	(2)-	1	−2	−1			4		−1		−2		1	

$$\text{consider } \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^2$$

$$= (2i\sin\theta)^4(2\cos\theta)^2$$

$$= 2^4 i^4 2^2 \sin^4\theta\cos^2\theta$$

$$\therefore \sin^4\theta\cos^2\theta = \frac{1}{26}\left[\left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^2\right]$$

$$= \frac{1}{2^6}\left[\begin{array}{l} x^6 - 2x^5\frac{1}{x} - 1x^4\frac{1}{x^2} + 4x^3\frac{1}{x^3} \\ \qquad\qquad - 1x^2\frac{1}{x^4} - 2x\frac{1}{x^5} + 1\frac{1}{x^6} \end{array}\right]$$

$$= \frac{1}{2^6}\left[\left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - 1\left(x^2 + \frac{1}{x^2}\right) + 4\right]$$

$$= \frac{1}{2^6}\left[2\cos 6\theta - 2(2\cos 4\theta) - 1(2\cos 2\theta) + 4\right]$$

$$= \frac{1}{2^5}\left[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\right]$$

5) Prove that $64\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$

6) Express $\cos^9\theta$ in a series of cosines of multiples of θ.

Ans: $\frac{1}{28}\left[\cos 9\theta + 9\cos 7\theta + 35\cos 5\theta + 84\cos 3\theta + 126\cos\theta\right]$

7) Prove that $\cos^5\theta\sin^4\theta = \frac{1}{2^8}\left[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta\right]$

8) Prove that $\cos^5\theta\sin^3\theta = \frac{-1}{2^7}\left[\sin 8\theta + \sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$

9) Prove that $\cos^5\theta\sin^7\theta = \frac{-1}{2^{11}}\left[\begin{array}{c}\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta \\ +5\sin 4\theta - 20\sin 2\theta\end{array}\right]$

10) Expand $\cos^6\theta$

Expansions of sinθ, cosθ & tanθ interms of θ

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \ldots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \ldots$$

Problems:

1) If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$ **show that** $\theta = \frac{1}{19}$ **radians.**

Proof:

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \ldots}{\theta} = 1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} - \ldots = \frac{2165}{2166}$$

Neglecting power of θ higher than θ^2,

$$\frac{\theta^2}{6} = 1 - \frac{2165}{2166} = \frac{1}{2166}$$

$$\theta^2 = \frac{6}{2166} = \frac{1}{361}$$

$$\theta = \frac{1}{19}\text{radian}$$

2) If $\cos\theta = \frac{1681}{1682}$ find θ in radians.

Solution:

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - ....$$

Neglecting powers of θ higher than θ^2,

$$1 - \frac{\theta^2}{2} = \frac{1681}{1682}$$

$$\frac{\theta^2}{2} = 1 - \frac{1681}{1682} = \frac{1}{1682}$$

$$\theta^2 = \frac{2}{1682} = \frac{1}{841}$$

$$\theta = \frac{1}{29}\text{radian}$$

3) Find θ if $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$. **Nov'17**

Solution:

$$\frac{\tan\theta}{\theta} = \frac{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + ...}{\theta}$$

$$= 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + ... = \frac{2524}{2523}$$

Neglecting powers of θ higher than θ^2,

$$1 + \frac{\theta^2}{3} = \frac{2524}{2523}$$

$$\frac{\theta^2}{3} = \frac{2524}{2523} - 1 = \frac{1}{2523}$$

$$\theta^2 = \frac{3}{2523} = \frac{1}{841}$$

$$\theta = \frac{1}{29}\text{radian.}$$

4) Find θ when $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$ **Nov'15**

Ans: $\frac{1}{29}$radian

5) Find θ if $\dfrac{\sin\theta}{\theta} = \dfrac{19493}{19494}$ **Nov'15**

6) Find $\underset{x\to 0}{\mathrm{lt}} \dfrac{x - \sin x}{x^3}$

Solution:

Consider $\dfrac{x - \sin x}{x^3} = \dfrac{x - \left(x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots\right)}{x^3}$

$$= \frac{x - x - \dfrac{x^3}{\lfloor 3} - \dfrac{x^5}{\lfloor 5} - \ldots}{x^3}$$

$$= \frac{\cancel{x^3}\left[\dfrac{1}{\lfloor 3} - \dfrac{x^2}{\lfloor 5} - \ldots\right]}{\cancel{x^3}}$$

$$\frac{x - \sin x}{x^3} = \frac{1}{\lfloor 3} - \frac{x^2}{\lfloor 5} + \ldots$$

$$\lim_{x\to\infty} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1}{6} - \frac{x^2}{\lfloor 5} + \ldots$$

$$= 1/6$$

7) Find $\underset{x\to 0}{\mathrm{lt}} \dfrac{\tan x - \sin x}{\sin^3 x}$

Solution:

$$\frac{\tan x - \sin x}{\sin^3 x} = \frac{\left(x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots\right) - \left(x + \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots\right)}{\left(x + \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots\right)^3}$$

$$= \frac{x - x + \dfrac{x^3}{3} + \dfrac{x^3}{\lfloor 3} + \dfrac{2x^5}{15} - \dfrac{x^5}{\lfloor 5} - \ldots}{x^3\left[1 - \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} - \ldots\right]^3}$$

$$= \frac{\left(\dfrac{1}{3} + \dfrac{1}{6}\right)x^3 + \dfrac{2x^5}{15} - \dfrac{x^5}{\lfloor 5} - \ldots}{x^3\left[1 - \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} - \ldots\right]^3}$$

$$\frac{\tan x-\sin x}{\sin^3 x}=\frac{\cancel{x^3}\left[\frac{1}{2}+\frac{2x^5}{15}-\frac{x^5}{\lfloor 5}-\ldots\right]}{\cancel{x^3}\left[1-\frac{x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}-\ldots\right]^3}$$

$$\underset{x\to 0}{\text{lt}}\frac{\tan x-\sin x}{\sin^3 x}=\underset{x\to 0}{\text{lt}}\frac{\cancel{x^3}\left[\frac{1}{2}+\frac{2x^5}{15}-\frac{x^5}{\lfloor 5}-\ldots\right]}{\cancel{x^3}\left[1-\frac{x^2}{\lfloor 3}+\frac{x^4}{\lfloor 5}-\ldots\right]^3}$$

$$=\frac{1/2}{1^3}=1/2$$

8) **Prove that** $\lim\limits_{x\to 0}\frac{\sin 2x-2\sin x}{x^3}=-1$

9) **Prove that** $\lim\limits_{x\to 0}\frac{\tan 2x-2\sin x}{4x^3}=3/4$

10) **Prove that** $\underset{x\to 0}{\text{lt}}\frac{5\sin x-\sin 5x}{5[\cos x-\cos 5x]}=0$

Proof:

Consider $$\frac{5\sin x-\sin 5x}{5[\cos x-\cos 5x]}=\frac{5\left[x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\ldots\right]-\left[5x-\frac{(5x)^3}{\lfloor 3}+\frac{(5x)^5}{\lfloor 5}+\ldots\right]}{5\left[\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\ldots\right)-\left(1-\frac{(5x)^2}{\lfloor 2}-\frac{(5x)^4}{\lfloor 4}+\ldots\right)\right]}$$

$$=\frac{\cancel{5x}-\cancel{5x}-\frac{5x^3}{\lfloor 3}+\frac{5^3x^3}{\lfloor 3}+\ldots.}{\cancel{5}-\cancel{5}-\frac{5x^2}{\lfloor 2}+\frac{5^2x^2}{\lfloor 2}+\ldots.}$$

$$=\frac{x^3\left[\frac{-5}{\lfloor 3}+\frac{5^3}{\lfloor 3}+\ldots.\right]}{x^2\left[\frac{-5}{\lfloor 2}+\frac{5^2}{\lfloor 2}+\ldots.\right]}=x\frac{\left[\frac{-5}{\lfloor 3}+\frac{5^3}{\lfloor 3}+\ldots.\right]}{\left[\frac{-5}{\lfloor 2}+\frac{5^2}{\lfloor 2}+\ldots.\right]}$$

$$\underset{x\to 0}{\text{lt}}\frac{5\sin x-\sin 5x}{5[\cos x-\cos 5x]}=0$$

HYPERBOLIC FUNCTIONS

For all values of x, real or complex the functions $\frac{e^x+e^{-x}}{2}$ and $\frac{e^x-e^{-x}}{2}$ are called respectively hyperbolic cosine and hyperbolic sine of x.

We write

$$\sinh x=\frac{e^x-e^{-x}}{2}, \qquad \cosh x=\frac{e^x+e^{-x}}{2}$$

Similarly

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}, \qquad \operatorname{sech} x=\frac{e^x+e^{-x}}{e^x+e^{-x}}$$

$$\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}} \quad \& \quad \operatorname{cosech} x=\frac{2}{e^x-e^{-x}}$$

Difference between Circular and Hyperbolic Functions:

Circular Functions	Hyperbolic Functions
$\sin x=\frac{e^{ix}-e^{-ix}}{2i}$	$\sinh x=\frac{e^x-e^{-x}}{2}$
$\cos x=\frac{e^{ix}+e^{-ix}}{2}$	$\cosh x=\frac{e^x+e^{-x}}{2}$
$\tan x=\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}$	$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$
$\cot x=\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}$	$\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$
$\sec x=\frac{2}{e^{ix}+e^{-ix}}$	$\operatorname{sech} x=\frac{2}{e^x+e^{-x}}$
$\operatorname{cosec} x=\frac{2i}{e^{ix}-e^{-ix}}$	$\operatorname{cosech} x=\frac{2}{e^x-e^{-x}}$

Problems:

1) Prove that cosix=coshx.

Proof:

$$\cos x=\frac{e^{ix}+e^{-ix}}{2}$$

Put x=ix

$$\cos ix=\frac{e^{i(ix)}+e^{-i(ix)}}{2}=\frac{e^{-x}+e^{x}}{2}=\cosh x$$

cosix=coshx

Hence the proof.

2) Prove that sinix=isinhx

Proof:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Put x=ix

$$\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2} = \frac{e^{-x} - e^{x}}{2i} \frac{\times i}{\times i}$$

$$= i\left[\frac{e^{-x} - e^{x}}{-2}\right]$$

$$= i\left[\frac{e^{-x} - e^{x}}{2}\right] = i\sinh x$$

sinix = isinhx

3) Prove that tanix=itanhx

Proof:

$$\tan ix = \frac{\sin ix}{\cos ix} = \frac{i\sinh x}{\cosh x} = i\tanh x$$

4) Prove that secix=sechx

Proof:

$$\sec ix = \frac{1}{\cos ix} = \frac{1}{\cosh x} = \operatorname{sech} x$$

$$\therefore \sec ix = \operatorname{sech} x$$

5) Prove that cosecix= –icosechx

Proof:

$$\text{cosec}\, ix = \frac{1}{\sin ix} = \frac{1}{i\sinh x} = -i\,\text{cosech}\, x$$

$$\therefore \text{cosec}\, ix = -i\,\text{cosech}\, x$$

6) Prove that cotix= –icothx

Proof:

$$\cot ix = \frac{1}{\tan ix} = \frac{1}{i\tanh x} = -i\coth x$$

$$\therefore \text{cotix} = -\text{icothx}$$

7) Prove that $\cosh^2 x - \sinh^2 x = 1$

Proof:

$$\cos^2 x + \sin^2 x = 1$$

Put x=ix,

$$\cos^2 ix + \sin^2 ix = 1 \Rightarrow (\cosh x)^2 + i^2(\sinh x)^2 = 1$$

$$\text{i.e. } \cosh^2 x - \sinh^2 x = 1$$

8) Prove that $\text{Sech}^2 x = 1 - \tanh^2 x$

Hint: ***$\text{Sec}^2 x = 1 + \tan^2 x$*** ***Replace x by ix***

9) Prove that $\text{cosech}^2 x = -\coth^2 x - 1$

Hint: ***$1 + \cot^2 x = \text{cosec}^2 x$.*** ***Replace x by ix.***

10) sinh2x= 2sinhxcoshx (sin2θ = 2sinθcosθ)

11) cosh2x= $\cosh^2 x + \sinh^2 x$ ($\cos^2\theta - \sin^2\theta = \cos 2\theta$)

12) $$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x} \qquad \left(\tan 2A = \frac{2\tan A}{1 \pm \tan^2 A}\right)$$

13) Resolve into real and imaginary parts of sin (α+β).

