MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

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DEPARTMENT OF MATHEMATICS

SUBJECT NAME: REAL ANALYSIS-II

SUBJECT CODE: SM26B

SEMESTER: VI

PREPARED BY: PROF.S.C.PREMILA

UNIVERSITY OF MADRAS
B.Sc. DEGREE COURSE IN MATHEMATICS
SYLLABUS WITH EFFECT FROM 2020-2021

CORE-XIV: REAL ANALYSIS-II
(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 6 **YEAR: III**
Credits : 4 **SEMESTER: VI**
Learning outcomes:

Students will acquire knowledge about

- The Real Numbers and the Analytic Properties of Real- Valued Functions.
- The Analytic concepts of Connectedness, Compactness, Completeness And Calculus.

UNIT I

Continuous Functions on Metric Spaces: Open sets- closed sets- Discontinuous function on R^1. Connectedness, Completeness and Compactness :More about open sets- Connected sets. Chapter 5 Section 5.4 to 5.6 Chapter 6 Section 6.1 and 6.2

UNIT II

Bounded sets and totally bounded sets: Complete metric spaces- compact metric spaces, continuous functions on a compact metric space, continuity of inverse functions, uniform continuity.

Chapter 6 Section 6.3 to 6.8

UNIT III

Calculus:Sets of measure zero, definition of the Riemann integral, existence of the Riemann integral-properties of Riemann integral.

Chapter 7 Section 7.1 to 7.4

UNIT IV

Derivatives- Rolle's theorem, Law of mean, Fundamental theorems of calculus.

Chapter 7 Section 7.5 to 7.8

UNIT V

Taylor's theorem- Pointwise convergence of sequences of functions, uniform convergence of sequences of functions.

Chapter 8 Section 8.5 Chapter 9 Section 9.1 and 9.2

Content and Treatment as in

"Methods of Real Analysis"- Richard R. Goldberg (Oxford and IBH Publishing Co)

Reference: -

1. Principles of Mathematical Analysis by Walter Rudin,TataMcGrawHill.
2. Mathematical Analysis Tom M Apostal,Narosa Publishing House.

Unit - I Metric Spaces

Introduction

A Metric Space is a set equipped with a distance function, also called a metric, which enables us to measure the distance between two elements in the set.

1.1 Definition And Examples

Definition 1.1.1 A **Metric Space** is a non empty set **M** together with a function **d : M × M →**

R satisfying the following conditions.

(i) $d(x , y) \geq 0$ for all $x , y \in M$
(ii) $d(x , y) = 0$ if and only if $x = y$
(iii) $d(x , y) = d(y , x)$ for all $x , y \in M$
(iv) $d(x , z) \leq d(x , y) + d(y , z)$ for all $x , y , z \in M$ **[Triangle Inequality] d** is called a **metric** or **distance function** on M and **d(x , y)** is called the distance between x and y in M. The metric space M with the metric d is denoted by (M , d) or simply by M when the underlying metric is clear from the context.

Example 1.1.2 Let **R** be the set of all real numbers. Define a function $d : M \times M \rightarrow R$ by $d(x , y) = | x - y |$. Then d is a metric on **R** called the usual metric on **R.**

Proof.

Let $x , y \in$ **R.**

Clearly $d(x , y) = | x - y | \geq 0$.

Moreover, $d(x , y)= 0 \Leftrightarrow |x - y | = 0$.

$\Leftrightarrow x - y = 0$.

$\Leftrightarrow x = y$

$d(x , y) = | x - y |$

$= | y - x |$

$= d(y ,x)$.

$\therefore d(x , y) = d(y , x)$.

Let $x , y , z \in$ **R.** $d(x , z)$

$= |x - z |$

$= |x - y + y - z|$

$\leq |x - y| + |y - z|$

$= d(x, y) + d(y, z)$.

$\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Hence d is a metric on **R.**

Note. When R is considered as a metric space without specifying its metric, it is the usual metric.

Example 1.1.2

Let M be any non-empty set. Define a function $d : M \times M \to \mathbf{R}$ by $d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

Then d is a metric on M called the **discrete metric** or **trivial metric** on M.

Proof.

Let $x, y \in M$.

Clearly $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.

Also, $d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

$= d(y, x)$.

Let $x, y, z \in M$.

We shall prove that $d(x, z) \leq d(x, y) + d(y, z)$.

Case (i) Suppose $x = y = z$.

Then $d(x, z) = 0$, $d(x, y) = 0$, $d(y, z) = 0$.

$\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Case (ii) Suppose $x = y$ and z distinct.

Then $d(x, z) = 1$, $d(x, y) = 0$, $d(y, z) = 1$. $\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Case (iii) Suppose $x = z$ and y distinct.

Then $d(x, z) = 0$, $d(x, y) = 1$, $d(y, z) = 1$.

$\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Case (iv) Suppose $y = z$ and x distinct.

Then $d(x, z) = 1$, $d(x, y) = 1$, $d(y, z) = 0$.

$\therefore d(x, z) \le d(x, y) + d(y, z)$.

Case (v) Suppose $x \ne y \ne z$.

Then $d(x, z) = 1$, $d(x, y) = 1$, $d(y, z) = 1$. $\therefore d(x, z) \le d(x, y) + d(y, z)$.

In all the cases, $d(x, z) \le d(x, y) + d(y, z)$.

Hence d is a metric on M.

1.2 OPEN SETS IN A METRIC SPACE

Definition 1.2.1 Let (M , d) be a metric space. Let $a \in M$ and r be a positive real number. The open ball or the open sphere with center a and radius r is denoted by $B_d(a, r)$ and is the subset of M defined by $B_d(a, r) = \{x \in M / d(a, x) < r\}$. We write $B(a, r)$ for $B_d(a, r)$ if the metric d under consideration is clear.

Note. Since $d(a, a) = 0 < r$, $a \in B_d(a, r)$.

Examples 1.2.2

1. In **R** with usual metric $B(a, r) = (a - r, a + r)$.
2. In $\mathbf{R}^2$ with usual metric $B(a, r)$ is the interior of the circle with center a and radius r.
3. In a discrete metric space M, $B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \le 1 \end{cases}$

Definition 1.2.3 Let (M , d) be a metric space. A subset A of M is said to be open in M if for each $x \in A$ there exists a real number $r > 0$ such that $B(x, r) \subseteq A$.

Note. By the definition of open set, it is clear that $\emptyset$ and M are open sets.

Examples 1.2.3

1. Any open interval (a , b) is an open set in **R** with usual metric.
   For,

   Let $x \in (a, b)$.

   Choose a real number r such that $0 < r \le \min\{x-a, b-x\}$.

   Then $B(x, r) \subseteq (a, b)$. $\therefore (a, b)$ is open in R.

2. Every subset of a discrete metric space M is open.
   For,

   Let A be a subset of M.

If $A = \emptyset$, then A is open.

Otherwise, let $x \in A$.

Choose a real number r such that $0 < r \leq 1$.

Then $B(x, r) = \{ x \} \subseteq A$ and hence A is open.

3. Set of all rational numbers **Q** is not open in **R**. For,
Let $x \in \mathbf{Q}$.

For any real number $r > 0$, $B(x, r) = (x - r, x + r)$ contains both rational and irrational numbers.

$\therefore B(x, r) \nsubseteq \mathbf{Q}$ and hence **Q** is not open.

Theorem 1.2.4 Let (M , d) be a metric space. Then each open ball in M is an open set.

Proof.

Let $B(a, r)$ be an open ball in M.

Let $x \in B(a, r)$.

Then $d(a, x) < r$.

Take $r_1 = r - d(a, x)$. Then $r_1 > 0$.

We claim that $B(x, r_1) \subseteq B(a, r)$.

Let $y \in B(x, r_1)$. Then $d(x, y) < r_1$.

Now, $d(a, y) \leq d(a, x) + d(x, y)$

$< d(a, x) + r_1$

$= d(a, x) + r - d(a, x) = r$.

$\therefore d(a, y) < r$.

$\therefore y \in B(a, r)$.

$\therefore B(x, r_1) \subseteq B(a, r)$.

Hence $B(a, r)$ is an open ball.

Theorem1.2.5 In any metric space M, the union of open sets is open.

Proof.

Let $\{A_\alpha\}$ be a family of open sets in M.

We have to prove $A = \cup A_\alpha$ is open in M.

Let $x \in A$.

Then $x \in A_\alpha$ for some α.

Since A_α is open, there exists an open ball $B(x , r)$ such that $B(x , r) \subseteq A_\alpha$.

$\therefore B(x , r) \subseteq A$.

Hence A is open in M.

Theorem 1.2.6 In any metric space M, the intersection of a finite number of open sets is open.

Proof.

Let $A_1, A_2, \ldots, A_n$ be open sets in M.

We have to prove $A = A_1 \cap A_2 \cap \ldots \cap A_n$ is open in M.

Let $x \in A$.

Then $x \in \forall\ A_i\ i = 1, 2, \ldots , n$.

Since each A_i is open, there exists an open ball $B(x , r_i)$ such that $B(x , r_i) \subseteq A_i$.

Take $r = \min \{ r_1 , r_2 , \ldots , r_n \}$.

Clearly $r > 0$ and $B(x , r) \subseteq B(x , r_i)\ \ \forall\ i = 1, 2, \ldots , n$.

Hence $B(x , r) \subseteq A_i \forall\ i = 1, 2, \ldots , n$.

$\therefore B(x , r) \subseteq A$.

$\therefore$ A is open in M.

Theorem 1.2.7 Let (M , d) be a metric space and $A \subseteq M$. Then A is open in M if and only if A can be expressed as union of open balls.

Proof.

Suppose that A is open in M.

Then for each $x \in A$ there exists an open ball $B(x ,r_x)$ such that $B(x , r_x) \subseteq A$.

$\therefore A = \bigcup_{x \in A} B(x , r_x)$.

Thus A is expressed as union of open balls.

Conversely, assume that A can be expressed as union of open balls.

Since open balls are open and union of open sets is open, A is open.

1.3 Interior of a set

Definition1.3.1 Let (M , d) be a metric space and A $\subseteq$ M. A point x $\in$ A is said to be an interior point of A if there exists a real number r > 0 such that B(x , r) $\subseteq$ A. The set of all interior points is called as interior of A and is denoted by **Int** A.

Note1.3.2 Int A $\subseteq$ A.

Example1.3.3 In **R** with usual metric, let A = [1 , 2]. 1 is not an interior points of A, since for any real number r > 0 , B(1 , r) = (1 – r , 1 + r) contains real numbers less than 1. Similarly, 2 is also not an interior point of A. In fact every point of (1 , 2) is a limit point of A. Hence **Int**A = (1 , 2).

Note1.3.4(1)**Int**$\emptyset$ = $\emptyset$ **and Int** M = M.

(2) A is open $\Leftrightarrow$**Int** A = A.

(3) A $\subseteq$ B $\Rightarrow$**Int** A$\subseteq$ **Int** B

Theorem1.3.5 Let (M , d) be a metric space and A $\subseteq$ M. Then **Int** A = Union of all open sets contained in A.

Proof.

Let G = $\bigcup${ B / B is an open set contained in A } We have to

prove **Int** A = G.

Let x $\in$ **Int** A .

Then x is an interior point of A.

$\therefore$ there exists a real number r > 0 such that B(x , r) $\subseteq$ A.

Since open balls are open, B(x , r) is an open set contained in A.

$\therefore$ B(x , r) $\subseteq$ G.

$\therefore$x $\in$ G .

$\therefore$**Int** A $\subseteq$ G ……………………………. (1)

Let x $\in$ G .

Then there exists an open se B such that B $\subseteq$ A and x $\in$ B.

Since B is open and x $\in$ B, there exists a real number r > 0 such that B(x , r) $\subseteq$ B $\subseteq$ A.

$\therefore$ x is an interior point of A.

$\therefore$ x $\in$ **Int** A .

$\therefore$ G $\subseteq$ **Int** A ………………………… (2) From (1) and (2), we get

Int A = G.

Note1.3.6 Int A is an open set and it is the largest open set contained in A.

Theorem1.3.7 Let M be a metric space and A , B $\subseteq$ M. Then

(1) **Int** (A $\cap$ B) = (**Int** A) $\cap$ (**Int** A) (2)

Int (A $\cup$ B) $\supseteq$ (**Int** A) $\cup$ (**Int** A) **Proof.**

(1) A $\cap$ B $\subseteq$ A $\Rightarrow$**Int** (A $\cap$ B) $\subseteq$ **Int** A .

Similarly, **Int** (A $\cap$ B) $\subseteq$**Int** B .

$\therefore$**Int** (A $\cap$ B) $\subseteq$ (**Int** A) $\cap$ (**Int** A) .... (a) **Int**A $\subseteq$ A and **Int** B $\subseteq$ B .

$\therefore$ (**Int** A) $\cap$ (**Int** A) $\subseteq$ A $\cap$ B

Now, (**Int** A) $\cap$ (**Int** A) is an open set contained in A $\cap$B .

But, **Int** (A $\cap$ B) is the largest open set contained in A $\cap$B .

$\therefore$(**Int** A) $\cap$ (**Int** A) $\subseteq$ **Int** (A $\cap$ B) .... (b)

From (a) and (b) , we get **Int** (A $\cap$ B) = (**Int** A) $\cap$ (**Int** A)

(2) A $\subseteq$ A $\cup$ B$\Rightarrow$**Int** A$\subseteq$ **Int** (A $\cup$ B)
Similarly, **Int B**$\subseteq$ **Int** (A $\cup$ B)

$\therefore$**Int** (A $\cup$ B) $\supseteq$ (**Int** A) $\cup$ (**Int** A)

Note1.3.8 Int (A $\cup$ B)need not be equal to(**Int** A) $\cup$ (**Int** A)

For,

In **R** with usual metric, let A = (0 , 1] and B = (1 , 2).

A $\cup$ B = (0 , 2).

$\therefore$**Int** (A $\cup$ B) = (0 , 2)

Now, **Int** A (0 , 1) and **Int** B = (1 , 2) and hence (**Int** A) $\cup$ (**Int** A) = (0 , 2) – { 2 }.

$\therefore$**Int** (A $\cup$ B)$\neq$(**Int** A) $\cup$ (**Int** A)

1.4 Subspace

Definition1.4.1 Let (M , d) be a metric space. Let M_1 be a nonempty subset of M. Then M_1 is also a metric space under the same metric d. We call (M_1 , d) is a subspace of (M, d).

Theorem1.4.2 Let M be a metric space and M_1 a subspace of M. Let A $\subseteq$ M_1. Then A is open in M_1 if and only if A = G $\cap$ M_1 where G is open in M.

Proof.

Let $B_1(a , r)$ be the open ball in M_1 with center a and radius r.

Then $B_1(a , r) = B(a , r) \cap M_1$ where $B(a , r)$ is the open ball in M with center a and radius r.

Let A be an open set in M_1.

Then $A = \bigcup_{x \in A} B_1(x , r(x))$

$= \bigcup_{x \in A} [B(x , r(x)) \cap M_1)]$

$= [\bigcup_{x \in A} B(x , r(x))] \cap M_1$

$= G \cap M_1$ where $G = \bigcup_{x \in A} B(x , r(x))$ which is open in M.

Conversely, let $A = G \cap M_1$ where G is open in M.

We shall prove that A is open in M_1.

Let $x \in A$.

Then $x \in G$ and $x \in M_1$.

Since G is open in M, there exists an open ball $B(x , r)$ such that $B(x , r) \subseteq G$.

$\therefore B(x , r) \cap M_1 \subseteq G \cap M_1$.

i.e. $B_1(a , r) \subseteq A$.

$\therefore$ A is open in M_1.

Example1.4.3 Consider the subspace $M_1 = [0 , 1] \cup [2 , 3]$ of **R**.

$A = [0 , 1]$ is open in M_1 since $A = (-\frac{1}{2} , \frac{3}{2}) \subseteq M_1$ where $(-\frac{1}{2} , \frac{3}{2})$ is open in **R**.

Similarly, $B = [2 , 3]$, $C = [0 , \frac{1}{2}]$, $D = (\frac{1}{2} , 1]$ are open in M_1.

Note that A, B, C, D are not open in **R**.

1.5 Closed Sets.

Definition1.5.1 A subset A of a metric space M is said to be closed in M if its complement is open in M.

Examples 1.5.2

1. In **R** with usual metric any closed interval [a , b] is closed. For,

$[a , b]^c = \mathbf{R} - [a , b] = (- \infty , a) \cup (b , \infty)$.

$(- \infty , a)$ and (b , ∞) are open sets in R and hence $(- \infty , a) \cup (b , \infty)$ is open in **R**. i.e. $[a , b]^c$ is open in **R**.

$\therefore$ [a , b] is open in **R**.

2. Any subset A of a discrete metric space M is closed since A^c is open as every subset of M is open.

Note. In any metric space M, ∅ and M are closed sets since $\emptyset^c = M$ and $M^c = \emptyset$ which are open in M. Thus ∅ and M are both open and closed in M.

Theorem 1.5.3 In any metric space M, the union of a finite number of closed sets is closed.

Proof.

Let $A_1, A_2, \ldots, A_n$ be closed sets in a metric space M.

Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$.

We have to prove A is open in M.

Now, $A^c = [A_1 \cup A2 \cup \ldots \cup A_n]^c$

$= A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ [By De Morgan's law.] Since

A_i is closed in M, A_i^c is open in M.

Since finite intersection of open sets is open, $A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ is open in M. i.e. A^c is open in M.

$\therefore$ A is closed in M.

Theorem 1.5.4 In any metric space M, the intersection of closed sets is closed.

Proof.

Let $\{A_\alpha\}$ be a family of closed sets in M.

We have to prove $A = \cap A_\alpha$ is open in M.

Now, $A^c = (\cap A_\alpha)^c$

$= \cup A_\alpha^c$ [ByDe Morgan's law.]

Since A_α is closed in M, A^c_α is open in M. Since union of open sets is open, $\cup A_\alpha^c$ is open.

i.e. A^c is open in M.

$\therefore$ A is closed in M.

Theorem 1.5.5 Let M_1 be a subspace of a metric space M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 if and only if $F_1 = F \cap M_1$ where F is a closed set in M.

Proof.

Suppose that F_1 is closed in M_1.

Then $M_1 - F_1$ is open in M_1.

$\therefore M_1 - F_1 = A \cap M_1$ where A is open in M.

Now, $F_1 = A^c \cap M_1$.

Since A is open in M, A^c is closed in M.

Thus, $F_1 = F \cap M_1$ where $F = A^c$ is closed in M.

Conversely, assume that $F_1 = F \cap M_1$ where F is closed in M.

Since F is closed in M, F^c is open in M.

$\therefore F^c \cap M_1$ is open in M_1.

Now, $M_1 - F_1 = F^c \cap M_1$ which is open in M_1.

$\therefore F_1$ is closed in M_1.

1.6 Closure.

Definition1.6.1 Let A be a subset of a metric space (M, d). The closure of A, denoted by $\overline{A}$, is defined as the intersection of all closed sets which contain A.

i.e. $\overline{A} = \cap \{B \mid B \text{ is closed in } M \text{ and } B \supseteq A\}$

Note 1.6.2

(1) Since intersection of closed sets is closed, $\overline{A}$ is a closed set.

(2) $\overline{A} \supseteq A$.

(3) $\overline{A}$ is the smallest closed set containing A.

(4) A is closed $\Leftrightarrow A = \overline{A}$.

(5) $\overline{\overline{A}} = \overline{A}$.

Theorem 1.6.3 Let (M, d) be a metric space. Let $A, B \subseteq M$. Then

(1) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(3) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof.

(1) Let $A \subseteq B$. $\overline{B} \supseteq B \supseteq A$.

Thus $\overline{B}$ is a closed set containing A.

But $\overline{A}$ is the smallest closed set containing A.

∴ $\bar{A} \subseteq \bar{B}$.

(2) $A \subseteq A \cup B$.

∴ by (1), $\bar{A} \subseteq \overline{A \cup B}$.

Similarly , $\bar{B} \subseteq \overline{A \cup B}$.

∴ $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ .... (a)

$\bar{A}$ is a closed set containing A and $\bar{B}$ is a closed set containing B.

∴ $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

∴ $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ .... (b) From (a) and (b) we get $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(3) $A \cap B \subseteq A$.

∴ $\overline{A \cap B} \subseteq \bar{A}$.

Similarly, $\overline{A \cap B} \subseteq \bar{B}$.

∴ $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Note1.6.4 $\overline{A \cap B}$ need not be equal to $\bar{A} \cap \bar{B}$.

For example, in **R** with usual metric take A = (0 , 1) and B = (1 , 2) .

$A \cap B = \emptyset \Rightarrow \overline{A \cap B} = \emptyset$.

But $\bar{A} \cap \bar{B}$ = [0 , 1] ∩ [1 , 2] = { 1 }.

∴ $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

1.7 Limit Point.

Definition 1.7.1 Let (M , d) be a metric space and $A \subseteq M$. A point $x \in M$ is said to be a limit point of A if every open ball with center x contains a point of A other than x.

i.e. $B(x , r) \cap (A - \{ x \}) \neq \emptyset$ for all r > 0.

The set of all limit points of A is denoted by A'.

Example 1.7.2 In **R** with usual metric let A = (0 , 1).

Every open ball with center 0, B(0 , r) = (-r , r) contains points of (0 , 1) other than 0.

∴ 0 is a limit point of A.

Similarly, 1 is a limit point of A and in fact every point of A is also a limit Point of A. For each real number x < 0, if we choose r such that $0 < r \leq -x$, then B(x , r)

2 contains no point of (0 , 1) , and hence x is not a limit point of limit point of A.

Similarly, every real number x > 0 is not a limit point of A.

Hence A^{I} = [0 , 1].

Example 1.7.3 In **R** with usual metric, **Z** has no limit point.

For,

Let x be any real number.

If x is an integer, then $B(x , \frac{1}{2}) = (x - \frac{1}{2} , x + \frac{1}{2})$ has no integer other than x. ∴ x is not a limit point of **Z** .

If x is not an integer, choose r such that $0 < r < |x-n|$ where n is the integer closest to x.

Then B(x , r) = (x – r , x + r) contains no integer.

Hence x is not a limit point of **Z.**

Thus no real number x is a limit point of **Z.**

∴ $\mathbf{Z}^{I} = \emptyset$.

Example 1.7.4 In **R** with usual metric, every real number is a limit point of **Q** .

For,

Let x be any real number.

Every open ball B(x , r) = (x – r , x + r) contains infinite number of rational numbers.

∴ x is a limit point of **Q**.

∴ $\mathbf{Q}^{I} = \mathbf{R}$.

Theorem 1.7.5 Let (M , d) be a metric space and A ⊆ M. Then x is a limit point of A if and only if every open ball with center x contains infinite number of points of A.

Proof.

Let x be a limit point of A.

We have to prove every open ball with center x contains infinite number of points of A.

Suppose not.

Then there exists an open ball B(x , r) contains only a finite number of points of A and hence of (A – { x }).

Let $B(x , r) \cap (A - \{ x \}) = \{x_1, x_2, \ldots , x_n\}$.

Let $r_1 = \min \{ d(x , x_i) / i = 1 , 2 , ... , n \}$.

Since $x \neq x_i$, $d(x , x_i) > 0 \ \forall \ i = 1 , 2 , ... , n$ and hence $r_1 > 0$.

Moreover, $B(x , r_1) \cap (A - \{ x \}) = \emptyset$. $\therefore$ x is not a

limit point of A.

This is a contradiction.

$\therefore$ every open ball with center x contains infinite number of points of A.

Conversely, assume that every open ball with center x contains infinite number of points of A.

Then, every open ball with center x contains infinite number of points of $A - \{ x \}$.

Hence x is a limit point of A.

Note 1.7.6 Any finite subset of a metric space has no limit points.

Theorem 1.7.7 Let M be a metric space and $A \subseteq M$. Then $\overline{A} = A \cup A'$.

Proof.

Let $x \in A \cup A'$.

We claim that $x \in \overline{A}$.

Suppose $x \notin \overline{A}$.

Then, $x \in M - \overline{A}$.

Since $\overline{A}$ is closed , $M - \overline{A}$ is open.

$\therefore$ there exists an open ball $B(x , r)$ such that $B(x , r) \subseteq M - \overline{A}$.

$\therefore B(x , r) \cap \overline{A} = \emptyset$.

$\therefore B(x , r) \cap A = \emptyset$. [$\because A \subseteq \overline{A}$].

$\therefore x \notin A \cup A'$, which is a contradiction.

$\therefore x \in \overline{A}$.

$\therefore A \cup A' \subseteq \overline{A}$ .... (1)

Let $x \in \overline{A}$.

We have to prove $x \in A \cup A'$.

If $x \in A$, then $x \in A \cup A'$.

Suppose $x \notin A$.

We claim that $x \in A'$.

Suppose $x \notin A'$.

Then there exists an open ball $B(x, r)$ such that $B(x, r) \cap (A - \{x\}) = \emptyset$.

$\therefore B(x, r) \cap A = \emptyset$. [$\because x \notin A$] $\therefore A \subseteq B(x, r)^c$.

Since $B(x, r)$ is open, $B(x, r)^c$ is closed.

Thus $B(x, r)^c$ is a closed set containing A.

But, $\overline{A}$ is the smallest closed set containing A.

Hence $\overline{A} \subseteq B(x, r)^c$.

Now, $x \notin B(x, r)^c$.

$\therefore x \notin \overline{A}$, which is a contradiction.

$\therefore x \in A'$ and hence $x \in A \cup A'$.

$\overline{A} \subseteq A \cup A'$ .... (2)

From (1) and (2), we get $\overline{A} = A \cup A'$.

Corollary1.7.8 A is closed if and only if A contains all its limit points.

Proof.

A is closed $\Leftrightarrow A = \overline{A}$.

$\Leftrightarrow A = A \cup A'$.

$\Leftrightarrow A' \subseteq A$.

Corollary 1.7.9 $x \in \overline{A} \Leftrightarrow B(x, r) \cap A \neq \emptyset \ \forall r > 0$.

Proof.

$x \in \overline{A} \Rightarrow x \in A \cup A'$.

$\therefore x \in A$ or $x \in A'$.

If $x \in A$, then $x \in B(x , r) \cap A$.

If $x \in A^{\prime}$, then $B(x , r) \cap (A - \{ x \}) \neq \emptyset \ \forall \ r > 0$.

Thus $B(x , r) \cap A \neq \emptyset \ \forall \ r > 0$.

Conversely, let $B(x , r) \cap A \neq \emptyset \ \forall \ r > 0$.

We have to prove $x \in \overline{A}$.

If $x \in A$, then $x \in \overline{A}$.

If $x \notin A$, then $A = A - \{ x \}$.

$\therefore B(x , r) \cap (A - \{ x \}) \neq \emptyset \ \forall \ r > 0$.

$\therefore$ x is a limit point of A.

$\therefore x \in A^{\prime}$.

$\therefore x \in \overline{A}$.

Corollary 1.7.10 $x \in \overline{A} \Leftrightarrow G \cap A \neq \emptyset$ for all open set G containing x.

Proof.

Let $x \in \overline{A}$.

We have to prove $G \cap A \neq \emptyset$ for all open set G containing x.

Let G be an open set containing x.

Then there exists an open ball $B(x , r)$ such that $B(x , r) \subseteq G$.

Since $x \in \overline{A}$, $B(x , r) \cap A \neq \emptyset$ and hence $G \cap A \neq \emptyset$.

Conversely, assume that $G \cap A \neq \emptyset$ for every open set containing x.

Then $B(x , r) \cap A \neq \emptyset \ \forall \ r > 0$.

$\therefore x \in \overline{A}$.

1.8 Bounded Sets in a Metric space.

Definition 1.8.1 Let (M , d) be a metric space. A subset A of M is said to be bounded if there exists a positive real number k such that $d(x , y) \leq k \ \forall \ x , y \in A$.

Example 1.8.2 Any finite subset A of a metric space (M , d) is bounded.

For,

Let A be any finite subset of M.

If $A = \emptyset$ then A is obviously bounded.

Let $A \neq \varnothing$ .Then $\{d(x , y)/x , y \in A\}$ is a finite set of real numbers.

Let $k = \max \{d(x , y)/x , y \in A\}$.

Clearly $d(x , y) \leq k$ for all $x , y \in A$.

$\therefore$ A is bounded.

Example 1.8.3 [0,1] is a bounded subset of **R** with usual metric since $d(x , y) \leq 1$ for all $x , y \in [0,1]$.

Example 1.8.4 $(0 , \infty)$ is an unbounded subset of **R.**

Example 1.8.5 Any subset A of a discrete metric space M is bounded since $d(x , y) \leq 1$ for all $x , y \in A$.

Note 1.8.6 Every open ball $B(x , r)$ in a metric space (M , d) is bounded.

For,

Let $s , t \in B(x , r)$.

$d(s , t) \leq d(s , x) + d(x , t) < r + r$.

$\therefore d(s , t) < 2r$.

Hence $B(x , r)$ is bounded.

Definition 1.8.7 Let (M , d) be a metric space and $A \subseteq M$. The diameter of A, denoted by $d(A)$, is defined by $d(A)= \text{l.u.b} \{d(x , y)/x , y \in A\}$.

Example 1.8.8 In R with usual metric the diameter of any interval is equal to the length of the interval. The diameter of $[0 , 1]$ is 1.

1.9 Complete Metric Spaces.

Definition 1.9.1 Let (M , d) be a metric space. Let (x_n) be a sequence in M. Let $x \in M$. We say that (x_n) converges to x if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x) < \varepsilon$ for all $n \geq N$. If (x_n) converges to x , then x is called a limit of (x_n) and we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Note 1.9.2 (1) $x_n \to x$ if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $x_n \in B(x , \varepsilon)$ $\forall$ $n \geq N$. Thus, the open ball $B(x , r)$ contains all but a finite number of terms of the sequence.

(2) $x_n \to x$ if and only if $(d(x_n , x)) \to 0$.

Theorem 1.9.3 The limit of a convergent sequence in a metric space is unique.

Proof.

Let (M , d) be a metric space and let (x_n) be a sequence in M.

Suppose that (x_n) has two limits say x and y.

Let $\varepsilon > 0$ be given.

Since $x_n \to x$, there exists a positive integer N_1 such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N_1$.

Since $x_n \to y$, there exists a positive integer N_2 such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N_2$.

Let $N = \max \{ N_1 , N_2 \}$.

Then, $d(x , y) \leq d(x , x_N) + d(x_N , y)$

$< \varepsilon/2 + \varepsilon/2$ ∴ $d(x ,$

$y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary , $d(x , y) = 0$.

∴ $x = y$.

Theorem1.9.4 Let (M, d) be a metric space and $A \subseteq B$. Then

(i) X is a limit point of A $\Leftrightarrow$ there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

(ii) $X \in \overline{A} \Leftrightarrow$ there exists a sequence (x_n) in A such that $x_n \to x$.

Proof.

(i) Let x be a limit point of A.

Then every open ball $B(x , r)$ contains infinite number of points of A.

Thus, for each natural number n , we can choose $x_n \in B(x , \frac{1}{n})$ such that

$x_n \neq x_1, x_2, x_3, \ldots , x_{n-1}$.

Now, (x_n) is a sequence of distinct points in A and $d(x_n , x) < \frac{1}{n}$ $\forall$ n.

∴ $(d(x_n , x)) \to 0$. ∴ x_n
$\to x$.

Conversely, assume that there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

We have to prove x is a limit point of A.

Let it be given an open ball $B(x , \varepsilon)$.

Since $x_n \to x$, there exists a positive integer N such that $d(x_n , x) < \varepsilon$ $\forall$ n

$\geq N$.

∴ $x_n \in B(x , \varepsilon)$ $\forall$ $n \geq N$.

Since x_n are distinct points of A, $B(x , \varepsilon)$ contains infinite number of points of A.

Thus, every open ball with center x contains infinite number of points of A.

Hence x is a limit point of A.

(ii) Let $x \in \overline{A}$.

Then $x \in A \cup A^{l}$.

If $x \in A$ then the constant sequence x, x, x, ….. is a sequence in A converges to x.

If $x \notin A$, then $x \in A^{l}$.

∴ x is a limit point of A.

∴ by (i), there exists a sequence (x_n) in A converges to x.

Conversely, assume that there exists a sequence (x_n) in A such that $x_n \rightarrow x$.

Then every open ball $B(x , \varepsilon)$ contains points in the sequence and hence points of A.

∴ $x \in \overline{A}$.

Definition 1.9.5 Let (M , d) be a metric space. Let (x_n) be a sequence in M. Then (x_n) is said to be a Cauchy sequence in M if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x_m) < \varepsilon$ for all $n , m \geq N$.

Theorem 1.9.6 Every convergent sequence in a metric space (M , d) is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence in M converges to $x \in M$.

We have to prove (x_n) is Cauchy.

Let $\varepsilon > 0$ be given.

Since $x_n \rightarrow x$, there exists a positive integer N such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N$.

∴ $d(x_n , x_m) \leq d(x_n , x) + d(x , x_m) < \varepsilon/2 +$

$\varepsilon/2$ for all $n , m \geq N$.