Solution:

$$\sin(\alpha + i\beta) = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta$$

$$= \sin\alpha\cos\beta + i\cos\alpha\sinh\beta$$

$$\text{Re. part of } \sin(\alpha + i\beta) = \sin\alpha\cosh\beta$$

$$\text{Im. part of } \sin(\alpha + i\beta) = \cos\alpha\sinh\beta$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \quad \to (1)$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \quad \to (2)$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad \to (3)$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \quad \to (4)$$

$$(1) + (2) \Rightarrow \sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$(1) - (2) \Rightarrow \cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$(3) + (4) \Rightarrow \cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$(3) - (4) \Rightarrow \sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

14) Cos(α+iβ)

$$\cos(\alpha+i\beta)=\cos\alpha\cos i\beta-\sin\alpha\sin i\beta$$

$$=\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta$$

$$R.P=\cos\alpha\cos\beta$$

$$I.P\ =-\sin\alpha\sinh\beta$$

15) tan(α+iβ)

$$\tan(\alpha+i\beta)=\frac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)}\times\frac{\cos(\alpha-i\beta)}{\cos(\alpha-i\beta)}$$

$$=\frac{\sin 2\alpha+\sin 2i\beta}{\cos 2\alpha+\cos 2i\beta}$$

$$=\frac{\sin 2\alpha+i\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}$$

$$\tan(\alpha+i\beta)=\frac{\sin 2\alpha}{\cos 2\alpha+\cosh 2\beta}+i\frac{\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}$$

$$R.P=\frac{\sin 2\alpha}{\cos 2\alpha+\cosh 2\beta}$$

$$Im.P=\frac{\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}$$

16) Show that $\sinh x=\frac{x}{\underline{|1}}+\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}+\ldots,\ \cosh x=1+\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}+\ldots$

Proof:

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots$$

$$e^{-x}=1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\ldots$$

$$\sinh x=\frac{e^x-e^{-x}}{2}=\frac{1}{2}\left[2x+\frac{2x^3}{\underline{|3}}+\ldots\right]=x+\frac{x^3}{\underline{|3}}+\ldots$$

$$\cosh x=\frac{e^x+e^{-x}}{2}=\frac{1}{2}\left[2x+\frac{2x^2}{\underline{|2}}+\frac{2x^4}{\underline{|4}}+\ldots\right]=1+\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}+\ldots$$

17) If $\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha$ prove that $\cos 2\theta\cosh 2\phi=3$

Proof:

$$\sin(\theta+i\phi)=\sin\theta\cosh\phi+i\cos\theta\sinh\phi=\tan\alpha+i\sec\alpha$$

$$\therefore \sin\theta\cosh\phi = \tan\alpha$$

$$\cos\theta\sinh\phi = \sec\alpha$$

But $\sec^2\alpha - \tan^2\alpha = 1$

$$\therefore \cos^2\theta\sinh^2\phi - \sin^2\theta\cosh^2\phi = 1$$

$$\text{i.e.}\left[\frac{1+\cos 2\theta}{2}\right]\left[-\left(\frac{1-\cosh 2\phi}{2}\right)\right]$$

$$-\left[\frac{1+\cos 2\theta}{2}\right]\left[-\left(\frac{1-\cosh 2\phi}{2}\right)\right] = 1$$

$$\text{i.e. } \frac{1}{4}\left[\begin{array}{l}(1+\cos 2\theta)(\cosh 2\phi - 1) \\ \quad -(1-\cos 2\theta)(1+\cosh 2\phi)\end{array}\right] = 1$$

$$\text{i.e. } \cosh 2\phi - 1 + \cos 2\theta\cosh 2\phi - \cos 2\theta$$

$$-[1+\cosh 2\phi - \cos 2\theta - \cos 2\theta\cosh 2\phi] = 4$$

$$-2 + 2\cos 2\theta\cosh 2\phi = 4$$

$$2\cos 2\theta\cosh 2\phi = 6$$

$$\therefore \cos 2\theta\cosh 2\phi = 3$$

Hence the proof.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \frac{1-\cos 2\theta}{2} = \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \frac{1+\cos 2\theta}{2} = \cos^2\theta$$

$$\cosh 2\phi = 1 + 2\sinh^2\phi$$

$$\Rightarrow \frac{\cosh 2\phi - 1}{2} = \sinh^2\phi$$

$$\cosh 2\phi = 2\cosh^2\phi - 1$$

$$\Rightarrow \frac{\cosh 2\phi + 1}{2} = \cosh^2\phi$$

18) If $\sin(x+iy) = r(\cos\theta + i\sin\theta)$

Prove that

(i) $r^2 = \frac{1}{2}(\cosh 2y - \cos 2x)$

(ii) $\tan\theta = \cot x \tanh y$

Proof:

Given $\sin(x+iy) = r(\cos\theta + i\sin\theta) = re^{i\theta}$

(i) $\sin(x-iy) = re^{-i\theta}$

$$\sin(x+iy)\sin(x-iy) = r^2$$

$$(\text{i.e.}) -\tfrac{1}{2}\left[\cos(x+iy+x-iy) - \cos(x+iy-x+iy)\right] = r^2$$

$$-\tfrac{1}{2}\left[\cos 2x - \cos 2iy\right] = r^2$$

$$\text{i.e. } \cosh 2y - \cos 2x = r^2$$

(ii) $\sin\left(x+iy\right) = r\cos\theta + ir\sin\theta$

$$\text{i.e. } \sin x \cosh y + i\cos x \sinh y = r\cos\theta + ir\sin\theta$$

$$\Rightarrow \sin x \cosh y = r\cos\theta \qquad \rightarrow(1)$$

$$\& \ \cos x \sinh y = r\sin\theta \qquad \rightarrow(2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r\sin\theta}{r\cos\theta} = \frac{\cos x \sinh y}{\sin x \cosh y}$$

$$\text{i.e. } \tan\theta = \cot x \tanh y.$$

19) If cos(x+iy) = cosθ+isinθ

Prove that $\mathbf{\cos 2x + \cosh 2y = 2}$

Proof:

$$\cos x \cos iy - \sin x \sin iy = \cos\theta + i\sin\theta$$

$$\cos x \cosh y = \cos\theta \qquad \rightarrow (1)$$

$$-\sin x \sinh y = \sin\theta \qquad \rightarrow (2)$$

Squaring and adding (1) & (2)

$$\cos^2\theta + \sin^2\theta = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$$1 = \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right)$$

$$1 = \frac{1}{4}\left[\left(1+\cos 2x\right)\left(1+\cosh^2 y\right) + \left(1-\cos 2x\right)\left(\cosh 2y - 1\right)\right]$$

$$4 = \begin{bmatrix} \cancel{1} + \cosh 2y + \cos 2x + \cos 2x\, \cancel{\cosh 2y} \\ + \cosh 2y \cancel{-1} \cancel{-\cos 2x} \cosh 2y + \cos 2x \end{bmatrix}$$

$$4 = 2\cosh 2y + 2\cos 2x$$

$$\therefore \cosh 2y + \cos 2x = 2$$

Hence the Proof.

20) If $\sin(A+iB) \Rightarrow x+iy$ then show that **Nov'17**

(i) $x = \sin A \cosh B$

(ii) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1$

(iii) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$

Proff:

Given that

$$x + iy = \sin A \cos iB + \cos A \sin iB$$

$$= \sin A \cosh B + i \cos A \sinh B$$

$$\therefore x = \sin A \cosh B \quad \rightarrow (1)$$

$$y = \cos A \sinh B \quad \rightarrow (2)$$

$$(1) \Rightarrow \cosh B = \frac{x}{\sin A} \qquad (2) \Rightarrow \sinh B = \frac{y}{\cos A}$$

using $\text{Cosh}^2 B - \sinh^2 B = 1, \dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1$

$$(1) \Rightarrow \sin A = \frac{x}{\cosh B} \qquad (2) \Rightarrow \cos A = \frac{y}{\sinh B}$$

Using $\sin^2 A + \cos^2 A = 1, \dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$

21) If $\tan\dfrac{x}{2} = \tan h\dfrac{x}{2}$ show that $\cos x \cosh x = 1$

Proof:

$$\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$$

Replace x by ix,

$$\cos ix = \frac{1 - \left(\tan i x/2\right)^2}{1 + \left(\tan i x/2\right)^2} = \frac{1 - i^2 \tanh^2 x/2}{1 + i^2 \tanh^2 x/2}$$

$$= \frac{1+\tanh^2 x/2}{1-\tanh^2 x/2} = \frac{1+\tan^2 x/2}{1-\tan^2 x/2}$$

$$= \frac{1}{\cos x}$$

$\therefore \cos ix \cos x = 1$

i.e. $\cosh x \cos x = 1$

22) If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$ Prove that $y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$.

23) Separate real and imaginary parts of **Nov'16**

(a) $\coth(x+iy)$ (b) $\operatorname{sech}(x+iy)$

Solution:

$$\cot ix = -i\coth x$$

$$\Rightarrow \coth x = i\cot ix$$

$$\coth(x+iy) = i\cot i(x+iy)$$

$$= i\cot i(ix-y)$$

$$= i\left[\frac{\cos(ix-y)}{\sin(ix-y)} \times \frac{\sin(ix+y)}{\sin(ix+y)}\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(a+B) - \sin(A-B)\right]$$

$$\sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$\therefore \coth(x+iy) = i\left[\frac{\frac{1}{2}\left[\sin(ix-y+ix+y) - \sin(ix-y-ix-y)\right]}{-\frac{1}{2}\left[\cos(ix-y+ix+y) - \cos(ix-y-ix-y)\right]}\right]$$

$$= -i\frac{\left[\sin 2ix - \sin(-2y)\right]}{\left[\cos 2ix - \cos(-2y)\right]}$$

$$\sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta$$

$$\therefore \coth(x+iy) = -i\left[\frac{i\sinh 2x + \sin 2y}{\cosh 2x - \cos 2y}\right]$$

$$= \frac{\sinh 2x}{\cosh 2x - \cos 2y} - i\frac{\sin 2y}{\cosh 2x - \cos 2y}$$

$$RP = \frac{\sinh 2x}{\cosh 2x - \cos 2y} \qquad IP = \frac{-\sin 2y}{\cosh 2x - \cos 2y}$$

5 Marks

1) If $\cos(A+iB) = x+iy$, then show that $\frac{x^2}{\cos^2 A} - \frac{y^2}{\sin^2 A} = 1$ and $\frac{x^2}{\cosh^2 A} + \frac{y^2}{\sinh^2 A} = 1$

Proof:

Given $\cos(A+iB) = x+iy$

$$\cos A \cos iB + \sin A \sin iB = x + iy$$

$$\cos A \cosh B + i \sin A \sinh B = x + iy$$

Equation RP, IP,

$$\cos A \cosh B = x; \ \sin A \sinh B = y$$

$$\Rightarrow \frac{x}{\cos A} = \cosh B \quad \to (1); \quad \Rightarrow \frac{y}{\sin A} = \sinh B \quad \to (2);$$

$$\text{and } \frac{x}{\cosh B} = \cos A \quad \to (3) \quad \text{and } \frac{y}{\sinh B} = \sin A \quad \to (4);$$

consider $\frac{x^2}{\cos^2 A} - \frac{y^2}{\sin^2 A} = \cosh^2 B - \sinh^2 B = 1$ (By(1) & (2))

consider $\frac{x^2}{\cos^2 B} + \frac{y^2}{\sinh^2 B} = \cos^2 A + \sin^2 A = 1$ (By(3) & (4))

Hence the proof.