∴ $d(x_n , x_m) < \varepsilon$ for all $n , m \geq N$.

Hence (x_n) is a Cauchy sequence.

Definition1.9.7 A metric space M is said to be complete if every Cauchy sequence in M converges to a point in M.

Example 1.9.8 R with usual metric is complete.

Theorem 1.9.9 A subset A of a complete metric space M is complete if and only if A is closed.

Proof.

Suppose that A is complete.

We have to prove A is closed.

For that it is enough to prove A contains all its limit points.

Let x be a limit point of A.

Then there exists a sequence (x_n) in A such that $x_n \to x$.

Since A is complete $x \in A$.

$\therefore$ A contains all its limit points.

Hence A is closed.

Conversely, assume that A is a closed subset of M.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) be a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

Thus (x_n) is a sequence in A such that $x_n \to x$.

$\therefore x \in \overline{A}$.

Since A is closed $\overline{A} = A$ and hence $x \in A$.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

$\therefore$ A is complete.

Note 1.9.10 Every closed interval [a , b] with usual metric is complete since it is a closed subset of the complete metric space **R**.

Theorem 1.9.11 [Cantor's Intersection Theorem]

Let M be a metric space. Then M is complete if and only if for every sequence (F_n) of nonempty closed subsets of M such that $F_1 \supseteq F_2 \supseteq \ldots F_n \supseteq \ldots$ and $(d(F_n)) \to 0$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof.

Let M be a complete metric space.

Let (F_n) be a sequence of nonempty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \ldots\ldots F_n \supseteq \ldots$ .... (1)

and $(d(F_n)) \to 0$, ... (2)

We have to prove $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

For each natural number n , we choose a point x_n in F_n.

By (1), x_n, x_{n+1}, x_{n+2}, ….. all lie in F_n.

i.e. $x_m \in F_n \ \forall\ m \geq n$. ……………………. (3) We claim that (x_n) is a Cauchy sequence in M.

Let $\varepsilon > 0$ be given.

Since $(d(F_n)) \to 0$, there exists a positive integer N such that $d(F_n) < \varepsilon \ \forall\ n \geq N$.

In particular, $d(F_N) < \varepsilon$. ……………………. (4) Now, let m , n $\geq$ N.

Then by (3), x_m , $x_n \in F_N$.

$\therefore d(x_m , x_n) < \varepsilon$. [By (4)]

Thus $d(x_m , x_n) < \varepsilon \ \forall$ m , n $\geq$ N.

$\therefore (x_n)$ is a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

We show that $x \in \bigcap_{n=1}^{\infty} F_n$.

For any natural number n, x_n, x_{n+1} , x_{n+2} is a sequence in F_n converges to x.

$\therefore x \in \overline{F_n}$.

Since F_n is closed, $F_n = \overline{F_n}$.

$\therefore x \in F_n$.

$\therefore x \in \bigcap_{n=1}^{\infty} F_n$.

Hence $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Conversely, assume that for every sequence (F_n) of nonempty closed subsets of M such that $F_1 \supseteq F_2 \supseteq \ldots F_n \supseteq \ldots.$ and $(d(F_n)) \to 0$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

We have to prove M is complete.

Let (x_n) be a Cauchy sequence in M.

We claim that $x_n \to x$ for some $x \in M$.

Define a decreasing sequence of sets $F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq \quad \supseteq F_n$ …… as follows

$F1 = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$.

$F2 = \{x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\}$.

….. …… …….. …..

….. …… ……. ……

$Fn = \{x_n, x_{n+1}, \ldots\ldots, \ldots\ldots\ldots\}$

…… ….. ……. …….

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \ldots\ldots\ldots \supseteq \quad \overline{} \supseteq F_n$ ……

Thus $(\overline{F_n})$ is a decreasing sequence of closed sets.

Since (x_n) is a Cauchy sequence, for given $\varepsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \varepsilon \ \forall\ n,m \geq N$. $\therefore d(F_N) < \varepsilon$.

Now, $F_n \subseteq F_N\ \forall\ n \geq N \Rightarrow d(F_n) < \varepsilon\ \forall\ n \geq N$.

But $d(F_n) = d(\overline{F_n})$.

$$\therefore d(\overline{F_n}) < \varepsilon\ \forall\ n \geq N \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$\therefore (\ d(\overline{F_n})\ ) \to 0$.

Hence by hypothesis, $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \emptyset$.

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$.

Then $x, x_n \in \overline{F_n}$. $\therefore d(x_n, x) \leq d(\overline{F_n})$.

$\therefore d(x_n, x) < \varepsilon\ \ \forall\ n \geq N$ [By (5)] $\therefore \quad \to x_n$

x .

$\therefore$ M is complete.

Note 1.9.12 In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point, since if it contains distinct points x and y, then $d(F_n) \geq d(x , y)$ for all n and hence ($d(F_n)$) does not converge to 0.

1.10 Baire's Category Theorem.

Definition 1.10.1 A subset A of a metric space M is said to be nowhere dense in M if

Int $\overline{A} = \emptyset$.

Definition 1.10.2 A subset A of a metric space M is said to be of first category in M if A can be expressed as a countable union of nowhere dense sets.

If A is not of first category, then we say it is of second category.

Example1.10.3 In **R** with usual metric, every finite subset A is nowhere dense.

Example 1.10.4 In **R** with usual metric, the subset **Q** is of first category.

For,

Since **Q** is countable it can be expressed as countable union of singleton sets and each singleton set is nowhere dense in **R.**. Thus, **Q** is countable union of nowhere dense sets. Hence **Q** is of first category.

Example 1.10.5 If M is a discrete metric space, then any nonempty subset A of M is not nowhere dense set. Also A is of second category.

Theorem 1.10.6 Let M be a metric space and $A \subseteq M$. Then A is nowhere dense if and only if each nonempty open set contains an open ball disjoint from A.

Proof.

Suppose that A is nowhere dense.

Let G be a nonempty open set.

Since A is nowhere dense, **Int** $\overline{A} = \emptyset$.

$\therefore$ $\overline{A}$ does not contain G.

$\therefore$ there exists $x \in G$ such that $x \notin \overline{A}$.

$x \notin \overline{A} \Rightarrow$ there exists an open ball $B(x , r_1)$ such that $B(x , r_1) \cap A = \emptyset$.

G is open $\Rightarrow$ there exists an open ball $B(x , r_2)$ such that $B(x , r_2) \subseteq G$.

Let $r = \min \{ r_1 , r_2 \}$.

Then G contains $B(x , r)$ and disjoint from A.

Conversely, assume every nonempty open set contains an open ball disjoint from A.

We claim that **Int** $\overline{A} = \emptyset$.

Let $x \in \overline{A}$.

We claim that x is not an interior point of $\overline{A}$.

Suppose x is an interior point.

Then there exists an open ball B(x , r) such that $B(x , r) \subseteq \overline{A}$.

Now, every open ball in B(x , r) intersects with A, which is a contradiction.

Hence x is not an interior point of $\overline{A}$.

$\therefore$ **Int** $\overline{A} = \emptyset$.

$\therefore$ A is nowhere dense set.

Theorem 1.10.7 [Baire's Category Theorem] Any

complete metric space is of second category.

Proof.

Let M be a complete metric space.

We claim that M is not of first category.

Let (A_n) be a countable collection of nowhere dense sets in M.

We shall prove that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Since M is open and A_1 is nowhere dense, there exists an open ball B_1 of radius less than 1 such that $B_1 \cap A_1 = \emptyset$.

Let F_1 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now, **Int** F_1 is open and A_2 is nowhere dense.

$\therefore$ **Int** F_1 contains an open ball B_2 of radius less than $\frac{1}{2}$ such that $B_2 \cap A_2 = \emptyset$.

Let F_2 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2.

Now, **Int** F_2 is open and A_3 is nowhere dense.

$\therefore$ **Int** F_2 contains an open ball B_3 of radius less than $\frac{1}{4}$ such that $B_3 \cap A_3 = \emptyset$.

Let F_3 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceeding like this we get a sequence of nonempty closed balls F_n such that

$F_1 \supseteq F_2 \supseteq \ldots\ldots F_n \supseteq \ldots.$ and $d(F_n) < \frac{1}{2^n}$.

$\therefore (d(F_n)) \to 0$ as $n \to \infty$.

Since M is complete, By Cantor's intersection theorem, there exists a point $x \in M$

Such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Moreover, $F_n \cap A_n = \emptyset \ \forall n$.

$\therefore x \notin A_n \ \forall n$.

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$.

Hence M is of second category.

Corollary 1.10.8 R is of second category.

Proof.

R is a complete metric space. Hence, **R** is of second category.

Unit II CONTINUITY

2.1 Continuity of functions.

Definition 2.1.1 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f : M_1 \to M_2$ is said to be **continuous at a** if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$. The function f is said to be continuous if it is continuous at every point of M_1.

Note 2.1.2 $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon \Leftrightarrow x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$.

$$\Leftrightarrow f(B(a, \delta)) \subseteq B(f(a), \varepsilon).$$

Theorem 2.1.3 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous if and only if $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.

Proof. Assume that f is continuous.

Let V be open in M_1.

We have to prove $f^{-1}(V)$ is open in M_1.

If $f^{-1}(V) = \phi$, then it is open.

Let $f^{-1}(V) \neq \phi$.

We shall prove that for each $x \in f^{-1}(V)$ there exists an open ball $B(x, \delta)$ such that $B(x, \delta) \subseteq f^{-1}(V)$.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

Since V is open, there exists an open ball $B(f(x), \varepsilon)$ such that $B(f(x), \varepsilon) \subseteq V$. ....(1)

Now, since f is continuous, there exists an open ball $B(x, \delta)$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

By (1), $f(B(x, \delta)) \subseteq V$ and hence $B(x, \delta) \subseteq f^{-1}(V)$.

$\therefore f^{-1}(V)$ is open.

Conversely, assume that $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.

To prove f is continuous, we shall prove that f is continuous at every point of M_1.

Let $x \in M_1$ and let $\varepsilon > 0$ be given.

We know that, $B(f(x), \varepsilon)$ is an open set in M_2.

By hypothesis, $f^{-1}(B(f(x), \varepsilon))$ is open in M_1.

Also, $x \in f^{-1}(B(f(x), \varepsilon))$.

$\therefore$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$.

$\therefore f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

$\therefore$ f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous on M_1.

Note 2.1.4 f is continuous if and only if inverse image of every open set is open.

Theorem 2.1.5 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous if and only if $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

Proof. Assume that f is continuous.

Let W be a closed set in M_2.

Then W^C is an open set in M_2.

By hypothesis, $f^{-1}(W^C)$ is open in M_1.

But $f^{-1}\left(W^C\right) = \left[f^{-1}(W)\right]^C$.

$\therefore \left[f^{-1}(W)\right]^C$ is open in M_1. $\therefore f^{-1}(W)$ is closed in M_1.

Conversely, assume that $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

To prove f is continuous, we shall prove that $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.
Let V be an open set in M_2.

$\therefore V^C$ is a closed set in M_2.

By hypothesis, $f^{-1}(V^C)$ is a closed set in M_1.

(i.e) $\left[f^{-1}(V)\right]^C$ is a closed set in M_1.

$\therefore$ $f^{-1}(V)$ is an open set in M_1.

Thus, inverse image of every open set is open under f. $\therefore$ f is continuous.

Note 2.1.6 f is continuous if and only if inverse image of every closed set is closed.

Theorem 2.1.7 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \to M_2$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof. Assume that f is continuous.

We have to prove $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$.

$\overline{f(A)}$ is a closed set in M_2.

Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed in M_1.

Since $\overline{f(A)} \supseteq f(A)$, $f^{-1}(\overline{f(A)}) \supseteq A$.

But $\overline{A}$ is the smallest closed set containing A.

$\therefore \overline{A} \subseteq f^{-1}(\overline{f(A)})$.

$\therefore f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove f is continuous, we shall prove that $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

Let W be a closed set in M_2.

By hypothesis, $f(\overline{f^{-1}(W)}) \subseteq \overline{ff^{-1}(W)}$.

$\subseteq \overline{W}$

$= W$ (Since W is closed.).

Thus, $f(\overline{f^{-1}(W)}) \subseteq \overline{W} = W$. $\therefore$ $\overline{f^{-1}(W)} \subseteq f^{-1}(W)$.

Also, $f^{-1}(W) \subseteq \overline{f^{-1}(W)}$. $\therefore$ $\overline{f^{-1}(W)} = f^{-1}(W)$.

Hence $f^{-1}(W)$ is closed.

$\therefore$ f is continuous.

Theorem 2.1.8 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $x \in M_1$. A function $f : M_1 \to M_2$ is continuous at x if and only if $x_n \to x$ in $M_1 \Rightarrow f(x_n) \to f(x)$ in M_2.

Proof.

Suppose that f is continuous at x.

Let (x_n) be a sequence in M_1 such that $x_n \to x$.

We shall prove that $f(x_n) \to f(x)$.

Let $\varepsilon > 0$ be given.

Since f is continuous at x, there exists $\delta > 0$ such that $d_1(y, x) < \delta \Rightarrow d_2(f(y), f(x)) < \varepsilon$ ………… (1).

Since $x_n \to x$, there exists positive integer N such that $d_1(x_n, x) < \delta \ \forall n \geq N$.

$\therefore d_2(f(x_n), f(x)) < \varepsilon \ \forall n \geq N$. [By (1)]

$\therefore f(x_n) \to f(x)$.

Conversely, assume that $x_n \to x \Rightarrow f(x_n) \to f(x)$.

We have to prove f is continuous at x.

Suppose not. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ $f(B(x, \delta)) \not\subseteq B(f(x), \varepsilon)$.

Thus for each natural number n, $f(B(x, \frac{1}{n})) \not\subseteq B(f(x), \varepsilon)$.

Choose x_n such that $x_n \in B(x_n, \delta)$ but $f(x_n) \not\subseteq B(f(x), \varepsilon)$. $\therefore d_1(x_n, x) < \frac{1}{n}$ for all n and $d_2(f(x_n), f(x)) \geq \varepsilon$ for all n.

$\therefore$ $x_n \to x$ and $f(x_n)$ does not converge to f(x).

This is a contradiction.

$\therefore$ f is continuous at x.

Problem 2.1.9 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then prove that any constant function $f : M_1 \to M_2$ is continuous.

Solution.

Let $f : M_1 \to M_2$ be given by f(x) = c where $c \in M_2$ is a constant.

We have to show that f is continuous.

Let V be an open set in M_2.

Now, $f^{-1}(V) = \begin{cases} \emptyset & \text{if } x \notin V \\ M_1 & \text{if } x \in V \end{cases}$.

In both cases, $f^{-1}(V)$ is an open set.

Thus, inverse image of every open set is open under f.

$\therefore$ f is continuous.

Problem 2.1.10 Let M_1, M_2, M_3 be metric spaces. If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are continuous, then prove that $g \circ f : M_1 \to M_2$ is also continuous.

i.e. composition of two continuous functions is continuous.

Solution.

Let W be an open set in M_3.

Since g is continuous, $g^{-1}(W)$ is open in M_2.

Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in M_1.

Now, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$.

$\therefore (g \circ f)^{-1}(W)$ is open in M_1.

Hence $g \circ f$ is continuous.

Problem 2.1.11 Let f be a continuous real valued function defined on a metric space M. Let A = $\{ x \in M | f(x) \geq a$ where $a \in \mathbf{R} \}$. Prove that A is closed.

Solution.

$A = \{ x \in M | f(x) \geq a \text{ where } a \in \mathbf{R} \}$

$= \{ x \in M | f(x) \in [a, \infty) \}$

$= f^{-1}([a, \infty))$.

Now, $[a, \infty)$ is a closed subset of **R.**

Since f is continuous, $f^{-1}([a, \infty))$ is a closed subset of M.

$\therefore$ A is closed.

Problem 2.1.12 Let $f : M \to \mathbf{R}$ and $f : M \to \mathbf{R}$ be continuous functions. Prove that $f+g : M \to \mathbf{R}$ is continuous.

Solution.

Let $x \in M$.

We show that f + g is continuous at x.

Let x_n be a sequence in M such that $x_n \to x$.

Since f and g are continuous, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. $\therefore f(x_n) + g(x_n) \to f(x) + g(x)$.

i.e. $(f+g)(x_n) \to (f+g)(x)$. $\therefore$ f+g is continuous at x.

Note 2.1.13 In a similar way, we can prove that f – g, fg, cf if $c \in \mathbf{R}$ and $\frac{f}{g}$ if $g(x) \neq 0 \ \forall \ x \in M$ are continuous.

2.2 Homeomorphism.

Definition 2.2.1 Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

A function $f : M_1 \to M_2$ is said to be a homeomorphism if the following holds.

(1) f is a bijection.
(2) f is continuous.
(3) f^{-1} is continuous.

M_1 and M_2 are said to be homeomorphic if there exists a homeomorphism between them.

Definition 2.2.2 A function $f : M_1 \to M_2$ is said to be an open mapping if for every open set G in M_1, f(G) is open in M_2.

i.e. image of every open set in M_1 under f is open in M_2.

Definition 2.2.3 A function $f : M_1 \to M_2$ is said to be a closed mapping if for every closed set F in M_1, f(F) is closed in M_2.

i.e. image of every closed set in M_1 under f is closed in M_2.

Theorem 2.2.4 Let $f : M_1 \to M_2$ be a bijection. Then the following are equivalent.

(1) f is a homeomorphism
(2) f is a continuous open map
(3) f is a continuous closed map **Proof.**

We shall prove that (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) .

Suppose that f is a homeomorphism.

Then f and f^{-1} are continuous.

We have to prove f is an open mapping.

Let G be an open set in M_1.

Since $f^{-1} : M_2 \to M_1$ is continuous, $(f^{-1})^{-1}(G)$ is open in M_1.

i.e. f(G) is open in M_2.

$\therefore$ f is an open map.

Conversely, assume that f is a continuous open map.

We prove that f^{-1} is continuous.

Let G be an open set in M_1.

Since f is an open mapping, f(G) is open in M_2.

i.e. $(f^{-1})^{-1}(G)$ is open in M_2.

$\therefore f^{-1}$ is continuous.

The proof of (1) $\Leftrightarrow$ (3) is similar.

Note 2.2.5 Let f : $M_1 \to M_2$ be a homeomorphism. Then a subset G of M_1 is open in M_1 if and only if f(G) is open in M_2.

For,

Since f is a homeomorphism, f is a continuous open mapping.

Since f is open mapping, G is open in $M_1 \Rightarrow$ f(G) is open in M_2.

Since f is continuous, f(G) is open in $M_2 \Rightarrow f^{-1}(f(G)) = G$ is open in M_1.

$\therefore$ G is open in $M_1 \Leftrightarrow$ f(G) is open in M_2.

Thus a homeomorphism f : $M_1 \to M_2$ gives not only a 1 – 1 correspondence between the elements of the two spaces but also a 1 – 1 correspondence between their open sets.

Note 2.2.6 Let f : $M_1 \to M_2$ be a homeomorphism. Then a subset F of M_1 is closed in M_1 if and only if f(F) is closed in M_2.

Example 2.2.7 The metric spaces (0 ,1) and (0 , ∞) with usual metric are homeomorphic.

For, Define f : (0 ,1) $\to$ (0 , ∞) by $f(x) = \frac{x}{1-x}$.

We show that f is 1 – 1 and on to.

Let x , y $\in$ (0 , 1).

$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y}$

$\Rightarrow x(1-y) = y(1-x)$

$\Rightarrow x - xy = y - xy$

$\Rightarrow x = y$.

Hence f is 1 – 1.

Let y $\in$ (0 , ∞).

Now, $f(x) = y \Rightarrow \frac{x}{1-x} = y$

$\Rightarrow x = y(1-x)$

$\Rightarrow x = y - xy$

$\Rightarrow x + xy = y$

$\Rightarrow x(1 + y) = y$

$\Rightarrow x = \frac{y}{1+y}$

$\therefore \frac{y}{1+y} \in (0, 1)$ is the pre image of y under f.

$\therefore$ f is on to. Thus f is a bijection and hence $f^{-1}: (0, \infty) \to (0, 1)$ by $f(x) = \frac{x}{1+x}$ is a bijection.

Also, f and f^{-1} are continuous.

$\therefore$ f is a homeomorphism.

2.3 Uniform Continuity.

Definition 2.3.1 Let (M_1, d_1) and (M_2, d_2) be a metric space. A function $f : M_1 \to M_2$ is said to be uniformly continuous on M_1, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

Note 2.3.2 Every uniformly continuous function is continuous but the converse need not be true.

Example 2.3.3 The function $f : [0, 1] \to \mathbf{R}$ given by $f(x) = x^2$ is uniformly continuous on [0 , 1].

For,

Let $\varepsilon > 0$ be given.

Let $x, y \in [0, 1]$.

Now, $|f(x) - f(y)| = |x^2 - y^2|$

$= |x + y| |x - y|$

$\leq 2 |x - y|$

Choose $\delta = \frac{\varepsilon}{2}$.

Then, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. $\therefore$ f is uniformly continuous on [0 , 1] .

2.4 Discontinuities of R

Definition 2.4.1

A function $f: R \to R$ is said to approach to a limit ℓ as x tends to a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \Longrightarrow |f(x) - \ell| < 0$ and we write $\lim_{x \to a} = \ell$.

Definition 2.4.2

A function f is that to have ℓ as the right limit at x=a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta \Longrightarrow |f(x) - \ell| < \varepsilon$ and we write $\lim_{x \to a+} = \ell$

Also we denote the right limit ℓ by f(a+)

A function f is that to have ℓ as the right limit at x=a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a - \delta \Longrightarrow |f(x) - \ell| < \varepsilon$ and we write $\lim_{x \to a-} = \ell$

Also we denote the right limit ℓ by f(a-)

Note 1

$\lim_{x \to a} f(x) = \ell$ if and only if $\lim_{x \to a+} = f(x) = \lim_{x \to a-} f(x) = \ell$.

i.e.

$\lim_{x \to a} f(x) = \ell$ if and only if the left and right limits of f(x) at x = a exist and are equal.

Note 2

The definition of continuity of f at x=a can be formulated as follows. f is continuous at a if and only if f(a+) = f(a-)=f(a) .

Note 3

If $\lim_{x \to a} f(x)$ does not exist then one of the following happens.

1. $\lim_{x \to a+} f(x)$ does not exists.
2. $\lim_{x \to a-} f(x)$ does not exists.
3. $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and are not equal.

Definition 2.4.3

If a function f is discontinuous at a then a is called a point of discontinuity for the function.

If a is a point of discontinuity of a function then any one of the following cases arises.

i. $\lim_{x \to a} f(x)$ exists but is not equal to f(a).
ii. $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and are not equal.
iii. Either $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ does not exists.

Definition 2.4.4

Let a be a point of discontinuity for f(x). a is said to be a point of discontinuity of the first kind if $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and both of them are finite and not equal. a is said to be a point of discontinuity of the second kind if either $\lim_{x \to a+} f(x)$ or $\lim_{x \to a-} f(x)$ does not exist.

Definition 2.4.5

Let $A \subseteq R$. A function $f : A \to R$ is called monotonic increasing if $x, y \in A$ and $x < y \Longrightarrow f(x) \leq f(y)$.

f is called monotonic decreasing if $x, y \in A$ and $x > y \Longrightarrow f(x) \geq f(y)$.

f is called monotonic if it is either monotonic increasing or monotonic decreasing.

Theorem 2.4.6

Let $f:[a, b] \to R$ be a monotonic increasing function. Then f has a left limit and a right limit at every point of (a, b). Also f has a right limit at a and f has a left limit at b. Further

$$x < y \Longrightarrow f(x+) \leq f(y-)$$

Similar result is true for monotonic decreasing functions.

Proof

Let $f : [a, b] \to R$ be monotonic increasing.

Let $x \in [a, b]$. Then $\{f(t) \mid a \leq t < x\}$ is bounded above by f(x).

We claim that $f(x-) = \ell$

Let $\varepsilon > 0$ be given. By definition of l.u.b there exists t such that $a \leq t < x$ and $\ell - \varepsilon < f(t) \leq \ell$.

$\therefore t < u < x \Longrightarrow \ell - \varepsilon < f(t) \leq f(u) \leq \ell$

($\because$ f is monotonic increasing)

$\Longrightarrow \ell - \varepsilon < f(u) \leq \ell$

$\therefore x - \delta < u < x \Longrightarrow \ell - \varepsilon < f(u) \leq \ell$ where $\delta = x - t$

$\therefore f(x-) = \ell$

Similarly we can prove that $f(x+) =$ g. l. b. $\{f(t) \mid x < t \leq b\}$.

Now we shall prove that $x < y \Longrightarrow f(x+) \leq f(y-)$ Let $x < y$.

Now, $f(x+) =$ g.l.b $\{f(t)/x < t \leq b\}$

$=$ g.l.b $\{f(t)/x < t \leq y\}$ (1)

($\because$ f is monotonic increasing)

Also $f(y-) =$ l.u.b $\{f(t)/a \leq t < y\}$

$=$ l.u.b $\{f(t)/x \leq t < y\}$ (2)

$\therefore f(x+) \leq f(y-)$ [by (1) and (2)]

The proof for monotonic decreasing functions is similar.

Theorem 2.4.7

Let f:[a, b] □ R be a monotonic function. Then the set of points of [a, b] at which f is discontinuous is countable.

Proof

We shall prove the theorem for a monotonic increasing function.

Let E = {x |x□[a , b] and f is discontinuous at x}.

Let x□E. Then f(x+) and f(x-) exists and f(x-) ≤ f(x) ≤ f(x+)

If f(x-) = f(x+) then f(x-) = f(x)=f(x+)

∴ f is continuous at x, which is a contradiction.

∴ f(x-) ≠ f(x+)

∴ f(x-) < f(x+)

Now choose a rational number r(x) such that f(x-) < r(x) < f(x+)

This defines a map r from E to Q which maps x to r(x).

We claim that r is 1-l.

Let $x_1 < x_2$.

∴ $f(x_1+) < f(x_2-)$.

Also $f(x_1-) < r(x_1) < f(x_1+)$

And $f(x_2-) < r(x_2) < f(x_2+)$

∴ $r(x_1) < f(x_1+) < f(x_2-) < r(x_2)$ Thus

$x_1 < x_2 \Longrightarrow r(x_1) < r(x_2)$. ∴ r : E□Q is 1 - l

∴ E is countable.

2.5 Connectedness

Definition 2.5.1 A separation of a metric space M is a pair A, B of nonempty disjoint open subsets of M whose union is M.

M is said to be a connected metric space if there is no separation for M.

Example 2.5.2 Any discrete metric space with more than one element is connected.

For,

Let M be a metric space with more than two elements.

Choose an element a ∈ M and let A = { a }.

Then A^c is a proper subset of M.

Now, A and A^c forms a separation of M.

∴ M is not connected.

Theorem 2.5.3 Let (M, d) be a metric space. Then M is connected if and only if $\emptyset$ and M are the only sets which are both open and closed in M.

Proof.

Suppose that M is connected.

We have to prove $\emptyset$ and M are the only sets which are both open and closed in M.

Suppose not.

Then there exists a proper subset A of M which is both open and closed in M.

Now, A and A^c forms a separation of M, which is a contradiction.

Conversely, assume that $\emptyset$ and M are the only sets which are both open and closed in M.

We have to prove M is connected.

Suppose not.

Then there exists a separation A, B of M.

A is a proper subset of M which is both open and closed in M, a contradiction.

∴ M is connected.

Theorem 2.5.4 Let (M , d) be a metric space. Then the following are equivalent.

(i) The sets A and B form a separation of M.

(ii) A and B are nonempty disjoint closed sets in M whose union is M.

(iii) A and B are nonempty disjoint sets in M whose union is M and $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Proof.

We shall prove that **(i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) (i) $\Rightarrow$ (ii).**

Suppose that A and B forms a separation of M.

Then A and B are nonempty disjoint sets in M whose union is M.

We have to prove A and B are closed in M.

Now, $A = B^c$ and $B = A^c$.

Since A and B are open in M, A^c and B^c are closed in M.

i.e., A and B are closed in M.

∴ **(i) $\Rightarrow$ (ii).**

The proof of **(ii)** $\Rightarrow$ **(i)** is similar.

(ii) $\Rightarrow$ **(iii).**

Suppose that A and B are nonempty disjoint closed sets in M whose union is M.

We have to prove $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Since B is closed, $B = B$.

$\therefore A \cap \overline{B} = A \cap B = \emptyset$.

Similarly, $\overline{A} \cap B = \emptyset$.

(iii) $\Rightarrow$ **(i).**

Suppose that A and B are nonempty disjoint sets in M whose union is M and

$A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

We have to prove A and B are closed in M.

Let $x \in \overline{A}$.

Since $\overline{A} \cap B = \emptyset$, $x \notin B$.

Since $A \cup B = M$, $x \in A$.

$\therefore \overline{A} \subseteq A$.

But $A \subseteq \overline{A}$.

$\therefore A = A$ and hence A is closed.

Similarly, B is closed.

Theorem 2.5.5 Let M be a connected metric space. Let A be a connected subset of M. If B is a subset of M such that $A \subseteq B \subseteq \overline{A}$ then B is connected. In particular, A is connected.

Proof.

Suppose B is not connected.

Then there exists a separation B_1 , B_2 of B.

Since B_1 and B_2 are open in B, $B_1 = G_1 \cap B$ and $B_2 = G_2 \cap B$, where G_1 and G_2 are open in M.

Now, $B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B$.

$\therefore B \subseteq G_1 \cup G_2$ and hence $A \subseteq G_1 \cup G_2$.

Take $A_1 = G_1 \cap A$ and $A_2 = G_2 \cap A$.

Then A_1 and A_2 are open in A.

Also, $A_1 \cup A_2 = (G_1 \cap A) \cup (G_2 \cap A)$

$= (G_1 \cup G_2) \cap A$

$= A$ [Since $A \subseteq G_1 \cup G_2$]

$A_1 \cap A_2 = (G_1 \cap A) \cap (G_2 \cap A)$

$= (G_1 \cap G_2) \cap A$

$\subseteq (G_1 \cap G_2) \cap B$ [Since $A \subseteq B$]

$= (G_1 \cap B) \cap (G_2 \cap B)$

$= B_1 \cap B_2$

$= \emptyset$.

Since A is connected, either $A_1 = \emptyset$ or $A_2 = \emptyset$. Without loss of generality , assume that $A_1 = \emptyset$.

i.e. $G_1 \cap A = \emptyset$.

Since G_1 is open, $G_1 \cap \overline{A} = \emptyset$.

$\therefore G_1 \cap B = \emptyset$. [Since $B \subseteq \overline{A}$]

i.e. $B_1 = \emptyset$, which is a contradiction.

$\therefore$ B is connected .

2.6 Connected subsets of R.

Theorem 2.6.1 A subspace of $\mathbf{R}$ is connected if and only if it is an interval.

Proof.

Suppose that A is a connected subset of $\mathbf{R}$.

We have to prove A is an interval.

Suppose not .

Then, there exists a , b , $c \in \mathbf{R}$ such that $a < b < c$ and a , $c \in A$ but $b \notin A$.

Define $A_1 = (-\infty , b) \cap A$ and $A_2 = (b , \infty) \cap A$.

Since $(-\infty , b)$ and (b , ∞) are open in $\mathbf{R}$, A_1 and A_2 are open in A.

Moreover, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$.

Clearly $a \in A_1$ and $c \in A_2$.

$\therefore A_1 \neq \emptyset$ and $A_2 \neq \emptyset$.

Thus, A is the union of a pair of nonempty disjoint open sets A_1 and A_2 .

$\therefore$ A is not connected, which is a contradiction.

Hence A is an interval.

Conversely, assume that A is an interval.

We have to prove A is connected.

Suppose not.

Then, there exists nonempty disjoint closed sets $A_1$and A_2 in A such that $A = A_1 \cup A_2$.

Choose $x \in A_1$ and $z \in A_2$. Since $A_1 \cap A_2 = \emptyset$, $x \neq z$. $\therefore x < z$ or $z < x$.

Without loss of generality we assume that $x < z$.

Now, $x , z \in A$ and A is an interval.

$\therefore [x , z] \subseteq A \subseteq A_1 \cup A_2$.

Hence every element of $[x , z]$ is either in A_1 or in A_2.

Let $y = \text{l.u.b.} \{ [x , z] \cap A_1 \}$.

Clearly $x \leq y \leq z$.

By the definition of l.u.b. , for each $\varepsilon > 0$ there exists $t \in [x , z] \cap A_1$ such that $y - \varepsilon < t \leq y$.

$\therefore (y - \ , y + \varepsilon) \cap ([x , z] \cap A_1) \neq \emptyset \ \forall \varepsilon > 0$.

$\therefore y \in \overline{[x , z] \cap A_1}$.

Since $[x , z] \cap A_1$ is closed in A , $y \in [x , z] \cap A_1$

$\therefore y \in A_1$. ………………… (1)

Again, by the definition of y, for each $\varepsilon > 0$ there exists $s \in A_2$ such that $y \leq s < y + \varepsilon$.