8) If $\tan(\theta + i\phi) = x + iy$, prove that $x^2 + y^2 + 2x\cot 2\theta = 1$ **Apr' 17**

Proof:

Given $\tan(\theta + i\phi) = x + iy$

$$\frac{\sin(\theta + iq)}{\cos(\theta + iq)} = x + iy$$

$$x + iy = \frac{\sin(\theta + i\phi) \times \cos(\theta - i\phi)}{\cos(\theta + i\phi)}$$

$$= \frac{\frac{1}{2}[\sin 2\theta + \sin 2i\phi]}{\frac{1}{2}[\cos 2\theta + \cos 2i\phi]}$$

$$x + iy = \frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} + i\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}$$

$$x = \frac{\sin 2\theta}{\cos 2\theta + \cosh 2\theta};\ y = \frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}$$

Consider,

$$x^2 + y^2 + 2x\cot 2\theta = \frac{\sin^2 2\theta}{[\cos 2\theta + \cosh 2\phi]^2} + \frac{\sin^2 h2\phi}{[\cos 2\theta + \cosh 2\phi]^2} + 2\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi}\frac{\cos 2\theta}{\sin 2\theta}$$

$$= \frac{\sin^2 2\theta + \sinh^2 2\phi + 2\cos^2 2\theta + 2\cos 2\theta \cosh 2\phi}{(\cos 2\theta + \cosh 2\phi)^2}$$

$$= \frac{1 + \sinh^2 2\phi + \cos^2 2\theta + 2\cos 2\theta \cosh 2\phi}{(\cos 2\theta + \cosh 2\phi)^2}$$

$$\sin^2 2\theta + \cos^2 2\theta = 1$$

$$\cosh^2\phi - \sinh^2\phi = 1 \Rightarrow 1 + \sinh^2\phi = \cosh^2\phi$$

$$\text{LHS} = \frac{\cosh^2 2\phi + \cos^2 2\theta + 2\cos 2\theta \cosh 2\phi}{(\cos 2\theta + \cosh 2\phi)^2}$$

$$= \frac{(\cos 2\theta + \cosh 2\phi)^2}{(\cos 2\theta + \cosh 2\phi)^2} = 1$$

Hence proved.

Inverse Hyperbolic Functions:

If sinhx=y then $x=\sinh^{-1}y$

& if coshx=y then $x=\cosh^{-1}y$

The values $\sinh^{-1}y$, $\cosh^{-1}y$, ... are called inverse hyperbolic functions.

Problems:

1) Prove that $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

Proof:

Let $\sinh^{-1} x = y$

$x = \sinh y$

$$x = \frac{e^y - e^{-y}}{2}$$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$$e^y = \frac{-(-2x) \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2 \times 1}$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$e^y = x \pm \sqrt{x^2+1}$

But e^y lies between 0 and ∞

$\therefore e^y = x \pm \sqrt{x^2+1}$

$\Rightarrow y = \log\left(x + \sqrt{x2+1}\right)$

i.e. $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

2) Prove that $\cosh^{-1} x = \pm\log\left(x + \sqrt{x^2-1}\right)$ **Apr' 17**

Proof:

Let $\cosh^{-1} x = y$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y}$$

$e^{2y} - 2xe^y + 1 = 0$

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2-4}}{2} = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$\text{consider } x-\sqrt{x^2-1} = x-\sqrt{x^2-1} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$= \frac{x^2-\left(x^2-1\right)}{x+\sqrt{x^2-1}} = \frac{1}{x+\sqrt{x^2-1}} = \left(x+\sqrt{x^2-1}\right)^{-1}$$

$$\therefore e^y = \left(x+\sqrt{x^2-1}\right)^{\pm 1}$$

$$y = \log\left(x+\sqrt{x^2-1}\right)^{\pm 1}$$

$$y = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$\text{(i.e.) } \cosh^{-1} x = \pm\log\left(x+\sqrt{x^2-1}\right)$$

3) Prove that $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ **Nov'17**

Proof:

Let $\tanh^{-1} x = y$

$$\therefore x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x = \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}} = \frac{e^{2y}-1}{e^{2y}+1}$$

$$x\left(e^{2y}+1\right) - e^{2y} + 1 = 0$$

$$xe^{2y} + x - e^{2y} + 1 = 0$$

$$(x-1)e^{2y} + (x+1) = 0$$

$$e^y = \left(\frac{1+x}{1-x}\right)^{1/2}$$

$$y = \log\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

4) Separate real and imaginary parts of $\tan^{-1}(x+iy)$.

Solution:

Let $\tan^{-1}(x+iy) = a+ib$ a=R.P; b = Im.p

$$x+iy = \tan(a+ib)$$

$$x-iy = \tan(a-ib)$$

(a) To Find Real part, take 2a=(a+ib) + (a–ib)

$$\tan 2a = \tan((a+ib)+(a-ib))$$

$$= \frac{\tan((a+ib)+(a-ib))}{1-\tan(a+ib)\tan(a-ib)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-(x^2-i^2y^2)}$$

$$\tan 2a = \frac{2x}{1-(x^2+y^2)}$$

$$a = \text{Re.part}=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)$$

(b) To find the Im.Part,

take, $2ib = (a+ib)-(a-ib)$

$$\tan 2ib = \tan((a+ib)-(a-ib))$$

$$= \frac{\tan(a+ib)-\tan(a-ib)}{1+\tan(a+ib)\tan(a-ib)}$$

$$= \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2-i^2y^2)} = \frac{2iy}{1+x^2+y^2}$$

$$i\tanh 2b = \frac{2iy}{1+x^2+y^2}$$

$$\tanh 2b = \frac{2y}{1+x^2+y^2}$$

$$b = \text{Impart} = \frac{1}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

Note:

But $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$\therefore b = \frac{1}{2}\left[\frac{1}{2}\log\left(\frac{1+\frac{2y}{1+x^2+y^2}}{1-\frac{2y}{1+x^2+y^2}}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y}\right)$$

$$= \frac{1}{4}\log\left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right)$$

Problems:

1) If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$, then show that

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$

(ii) $\phi = \frac{1}{2}\log\tan\left[\frac{\pi}{4}+\frac{\alpha}{2}\right]$

Proof:

Given $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$

But, If $\tan(\theta+i\phi) = x+iy, \Rightarrow \theta+i\phi = \tan^{-1}(x+iy)$

$$\theta = \frac{1}{2}\tan^{-1}\frac{2x}{1-(x^2+y^2)} \ \& \ \phi = \frac{1}{4}\log\left(\frac{x^2+(y+1)^2}{x^2+(y+1)^2}\right)$$

(i) Here x=cosα, y=sinα

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{2\cos\alpha}{0}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\infty = \frac{1}{2}\left(n\pi + \pi/2\right) = n\,\pi/2 + \pi/4$$

$$\theta = n\,\pi/2 + \pi/4$$

(ii) $$\phi = \frac{1}{4}\log\left(\frac{\cos^2\alpha + (\sin\alpha + 1)^2}{\cos^2\alpha + (\sin\alpha - 1)^2}\right)$$

$$= \frac{1}{4}\log\left(\frac{\cos^2\alpha + \sin^2\alpha + 2\sin\alpha + 1}{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha + 1}\right)$$

$$= \frac{1}{4}\log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right) = \frac{1}{4}\log\left(\frac{1+\cos\left(\pi/2+\alpha\right)}{1-\cos\left(\pi/2+\alpha\right)}\right)$$

$$= \frac{1}{4}\log\left[\frac{2\sin^2\left(\frac{\pi/2+\alpha}{2}\right)}{2\cos^2\left(\frac{\pi/2+\alpha}{2}\right)}\right] = \frac{1}{4}\log\tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$= \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

2) If $\tanh\frac{u}{2} = \tan\alpha/2$ then show that $u = \log\tan\left(\pi/4 + \alpha/2\right)$

Proof:

Given $$\tanh\frac{u}{2} = \tan\alpha/2$$

$$u/2 = \tan^{-1}\left(\tan\alpha/2\right)$$

$$u = \tan^{-1}\left(\tan\alpha/2\right)$$

$$= 2\left[\frac{1}{2}\log\left(\frac{1+\tan\alpha/2}{1-\tan\alpha/2}\right)\right] \qquad \left(\because \tan^{-1}x = 1/2\log\left(\frac{1+x}{1-x}\right)\right)$$

$$= \log\left(\frac{\tan\pi/4 + \tan\alpha/2}{1-\tan\pi/4\tan\alpha/2}\right) \qquad \left(\because \tan\pi/4 = 1\right)$$

$$= \log\tan\left(\pi/4 + \alpha/2\right)$$

Logarithm of a Complex Number

Principal and general value of logarithm

Let $z = re^{i\theta}$

$$\log z = \log\left(re^{i\theta}\right)$$

$$= \log r + \log e^{i\theta}$$

$$\boxed{\log z = \log r + i\theta}$$

log z is known as principal value of logarithm.

Let $z = re^{i\theta}e^{2n\pi i}$ $\qquad (e^{2n\pi i} = \cos 2n\pi + i\sin 2n\pi$

$\log z = re^{i\theta}e^{2n\pi i}$ $\qquad = 1 + 0 = 1)$

$$= \log r + \log e^{i\theta} + \log e^{2n\pi i}$$

$$\log z = \log r + i\theta + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, ...)$$

log z is known as general value of logarithm.

Problems:

1) Find logi

Solution:

$i = \cos \pi/2 + i\sin \pi/2$ $\qquad (\because \cos \pi/2 = 0$

$= e^{i\pi/2}$ $\qquad \sin \pi/2 = 1)$

$$\log i = \log e^{i\pi/2} = i\pi/2$$

$$\log i = i\pi/2$$

2) Find logi

$$i = e^{i\pi/2}$$

$$= e^{i\pi/2}.e^{2n\pi i} \qquad (\because e^{2n\pi i} = 1)$$

$$i = e^{i\pi/2 + 2n\pi i}$$

$$\log i = i\pi/2 + 2n\pi i \qquad n = 0, \pm 1, \pm 2, ...$$

$$= i\pi/2(4n+1)$$

3) Find $\log\sqrt{i}$

$$\sqrt{i} = \left(\cos\pi/2 + i\sin\pi/2\right)^{1/2} = \cos\pi/4 + i\sin\pi/4 = e^{i\pi/4}$$

$$= e^{i\pi/4}e^{2n\pi i}, \quad n = 0, \pm 1, \pm 2, ...$$

$$= e^{i(\pi/4 + 2n\pi)}$$

$$\log\sqrt{i} = i\left(\pi/4 + 2n\pi\right), \quad n = 0, \pm 1, ...$$

4) Find real and imaginary part of Log(a+ib)

Solution:

$$a + ib = re^{i\theta} \qquad \text{where} \quad r = \sqrt{a^2 + b^2} \qquad b = \tan^{-1}\left(b/a\right)$$

$$= re^{i\theta}e^{2n\pi i}$$

$$= re^{i(\theta + + 2n\pi)}$$

$$\log(a + ib) = \log r + \log e^{i(\theta + 2n\pi)}$$

$$= \log r + i(\theta + 2n\pi)$$

$$\log(a + ib) = \log\sqrt{a^2 + b^2} + i\tan^{-1}\left(b/a\right) + i2n\pi$$

$$\text{R.P of } \log(a + ib) = \log\sqrt{a^2 + b^2}$$

$$\text{Im.P of } \log(a + ib) = \tan^{-1}\left(b/a\right) + 2n\pi$$

5) Separate $\log(1+i)$ into real time parts.

Let $\quad (1 + i) = re^{i\theta}$

$$\log(1 + i) = \log r + \log e^{i\theta}$$

$$\log(1 + i) = \log r + i\theta \qquad r = \sqrt{1^2 + 2^2} \qquad \theta = \tan^{-1}\left(1/1\right)$$

$$= \log\sqrt{2} + i\tan^{-1}(1)$$

$$\log(1 + i) = 1/2 \log 2 + i\pi/4 \qquad \left(\because \tan\pi/4 = 1\right)$$

$$\text{R.P} = 1/2 \log 2$$

$$\text{Im.p} = \pi/4$$

6) Write Log(1–i)

7) Find Log(1+2i)

8) Show that $\log\left(\dfrac{a + ib}{a - ib}\right) = 2i\tan^{-1}\left(b/a\right)$

Proof:

Let $(a+ib) = re^{i\theta}$ $\Rightarrow r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\left(b/a\right)$

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\left(\frac{re^{i\theta}}{re^{-i\theta}}\right) = \log e^{2i\theta}$$

$$= 2i\theta \Rightarrow 2i\tan^{-1}\left(b/a\right)$$

$$\text{i.e.}\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(b/a\right)$$

8) Find real and ima part of (x+iy)$^{a+ib}$

Solution:

$$(x+iy)^{a+ib} = e\log^{(x+iy)a+ib}$$

$$= e^{(a+ib)\log(x+iy)}$$

$$= e^{a+ib\left[\log\sqrt{x^2+y^2}+i\tan-1 y/x+2n\pi i\right]}$$

$$= e^{a+ib\left[\log r+i(\theta+2n\pi)\right]}$$

$$= e^{a\log r-b(\theta+2n\pi)}e^{i(b\log r+a(\theta+2n\pi))}$$

$$= e^{a\log r-b(\theta+2n\pi)}\begin{bmatrix}\cos\left(b\log r-a\left(\theta+2n\pi\right)\right)\\ +i\sin\left(b\log r-a\left(\theta+2n\pi\right)\right)\end{bmatrix}$$

$$\text{R.Part} = e^{a\log r-b(\theta+2n\pi)}\cos\left(b\log r+a\left(\theta+2n\pi\right)\right)$$

$$\text{I.Part} = e^{a\log r-b(\theta+2n\pi)}\sin\left(b\log r+a\left(\theta+2n\pi\right)\right)$$

9) Find i^i

Solution:

$$i^i = e\log i^i$$

$$= e\log i^i = e^{i\left[i\pi/2\right]} = e^{-\pi/2} \quad \left(\because \log i = i\,\pi/2\right)\text{From problem (1)}$$

UNIT - 5

DIFFERENTIAL CALCULUS

Introduction:

The mathematical study of change like motion, growth or decay is calculus. The Rate of change of given function is derivative or differential.