$\therefore (y - \ , y + \varepsilon) \cap A_2 \neq \emptyset \ \forall \varepsilon > 0$.

$\therefore y \in \overline{A_2}$.

Since A_2 is closed in A, $y \in A_2$ …………. (2) $\therefore y \in A_1 \cap A_2$ [

By (1) & (2)].

This is a contradiction to $A_1 \cap A_2 = \emptyset$.

Hence A is connected.

2.7 Connectedness and continuity.

Theorem 2.7.1 Let M_1 be a connected metric space. Let M_2 be any metric space. Let $f : M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.

i.e. continuous image of a connected set is connected.

Proof.

Let $f (M_1) = A$ so that f is a continuous function from M_1 on to A.

We claim that A is connected.

Suppose A is not connected.

Then, there exists a proper subset B of A which is both open and closed in A.

Hence $f^{-1}(B)$ is a proper subset of M_1 which is both open and in M_1.

$\therefore M_1$ is not connected which is a contradiction.

Hence A is connected.

Theorem 2.7.2 [intermediate value Theorem]

Let f be a real valued continuous function defined on an interval $\mathbf{I}$. Then f takes every value between any two value it assumes.

Proof.

Let $a , b \in \mathbf{I}$ and let $f(a) \neq f(b)$.

Without loss of generality we assume that $f(a) < f(b)$.

Let c be a real number such that $f(a) < c < f(b)$.

The interval $\mathbf{I}$ is a connected subset of $\mathbf{R}$.

Since f is continuous, $f(\mathbf{I})$ is a connected subset of $\mathbf{R}$.

Hence $f(\mathbf{I})$ is an interval.

Also $f(a) , f(b) \in f(\mathbf{I})$. $\therefore [f(a) , f(b)] \subseteq f(\mathbf{I})$. $\therefore c \in f(\mathbf{I})$. [Since $f(a) < c < f(b)$] $\therefore c = f(x)$ for some $x \in \mathbf{I}$.

Unit III Compactness

3.1 Compact Metric Spaces.

Definition 3.1.1 Let M be a metric space. A collection of open sets $\{G_\alpha\}$ is said to be an **open cover** for M if $\cup\, G_\alpha = M$. A sub collection of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

Definition 3.1.2 A metric space M is said to be **compact** if every open cover for M has a finite subcover.

i.e. for each collection of open sets $\{G_\alpha\}$ such that $\cup\, G_\alpha = M$, there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} = M$.

Theorem 3.1.3 Let M be a metric space. Let $A \subseteq M$. Then A is compact if and only if for every collection $\{G_\alpha\}$ of open sets in M such that $\cup\, G_\alpha \supseteq A$ there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

i.e. A is compact if and only if every open cover for A by sets open in M has a finite subcover.

Proof.

Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a collection of open sets in M such that $\cup\, G_\alpha \supseteq A$.

Then $(\cup\, G_\alpha) \cap A = A$.

$\therefore \cup\, (G_\alpha \cap A) = A$.

Since G_α is open in M, $G_\alpha \cap A$ is open in A.

$\therefore \{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact, this open cover has a finite subcover say

$\{G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots, G_{\alpha_n} \cap A\}$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha i} \cap A) = A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

Conversely, assume that for every collection $\{G_\alpha\}$ of open sets in M such that $\cup G_\alpha \supseteq A$ there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that
$\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

We have to prove A is compact.

Let $\{H_\alpha\}$ be an open cover for A.

Then H_α is open in A $\forall\ \alpha$.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M $\forall\ \alpha$.

Now $\cup H_\alpha = A \Rightarrow \cup (G_\alpha \cap A) = A$.

$\Rightarrow (\cup G_\alpha) \cap A = A$.

$\Rightarrow \cup G_\alpha \supseteq A$.

Hence by our assumption, there exists a finite sub collection
$\{G\alpha_1, G\alpha_2, \ldots, G\alpha_n\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha i} \cap A) = A$.

$\bigcup_{I=1}^{n} H_{\alpha I} = A$.

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}\}$ is a finite subcover of the given open cover $\{H_\alpha\}$ of A.

$\therefore$ A is compact.

Theorem 3.1.4 Any compact subset A of a metric space (M , d) is closed.

Proof.

We shall prove that A^c is open.

Let $y \in A^c$.

Now, for each $x \in A$, $x \neq y$.

$\therefore d(x, y) = r_x > 0$ and $B(x, \frac{r_x}{2}) \cap B(y, \frac{r_x}{2}) = \emptyset$.

Clearly the collection $\{ B(x, \frac{r_x}{2}) / x \in A \}$ is an open cover for A by sets open in M.

Since A is compact, there exists $x_1, x_2, \ldots, x_n \in A$ such that

$$\bigcup_{i=1}^{n} B(x, \frac{r_{x_i}}{2}) \supseteq A \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $V_y = \bigcap_{i=1}^{n} B(y, \frac{r_{x_i}}{2})$.

Then V_y is an open set containing y.

Since $B(x, \frac{r_{x_i}}{2}) \cap B(y, \frac{r_{x_i}}{2}) = \emptyset$, $V_y \cap B(x, \frac{r_{x_i}}{2}) = \emptyset \ \forall$ i = 1, 2, …. , n .

$\therefore V_y \cap [\bigcup_{i=1}^{n} B\left(x, \frac{r_{x_i}}{2}\right)] = \emptyset$.

$\therefore V_y \cap A = \emptyset$.   [By (1)]

$\therefore V_y \subseteq A^c$.

Thus, for each $y \in A^c$ there exists an open set V_y containing y such that $V_y \subseteq A^c$

.

$\therefore A^c = \bigcup_{y \in A^c} V_y$.

$\therefore A^c$ is open .

Hence A is closed.

Theorem 3.1.5 Any compact subset A of a metric space M is bounded.

Proof.

Let $x \in A$.

Now, { B(x , n) / n $\in \mathbf{N}$ } is an open cover for A by sets open in M.

Since A is compact, there exists natural numbers $n_1, n_2, \ldots, n_k$, such that $\bigcup^{k}_{i=1} B(x, n_k) \supseteq A$.

Let N = max { $n_1, n_2, \ldots, n_k$}.

Then $\bigcup^{k}_{i=1} B(x, n_k) = B(x, N)$.

$\therefore B(x, N) \supseteq A$.

Since B(x , N) is bounded and subset of a bounded set is bounded, A is bounded.

Theorem 3.1.6 A closed subset A of a compact metric space M is compact.

Proof.

Let $\{G_\alpha\}$ be a collection of open sets in M such that $\cup G_\alpha \supseteq A$.

$\therefore A^c \cup \cup G_\alpha = M$.

Since A is closed, A^c is open.

$\therefore \{G_\alpha\} \cup \{A^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say

$\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}, A^c\}$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha i}) \cup A^c = M$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

Hence A is compact.

Theorem 3.1.7 [Heine Borel Theorem]

Any closed interval [a , b] is a compact subset of **R.**

Proof.

Let $\{G_\alpha\}$ be a collection of open sets in **R** such that $\cup G_\alpha \supseteq \mathbf{R}$. Let S = { $x \in [a , b]$ / [a , x] can be covered by a finite number of G_α's. }

Clearly $a \in S$ and hence $S \neq \emptyset$.

Since S is bounded above by b , l.u.b of S exists.

Let c = l.u.b of S.

Clearly $c \in [a , b]$.

$\therefore c \in G_{\alpha_1}$ for some index α_1.

Since G_{α_1} is open , there exists $\varepsilon > 0$ such that $B(x , \varepsilon) \subseteq G_{\alpha_1}$.

i.e. $(c - \varepsilon , c + \varepsilon) \subseteq G_{\alpha_1}$.

Choose $x_1 \in [a , b]$ such that $x_1 < c$ and $[x_1 , c] \subseteq G_{\alpha_1}$.

Since $x_1 < c$, $[a , x_1]$ is covered by a finite number of G_α's.

These finite number of G_α's together with G_{α_1} covers [a , c].

$\therefore$ by the definition of S , $c \in S$.

Now, we claim that c = b.

Suppose $c \neq b$.

Then choose $x_2 \in [a , b]$ such that $x_2 > c$ and $[c , x_2] \subseteq G_{\alpha_1}$.

Since [a , c] is covered by a finite number of G_α's , these finite number of G_α's together with $G_{\alpha 1}$ covers [a , x_2].

∴ $x_2 \in S$, which is a contradiction to c is l.u.b of S [∵$x_2 > c$].

Hence c = b.

∴ [a , x] can be covered by a finite number of G_α's.

∴ [a , b] is a compact subset of **R** .

Theorem 3.1.8 A subset A or **R** is compact if and only if A is closed and bounded.

Proof.

If A is compact, then A is closed and bounded.

Conversely, assume that A is closed and bounded subset of **R** .

Since A is bounded, A has a lower bound and an upper bound say a and b respectively.

Then $A \subseteq [a , b]$.

Since A is closed in **R** , $A \cap [a , b]$ is closed in [a , b] . I.e. A is closed in [a , b].

Thus, A is a closed subset of the compact space [a , b].

Hence A is compact.

3.2 Compactness and Continuity.

Theorem 3.2.1 Let M_1 be a compact metric space and M_2 be any metric space. Let $f : M_1 \rightarrow M_2$ be a continuous function. Then f(M_1) is compact.

i.e. Continuous image of a compact metric space is compact.

Proof.

Without loss of generality we assume that f(M_1) = M_2.

Let $\{G_\alpha\}$ be a collection of open sets in M_2 such that $\cup G_\alpha = M_2$.

∴$\cup G_\alpha$ = f(M_2).

∴ $f^{-1}(\cup G_\alpha) = M_1$. ∴$\cup f^{-1}(G_\alpha) = M_1$.

Since f is continuous, $f^{-1}(G_\alpha)$ is open in $M_1 \forall \alpha$.

$\therefore \{ f^{-1}(G_\alpha) \}$ is an open cover for M_1.

Since M1 is compact, this open cover has a finite subcover say $\left\{ f^{-1}\left(G_{\alpha 1}\right), f^{-1}\left(G_{\alpha 2}\right), \ldots\ldots , f^{-1}\left(G_{\alpha n}\right) \right\}$.

$\therefore f^{-1}(\bigcup_{i=1}^{n} G_{\alpha i}) = M_1$.

$\bigcup_{i=1}^{n} G_{\alpha_i} = f(M_1) = M_2$.

Thus $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ is a finite subcover for the given open cover $\{G_\alpha\}$ of M_2.

Hence M_2 is compact.

Corollary 3.2.2 Let f be a continuous map from a compact metric space M_1 into any metric space M_2. Then $f(M_1)$ is closed and bounded.

Proof.

Since f is continuous, $f(M_1)$ is compact and hence closed and bounded.

Theorem 3.2.3 Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof.

Let $\varepsilon > 0$ be given.

Let $x \in M_1$.

Since f is continuous at x, for $\varepsilon/2 > 0$, there exists $\delta_x > 0$ such that

$d_1(x, y) < \delta_x \Rightarrow d_2(f(x), f(y)) < \varepsilon/2$ ……………….. (1) Clearly, $\{ B(x, \frac{\delta_x}{2}) / x \in M_1 \}$ is an open cover for M_1.

Since M_1 is compact, there exists $x_1, x_2, \ldots., x_n \in M_1$ such that

$\bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{2}) = M_1$.

Let $\delta = \min \{ \frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots.., \frac{\delta_{x_n}}{2} \}$.

Now, we shall prove that $d_1(p , q) < \delta \Rightarrow d_2(f(p) , f(q)) < \varepsilon \ \forall\ p , q \in M_1$.

Let $p , q \in M_1$ such that $d_1(p , q) < \delta$

$p \in M_1 \Rightarrow P \in \bigcup_{i=1}^{n} B(x_i , \frac{\delta_{x_i}}{2})$

$\Rightarrow p \in B(x_i , \frac{\delta_{x_i}}{2})$ for some i such that $1 \leq i \leq n$

$\Rightarrow$ $\delta_{xi} < \delta_{xi}\ d_1(\overline{p} , x_i) < 2$

$\therefore$ by (1), $d_2(f(p) , f(x_i)) < \varepsilon/2$ ………………… (2)

Similarly, $d_2(f(q) , f(x_i)) < \varepsilon/2$ ………………… (3)

Now, $d_2(f(p) , f(q)) \leq d_2(f(p) , f(x_i)) + d_2(f(x_i) , f(q))$

$< \varepsilon/2 + \varepsilon/2$ [By (2) and (3)] $\therefore d_2(f(p) , f(q)) < \varepsilon$.

Thus, $d_1(p , q) < \delta \Rightarrow d_2(f(p) , f(q)) < \varepsilon \ \ \forall\ p , q \in M_1$.

Hence f is uniformly continuous.

3.3 Equivalent forms of Compactness.

Definition 3.3.1 A collection $\mathbf{F}$ of subsets of a set M is said to have finite intersection property if the intersection of any finite number of elements of $\mathbf{F}$ is nonempty.

Theorem3.3.2 A metric space M is compact if and only if every collection of closed sets in M with finite intersection property has nonempty intersection.

Proof.

Suppose that M is compact.

Let $\{F_\alpha\}$ be a collection of closed subsets of M with finite intersection property.

We have to prove $\bigcap F_\alpha \neq \emptyset$.

Suppose $\bigcap F_\alpha = \emptyset$.

Then $(\bigcap F_\alpha)^c = M$.

$\therefore \bigcup F_\alpha^c = M$. [By De Morgan's laws]

Since each F_α is closed, each F_α^c is open.

Thus, $\{ F_\alpha^c \}$ is an open cover for M.

Since M is compact, this open cover has a finite subcover say

$\{F_{\alpha_1}{}^c, F_{\alpha_2}{}^c, \ldots, F_{\alpha_n}{}^c\}$.

$\therefore \bigcup_{i=1}^{n} F_{\alpha_i}{}^c = M$.

$\therefore (\bigcap_{i=1}^{n} F_{\alpha_i})^c = M$.

$\therefore \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$.

This is a contradiction to the collection $\{F_\alpha\}$ has finite intersection property.

$\therefore \bigcap F_\alpha \neq \emptyset$.

Conversely, assume that every collection of closed sets in M with finite intersection property has nonempty intersection.

We have to prove M is compact.

Let $\{G_\alpha\}$ be an open cover for M.

$\therefore \bigcup G_\alpha = M$.

$\therefore (\bigcup G_\alpha)^c = \emptyset$.

$\therefore \bigcap G_\alpha{}^c = \emptyset$.

Since each G_α is open, each $G_\alpha{}^c$ is closed.

Hence $\mathbf{F} = \{ G_\alpha{}^c \}$ is a collection of closed sets whose intersection is empty. $\therefore$ by hypothesis,

this collection does not have finite intersection property.

Hence there exists a finite sub collection $\{G_{\alpha_1}{}^c, G_{\alpha_2}{}^c, \ldots, G_{\alpha_n}{}^c\}$ such that $\bigcap_{i=1}^{n} G_{\alpha_i}{}^c = \emptyset$

.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i})^c = \emptyset$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} = M$.

Thus the given open cover $\{G_\alpha\}$ of M has a finite subcover $\{ G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n} \}$.

Hence M is compact.

Definition 3.3.3 A metric space M is said to be totally bounded if for every

$\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, \ldots, x_n \in M$ such that

$B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots B(x_n, \varepsilon) = M$.

A nonempty subset A of a metric space M is said to be totally bounded if the subspace A is totally bounded metric space.

Theorem 3.3.4 Any compact metric space is totally bounded.

Proof.

Let M be a compact metric space.

We have to prove M is totally bounded.

Let $\varepsilon > 0$ be given.

Now, $\{ B(x, \varepsilon) / x \in M \}$ is an open cover for M.

Since M is compact, there exists points $x_1, x_2, \ldots, x_n \in M$ such that

$M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon)$.

Hence M is totally bounded.

Theorem 3.3.5 Any totally bounded subset A of a metric space M is bounded.

Proof.

Let A be a totally bounded subset of a metric space M.

Then for given $\varepsilon > 0$, there exists points $x_1, x_2, \ldots, x_n \in A$ such that

$A = B_1(x_1, \varepsilon) \cup B_1(x_2, \varepsilon) \cup \ldots\ldots \cup B_1(x_n, \varepsilon)$ where $B_1(x_i, \varepsilon)$ are open balls in A.

Since open balls are bounded sets and finite union of bounded sets is bounded, A is bounded.

Note3.3.6 The converse of the above theorem is not true. For,

Let M be an infinite set with discrete metric.

Then M is bounded.

Also, $B(x, 1) = \{ x \}$ for all $x \in M$.

Since M is infinite, M cannot be expressed as finite union of open balls of radius 1.

Hence M is not totally bounded.

Definition 3.3.7 Let (x_n) be a sequence in a metric space M. If $n_1 < n_2 < \ldots < n_k < \ldots\ldots$ is a sequence of positive integers, then (x_{nk}) is a subsequence of (x_n).

Theorem 3.3.8 A metric space M is totally bounded if and only if every sequence in M contains a Cauchy subsequence.

Proof.

Suppose that every sequence in M contains a Cauchy subsequence.

We have to prove M is totally bounded.

Let $\varepsilon > 0$ be given.

Choose $x_1 \in M$.

If $B(x_1, \varepsilon) = M$, then M is totally bounded.

If $B(x_1, \varepsilon) \neq M$, Then choose $x_2 \in B(x_1, \varepsilon) - M$ so that $d(x_1, x_2) \geq \varepsilon$.

If $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = M$, then M is totally bounded.

Otherwise, choose $x_3 \in [B(x_1, \varepsilon) \cup B(x_2, \varepsilon)] - M$ so that $d(x_3, x_1) \geq \varepsilon$ and $d(x_3, x_2) \geq \varepsilon$.

We proceed this process and if the process is terminated at a finite stage means M is totally bounded.

Suppose not, then we get a sequence (x_n) in M such that $d(x_n, x_m) \geq \varepsilon$ if $n \neq m$

.

$\therefore$ (x_n) cannot be a Cauchy sequence, which is a contradiction.

Conversely, suppose that M is totally bounded.

Let $S_1 = \{ x_{11}, x_{12}, \ldots, x_{1n}, \ldots \}$ be a sequence in M.

If one of the terms in the sequence is repeated infinitely, then S_1 contains a constant subsequence which is in fact a Cauchy sequence.

So, we assume that no terms of S_1 is repeated infinitely so that the range of S_1 is infinite.

Since M is totally bounded, M can be covered by a finite number of open balls of radius $\frac{1}{2}$.

Hence one of these balls contains infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = \{ x_{21}, x_{22}, \ldots, x_{2n}, \ldots \}$ which lies within an open ball of radius $\frac{1}{2}$.

Similarly, S_2 contains a subsequence $S_3 = \{ x_{31}, x_{32}, \ldots, x_{3n}, \ldots \}$ which lies within an open ball of radius $\frac{1}{3}$.

We repeat the process of forming successive subsequences and finally we take the diagonal sequence $S = \{ x_{11}, x_{22}, \ldots, x_{nn}, \ldots \}$.

We claim that S is a Cauchy subsequence of S_1.

If m > n then both x_{mm} and x_{nn} lie within an open ball of radius $\frac{1}{n}$.

$\therefore d(x_{mm}, x_{nn}) < \frac{2}{n}$.

$\therefore d(x_{mm}, x_{nn}) < \varepsilon \ \forall \ m, n \geq \frac{2}{\varepsilon}$.

Hence S is a Cauchy subsequence of S_1.

Thus every sequence in M has a convergent subsequence.

Corollary3.3.9 A nonempty subset of a totally bounded set is totally bounded.

Proof.

Let A be a totally bounded subset of a metric space M.

Let B be a nonempty subset of A.

Let (x_n) be a sequence in B.

Since $B \subseteq A$, (x_n) is a sequence in A.

Since A is totally bounded, (x_n) has a Cauchy subsequence.

Thus every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded.

3.4 Sequentially Compact.

Definition 3.4.1 A metric space M is said to be sequentially compact if every sequence in M has a convergent subsequence.

Theorem 3.4.2 Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{nk}) converges to x , then (x_n) converges to x.

Proof.

Suppose that (x_n) has a subsequence (x_{nk}) which converges to x.

We have to prove $x_n \rightarrow x$.

Let $\varepsilon > 0$ be given.

Since (x_n) is a Cauchy sequence, there exists a positive integer N such that $d(x_n, x_m) < \frac{\varepsilon}{2}\ \forall\ n, m \geq N_1$ .... (1) Since $x_{nk} \to x$, there exists a positive integer N_2 such that $d(x_{nk}, x) < \frac{\varepsilon}{2}\ \forall\ n_k \geq N_2$ .... (2)

Let $N = \max\{N_1, N_2\}$. Fix $n_k \geq N$.

Now. $d(x_n, x) \leq d(x_n, x_{nk}) + d(x_{nk}, x)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\ \forall\ n \geq N$$

$\therefore d(x_n, x) < \varepsilon\ \forall\ n \geq N$.

$\therefore x_n \to x$.

Definition 3.4.3 A metric space M has Bolzano – Weierstrass property if every infinite subset of M has a limit point.

Theorem 3.4.4 In a metric space M the following are equivalent.

(i) M is compact.
(ii) M has Bolzano – Weierstrass property
(iii) M is sequentially compact
(iv) M is totally bounded and complete.

Proof.

(i) $\Rightarrow$ (ii)

Let M be compact metric space.

Let A be an infinite subset of M.

Suppose that A has no limit point.

Let $x \in M$.

Since x is not a limit point if A, there exists an open ball $B(x, r_x)$ such that

$B(x, r_x) \cap (A - \{x\}) = \emptyset$.

$B(x, r_x)$ contains at most one point of A (contains x if $x \in A$).

Now, $\{B(x, r_x) / x \in M\}$ is an open cover for M.

Since M is compact, there exists points $x_1, x_2, \ldots, x_n \in M$ such that

$M = B(x_1, r_{x1}) \cup B(x_2, r_{x2}) \cup \ldots \cup B(x_n, r_{xn})$.

$\therefore A \subseteq B(x_1 , r_{x1}) \cup B(x_2 , r_{x2}) \cup \ldots\ldots \cup B(x_n , r_{xn})$.

Since each $B(x_1 , r_{xi})$ has at most one point of A, A must be finite.

This is a contradiction to A is infinite.

Hence A has a limit point.

(ii) ⇒ (iii)

Suppose that M has Bolzano – Weierstrass property.

We have to prove M is sequentially compact.

Let (x_n) be a sequence in M.

If the range of (x_n) is finite , then a term of the sequence is repeated infinitely and hence (x_n) has a constant subsequence which is convergent.

Otherwise (x_n) has infinite number of distinct terms.

By hypothesis, this infinite set has a limit point say x.

$\therefore$ for any $r > 0$, the open ball $B(x , r)$ contains infinite number of terms of the sequence (x_n).

Choose a positive integer n_1 such that $x_{n1} \in B(x , 1)$.

Now, choose $n_2 > n_1$ such that $x_{n2} \in B(x , \frac{1}{2})$.

In general, for each positive integer k we choose $n_k > n_{k-1}$ such that $x_{nk} \in B(x , \frac{1}{k})$.

Then (x_{nk}) is a subsequence of (x_n) and $d(x_{nk} , x) < \frac{1}{k} \; \forall \; k$.

$\therefore x_{nk} \to x$.

Thus (x_{nk}) is a convergent subsequence of (x_n).

Hence M is sequentially compact.

(iii) ⇒ (iv)

Suppose that M is sequentially compact.

Then every sequence in M has a convergent subsequence.

We have every Cauchy sequence is convergent.

Thus, every sequence in M has a Cauchy subsequence.

Hence M is totally bounded.

Now, we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis, (x_n) contains a convergent subsequence (x_{nk}).

Let $x_{nk} \to x$.

Then $x_n \to x$.

$\therefore$ M is complete.

(iv) ⇒ (i)

Suppose that M is totally bounded and complete.

We have to prove M is compact.

Suppose not.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Take $r_n = \frac{1}{2^n}$.

Since M is totally bounded, M can be covered by a finite number of open balls of radius r_1 .

Since M is not covered by a finite number of G_α's , at least one of these open balls say $B(x_1 , r_1)$ cannot be covered by finite number of G_α's .

Now, $B(x_1 , r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1 , r_1)$ such that $B(x_2 , r_2)$ cannot be covered by finite number of G_α's .

Proceeding like this we get a sequence (x_n) in M such that $B(x_n , r_n)$ cannot be covered by finite number of G_α's and $x_{n+1} \in B(x_n , r_n)$.

Let m and n be positive integers with n < m.

Now, $d(x_n , x_m) \le d(x_n , x_{n+1}) + d(x_{n+1} , x_{n+2}) + \dots\dots\dots\dots + d(x_{m-1} , x_m)$

$$< r_n + r_{n+1} + \dots\dots + r_{m-1}$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots\dots\dots\dots + \frac{1}{2^{m-1}}$$

$$< \frac{1}{2^{n-1}}\left(\frac{1}{2^n} + \frac{1}{2^n} + \dots\dots\right)$$

$< \frac{1}{2^{n-1}}$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

Now, $x \in G_\alpha$ for some α.

Since G_α is open, there exists $\varepsilon > 0$ such that $B(x , \varepsilon) \subseteq G_\alpha$.

We have $x_n \to x$ and $r_n = \frac{1}{2^n} \to 0$.

$\therefore$ there exists a positive integer N such that

$d(x_n , x) < \frac{\varepsilon}{2}$ and $r_n < \frac{\varepsilon}{2}$ $\forall\, n \geq N$.

Fix $n \geq N$.

We claim that $B(x_n , r_n) \subseteq B(x , \varepsilon)$. $y \in B(x_n ,$

$r_n) \Rightarrow d(x_n , y) < r_n < \frac{\varepsilon}{2}$

$$\Rightarrow d(x_n , x) + d(x_n , y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\Rightarrow d(x , y) < \varepsilon$$

$$\Rightarrow y \in B(x , \varepsilon) .$$

$\therefore$ $B(x_n , r_n) \subseteq B(x , \varepsilon) \subseteq G_\alpha$.

Thus, $B(x_n , r_n)$ is covered by a single G_α , which is a contradiction.

Hence M is compact.

UNIT-IV

DERIVATIVES

CONTinuity and Differentiation
Let X, Y be the metric spaces. Suppose $E \subset X, f$ maps E into Y and p is a limit point of E we write $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q.$$

If there is a point $q \in Y$ with the following property, for every $\epsilon > 0$ there exists $S > 0$ such that $d_y(f(x), q) < \epsilon \forall x \in E$ for which $0 < d_X(x, p) < S$. (i.e.)

$$\lim_{x \to p} f(x) = q.$$

if given $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x, p) < S \Rightarrow d_Y(f(x), q) < \epsilon$.

Definition 3.1 *Let X and Y be any two metric spaces and $E \subset X$. Let f and g be any complex functions defined on E then we define $f + g$ as follows.* $(f + g)(x) = f(x) + g(x)$

Theorem 3.2 *Let X and Y be any two metric spaces and $E \subset X$. p is a limit point of E. Then*

$$\lim_{x \to p} f(x) = q \textit{ iff } \lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and

$$\lim_{n \to \infty} p_n = p.$$

Proof: Suppose

$$\lim_{x \to p} f(x) = q$$

$\Rightarrow$ Given $\epsilon > 0$, there exists $S > 0$ such that $0 < d_X(x, p) < S \Rightarrow d_Y(f(x), q) < \epsilon \; \forall x \in E$.....(1)
$\{p_n\}$ is a sequence of points in E such that $\{p_n\} \to p$ as $n \to \infty (p_n \neq p)$ (This is possible $\because p$ is a limit point of E) $\Rightarrow$ there exists N depending on S such that $d_X(p_n, p) < S \; \forall n \geq N$. Now By (1) we have, $d_Y(f(p_n), q) < \epsilon \; \forall n \geq N$ (i.e.)

$$\lim_{n \to \infty} f(p_n) = q.$$

Conversely, Suppose

$$\lim_{n \to \infty} f(p_n) = q$$

for every $\{p_n\}$ in E such that $p_n \neq p$ and

$$\lim_{n \to \infty} p_n = p$$

To Prove

$$\lim_{x\to p} f(x) = q$$

Suppose this result is false, for some $\epsilon > 0$ and for every $S > 0$ such that $d_X(x,p) < S \Rightarrow d_Y(f(x),q) \geq \epsilon$. Let $S_n = \frac{1}{n}$, $n = 1,2,3...$ For $S > 0$ without loss of generality choose a point $p \in E$ such that $d_X(p_1,p) < S_1(= 1) \Rightarrow d_Y(f(p_1),q) \geq \epsilon$. Similarly, for $S_2 > 0$ choose a point $p_2 \in E$ such that $d_X(p_2,p) < S_1 = (1/2) \Rightarrow d_Y(f(p_2),q) \geq \epsilon$. Proceeding for $S_n > 0$, choose a point $p_n \in E$ such that $d_X(p_n,p) < S_1(= 1/n) \Rightarrow d_Y(f(p_n),q) \geq \epsilon$. $\therefore$ we have a sequence $\{p_n\}$ in E such that $d_X(p_n,p) < \frac{1}{n} \Rightarrow d_Y(f(p_n),q) \geq \epsilon$. Now $\{p_n\} \to p$ as $n \to \infty$ [$\because 1/n \to 0$ as $n \to \infty$]. But $f(p_n)$ does not converge to q $\therefore$ our assumption is wrong. Hence for every $\epsilon > 0$ there exists $S > 0$ such that $d_X(x,p) < S \Rightarrow d_Y(f(x),q) < \epsilon \ \ \forall x \in E$.

$$\therefore \lim_{x\to p} f(x) = q.$$

Corollary 3.3 *If f has a limit at p then this limit is unique.*
Proof: Suppose q is a limit of f at p. (i.e.)

$$\lim_{x\to p} f(x) = q.$$

$\therefore$ By the previous theorem, we have

$$\lim_{n\to\infty} f(p_n) = q$$

for every $\{p_n\}$ in E such that $p_n \neq p$ and $p_n \to p$. But we know that, Every convergence sequence converges to a unique limit. $\therefore f$ has a unique limit at p.

Definition 3.4 *Suppose we have two complex f and g then $f \pm g, fg, \lambda f$, $\frac{f}{g}(g \neq 0)$ are defined on a set E as follows.*

1. $(f+g)(x) = f(x) + g(x)$.
2. $(f \cdot g)(x) = f(x) \cdot g(x)$
3. $(\lambda f)(x) = \lambda f(x)$
4. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$.

Similarly we define $\bar{f}, \bar{g}$ map E into $\mathbb{R}^k$. Then we can define $\bar{f} \pm \bar{g}, \bar{f}\bar{g}, \lambda\bar{f}$, $\frac{\bar{f}}{\bar{g}}, (\bar{g} \neq 0)$.

Definition 3.5 ***Continuous at a point:*** *Suppose X, Y are metric spaces and $E \subset X, p \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\epsilon > 0$, there exists a $S > 0 \Rightarrow 0 < d_X(x,p) < S \Rightarrow d_Y(f(x),f(p)) < \epsilon\ \forall x \in E$.*

Remark 3.6 *Suppose f is continuous at $p \Rightarrow$ for every $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \epsilon\ \forall x \in E \Rightarrow x \in N_S(p) \Rightarrow f(x) \in N_\epsilon(f(p))\ \forall x \in E \Rightarrow f(N_S(p)) \subset N_\epsilon(f(p))$.*

Theorem 3.7 *Let X, Y be metric space and $E \subset X$. p is a limit point of E and $f : E \to Y$. Then f is continuous at p iff*

$$\lim_{x \to p} f(x) = f(p)$$

Proof: Suppose f is continuous at p. $\Leftrightarrow$ for every $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \epsilon\ \ \forall x \in E \Leftrightarrow$

$$\lim_{x \to p} f(x) = f(p)$$

Theorem 3.8 *Suppose X, Y, Z are metric space and $E \subset E$. f maps E into Y, g maps the range of f into Z and h is a mapping of E into Z defined by $h(x) = g(f(x))$. If f is continuous at $p \in E$ and if g is continuous at $f(p)$ then h is continuous at p. (The function h is called composite of f and g and we write as $h = g \circ f$)*

Proof: Let $\epsilon > 0$ be given and g is continuous at $f(p)$. $\therefore \eta > 0$ such that $d_Y(y, f(p)) < \eta \Rightarrow d_Z(g(y), g(f(p))) < \epsilon,\ y \in f(E)$...... (1)

Since f is continuous at p for this $\eta > 0$, there exists $S > 0$ such that $d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \eta\ \ \forall x, y \in E$

$$\begin{aligned} (i.e.) d_Y(f(x), f(p)) &< \eta, f(X) \in f(E) \\ \Rightarrow d_Z(g(f(x)), (g(f(p)) &< \epsilon \text{ by } (1) \\ \Rightarrow d_Z(g \circ f(x), (g \circ f)(p)) &< \epsilon \\ \Rightarrow d_Z(h(x), h(p)) &< \epsilon\ (h = g \circ f). \end{aligned}$$

$\therefore$ we have, $d_X(x,p) < S \Rightarrow d_Z(h(x), h(p)) < \epsilon\ \forall x \in E \Rightarrow h$ is continuous at p.