The concept of derivative is essential in day to day life. Also applicable in Engineering, Science, Economics, Medicine etc.

Successive Differentiation:

Let $y = f(x)$ --(1) be a real valued function.

The first order derivative of y denoted by $\frac{dy}{dx}$ or y' or y_1 or Δ^1

The Second order derivative of y denoted by $\frac{d^2y}{dx^2}$ or y'' or y_2 or Δ^2

Similarly differentiating the function (1) n-times, successively,

the n^{th} order derivative of y *exists* denoted by $\frac{d^ny}{dx^n}$ or y^n *or* y_n *or* Δ^n

The process of finding 2nd and higher order derivatives is known as Successive Differentiation.

n^{th} derivative of some standard functions:

1. $y = e^{ax}$

Sol : $y_1 = a\, e^{ax}$

$y_2 = a^2 e^{ax}$

Differentiating Successively

$y_n = a^n e^{ax}$

ie. $D^n[e^{ax}] = a^n e^{ax}$

For, $a = 1$ $\quad D^n[e^x] = e^x$

2. $y = \log(ax+b)$

Solution: $y_1 = \dfrac{a}{ax+b}$

$$y_2 = \frac{(-1)a.a}{(ax+b)^2} = \frac{(-1)^1 a^2}{(ax+b)^2}$$

$$y_3 = \frac{(-1)(-2)a^2.a}{(ax+b)^3} = \frac{(-1)^2(1)(2)a^3}{(ax+b)^3} = \frac{(-1)^{3-1}(3-1)!a^3}{(ax+b)^3}$$

$$D^n[\log(ax+b)] = y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Similarly $D^n[\log x] = y_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

3. $y = (ax+b)^m$

Solution: $y_1 = m(ax+b)^{m-1}a$

$$y_2 = m(m-1)(ax+b)^{m-2}a^2$$

$$y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$$

Similarly

$$y_n = m(m-1)(m-2).......(m-n+1)(ax+b)^{m-n}a^n$$

…(*)

Case (i) :- If $m = n$ in (*)

$$D^n[(ax+b)^n] = n(n-1)(n-2)\ldots\ldots 3\ 2\ 1\ .\ a^n$$

$$= n!\, a^n$$

$$D^n[x^n] = n!$$

Case (ii) :- If $m > n$ in (*)

$$D^n[(ax+b)^m] = \frac{m(m-1)\ldots\ldots(m-n+1)(m-n)(m-n-1)\ldots.321}{(m-n)(m-n-1)\ldots\ldots 3.2.1}(ax+b)^{m-n}a^n$$

$$D^n[(ax+b)^m] = \frac{m!}{(m-n)!}(ax+b)^{m-n}a^n$$

$$D^n[x^m] = \frac{m!}{(m-n)!}x^{m-n}a^n$$

Case iii :- If $m < n$ in (*)

$$D^n[(ax+b)^m] = 0$$

Case iv :- If $m = -1$ in (*)

$$D^n\left[\frac{1}{ax+b}\right] = (-1)(-2)\ldots\ldots.(-n)(ax+b)^{-1-n}a^n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = \frac{(-1)^n p(p+1)\ldots\ldots\ldots(p+n-1)a^n}{(ax+b)^{p+n}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = (-1)^n\frac{(p+n-1)!}{(p-1)!}\frac{a^n}{(ax+b)^{p+n}}$$

If $a=1$, $$D^n\left[\frac{1}{x^p}\right] = (-1)^n\frac{(p+n-1)!}{(p-1)!}\frac{1}{x^{p+n}}$$

4. $y = Cos\,(ax + b)$

$$y_1 = -\,Sin\,(ax + b).\,a = a\,Cos\,(ax+b+\pi/2)$$
$$y_2 = -\,Sin\,(ax + b+\pi/2\,).a^2 = a^2\,Cos\,(ax+b+2\,\pi/2)$$

.

.

$$y_n = D^n\,[Cos\,(ax+b)] = a^n\,Cos\,(ax+b+n\,\pi/2)$$

If a=1, b=0

$$D^n\,[Cos\,x] = Cos\,(x + n\,\pi/2)$$

5. $y = Sin\,(ax + b)$

$$y_n = D^n\,[Sin\,(ax+b)] = a^n\,Sin\,(ax+b+n\,\pi/2)$$

If a=1, b=0

$$D^n\,[Sin\,x] = Sin\,(x + n\,\pi/2)$$

6. $y = e^{ax} Sin\,(bx + c)$

$$y_1 = a\,e^{ax}\,Sin\,(bx + c) + b\,e^{ax} Cos\,(bx + c)$$
$$= e^{ax}\,[a\,Sin\,(bx + c) + b\,Cos\,(bx + c)]$$

Put $a = r\cos\theta \quad b = r\,Sin\,\theta$ *then* $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$

$$y_1 = e^{ax}\,[r\cos\theta\,Sin\,(bx + c) + r\,Sin\,\theta\,Cos\,(bx + c)]$$
$$y_1 = r\,e^{ax}\,[Sin\,(bx + c + \theta)]$$
$$y_2 = r\,[\,e^{ax}\,a\,Sin\,(bx + c + \theta) + e^{ax}\,b\,Cos\,(bx + c + \theta)]$$

Put $a = r\cos\theta \quad b = r\,Sin\,\theta$ *then* $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$

$$y_2 = r\,e^{ax}\,[\,r\cos\theta\,Sin\,(bx + c + \theta) + r\,Sin\,\theta\,Cos\,(bx + c + \theta)]$$
$$y_2 = r^2\,e^{ax}[Sin\,(bx + c + 2\theta)]$$

Similarly,

$$y_n = r^n e^{ax}[Sin(bx + c + n\theta)]$$

$$y_n = D^n[e^{ax} Sin(bx + c)] = (a^2+b^2)^{n/2} e^{ax}[Sin(bx + c + n\tan^{-1}\frac{b}{a})]$$

For $a=b=1, c=0$

$$D^n[e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$$

7. $y = e^{ax}[Cos(bx + c)]$

$$y_n = D^n[e^{ax} Cos(bx + c)] = (a^2+b^2)^{n/2} e^{ax}[Cos(bx + c + n\tan^{-1}\frac{b}{a})]$$

For $a=b=1, c=0$

$$D^n[e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$$

8. $y = a^{mx}$

$$y_1 = a^{mx}(log\, a^m) = a^{mx}(m\,log\, a)$$

$$y_2 = a^{mx}(m\,log\, a)^2$$

Differentiating Successively

$$y_n = a^{mx}(m\,log\, a)^n$$

For $m=1$, $D^n[a^x] = a^x (log\, a)^n$

Leibnitz's Theorem :

It provides a useful formula for computing the n^{th} derivative of a product of two functions.

Statement : If u and v are any two functions of x *with* u_n and v_n as their n^{th} *derivative. Then the* n^{th} *derivative of uv is*

$$(uv)_n = u_0 v_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots + {}^nC_{n-1} u_{n-1} v_1 + u_n v_0$$

Note : We can interchange u & v $(uv)_n = (vu)_n,$

${}^nC_1 = n$, ${}^nC_2 = n(n-1)/2!$, ${}^nC_3 = n(n-1)(n-2)/3!$ ...

1. Find the n^{th} derivations of $e^{ax}\cos(bx + c)$

Solution: $y_1 = e^{ax} - b\sin(bx + c) + a\,e^{ax}\cos(bx + c)$, by product rule.

.i.e, $y_1 = e^{ax}\left[a\cos(bx + c) - b\sin(bx + c)\right]$

Let us put $a = r\cos\theta$, and $b = r\sin\theta$.

$\therefore\ a^2 + b^2 = r^2$ and $\tan\theta = b/a$

.ie., $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Now, $y_1 = e^{ax}\left[r\cos\theta\cos(bx + c) - r\sin\theta\sin(bx + c)\right]$

Ie., $y_1 = r\,e^{ax}\cos(\theta + bx + c)$

where we have used the formula cos A cos B – sin A sin B = cos (A + B)

Differentiating again and simplifying as before,

$y_2 = r^2 e^{ax}\cos(2\theta + bx + c)$.

Similarly $y_3 = r^3 e^{ax}\cos(3\theta + bx + c)$.

....

Thus $y_n = r^n e^{ax}\cos(n\theta + bx + c)$

Where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Thus $\mathbf{D^n\,[e^{ax}\cos(bx + c)]}$

$$= \left[\left(\sqrt{a^2 + b^2}\right)^n e^{ax}\cos\left[n\tan^{-1}(b/a) + bx + c\right]\right]$$

2. Find the n^{th} derivative of log $\sqrt{4x^2+8x+3}$

Solution : Let y = log $\sqrt{4x^2+8x+3}$ = log $(4x^2 + 8x + 3)^{1/2}$

ie., $y = \frac{1}{2}$ log $(4x^2 + 8x + 3)$ $\because$ log x^n = n log x

$y = \frac{1}{2}$ log { (2x + 3) (2x+1)}, by factorization.

$\because$ $y = \frac{1}{2}$ {log (2x + 3) + log (2x + 1)}

Now $y_n = \frac{1}{2}\left\{\frac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n} + \frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right\}$

Ie., $y_n = 2^{n-1}\,(-1)^{n-1}\,(n-1)!\left\{\frac{1}{(2x+3)^n} + \frac{1}{(2x+1)^n}\right\}$

3. Find the n^{th} derivative of log $_{10}$ $\{(1-2x)^3\,(8x+1)^5\}$

Solution : Let y = log $_{10}$ $\{1-2x)^3\,(8x+1)^5\}$

It is important to note that we have to convert the logarithm to the base e by the property:

$$\log_{10} x = \frac{\log_e x}{\log_e 10}$$

Thus $y = \frac{1}{\log_e 10}\log_e\left\{(1-2x)^3(8x+1)^5\right\}$

Ie., $y = \frac{1}{\log_e 10}\{3\log(1-2x)+5\log(8x+1)\}$

$\therefore y_n = \frac{1}{\log_e 10}\left\{3.\frac{(-1)^{n-1}(n-1)!(-2)^n}{(1-2x)^n} + 5\frac{(-1)^{n-1}(n-1)!8^n}{(8x+1)^n}\right\}$

Ie., $y_n = \frac{(-1)^{n-1}(n-1)!2^n}{\log_e 10}\left\{\frac{3(-1)^n}{(1-2x)^n} + \frac{5(4)^n}{(8x+1)^n}\right\}$

4. Find the n^{th} derivative of $e^{2x}\cos^2 x \sin x$

Solution : >> let $y = e^{2x}\cos^2 x \sin x = e^{2x}\left[\frac{1+\cos 2x}{2}\right]\sin x$

ie., $y = \frac{e^{2x}}{2}(\sin x + \sin x \cos 2x)$

$= \frac{e^{2x}}{2}\left\{\sin x + \frac{1}{2}[\sin 3x + \sin(-x)]\right\}$

$= \frac{e^{2x}}{4}(2\sin x + \sin 3x - \sin x) \because \sin(-x) = -\sin x$

$\therefore y = \frac{e^{2x}}{4}(\sin x + \sin 3x)$

Now $y_n = \frac{1}{4}\left\{D^n\left(e^{2x}\sin x\right) + D^n\left(e^{2x}\sin 3x\right)\right\}$

Thus $y_n = \frac{1}{4}\left\{\left(\sqrt{5}\right)^n e^{2x}\sin\left[n\tan^{-1}(1/2) + x\right] + \left(\sqrt{13}\right)^n e^{2x}\sin\left[n\tan^{-1}(3/2) + 3x\right]\right\}$