Theorem 3.9 *A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(E)$ is open in X for every open get E in Y.*

Proof: Suppose f is continuous on X. Let V be a open get in Y. To Prove: $f^{-1}(V)$ is open in X. Let $p \in f^{-1}(V)$; $p \in f^{-1}(V) \Rightarrow f(p) \subset V$. Since V is open, there exists $\epsilon > 0$ such that $N_\epsilon(f(p)) \subset V$....... (1)

Since f is continuous at p, for $\epsilon > 0$ there exists $S > 0$ such that $f(N_S(p)) \subset N_\epsilon(f(p))$...... (2)

From (1) and (2), $\Rightarrow f(N_S(p)) \subset V \Rightarrow N_S(p) \subset f^{-1}V \Rightarrow p$ is an interior point of $f^{-1}(V)$. Since p is arbitrary, $f^{-1}(V)$ is open in X. Conversely: Suppose $f^{-1}(V)$ is open in X for every open set V in Y. To Prove: f is continuous at $p, p \in X$. Let $\in > 0$ be given. Consider an open set $N_\epsilon(f(p))$ in Y, $f^{-1}(N_\epsilon(f(p)))$ is open in X. Now, $\Rightarrow p \in f^{-1}(N_\epsilon(f(p))) \Rightarrow p$ is an interior point of $f^{-1}(N_\epsilon(f(p))) \Rightarrow$ there exists $S > 0$ such that $N_S(p) \subset f^{-1}(N_\epsilon(f(p))) \Rightarrow f(N_S(p)) \subset N_\epsilon(f(p)) \Rightarrow f$ is continuous at p.

Corollary 3.10 *A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.*
Proof: Let C be a closed set in $Y.C^c$ is open in $Y \Rightarrow f^{-1}(C^c)$ is open in X. (by Theorem 3.9) $\Rightarrow [f^{-1}(C)]^c$ is open in $X \Rightarrow f^{-1}(C)$ is closed in X. Conversely: Suppose $f^{-1}(C)$ is closed in X for every closed set C in Y. To Prove: f is continuous on X. Let A be an open set in $Y \Rightarrow A^c$ is closed in $Y \Rightarrow f^{-1}(A^c)$ is closed in X. (by our assumption) $\Rightarrow [f^{-1}(A)]^c$ is closed in $X \Rightarrow f^{-1}(A)$ is open in X. $\Rightarrow f$ is continuous on X. (by the previous theorem)

Theorem 3.11 *Let f and g be complex continuous function in a metric space X, then $f+g, f\cdot g, \frac{f}{g}(g \neq 0)$ are continuous on X.*
Proof: At isolated point of X there is nothing prove. Fix a point $p \in X$ and suppose p is a limit point of X. Since f and g are continuous at p.

$$\lim_{x\to p} f(x) = f(p); \ \lim_{x\to p} g(x) = g(p)$$

Now,

$$\lim_{x\to p}(f+g)(x) = \lim_{n\to\infty}(f+g)p_n$$

where $p_n \to p$ as $n \to \infty$ and $p_n \neq p$

$$\begin{aligned}
\lim_{x\to p}(f+g)(x) &= \lim_{n\to\infty}(f(p_n)+g(p_n)) \\
&= \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n) \\
&= f(p)+g(p)
\end{aligned}$$

similarly the other results follow.

Theorem 3.12 *Let $f_1, f_2, ..., f_k$ be real functions in a metric space X. Let $\bar{f}$ be the mapping X into $\mathbb{R}^k$. defined by $\bar{f}(x) = (f_1(x), f_2(x), ..., f_k(x))x \in X$. Then*
(a) $\bar{f}$ is continuous iff each of the functions $f_1, f_2, ..., f_k$ is continuous.
(b) $\bar{f}$ and $\bar{g}$ are continuous mapping of X into $\mathbb{R}^k$ then $\bar{f}+\bar{g}, \bar{f}\cdot\bar{g}$ are continuous on $X(f_1, f_2, ..., f_k$ are called components of $\bar{f})$.
Proof: Suppose $\bar{f}$ is continuous at every $p \in X$. Then given $\epsilon > 0$ there exists $S > 0$ such that

$$\begin{aligned}
|\bar{f}(x) - \bar{f}(p)| &< \epsilon \ \text{ if } 0 < d_X(x,p) < S \\
\Rightarrow \left(\sum_{i=1}^{k}(f_i(x)-f_i(p))^2\right)^{1/2} &< \epsilon \text{ if } 0 < d_X(x,p) < S \\
\Rightarrow |f_i(x)-f_i(p)| < \left(\sum_{i=1}^{k}(f_i(x)-f_i(p))^2\right)^{1/2} &< \epsilon \ \forall i = 1,2,...,k \\
\Rightarrow |f_i(x)-f_i(p)| < \epsilon \ \forall i = 1,2,...,k &\text{ if } 0 < d_X(x,p) < S
\end{aligned}$$

$\Rightarrow$ each f_i is continuous at p, $(1 \le i \le k,\ p \in X) \Rightarrow$ each f_i is continuous on X, $(1 \le i \le k)$. Conversely, Suppose f_i is continuous on X for each $i = 1, ..., k \Rightarrow f_i$ is continuous at every $p \in X \Rightarrow$ Given $\epsilon > 0$ there exists $S_i > 0$ such that $0 < d_X(x,p) < S_i \Rightarrow |f_i(x) - f_i(p)| < \frac{\epsilon}{\sqrt{k}}\ \forall i = 1, 2, ..., k$. Let $S = min(S_1, S_2, ..., S_k)$. Now,

$$\begin{aligned}
0 < d_X(x,p) < S_i \Rightarrow |f_i(x) - f_i(p)| &< \frac{\epsilon}{\sqrt{k}}\ \forall i = 1, 2, ..., k \\
\Rightarrow |f_i(x) - f_i(p)|^2 &< \frac{\epsilon^2}{(\sqrt{k})^2} \\
\Rightarrow \sum_{i=1}^{k} |f_i(x) - f_i(p)|^2 &< \frac{\epsilon^2}{k} \cdot k \\
&= \epsilon^2 \\
\Rightarrow \sqrt{\sum_{i=1}^{k} |f_i(x) - f_i(p)|^2} &< \epsilon \\
\Rightarrow |\bar{f}(x) - \bar{f}(p)| &< \epsilon \\
(i.e.) 0 < d_X(x,p) < S \Rightarrow |\bar{f}(x) - \bar{f}(p)| &< \epsilon
\end{aligned}$$

$\Rightarrow \bar{f}$ is continuous at every $p \in X \Rightarrow \bar{f}$ is continuous on X

(b) Let $\bar{f} = (f_1, f_2, ..., f_k)$ and $\bar{g} = (g_1, g_2, ..., g_k)$. Now, $\bar{f} + \bar{g} = (f_1 + g_1, f_2 + g_2, ..., f_k + g_k)$; $\bar{f} \cdot \bar{g} = (f_1 \cdot g_1, f_2 \cdot g_2, ..., f_k \cdot g_k)$. Given $\bar{f}$ and $\bar{g}$ are continuous. by (a), each f_i, g_i are continuous $(i \le i \le k)$ (by Theorem 3.11) $\Rightarrow f_i + g_i, f_i \cdot g_i$ are continuous. (by (a))

Theorem 3.13 *Let* $\bar{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$ *define* $\phi_i : \mathbb{R}^k \to \mathbb{R}$ *by* $\phi_i(\bar{x}) = x_i$, $(i = 1, 2, ..., k)$. ϕ_i *is called the coordinate function, then* ϕ_i *is continuous.*
Proof: Let $\bar{x}, \bar{y} \in \mathbb{R}^k$. Given $\epsilon > 0$ choose $S = \epsilon$ such that

$$\begin{aligned}
|\bar{x} - \bar{y}| &< S \\
\Rightarrow |\phi_i(\bar{x}) - \phi_i(\bar{y})| &= |x_i - y_i| \\
&< \left(\sum_{i=1}^{k} |x_i - y_i|^2 \right)^{1/2} \\
&= |\bar{x} - \bar{y}| \\
&< \epsilon
\end{aligned}$$

$\Rightarrow \phi_i$ is continuous on $\mathbb{R}^k$

Theorem 3.14 *Every polynomial in* $\mathbb{R}^k$ *is continuous.*
Proof: By the above theorem $\phi_i : \mathbb{R}^k \to \mathbb{R}$ is continuous for every i. Now, $\phi_i^2(\bar{x}) = \phi_i(\bar{x}) \cdot \phi_i(\bar{x}) = x_i \cdot x_i = x_i^2\ \forall i$. In general $\phi_i^{n_i}(\bar{x}) = x_i^{n_i}\ \forall i$. By

Theorem 3.11, $\phi_i^{n_i}$ is continuous. Now,

$$\begin{aligned}(\phi_1^{n_1}\cdot\phi_2^{n_2}\cdots\phi_k^{n_k})\bar{x}& \\ &=\phi_1^{n_1}(\bar{x})\cdot\phi_2^{n_2}(\bar{x})\cdots\phi_k^{n_k}(\bar{x})\\ &=x_1^{n_1}\cdot x_2^{n_2}\cdots x_k^{n_k}\end{aligned}$$

Now $\phi_1^{n_1}\cdot\phi_2^{n_2}\cdots\phi_k^{n_k}$ is a monomial function, where $n_1, n_2, ..., n_k$ are positive integers. Every monomial function is continuous $C_{n_1,n_2,...,n_k}$ is a complex constant $\Rightarrow C_{n_1,n_2,...,n_k}\cdot x_1^{n_1}\cdot x_2^{n_2}\cdots x_k^{n_k}$ is continuous on $\mathbb{R}^k$. $\Rightarrow \sum C_{n_1,n_2,...,n_k}\cdot x_1^{n_1}\cdot x_2^{n_2}\cdots x_k^{n_k}$ is continuous on $\mathbb{R}^k$. $\Rightarrow$ Every polynomial is continuous on $\mathbb{R}^k$.

Continuity and Compact: A mapping $\bar{f}$ on a set E into X is said to be bounded, if there is a real number m such that $|\bar{f}(x)| < m\ \forall x \in X$.

Theorem 3.15 *Suppose f is continuous function on a compact metric space X into a metric space Y. Then $f(X)$ is compact. (i.e., continuous image of a compact metric space is compact)*
Proof: Given that X is compact. To Prove: $f(X)$ is compact. Let $\{V_\alpha\}$ be an open cover for $f(X) \Rightarrow$ each V_α is open in Y. Now, Given f is continuous $\Rightarrow f^{-1}(V_\alpha)$ is open in X for each $\alpha \Rightarrow \{f^{-1}(V_\alpha)\}$ is open cover for X. Since X is compact, there exists finitely may indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$\begin{aligned}X &\subset f^{-1}(V_{\alpha_1})\cup f^{-1}(V_{\alpha_2})\cup\cdots\cup f^{-1}(V_{\alpha_n})\\ &=\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\\ \Rightarrow f(X) &\subset \bigcup_{i=1}^{n} ff^{-1}(V_{\alpha_i}) \subset \bigcup_{i=1}^{n} V_{\alpha_i}\end{aligned}$$

$\Rightarrow \{V_\alpha\} \Rightarrow$ has a finite sub cover. $\therefore f(X)$ is compact.

Theorem 3.16 *If $\bar{f}$ is continuous mapping of a compact metric space X into $\mathbb{R}^k$. Then $\bar{f}(X)$ is closed and bounded. $\therefore \bar{f}$ is bounded.*
Proof: Given $\bar{f}$ is continuous and X is compact. $\Rightarrow \bar{f}(x)$ is a compact subset of $\mathbb{R}^k$. $\Rightarrow \bar{f}(x)$ is closed and bounded. (by Heine Borel theorem) Now, in particular $\Rightarrow \bar{f}(x)$ is bounded $\Rightarrow \bar{f}$ is bounded.

Theorem 3.17 *Suppose f is a continuous real function on a compact metric space X and $M = \sup_{p\in X} f(p)$ and let $m = \inf_{p\in X} f(p)$. Then, there exists a points $p, q \in X$ such that $f(p) = m_1,\ f(q) = m_2$ (i.e., f attains maximum M at p and minimum m at q)*
Proof: We know that, If E is bounded and $y = \sup E$ and $X = \inf E$ then $x, y \in \bar{E}$. Since f is continuous and X is compact $\Rightarrow f(X)$ is closed and bounded [By the above Theorem 3.16] and since $f(X)$ is bounded. $m, M \in \overline{f(X)} = f(X)$ ($\because f(X)$ is closed) $\Rightarrow m, M \in f(X) \Rightarrow$ there exists $p, q \in X$ such that $M = f(p),\ m = f(q)$.

Theorem 3.18 *Suppose f is continuous $1-1$ mapping of a compact metric space X into a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(X)) = X$ is a continuous mapping of Y onto X.*
Proof: Suppose f is a continuous $1-1$ mapping of a compact metric space X into a metric space Y and also $f^{-1}(f(X)) = X$. To Prove: f^{-1} is continuous on Y, it is enough to prove that $(f^{-1})(V)$ is open in Y for every open set V in X. Let V be a open set in $X \Rightarrow V^c$ is closed in X. Since X is compact, V^c is compact in X. Since f is continuous, $f(V^c)$ is compact in $Y \Rightarrow f(V^c)$ is closed in $Y \Rightarrow (f(V^c))^c$ is closed in $Y \Rightarrow f(V)$ is open in Y. ($\because f$ is $1-1$ and onto) $\Rightarrow (f^{-1}(V))^{-1}$ is open in $Y \Rightarrow f^{-1}$ is continuous on Y.

Definition 3.19 ***(Uniformly Continuous)*** *Let X and Y be any two metric space then the $f: X \to Y$ is said it to be uniformly continuous on X if for every $\epsilon > 0$ there exists a $S > 0$ such that $d_X(p,q) < S \Rightarrow d_Y(f(p), f(q)) < \epsilon$ $\forall p, q \in X$.*

Theorem 3.20 *Let f be a continuous mapping of a compact metric space X into a metric space Y then f is uniformly continuous. (i.e.) Continuous function defined on a compact metric space is uniformly continuous.*
Proof: Let $\epsilon > 0$ be given let f is continuous on $X \Rightarrow f$ is continuous at every point $p \in X$. Now, f is continuous at $p \Rightarrow$ there exists a positive real $\phi(p)$ such that $d_X(p,q) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \epsilon \; \forall q \in X$....... (1)
Let $J(p) = N_{\frac{\phi(p)}{2}}\{p\} \Rightarrow J(p)$ is a closed in $X \Rightarrow J(p)$ is a open in X. $\therefore$ $\{J(p) | p \in X\}$ is an open cover for X. Since X is compact, there exists finitely may $p \in S$. $p_1, p_2, ..., p_n$ such that $X \subset \bigcup_{i=1}^{n} J(p_i)$. Let $S = min\{(\frac{\phi(p)}{2}, ..., \frac{\phi(p)}{2})\}$. Clearly, $S > 0$. Let p, q be points in X such that $d_X(p,q) < S$. Now,

$$\begin{aligned}
p \in X \subset \bigcup_{i=1}^{n} J(p_i) & \\
\Rightarrow p \in J(p_m) & \text{ for some } m, 1 \leq m \leq n \\
\Rightarrow d_X(p, p_m) &< \frac{\phi(p_m)}{2} < \phi(p_m) \\
\Rightarrow d_Y(f(p), f(p_m)) &< \epsilon/2 ....(2) \; (by (1)) \\
\text{Now } d_X(q, p_m) &< d_X(q,p) + d(p, p_m) \\
&< S + \frac{\phi(p_m)}{2} \\
&< \frac{\phi(p_m)}{2} + \frac{\phi(p_m)}{2} \\
&= \phi(m) \\
(i.e.) \; d_X(q, p_m) &< \phi(p_m) \\
\Rightarrow d_Y(f(q), f(p_m)) &< \epsilon/2 \text{ by}(1)........(3)
\end{aligned}$$

$$\Rightarrow d_Y(f(p), f(q)) < d_Y(f(q), f(p_m)) + d_Y(f(p_m)f(q))$$
$$= \epsilon/2 + \epsilon/2 \text{ (by (2) and (3))}$$
$$\therefore d_X(p, q) < S \Rightarrow d_Y(f(p), f(q)) < \epsilon$$

$\Rightarrow f$ is uniformly continuous on X.