$\therefore y_n = \frac{e^{2x}}{4}\left\{\left(\sqrt{5}\right)^n \sin\left[n\tan^{-1}(1/2) + x\right] + \left(\sqrt{13}\right)^n \sin\left[n\tan^{-1}(3/2) + 3x\right]\right\}$

5. Find the n^{th} derivative of $e^{2x}\cos^3 x$

Solution : Let $y = e^{2x}\cos^3 x = e^{2x}.\ \frac{1}{4}(3\cos x + \cos 3x)$

Ie., $y = \frac{1}{4}(3e^{2x}\cos x + e^{2x}\cos 3x)$

$\therefore y_n = \frac{1}{4}\{3D^n(e^{2x}\cos x) + D^n(e^{2x}\cos 3x)\}$

$y_n = \frac{1}{4}\left\{\left(3\sqrt{5}\right)^n e^{2x}\cos\left[n\tan^{-1}(1/2) + x\right] + \left(\sqrt{13}\right)^n e^{2x}\cos\left[n\tan^{-1}(3/2) + 3x\right]\right\}$

Thus $y_n = \frac{e^{2x}}{4}\left\{3\left(\sqrt{5}\right)^n \cos\left[n\tan^{-1}(1/2) + x\right] + \left(\sqrt{13}\right)^n \cos\left[n\tan^{-1}(3/2) + 3x\right]\right\}$

6. **Find the n^{th} derivative of** $\dfrac{x^2}{(2x+1)(2x+3)}$

Solution : $y = \dfrac{x^2}{(2x+1)(2x+3)}$ is an improper fraction because; the degree of the numerator being 2 is equal to the degree of the denominator. Hence we must divide and rewrite the fraction.

$$y = \frac{x^2}{4x^2+8x+3} = \frac{1}{4}\cdot\frac{4x^2}{4x^2+8x+3}$$ for convenience.

$$4x^2+8x+3 \left| \begin{array}{l} \overline{1} \\ 4x^2 \\ \underline{4x^2+8x+3} \\ -8x-3 \end{array} \right.$$

$$\therefore \qquad y = \frac{1}{4}\left[1 + \frac{-8x-3}{4x^2+8x+3}\right]$$

Ie., $y = \dfrac{1}{4} - \dfrac{1}{4}\left[\dfrac{8x+3}{4x^2+8x+3}\right]$

The algebraic fraction involved is a proper fraction.

Now $\quad y_n = 0 - \dfrac{1}{4}D^n\left[\dfrac{8x+3}{4x^2+8x+3}\right].$

Let $\dfrac{8x+3}{(2x+1)(2x+3)} = \dfrac{A}{2x+1} + \dfrac{B}{2x+3}$

Multiplying by (2x + 1) (2x + 3) we have, 8x + 3 = A (2x + 3) + B (2x + 1) ....(1)

By setting 2x + 1 = 0, 2x + 3 = 0 we get x = -1/2, x = -3/2.

Put x = -1/2 in (1): -1 -1 + A (2) $\Rightarrow$ A = -1/2

Put x = -3/2 in (1): -9 = B (-2) $\Rightarrow$ B = 9/2

$$\therefore \qquad y_n = -\frac{1}{4}\left\{-\frac{1}{2}D^n\left[\frac{1}{2x+1}\right] + \frac{9}{2}D^n\left[\frac{1}{2x+3}\right]\right\}$$

$$= -\frac{1}{8}\left\{(-1)\cdot\frac{(-1)^n n!2^n}{(2x+1)^{n+1}} + 9\cdot\frac{(-1)^n n!2^n}{(2x+3)^{n+1}}\right\}$$

ie., $y_n = \frac{(-1)^{n+1} n!2^n}{8}\left\{\frac{1}{(2x+1)^{n+1}} + \frac{9}{(2x+3)^{n+1}}\right\}$

7. Find the n^{th} derivative of $\frac{x^4}{(x+1)(x+2)}$

Solution : $y = \frac{x^4}{(x+1)(x+2)}$ is an improper fraction.

(deg of nr. = 4 > deg. of dr. = 2)

On dividing x^4 by $x^2 + 3x + 2$, We get

$y = (x^2 - 3x + 7) + \left[\frac{-15x-14}{x^2+3x+2}\right]$

$\therefore \quad y_n = D^n (x^2-3x+7) - D^n \left[\frac{15x-14}{x^2+3x+2}\right]$

But $D = (x^2 - 3x + 7) = 2x - 3$, $D^2 (x^2 - 3x + 7) = 2$

$D^3(x^2 - 3x + 7) = 0$......... $D^n (x^2 - 3x + 7) = 0$ if $n > 2$

Hence $y_n = -D^n \left[\frac{15x+14}{(x+1)(x+2)}\right]$

Now, let $D^n \frac{15x+14}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

=> $15x + 14 = A(x+2) + B(x+1)$

Put $x = -1$; $-1 = A(1)$ or $A = -1$

Put $x = -2$; $-16 = B(-1)$ or $B = 16$

$$Y_n = \left\{-D^n\left[\frac{1}{x+1}\right] + 16D^n\left[\frac{1}{x+2}\right]\right\}$$

$$= \frac{(-1)^n n!\ 1^n}{(x+1)^{n+1}} - 16\frac{(-1)^n n!\ 1^n}{(x+2)^{n+1}}$$

$$y_n = (-1)^n\ n!\ \left\{\frac{1}{(x+1)^{n+1}} - \frac{16}{(x+2)^{n+1}}\right\} n > 2$$

8. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{n}\right\}$$

Solution : Let y = $\frac{\log x}{x} = \log x.\frac{1}{x}$ and let u = log x, v = $\frac{1}{x}$

We have Leibnitz theorem,

$(uv)_n = uv_n + n_{C_1} u_1 v_{n-1} + n_{C_2} u_2 v_{n-2} + ... + u_n v$ …… (1)

Now, u = log x $\therefore$ $u_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

$v = \frac{1}{x}$ $\therefore$ $v_n = \frac{(-1)^n n!}{x^{n+1}}$

Using these in (1) by taking appropriate values for n we get,

$$D_n = \left(\frac{\log x}{x}\right) = \log x..\frac{(-1)^n n!}{x^{n+1}} + n\frac{1}{x}.\ \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+ \frac{n(n-1)}{1.\ \ 2}\left(-\frac{1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$+ ...... + \frac{(-1)^{n-1}(n-1)!}{x^n}.\ \frac{1}{x}$$

Ie.. = log x $\frac{(-1)^n n!}{x^{n+1}} + \frac{(-1)^{n-1} n!}{x^{n+1}}$

$$-\frac{(-1)^{n-2} n!}{2x^{n+1}} + ... + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$-\frac{(-1)^{n-2} n!}{x^{n+1}}\left[\log x(-1)^{-1} - \frac{(-1)^{-2}}{2} + ... + \frac{(-1)^{-1}(n-1)!}{n^1}\right]$$

Note : $(-1)^{-1} = \frac{1}{-1} = -1; (-1)^{-2} = \frac{1}{(-1)^2} = 1$

Also $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$

$$\therefore \frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} ... - \frac{1}{n}\right]$$

9. If y_n= D^n (x^n logx)

Prove that $y_n = n\, y_{n-1}+(n-1)!$ and hence deduce that

$y_n = n\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$

Solution : $y_n = D^n(x^n \log x) = D^{n-1}\{D(x^n \log x\}$

$= D^{n-1}\left\{x^n . \frac{1}{x} + nx^{n-1}\log x\right\}$

$= D^{n-1}(x^{n-1}) + nD^{n-1}(x^{n-1} \log x\}$

$\therefore$ $y_n = (n-1)! + ny_{n-1}$. This proves the first part.

Now Putting the values for n = 1, 2, 3...we get

$y_1 = 0! + 1\ \ y_0 = 1 + \log x = 1! (\log x + 1)$

$y_2 = l! + 2y_1 = l+2 (l + \log x)$

ie., $y_2 = 21og\ x + 3 = 2(\log x + 3/2) = 2!\left(\log x + 1 + \frac{1}{2}\right)$

$y_3 = 2! + 3y_2 = 2 + 3(2 \log x + 3)$

ie., $y_3 = 61og\ x+ll = 6 (\log x + ll/6) = 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$

……………………………………………………………………………..

$$y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$$

10. If y = a cos (log x) + b sin (log x), show that

$x^2y_2 + xy_1 + y = 0$. Then apply Leibnitz theorem to differentiate this result n times.

or

If y = a cos (log x) + b sin (log x), show that

$x^2y_{n+2} + (2n+l)xy_{n+l}+(n^2+1)y_n = 0$. [July-03]

Solution : y = a cos (log x) + b sin (log x)

Differentiate w.r.t x

$\therefore$ y_1 = -a sin (log x) $\frac{1}{x}$ + b cos (log x). $\frac{1}{x}$

(we avoid quotient rule to find y_2) .

=> xy_1 = - a sin (log x) + b cos (log x)

Differentiating again w.r.t x we have,

xy_2 + 1 y_1 = - a cos (log x) + b sin (log x) $\frac{1}{x}$

or $x^2y_2 + xy_1$ = - [a cos (log x) + b sin (log x)] = -y

$\therefore$ $x^2y_2+xy_1+y = 0$

Now we have to differentiate this result n times.

ie., $D^n (x^2y_2) + D^n (xy_1) + D^n (y) = 0$

We have to employ Leibnitz theoreom for the first two terms.

Hence we have,

$$\left\{x^2 . D^n (y_2) + n. 2x.D^{n-1} (y_2) + \frac{n(n-1)}{1 . 2} . 2 .D^{n-2} (y^{2)})\right\}$$

$$\{x. D^n (y_1) + n. \ 1 \ . \ D^{n-1} (y_1)\} + y_n = 0$$

ie., $\{x^2y_{n+2} + 2n\,x\,y_{n+1} + n(n-1)y_n\} + \{xy_{n+1}+ny_n\}+y_n = 0$

ie., $x^2y_{n+2} + 2n\,x\,y_{n+1} + n^2y_n - ny_n + xy_{n+1}+ny_n+y_n = 0$

ie., $x^2y_{n+2} + (2n+l)xy_{n+l} + (n^2+l)y_n = 0$

11. If $\cos^{-1}$ (y/b) = log $(x/n)^n$, then show that

$x^2y_{n+2} + (2n+l)\,xy_{n+l} + 2n^2y_n = 0$

Solution : By data, $\cos^{-1}$ (y/b) = n log (x/n) $\therefore$ log(a^m) = m log a

=> $\frac{y}{b}$ = cos [n log (x/n)]

or y = b . cos [n log (x/n)]

Differentiating w.r.t x we get,

y_1 = -b sin [n log (x/n)] $\cdot n \cdot \frac{1}{(x/n)} \cdot \frac{1}{n}$

or xy_1 = - n b sin [n log (x/n)]

Differentiating w.r.t x again we get,

xy_2 + 1. y_1 = - n . b cos [n log (x/n)] n $\frac{1}{(x/n)} . \frac{1}{n}$

or x (xy_2+y_1) = n^2b cos [n log (x/n)] =-n^2y, by using (1).

or $x^2y_2 + xy_l + n^2y = 0$

Differentiating each term n times we have,

$$D(x^2y_2) + D^n(xy_1) + n^2D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{x^2 y_{n+2} + n.\, 2x.\, y_{n+1} + \frac{n(n-1)}{1.\,2}.\, 2\, .\, y_n\right\}$$
$$+\{xy_{n+1} + n.\, 1\, .y_n\} + n^2 y_n = 0$$

ie $x^2y_{n+2} + 2\,x\,y_{n+1} + n^2y_n + xy_{n+1} + ny_n + n^2y_n=0$

or $x^2\,y_{n+2} + (2n + l)\,xy_{n+1} + 2n^2y_n = 0$

12. If y = sin(log (x^2 + 2 x + 1)),

or [Feb-03]

If $\sin^{-1}$ y = 2 log (x + 1), show that

$(x+l)^2y_{n+2} + (2n+1)(x+1)y_{n+l} + (n^2 + 4)y_n = 0$

Solution : By data y = sin log (x^2 + 2 x + 1)

$\therefore$ y_1 = cos log (x^2 + 2 x + 1) $\frac{1}{(x+1)^2} 2x+2$

ie., y1 = cos log (x^2 + 2 x + 1) $\frac{1}{x^2+2x+1}$ 2 (x + 1)

ie., $y_1 = \frac{2\cos\log(x^2+2x+1)}{(x+1)}$

or (x + 1) y1 = 2 cos log (x^2 + 2 x + 1)

Differentiating w.r.t x again we get

$(x+1)y_2 + 1 \;\; y_1 = -2 \sin \log (x^2 + 2x + 1) \; \dfrac{1}{(x+1)^2} . \; 2(x+1)$

or $(x + 1)^2 y_2 + (x+1)\, y_1 = -4y$

or $(x+l\,)^2 y_2 + (x+l)\, y_1 + 4y = 0$,

Differentiating each term n times we have,

$$D^n [(x + 1)^2 y_2] + D^n [(x+ 1)y_1] + D^n [y] = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{(x+1)^2 y_{n+2} + n.\; 2(x+1).\;\; y_{n+1} + \frac{n(n-1)}{1.\;2} .2\;.y_n\right\}$$

$$+ \{(x+l)\, y_{n+1} + n. \;\; 1 \;\; .y_n\} + 4yn = 0$$

ie., $(x+l)^2 y_{n+2} + 2n\,(x+1)y_{n+1}$

$+ n^2 y_n - ny_n + (x+l)y_{n+l} + ny_n + 4y_n = 0$

ie., $(x+l)^2 y_{n+2} + (2n + l)\,(x + l)\, y_{n+1} + (n^2 + 4)y_n = 0$

13. If = log $\left(x+\sqrt{1+x^2}\right)$ prove that

$(1 + x^2)\, y_{n+2} + (2n + 1)\, xy_{n+1} + n^2 y_n = 0$

>> By data, $y = \log \left(x+\sqrt{1+x^2}\right)$

$\therefore y_1 = \dfrac{1}{(x+\sqrt{1+x^2})}\left\{1+\dfrac{1}{2\sqrt{1+x^2}}.\;2x\right\}$

Ie., $y_1 \dfrac{1}{(x+\sqrt{1+x^2})} \;\; \dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{1+x^2}}$

or $\sqrt{1+x^2}\, y_1 = 1$

Differentiating w.r.t.x again we get

$\sqrt{1+x^2 y_2} + \dfrac{1}{2\sqrt{(1+x^2)}} .2x.y_1 = 0$

or $(1+x^2)y_2 + xy_1 = 0$

Now $D^n [(l+x^2)y_2] + D^n[xy_1] = 0$

Applying Leibnitz theorem to each term we get,

$$\left\{(1+x^2)y_{n+2}+n.\,2x\,.y_{n+1}+\frac{n(n-1)}{1.2}.2\,.y_n\right\}$$

+ [x . y_{n+1}+n .1 y_n] = 0

Ie., (1 + x^2) y_{n+2} + 2 n x y_{n+1} + n^2y_n – ny_n + xy_{n+l}+ ny_n = 0

or (l+x^2)y_{n+2} + (2n + l)xy_{n+1}+n^2y_n = 0

14. If x = sin t and y = cos mt, prove that

(l-x^2)y_{n+2}-(2n+1)xy_{n+l} + (m^2-n^2)y_n = 0. [Feb-04]

Solution : By data x = sin t and y = cos mt

x = sin t => t = $\sin^{-1}$ x and y = cos mt becomes

y = cos [m $\sin^{-1}$x)

Differentiating w.r.t.x we get

y_1 = - sin (m $\sin^{-1}$x) $\frac{m}{\sqrt{1-x^2}}$

or $\sqrt{1+x^2}\,y_1$ = - m sin (m $\sin^{-1}$x)

Differentiating again w.r.f .x we get,

$$\sqrt{1-x^2y_2}+\frac{1}{2\sqrt{1-x^2}}(-2x)y_1=-m\,\cos(m\sin^{-1}x).\,\frac{m}{\sqrt{1-x^2}}$$

or (1 -x^2)y_2-xy_l = -m^2y

or (1 -x^2)y_2 –xy_1 +m^2y = 0

Thus (1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0

15. If x = tan (log y), find the value of

(l+x^2)y_{n+1} + (2nx-l) y_n+n(n-1)y_{n-1} [July-04]

Solution : By data x = tan(log y) => $\tan^{-1}$ x = log y or y = $e^{\tan^{-1}}$ x Since the desired relation involves y_{n+1}, y_n and y_{n-1} we can find y_1 and differentiate n times the result associated with y_1 and y.

Consider y = $e^{\tan^{-1}x}$ · $\therefore$ y. = $e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}$

or (1 +x^2)y_1 = y

Differentiating n times we have

$D^n[(l+x^2)y_1]=D^n[y]$

Anplying Leibnitz theorem onto L.H.S, we have,

$\{(l+x^2)D^n(y_1) + n .2x \ .D^{n-1} (y_1)$

$$+\frac{n(n-1)}{1.2}.2 \ .D^{n-2}(y_1)\} = y_n$$

Ie., $(1+x^2)y_{n+1}+2n\, x\, y_n + n\,(n-1)\, y_{n-1}-y_n=0$

Or $(l+x^2)y_{n+1} + (2nx-l)y_n + n(n-l)y_{n-1} = 0$

6) If $2x = y^{1/m} + y^{-1/m}$ Prove that

(i) $\left(x^2-1\right)y_2 + xy_1 - m^2y = 0$

(ii) $\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0$

Proof:

$$2x = y^{1/m} + y^{-1/m}$$

Diff, $$2 = \frac{1}{m}y^{1/m-1}y_1 - \frac{1}{m}y^{-1/m-1}y_1$$

$$= \frac{1}{m}\frac{y^{1/m}}{y}y_1 - \frac{1}{m}\frac{y^{-1/m}}{y}y_1$$

$$2 = \frac{y_1}{m}\left[y^{1/m} - y^{1/m}\right]$$

Squ. $$4m^2y^2 = y_1^2\left[y^{1/m} - y^{-1/m}\right]^2 = y_1^2\left[\left(4y^{1/m} + y^{-1/m}\right)^2 - 4\right]$$

$$4m^2y^2 = y_1^2\left[(2x)^2 - 4\right]$$

$$4m^2y^2 = y_1^2\left[4x^2 - 4\right]$$

i.e. $m^2y^2 = y_1^2\left[x^2-1\right]$

Diff, $m^2(2yy_1) = y_1^2(2x) + \left(x^2-1\right)(2y,y_2)$

$\div 2y_1$ $\left(x^2-1\right)y_2 + xy_1 - m^2y = 0$

Diff n times, $\left[\left(x^2-1\right)y_2\right]_n + \left[xy_1\right]_n - \left[m^2y\right]_n = 0$

i.e. $$(x^2-1)y_{n+2} + 2nxy_{n+1} + (n^2-n)y_n + xy_n + 1 + ny_n - m^2y_n = 0$$

$$\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0$$

Hence the Proof.

Jacobians:

If $(u_1,u_2,...,u_n)$ are funs of n variables, $(x_1,x_2,...,x_n)$.

$$\therefore J(u_1,u_2,...,u_n) = \frac{\partial(u_1,u_2,...,u_n)}{\partial(x_1,x_2,...,x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Problems:

1) **If $x = r\cos\theta,\ y = r\sin\theta$ find J(x, y) or $\dfrac{\partial(x,y)}{\partial(r,\theta)}$** **Nov'15, 16, Apr'17**

Solution:

$$J(x,y) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r$$

2) **If x=u(1+v), y=v(1+u) find J(x, y).**

Solution:

$$J(x,y) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+u)(1+v) - uv$$

$$= 1 + u + v$$

3) **u=xyz, v=xy+yz+zx, w=x+y+z find J(u, v, w)**

Solution:

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz\left[x+z-(y+x)\right] - xz\left[(y+z)-(y+x)\right] + xy\left[(y+z)-(x+z)\right]$$

$$= yz[z-y] - xz[z-x] + xy[y-x]$$

$$= z^2y - y^2z - xz^2 + x^2z + xy^2 - x^2y$$

4) u=x+y+z, uv=y+z, uvw=z find J(x, y, z) **Nov'18**

Solution:

$$u = x + y + z \qquad uv = y + z \qquad uvw = z$$

$$u = x + uv \qquad uv = y + uvw$$

$$x = u - uv \qquad uv - uvw = y \qquad z = uvw$$

$$J(x,y,z) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & uv \\ vw & uw & uv \end{vmatrix}$$

$$= uv\left[u(1-v) + uv\right] = u^2 v$$

5) $\mathbf{x = 2u \;\; y = 3v^2 \;\; z = 4w^2}$ **find** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

6) $\mathbf{x = u + v + w,\; y = u + v - w,\; z = u - v + w}$ **find** $J(x,y,z)$

7) If $\mathbf{x = e^r \sec\theta,\; y = e^r \tan\theta}$ **find** $\dfrac{\partial(x, y)}{\partial(r,\theta)}$

8) Prove that J(x,y)×J(u,v)=1 if x=uv, y=u+v.

8) If $\mathbf{x = r\cos\theta,\; y = r\sin\theta}$ **find** $\dfrac{\partial(r, \theta)}{\partial(x,y)}$

Solution:

To find J(r,θ). let us find r and θ.

$$x = r\cos\theta \quad \to (1) \qquad y = r\sin\theta \quad \to (2)$$

Squaring and adding (1) and (2), we get

$$x^2 = r^2\cos^2\theta$$

$$y^2 = r^2\sin^2\theta$$

$$x^2 + y^2 = r^2\left(\cos^2\theta + \sin 2\theta\right)$$

$$\frac{\partial r}{\partial x} = \frac{1(\cancel{2}x)}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1(2y)}{2\sqrt{x^2+y^2}} = \frac{y}{r}$$

i.e. $x^2 + y^2 = r^2$

$$\boxed{r = \sqrt{x^2+y^2}}$$

Dividing (2) by (1)

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\Rightarrow \tan\theta = y/x$$

$$\boxed{\theta = \tan^{-1} y/x}$$

$$\frac{\partial r}{\partial x} = \frac{1(\cancel{2}x)}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1(2y)}{2\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\boxed{J(r,\theta) = \frac{1}{r}}$$

Maxima and Minima of funs if two variables:

Working Rules:

(i) Take the fun as f(x, y)

(ii) Find f_x, f_y, f_{xx}=r, f_{yy}=t, f_{xy}=s.

(iii) Equate f_x & f_y to zero, & solve those eqns for x & y.

(iv) The pt (x,y) is critical pt or stationary point.

(v) Find rt–S^2 at critical pt.

If $rt - S^2 > 0$ and r > 0, f is minimum at (x, y).

If $rt - S^2 > 0$ and r < 0, f is maximum (x, y).

If $rt - S^2 < 0$, f & (x,y) is saddle point neither min nor max at (x, y).

If rt – S^2 =0, we cannot say f is max or min at (x, y).

Problems:

1) Find the maximum and minimum values of $\mathbf{f(x,y) = 2(x^2 - y^2) - x^4 + y^4}$

Solution:

$$f(x,y) = 2x^2 - 2y^2 - x^4 + y^4$$

$$f_x = \frac{\partial f}{\partial x} = 4x - 4x^3$$

$$f_y = \frac{\partial f}{\partial y} = -4y + 4y^3$$

$$f_{xx} = r = \frac{\partial^2 f}{\partial x^2} = -4y + 4y^3$$

$$f_{yy} = t = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$$

$$f_{xy} = s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

To find maxima and minima, Put f_x=0 & f_y=0

$$\begin{array}{ll} 4x - 4x^3 = 0 & -4y + 4y^3 = 0 \\ 4x\left(1 - x^2\right) = 0 & 4y\left[-i + y^2\right] = 0 \\ x = 0,\ x^2 = 1 \Rightarrow x = \pm 1 & y = 0,\ y^2 = 1 \\ x = 0, 1, -1 & y = 0,\ y = \pm 1, -1. \\ \therefore x = 0,\ \pm 1 & \\ \quad y = 0,\ \pm 1 & \end{array}$$

The critical points are $(0,0)(\pm 1,0),(\pm 1,\pm 1),(0,\pm 1)$.

At (0, 0)

$$\left[rt - S^2\right]_{(0,0)} = \left[\left(4 - 12x^2\right)\left(-4 + 12y^2\right) - 0\right]_{(0,0)}$$

$$= -16 < 0$$

$\therefore$ (0, 0) is a saddle pt. f is neither min nor max at (0, 0).

At (0, ±1)

$$\left[rt - S^2\right]_{(0,\pm 1)} = \left[\left(4 - 12x^2\right)\left(-4 + 12y^2\right) - 0\right]_{(0,\pm 1)}$$

$$= (4)\left(-4 + 12(1)\right)$$

$$= 4(8) = 32 > 0$$

$$\left[r\right]_{(0,\pm 1)} = \left[4 - 12x^2\right]_{(0,\pm 1)} = 4 > 0$$

$$rt - s^2 > 0 \,\&\, r > 0 \text{ at } (0, \pm 1)$$

$$\boxed{\text{f is min at } (0, \ \pm 1)} \quad f(0, \pm 1) = -1 \text{ (min value)}$$

At (±1, 0)

$$\left[rt - S^2\right]_{(\pm 1,0)} = \left[\left(4 - 12x^2\right)\left(-4 + 12y^2\right) - 0\right]_{(\pm 1,0)}$$

$$= \left[\left(4 - 12(1)\right)\left(-4 + 12(0)\right)\right]$$

$$= (-8)(-4) = 32 > 0$$

$$[r]_{(\pm 1,0)} = \left[4 - 12x^2\right]_{(\pm 1,0)}$$

$$= 4 - 12(1) = -8 < 0$$

$$rt - S^2 > 0 \quad \& \quad r < 0 \quad \text{at} \quad (\pm 1, 0)$$

$$\therefore \boxed{\text{f is maximum at } (\pm 1, 0)} \quad f(\pm 1, 0) = 1 \text{ (max value)}$$

At (±1, ±1)

$$\left[rt - S^2\right]_{(\pm 1, \pm 1)} = \left[4 - 12(1)\right]\left[-4 + 12(1)\right] = (-8)(8) = -64$$

$$rt - S^2 < 0$$

$$\therefore (\pm 1, \pm 1) \text{ is a saddle point.}$$

$\therefore$ f is neither minimum nor maximum at $(\pm 1, \pm 1)$

2) Investigate maxima of the function.

$$\mathbf{f(x,y) = x^3y^2(6 - x - y)}$$

Solution:

$$f(x,y) = x^3y^2(6 - x - y) = 6x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 18x^2y^2 - 4x^3y^2 - 3x^3y^3$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2$$

$$f_{xx} = r = 36xy^2 - 12x^2y^2 - 6xy^3$$

$$f_{yy} = t = 12x^3 - 2x^4 - 6x^3y$$

$$f_{xy} = s = 36x^2y - 8x^3y - 9x^2y^2$$

To find maxima,

Put $f_x = 0$, $f_y = 0$

$$18x^2y^2 - 4x^3y^3 + 3x^2y^3 = 0 \qquad\qquad 12x^3y - 2x^4y - 3x^3y^2 = 0$$
$$x^2y^2[18 - 4x - 3y] = 0 \qquad\qquad x^3y[12 - 2x - 3y] = 0$$
$$x = 0, y = 0,\ 4x + 3y = 18 \qquad\qquad x = 0, y = 0,\ 2x + 3y = 12$$

Solving

$$4x + 3y = +18$$
$$2x + 3y = +12$$
$$x = 3,\ y = 2$$

The critical pts are (0,0), (3,2)

At (0, 0)

$$\left[rt - s^2\right]_{(0,0)} = 0$$

$\therefore$ We cannot say f is max or min at (0, 0)

At (3, 2)

$$\left[rt - s^2\right]_{(3,2)} = +ve, \qquad [r]_{(3,2)} < 0$$

$\therefore$ f attains maximam at (3, 2)

Max value

$$f(3,2) = 32(6-3)$$
$$= 27 \times 4 = 108$$

9) Find maximum and minimum of xy(a – x – y).

Solution:

Let $f(x, y) = xy(a-x-y)$

$= xya - x^2y - xy^2$

$f_x = ay - 2xy - y^2$

$f_y = ax - x^2 - 2xy$

$f_{xx} = r = -2y$

$f_{yy} = t = -2x$

$f_{xy} = s = a - 2x - 2y$

To find max & min value

Put $f_x = 0$, $f_y = 0$

$ay - 2xy - y^2 = 0$ $\qquad$ ax–x2–2xy=0

$y[a-2x-y]=0$ $\qquad$ $x[a-x-2y]=0$

$y=0 \quad 2x+y=+a$ $\qquad$ $x=0 \quad 2y+x=+a$

x=0, y=0, solving 2x+y=+a for x & y,

x+2y=+a

$x = a/3,\ y = a/3$

If x=0, 2x+2y = a $\Rightarrow$ y=a

If y=0, 2y+x = a $\Rightarrow$ x=a

$\therefore$ If $x = 0, y = a$ & $y = 0 \Rightarrow x = a,\ x = a/3 \Rightarrow y = a/3, x = 0, y = 0$

The critical pts are $(0,a), (a,0), \left(a/3, a/3\right)$ & $(0,0)$.

At (0, 0)

$$\left[rt - s^2\right]_{(0,0)} = -a^2 < 0$$

$\therefore$ (0, 0) is a saddle pt.

f is neither min nor maximum at (0, 0)

At (0, a)

$$\left[rt - s^2\right]_{(0,a)} = -a^2 < 0$$

$\therefore$ (0, a) is a saddle pt.

f is neither min nor maximum at (0, a)

At (a, 0)

$$\left[rt - s^2\right]_{(a,0)} = -a^2 < 0$$

$\therefore$ (a, 0) is a saddle pt.

f is neither min nor maximum at (a, 0)

At $\left(a/3, a/3\right)$

$$\left[rt - s^2\right]_{\left(a/3, a/3\right)} = a^2/3 > 0$$

$$[r]_{\left(a/3, a/3\right)} = -2a/3 < 0$$

$rt - s^2 > 0$ & $r < 0$ at $\left(a/3, a/3\right)$

$\therefore$ f has maximum value at $\left(a/3, a/3\right)$

Max. value

$$f(x, y) = xy(a-x-y)$$

$$f\left(a/3, a/3\right) = \frac{a}{3}.\frac{a}{3}\left(a - a/3 - a/3\right) = \frac{a^2}{9}\left(a/3\right) = a^3/27$$

Max value is $a^3/27$

4) f(x, y) = $x^4+y^4+-4xy+1$

5) f(x, y) = $x^2+5y^2-6x+10y+12$

6) u=$x^3y^2(1-x-y)$ find max.

7) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

8) $x^2+y^2+6x+12$

9) $x^2+y^2-4x-2y+10$

10) $x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$

11) x^3+y^3-3xy

12) f(x)=x^2-2x+2, Find Min in[0,3]

1) Find the maximum and minimum values of the function f(x, y) = $x^2y^2-x^2-y^2$. Nov' 15

Solution:

$$f(x, y) = x^2y^2-x^2-y^2$$

$$\frac{\partial f}{\partial x} = f_x = 2xy^2 - 2x$$

$$\frac{\partial f}{\partial y} = f_y = 2x^2y - 2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 2y^2 - 2$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = 2x^2 - 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \text{ or } f_{yx} = 4xy$$

Step:1

Put $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow 2xy^2 - 2x = 0$$

$$2x\left(y^2 - 1\right) = 0$$

$$2x = 0,\ y^2 = 1$$

$$\Rightarrow x = 0, \Rightarrow y = \pm 1$$

Critical point are (0,0), (0,±1), (±1, 0), (±1, ±1)

Step:2

$$rt - s^2 = \left(2y^2 - 2\right)\left(2x^2 - 2\right) - \left(4xy\right)^2$$

$$= \left(2y^2 - 2\right)\left(2x^2 - 2\right) - 16x^2y^2$$

$$= 4x^2y^2 - 4y^2 - 4x^2 + 4 - 16x^2y^2$$

$$rt - s^2 = -12x^2y^2 - 4y^2 - 4x^2 + 4$$

Step:3

At(0,0)

$$\left[rt - s^2\right]_{(0,0)} = 4 > 0, \qquad \left[r\right]_{(0,0)} = -2 < 0$$

r is negative $\therefore$ f(x, y) is maximum at (0, 0).

Maximum value

$$[f]_{(0,\,0)} = 0.$$

maximum value of f is 0.

At [0, ±1] and [±1, 0]

$$\left[rt - s^2\right]_{(0,\pm 1)} = 0$$

$$\left[rt - s^2\right]_{(\pm 1,0)} = 0$$

Further investigation is needed at these points

At [±1, ±1]

$$\left[rt - s^2\right]_{(\pm1,\pm1)} = (0)(0) - 4 = -4 < 0$$

Since $rt – s^2$ is negative

(±1, ±1) is a saddle point.

4) Find the maximum and minimum values of **Apr'17.**

$$\mathbf{f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}}$$

Solution:

$$f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$$

$$\frac{\partial f}{\partial x} = f_x = 2x - \frac{2}{x^2}$$

$$\frac{\partial f}{\partial y} = f_y = 2y - \frac{2}{y^2}$$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 2 + \frac{4}{x^3}$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = 2 + \frac{4}{y^3}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} \text{or} \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = 0$$

Step:1

Put $f_x = 0$, $f_y = 0$

$$2x - \frac{2}{x^2} = 0 \qquad 2y - \frac{2}{y^2} = 0$$

$$x^3 - 1 = 0 \qquad y^3 - 1 = 0$$

$$(x-1)(x^2 + x + 1) = 0 \qquad (y-1)(y^2 + y + 1) = 0$$

$$x = 1 \qquad y = 1$$

$x^2 + x + 1 = 0$ and $y^2 + y + 1 = 0$ given imaginary roots.

Omitting those imaginary roots, the only critical point is (1, 1)

Step:1

$$rt - s^2 = \left(2 + \frac{4}{x^3}\right)\left(2 + \frac{4}{y^3}\right) - 0$$

Step: 2

At (1,1)

$$\left[rt - s^2\right]_{(1,1)} = (2+4)(2+4) = 36 > 0$$

$$rt - s^2 > 0$$

$$[r]_{(1,1)} = \left[2 + \frac{4}{x^3}\right]_{(1,1)} = 6 > 0$$

$\therefore$ f(x, y) attains minimum value at (1, 1)

Minimum value:

$$[f]_{(1,1)} = 1^2 + 1^2 + \frac{2}{1} + \frac{2}{1} = 6$$

Minimum value of f(x, y) is 6.

Method of Lagrange's Multipliers:

(Subject to the constraints)

Working rule:

(i) Form a new function f(x,y, z)

Where $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

(ii) Find F_x, F_y, F_λ and equate them to zero.

(iii) Solve those eqns for x, y, λ.

Problems:

1) Find the maxima and minima of $f(x,y) = 3x^2 + 4y^2 - xy$ if 2x+y=21.

Solution:

$f(x,y) = 3x^2+4y^2-xy$

Let $g(x, y) = 2x+y-21$

Let $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$F = 3x^2 + 4y^2 - xy - \lambda(2x + y - 21)$$

$$F_x = 6x - y = 2\lambda$$

$$F_y = 8y - x - \lambda$$

$$F_\lambda = -(2x + y - 21)$$

To find max & min,

Put $F_x=0$, $F_y=0$, $F_\lambda=0$

$$6x - y - 2\lambda = 0 \quad \rightarrow (1)$$
$$8y - x - \lambda = 0 \quad \rightarrow (2)$$
$$-(2x + y - 21) = 0 \quad \rightarrow (3)$$

Solving (1) & (2) for x, y by eliminating λ.

$$(1)+(2)\times 2 \quad \Rightarrow 8x - 17y = 0$$
$$(3)\times 4 \quad \Rightarrow \underline{8x + 4y = 84}$$
$$-21y = -84$$
$$y = 4$$

$$y = 4 \quad \Rightarrow \quad x = 17/2$$

At $\left(17/2, 4\right)$

$$f\left(17/2, 4\right) = 3\left(\frac{289}{4}\right) + 4(16) - 34 = \frac{987}{4}$$

Consider the constraint 2x+y=21

Put x=0 ⇒ y=21

∴ (0, 21) satisfies the constraint 2x+y=21.

At (0, 21)

$$f(0, 21) = 4\,(441) = 1764$$

$$f\,(0, 21) > f\left(17/2, 4\right)$$

∴ f is minimum at $\left(17/2, 4\right)$

& the min value is $\frac{987}{4}$

2) Find the max & min of $f(x, y)=2xy-3y^2-x^2$ subject to x+y=16

Solution:

$$f(x, y) = 12xy - 3y^2 - x^2$$

Let $g(x,y) = x+y-16$

$$F(x, y, \lambda) = 12xy - 3y^2 - x^2 - \lambda(x+y-16=0)$$

$F_x = 12y - 2x - \lambda$

$F_y = 12x - 6y - \lambda$

$F_\lambda = -(x+y-16) = 0$

To fin max & min,

Put $F_x = 0$, $F_y = 0$ & $F_\lambda = 0$

$$\begin{aligned}
&12y - 2x - \lambda = 0 && \rightarrow (1)\\
&12x - 6y - \lambda = 0 && \rightarrow (2)\\
&x + y - 16 = 0 && \rightarrow (3)\\
&(2) - (1) \Rightarrow 14x - 18y = 0 &&\\
&\qquad \text{i.e.} \quad 7x - 9y = 0 &&\\
&(3) \times 7 \;\Rightarrow\; 7x + 7y = 112 &&\\
&\qquad \Rightarrow\; x = 9,\; y = 7 &&
\end{aligned}$$

At (9, 7)

$f(9, 7) = 12(63) - 3(49) - 81 = 528$

Consider the constraint $x+y=16$

Put $x=0$ $\Rightarrow y=16$

$\therefore$ (0, 16) satisfies $x+y=16$

At (0, 16)

$f(0, 16) = -3(256) = -768$

$\therefore f(9, 7) > f(0, 16)$

f is max at (9, 7) & the max value is 528.

3) Show that the maximum value of $x^2y^2z^2$ subject to $x^2 + y^2 + z^2 = a^2$ is $\left(a^2/3\right)^3$ Nov'17

Proof:

Let $f(x, y, z) = x^2y^2z^2$

& $g(x, y, z) = x^2+y^2+z^2-a^2$

$F(x, y, \lambda) = f(x, y, z) - \lambda g(x, y, z)$

$= x^2y^2z^2 - \lambda(x^2+y^2+z^2-a^2)$

$F_x=0 \Rightarrow 2xy^2z^2-2\lambda x = 0$

$2x(y^2z^2-\lambda) = 0$

$x=0 \quad \& \quad \lambda=y^2z^2$

$F_y=0 \Rightarrow 2yx^2z^2-2\lambda y = 0$

$2y(x^2z^2-\lambda) = 0$

$y=0 \quad \& \quad \lambda=x^2z^2$

$Fz=0 \Rightarrow 2zx^2z^2-2\lambda z = 0 \qquad F\lambda=0 \Rightarrow x^2+y^2+z^2=a^2$

$\therefore z=0 \; \& \; x^2y^2=\lambda$

$\lambda = x^2y^2 = y^2z^2 = z^2x^2$

$x^2y^2 = y^2z^2 \Rightarrow x^2 = z^2$

$y^2z^2 = z^2x^2 \Rightarrow y^2 = x^2$

$\therefore x^2 = y^2 = z^2$

Consider the constraint,

$$x^2+y^2+z^2=a^2$$

$$x^2+x^2+x^2=a^2 \qquad \left(\therefore y^2=x^2;\; z^2=x^2\right)$$

$$3x^2=a^2$$

$$x^2=\frac{a^2}{3}$$

$$\therefore x^2=y^2=z^2=a^2/3$$

The max value of f is

$$f(x,y,z)=f\left(\pm\sqrt{a^2/3},\pm\sqrt{a^2/3},\pm\sqrt{a^2/3}\right)=\frac{a^2}{3}.\frac{a^2}{3}.\frac{a^2}{3}=\left(\frac{a^2}{3}\right)^3$$

4) Find min of $a^3x^2+b^3y^2+c^3z^2$ with the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Solution:

Let

$$f(x,y,z)=a^3x^2+b^3y^2+c^3z^2$$

$$\&g(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

$$F=a^3x^2+b^3y^2+c^3z^2-\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$$F_x = 0 \Rightarrow 2a^3x + \lambda/x^2 = 0 \Rightarrow \lambda = -2a^3x^3$$

$$F_y = 0 \Rightarrow 2b^3y + \lambda/y^2 = 0 \Rightarrow \lambda = -2b^3y^3$$

$$F_z = 0 \Rightarrow 2c^3z + \lambda/z^2 = 0 \Rightarrow \lambda = -2c^3z^3$$

$$F\lambda = 0 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

$$\lambda = -2a^3x^3 = --2b^3y^3 = --2c^3z^3$$

$$\text{i.e. } a^3x^3 = b^3y^3 = c^3z^3$$

$$ax = by = cz$$

$$ax = by \qquad by = cz$$

$$\Rightarrow x = b/a\, y \qquad \Rightarrow z = b/c\, y$$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \frac{1}{b/a\, y} + \frac{1}{y} + \frac{1}{b/c\, y} = 1$$

$$\frac{a}{by} + \frac{1}{y} + \frac{c}{by} = 1$$

$$\frac{1}{y}\left[\frac{a}{b} + 1 + \frac{c}{b}\right] = 1$$

$$y = \frac{a+b+c}{b}$$

$$x = \frac{b}{a}\left(\frac{a+b+c}{c}\right) = \frac{a+b+c}{a}$$

$$z = b/c\, y = b/c\left(\frac{a+b+c}{b}\right) = \frac{a+b+c}{c}$$

At $\left(\boldsymbol{\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}}\right)$

$$f(x,y,z) = a^3\frac{(a+b+c)^2}{a^2} + b^3\frac{(a+b+c)^2}{b^2} + c^3\frac{(a+b+c)^2}{c^2}$$

$$= a(a+b+c)^2 + b(a+b+c)^2 + c(a+b+c)^2$$

$$= (a+b+c)^2(a+b+c)$$

$$= (a+b+c)^3$$

The min value of f is $(a+b+c)^3$.

CURVATURE

1. 1 Curvature and radius of curvature

The curvedness of a curve at a point ρ on it is measured by the rate of change of Ψ with respect to s, where Ψ is the angle made by the tangent at ρ with the x-axis and s is the arcual distance of ρ from a fixed point Q on the curve, that is by dΨ/ds.

This rate is called the curvature of the curve at ρ.

Curvature of a circle

Consider a circle as in the figure whose centre is C and radius a. Let Ψ be the angle made by the tangent at any point ρ with the x-axis. If the arcual distance of ρ from O is s, then s = aΨ. This is the intrinsic eqn of the circle.

Differentiating this w.r.t 's', we get

$$1 = a\frac{d\Psi}{ds}.$$

$$\therefore \frac{d\Psi}{ds} = \frac{1}{a}$$

So, in the case of circle, the curvature is a constant which is the reciprocal of the radius.

1.2 Radius of curvature

The reciprocal of Curvature of a curve at a point is called the radius of curvature of the curve at the point. So it is $\frac{ds}{d\Psi}$.

The radius of Curvature of a circle is its radius.

Notation

Radius of Curvature is denoted by ρ.

Remark :1

In the case of a straight line the change of Ψ is zero and hence $\frac{d\Psi}{ds} = 0, \rho = \frac{ds}{d\Psi} = \infty$

Remark : 2

If the curve is such that, as 's' increases, Ψ increases, then $\frac{d\Psi}{ds}$ is +ve and, so ρ is +ve.

ie) if the curve is concave, ρ is +ve otherwise is –ve In general, ρ is given as its absolute value, namely $|\rho|$.

1.3. Cartesian formula for the radius of curvature

We know that $\frac{dy}{dx} = \tan\Psi$

$$\therefore \frac{d^2y}{dx^2} = \sec^2\Psi.\frac{d\Psi}{dx} = \sec^2\Psi\frac{d\Psi}{ds}\frac{ds}{dx}.$$

$$\therefore \frac{ds}{d\Psi} = \frac{\sec^3 \Psi}{\frac{d^{2y}}{dx^2}} \; as \frac{dx}{ds} = \cos \Psi$$

$$= \frac{(1 + \tan^2 \Psi)^{3/2}}{\frac{d^2 y}{dx^2}}$$

$$= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}}$$

Examples:

1. What is the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1)?

Soln:

Given the curve $x^4 + y^4 = 2$

Differentiating the above equation, we get

$$4x^3 + 4y^3 \frac{dy}{dx} = 0.$$

$$4x^3 = -4y^3 \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = -\frac{x^3}{y^3}.$$

Differentiating this once again, we get

$$\frac{d^2y}{dx^2} = \frac{3\left(x^3\frac{dy}{dx} - x^2y\right)}{y^4}.$$

At the point (1,1), $\frac{dy}{dx} = -1, and\ \frac{d^2y}{dx^2} = -6.$

$$\therefore \rho = \frac{(1+1)^{3/2}}{6} = -\frac{\sqrt{2}}{3}.$$

2. Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Soln:

Given $y = c\cosh\frac{x}{c}$

Differentiating the above equation, we get

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

Now,` $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left(1+\sinh^2\frac{x}{c}\right)^{3/2} = \cosh^3\frac{x}{c}$

Also $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}.$

Here $\rho = \dfrac{\cosh^3 \frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c} = \frac{y^2}{c}$

Again at any point (x,y)

the normal $= y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} = y\cosh\frac{x}{c} = \frac{y^2}{c}$

$\therefore$ Radius of curvature = length of the normal.

3. If a curve is defined by the parametric equation x=f (θ) and y=ϕ(θ), prove that the curvature is $\frac{1}{\rho} = \frac{x'y''-y'x''}{(x'2+y'2)^{3/2}}$

Soln:

where dashes denote differentiation with respect to θ.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{d\theta}{dx}$$

$$= \frac{y''x'-y'x''}{x'^2}\,\frac{1}{x'}$$

$$= \frac{y''x'-y'x''}{x'^3}$$

$$\therefore \frac{1}{\rho} = \frac{\frac{d^{2y}}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{y''x' - y'x''}{x'^3\left[1+\frac{y'^2}{x'^2}\right]^{3/2}}$$

$$= \frac{x'y'' - y'x''}{(x'^2+y'^2)^{3/2}}.$$

4. Prove that the radius of curvature at any point of the cycloid x =a (θ + sinθ) and y = a (1 - cosθ) is 4 a cos $\frac{\theta}{2}$.

Soln:

From the given equations ,

x =a (θ + sinθ)

differentiation with respect to θ.

$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{d^2x}{d\theta^2} = -a\sin\theta$$

y = a (1 - cosθ)

differentiation with respect to θ.

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{d^2y}{d\theta^2} = a\cos\theta.$$

Substituting the values in the formula obtained in the previous example, we get

$$\frac{1}{\rho} = \frac{a(1+\cos\theta)a\cos\theta - a\sin\theta(-a\sin\theta)}{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{a^2(1+\cos\theta)}{a^3\left[2(1+\cos\theta)\right]^{3/2}}$$

$$= \frac{2\cos^2\theta/2}{a\left[4\cos^2\theta/2\right]^{3/2}} = \frac{1}{4a\cos^{\theta/2}}$$

$$\therefore \rho = 4a\cos\frac{\theta}{2}.$$

5. Find ρ at the point 't' of the curve x = a (cos t + t sin t); y = a (sin t – t cos t)

Soln:

Given the curve

x = a (cos t + t sin t); y = a (sin t – t cos t)

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t.$$

$$\therefore \frac{dy}{dx} = \tan t.$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\frac{dt}{dx} = \sec^2 t\frac{1}{at\cos t} = \frac{1}{at\cos^3 t}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\frac{1+\tan^2 t)^{3/2}}{1}}{at\cos^3 t} = at.$$

(The formula of Ex.3 can also be employed)

Exercise 1:

1. Find the radius of curvature for the curves

(a) $y = e^x$ at the point where it crosses the y – axis

(b) $\sqrt{x} + \sqrt{y} = 1 at (1/4, 1/4)$

(c) $y^2 = x^3 + 8$ at the point (-2, 0).

(d) xy = 30 at the point (3,10)

(e) $(x^2 + y^2)^2 = a^2 (y^2 - x^2)$ at the point (0, a)

Polar form.

Let r = f(θ) be the given curve in polar coordinates.

∴ x = r cos θ and y = r sin θ, may be regarded as the parametric equations of the given curve the parameter being θ.

$$\therefore \frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$$

and $\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$

$$\therefore \frac{d^2x}{d\theta^2} = \cos\theta \frac{d^2r}{d\theta^2} - 2\sin\theta \frac{dr}{d\theta} - r\cos\theta \, and$$

$$\frac{d^2y}{d\theta^2} = \sin\theta \frac{d^2r}{d\theta^2} - 2\cos\theta \frac{dr}{d\theta} - r\sin\theta$$

Substituting these values in the formula for ρ in parametric from and simplifying we get

$$\rho = \frac{(r^2 + r_1^{\,2})^{\frac{3}{2}}}{r^2 + 2r_1^{\,2} - rr_2} \, where \; r_1 = \frac{dr}{d\theta} \, and \; r_2 = \frac{d^2r}{d\theta^2}.$$