Theorem 3.21 *Let E be a non-compact set in $\mathbb{R}^1$. Then*
(a) there exists a continuous function on E which is not bounded,
(b) there exists continuous and bounded function on which has no maximum if in addition E is bounded,
(c) there exists a continuous function on E which is not uniformly continuous.
Proof: Case(i): Suppose E is bounded.
(a) To Prove: f is continuous but not bounded. Since E is bounded, there exists a limit point of x_0 of E such that $x_0 \notin E$. [$\because E$ is not closed]. Define a map $f : E \to \mathbb{R}^1$ by $f(x) = \frac{1}{x-x_0}$, $x \in E$. $\therefore f$ is continuous on E. To Prove: f is unbounded on E. Since x_0 is a limit point of E. $N_r(x_0) \cap E \neq \emptyset$ $\forall r > 0 \Rightarrow$ there exists x_1 such that $x_1 \in N_r(x_0) \cap E \Rightarrow x_1 \in N_r(x_0)$ and $x_1 \in E$

$$\Rightarrow |x_1 - x_0| < r \text{ and } x_1 \in E$$
$$\Rightarrow \frac{1}{|x_1 - x_0|} > \frac{1}{r} \text{ and } x_1 \in E$$
$$\Rightarrow |f(x_1)| > \frac{1}{r} \text{ and } x_1 \in E \ \forall r > 0$$

$\forall r > 0$ there exists $x \in E$ such that $|f(x)| > \frac{1}{r} \Rightarrow f$ is unbounded on E.
(b) Define $g : E \to R$ by $g(x) = \frac{1}{1+(x-x_0)^2}$, $x \in E$. Clearly, g is continuous. Now, $0 < g(x) < 1 \Rightarrow g(x)$ is a bounded function. Clearly, $\sup_{x\in E} g(x) = 1$. But $g(x) < 1 \ \ \forall x \in E$. $\therefore g$ has no maximum on E.
(c) Let $f : E \to R$ be defined by $f(x) = \frac{1}{x-x_0}$, $x \in E$, where x_0 is a limit point of E. Clearly, f is continuous on E. Let $\epsilon > 0$ be given. Let $S > 0$ be arbitrary choose a point $x \in E$ such that $|x - x_0| < S$ and taking t very close to x_0 so as to satisfy $|t - x| < S$. Then,

$$|f(t) - f(x)| = \left|\frac{1}{t - x_0} - \frac{1}{x - x_0}\right|$$
$$= \left|\frac{x - x_0 - t + x_0}{(t - x_0)(x - x_0)}\right|$$
$$= \frac{|x - t|}{|t - x_0||x - x_0|}$$
$$> \frac{1}{t - x_0} > \epsilon$$

(If we choose $x \in (x_0 - S, x_0), t \in (x_0, x_0 + S)$ and $|x - t| < S$ or $t \in (x_0 - S, x_0), x \in (x_0, x_0 + S)$ and $|x - t| < S \Rightarrow |t - x| > |x - x_0|$) So we

have taken t very close to x_0 and we made the difference $|f(t) - f(x)| > \epsilon$ although $|t - x| < S$. Since this is true for every $S > 0 \Rightarrow f$ is not uniformly continuous.
Case(ii): Suppose E is not bounded.
(a) Define $f : E \to R$ by $f(x) = x$. Clearly, f is continuous on E and f is not bounded on E. $\therefore$ there exists function on E which is not bounded.
(b) Define $g : E \to R$ by $g(x) = \frac{x^2}{1+x^2} \Rightarrow g$ is continuous. Now, as $x^2 < 1 + x^2 \Rightarrow g(x) = \frac{x^2}{1+x^2} < 1$. $\therefore 0 < g(x) < 1 \quad \forall x \in E$. $\therefore g$ is a bounded. $\therefore g$ is a continuous and bounded function. $\sup_{x \in E} g(x) = 1$. But g has no maximum on E.
(c) If the boundedness is omitted then the result fails. Let E be the set of all integers. Then every function defined on E is uniformly continuous on $E \Rightarrow$ for every $\epsilon > 0$ choose $S < 1$ such that $|X - Y| < S \Rightarrow |f(x) - f(y)| = 0 < \epsilon$

Continuity and Connectedness:

Theorem 3.22 *If f is a continuous mapping on a metric space X into a metric space Y and E is a connected subset of X. Then $f(E)$ is connected. i.e., continuous image of a connected subset of a metric space is connected.*
Proof: Given E is connected subset of X. To Prove: $f(E)$ is a connected subset of Y. Suppose $f(E)$ is not connected. $\Rightarrow f(E) = A \cup B$ where A and B are non-empty separated sets. Put $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$

$$\begin{aligned}
G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\
&= E \cap (f^{-1}(A) \cup f^{-1}(B)) \\
&= E \cap (f^{-1}(A \cup B)) \\
&= E \cap E \\
G \cup H &= E
\end{aligned}$$

Clearly $G \neq \emptyset \quad H \neq \emptyset \quad (\because A \neq \emptyset, B \neq \emptyset)$. Claim: G and H are separated sets. i.e., To Prove $\bar{G} \cap H = \emptyset, G \cap \bar{H} = \emptyset$. Now

$$\begin{aligned}
G &= E \cap f^{-1}(A) \\
\Rightarrow G &\subset f^{-1}(A) \subset f^{-1}(\bar{A}) \\
\Rightarrow \bar{G} &\subset \overline{f^{-1}(\bar{A})} = f^{-1}(\bar{A}) \ [\because \bar{A} \text{ is closed and} \\
&\qquad\qquad\qquad\qquad f \text{ is continuous } \Rightarrow f^{-1}(\bar{A})] \\
\Rightarrow f(\bar{G}) &\subset ff^{-1}(\bar{A}) \subset \bar{A} \\
\Rightarrow f(\bar{G}) &\subset \bar{A} \\
H &= E \cap f^{-1}(B) \\
\Rightarrow H &\subset f^{-1}(B) \ \Rightarrow f(H) \subset ff^{-1}(B) = B \\
\Rightarrow f(H) &\subset B \\
\Rightarrow f(\bar{G}) \cap f(H) &\subset \bar{A} \cap B = \emptyset \ (\because A \text{ and } B \text{ are separated sets}) \\
\Rightarrow f(\bar{G}) \cap f(H) &= \emptyset \\
\Rightarrow f(\bar{G} \cap H) &= \emptyset \\
\Rightarrow \bar{G} \cap H &= \emptyset \\
\text{similarly, } G \cap \bar{H} &= \emptyset
\end{aligned}$$

$\therefore$ G and H are separated sets. $\Rightarrow$ E can be expressed as a union of two non-empty separated sets. $\Rightarrow$ E is not connected. $\Rightarrow\Leftarrow$ to E is connected. $\therefore$ $f(E)$ is connected.

Theorem 3.23 ***Intermediate Value Theorem:*** *Let f be a continuous real valued function on $[a, b]$. If $f(a) < f(b)$ and c is the number such that $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that $f(x) = c$.*
Proof: Every interval in $\mathbb{R}$ is connected and f is continuous. By the previous theorem, $f[a, b]$ is connected in $\mathbb{R}$. $\Rightarrow$ $f[a, b]$ is interval in $\mathbb{R}$. Let $f(a), f(b) \in f[a, b] \Rightarrow [f(a), f(b)] \subset f[a, b]$. Now, $f(a) < c < f(b) \Rightarrow c \in f[a, b] \Rightarrow c = f(x)$ for some $x \in [a, b]$.

Remark 3.24 *Converse not true.*
Proof: If any two points x_1 and x_2 and for any member c between $f(x_1)$ and $f(x_2)$ there is a point x in $[x_1, x_2]$ such that $f(x) = c$ then f may be discontinuous. For example:

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Choose $x_1 \in (-\frac{\pi}{2}, 0), x_2 \in (0, \frac{\pi}{2})$. Clearly $x_1 < x_2$; $f(x_1)$ =negative $f(x_2)$=positive. $\therefore$ $f(0) = 0$. f is continuous all the points except at 0.

Differentiation:

Definition 3.25 *Let f be real value function defined on $[a,b]$, for any $x \in [a,b]$ form the quotient $\phi(t) = \frac{f(t)-f(x)}{t-x}, a < t < b, t \neq x$, and defined*

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

provided the limit exists.

Remark 3.26 *1. If f' is defined at a point, we say that f is differentiable at x.*
2. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E.

Theorem 3.27 *Let f be defined on $[a,b]$. If f is differentiable at a point x in $[a,b]$, then f is continuous at x.*
Proof: Given f is differentiable at x. (i.e.)

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

To Prove: f is continuous at x (i.e.)To Prove

$$\lim_{t \to x} f(t) = f(x)$$

Now

$$\begin{aligned}
f(t) - f(x) &= \frac{f(t) - f(x)}{t - x}(t - x) \\
\lim_{t \to x}(f(t) - f(x)) &= \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x}(t - x) \right] \\
&= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \lim_{t \to x}(t - x) \\
&= f'(x) \cdot 0 \\
&= 0 \\
\lim_{t \to x}(f(t) - f(x)) &= 0 \\
\text{(or)} \ \lim_{t \to x} f(t) &= f(x)
\end{aligned}$$

$\therefore f$ is continuous at x.

Remark 3.28 *Converse of above theorem is not true. For example $f(x) = |x|$ is continuous but not differentiable at origin.*

Theorem 3.29 *Suppose f and g are defined on $[a,b]$ and are differentiable at at point x in $[a,b]$ then $f + g, fg, \frac{f}{g}$ are differentiable at x.*
(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
(c) $(\frac{f}{g})'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)},\ g(x) \neq 0.$
Proof: Given f and g are differentiable at x.

$$(i.e.)f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \text{ and } g'(x) = \lim_{t\to x} \frac{g(t)-g(x)}{t-x} \text{ exists.}$$

(a)

$$\begin{aligned}\phi(t) &= \frac{(f+g)(t)-(f+g)(x)}{t-x}\\ &= \frac{f(t)+g(t)-(f(x)+g(x))}{t-x}\\ \phi(t) &= \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}\end{aligned}$$

Taking limits as $t \to x$

$$\begin{aligned}\lim_{t\to x}\phi(t) &= \lim_{t\to x}\left\{\frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}\right\}\\ &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} + \lim_{t\to x}\frac{g(t)-g(x)}{t-x}\\ (i.e.)(f+g)'(x) &= f'(x)+g'(x)\end{aligned}$$

(i.e.) $(f+g)$ is differentiable at x.
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. Let $h = fg$. Now,

$$\begin{aligned}(h(t)-h(x)) &= (fg)(t)-(fg)(x)\\ &= f(t)g(t)-f(x)g(x)\\ &= f(t)g(t)-f(t)g(x)+f(t)g(x)-f(x)g(x)\\ &= f(t)(g(t)-g(x))+g(x)(f(t)-f(x))\\ \frac{h(t)-h(x)}{t-x} &= f(t)\frac{(g(t)-g(x))}{t-x} + g(x)\frac{(f(t)-f(x))}{t-x}\\ \lim_{t\to x}\frac{h(t)-h(x)}{t-x} &= \lim_{t\to x}\left\{f(t)\frac{g(t)-g(x)}{t-x} + g(x)\frac{f(t)-f(x)}{t-x}\right\}\\ &= \lim_{t\to x}f(t)\lim_{t\to x}\frac{g(t)-g(x)}{t-x} + \lim_{t\to x}g(x)\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\ h'(x) &= f(x)g'(x)+g(x)f'(x)\\ (fg)'(x) &= f(x)g'(x)+g(x)f'(x)\end{aligned}$$

fg is differentiable at x.

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}$. Let $h = \frac{f}{g}$.

$$\begin{aligned}
(h(t)-h(x)) &= \frac{f}{g}(t) - \frac{f}{g}(x)\\
&= \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}\\
&= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{g(t)g(x)}\\
&= \frac{g(x)(f(t)-f(x)) - f(x)(g(t)-g(x))}{g(t)g(x)}\\
\frac{h(t)-h(x)}{t-x} &= \frac{g(x)(f(t)-f(x)) - f(x)(g(t)-g(x))}{g(t)g(x)(t-x)}\\
\lim_{t\to x}\frac{h(t)-h(x)}{t-x} &= \lim_{t\to x}\frac{g(x)}{g(t)g(x)}\left(\frac{f(t)-f(x)}{t-x}\right) - \lim_{t\to x}\frac{f(x)}{g(t)g(x)}\left(\frac{g(t)-g(x)}{t-x}\right)\\
&= \frac{g(x)}{g^2(x)}\lim_{t\to x}\frac{f(t)-f(x)}{t-x} - \frac{f(x)}{g^2(x)}\lim_{t\to x}\frac{g(t)-g(x)}{t-x}\\
h'(x) &= \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}\\
\left(\frac{f}{g}\right)'(x) &= \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}
\end{aligned}$$

Since $f'(x), g'(x)$ exists and $g(x) \neq 0$, $\left(\frac{f}{g}\right)'(x)$ exists.

Example 3.30 *(1) The derivative of any constant is zero.*
(2) $f(x) = x \Rightarrow f'(x) = 1$
(3) $f(x) = n \Rightarrow f'(x) = nx^{n-1}$

Theorem 3.31 ***Chain Rule:*** *Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point x in $[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t)), a \leq t \leq b$ then h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$.*
Proof: Given

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \text{ exists, } t \in [a,b].$$

Let $h(t) = g(f(t))$. To Prove: $h'(x) = g'(f(x))f'(x)$. Since f is differentiable at $x \in [a,b]$

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \text{ exists, } \ t \in [a,b] \text{ exists.}$$

$$(i.e.)\ f'(x) + u(t) = \frac{f(t)-f(x)}{t-x},\ t \in [a,b] \text{ where } \lim_{t\to x} u(t) = 0$$

$$\Rightarrow (f'(x)+u(t))(t-x) = f(t) - f(x)......(1)$$

Let $y = f(x)$. Now g is differentiable at $y(= f(x))$

$$\begin{aligned}
g'(y) &= \lim_{s \to y} \frac{g(s) - g(y)}{s - y}, s \in I \\
(i.e.)\ g'(y) + v(s) &= \frac{g(s) - g(y)}{s - y}, s \in I \text{ where } \lim_{s \to y} v(s) = 0 \\
(g'(y) + v(s))(s - y) &= g(s) - g(y) ....(2)
\end{aligned}$$

Let $s = f(t)$. Now,

$$\begin{aligned}
h(t) - h(x) &= g(f(t)) - g(f(x)) \\
&= (g'(f(x)) + v(s))(s - y)\ (by(2)) \\
h(t) - h(x) &= g'(f(x) + v(s))(f(t) - f(x)) \\
&= g'(f(x) + v(s))(f'(x) + u(t))(t - x)\ (by(1)) \\
\frac{h(t) - h(x)}{t - x} &= g'(f(x) + v(s))(f'(x) + u(t)) \\
\lim_{t \to x} \frac{h(t) - h(x)}{t - x} &= \lim_{t \to x} \{g'(f(x) + v(s))(f'(x) + u(t))\} \\
h'(x) &= \lim_{t \to x} g'(f(x) + v(s)) \lim_{t \to x} (f'(x) + u(t)) \\
&= \lim_{s \to y} (g'(f(x)) + v(s)) f'(x) \\
&= g'(f(x)) f'(x) \\
\therefore h'(x) &= g'(f(x)) f'(x)
\end{aligned}$$

Example 3.32 *Let*

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find $f'(x)(x \neq 0)$, and show that $f'(0)$ does not exist.
Solution:

$$\begin{aligned}
f(x) &= x \sin \frac{1}{x} \\
f'(x) &= x \cos \left(\frac{1}{x}\right) (\frac{-1}{x^2}) + \sin \left(\frac{1}{x}\right) \\
&= -\frac{1}{x} \cos \left(\frac{1}{x}\right) + \sin \left(\frac{1}{x}\right) \\
&= \sin \left(\frac{1}{x}\right) - \left(\frac{1}{x}\right) \cos \left(\frac{1}{x}\right), x \neq 0.
\end{aligned}$$

since $x \neq 0 f'(x)$ exists. To Prove: $f'(0)$ does not exists.

$$\begin{aligned}
f'(0) &= \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} \\
&= \lim_{t \to 0} \frac{t \sin \frac{1}{t} - 0}{t - 0} \\
&= \lim_{t \to 0} \sin \frac{1}{t} \text{ which does not exists.}
\end{aligned}$$

$\therefore f'(0)$ does not exists.

Example 3.33 *Let*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find $f'(x)(x \neq 0)$, show that $f'(0) = 0$

Solution: Let

$$\begin{aligned}
f(x) &= x^2 \sin \frac{1}{x} \\
f'(x) &= x^2 (\cos \left(\frac{1}{x}\right)) \left(\frac{-1}{x^2}\right) + 2x \cdot \sin \frac{1}{x} \\
&= 2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0 \\
f'(0) &= \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} \\
&= \lim_{t \to 0} \frac{x^2 \sin \frac{1}{t} - 0}{t - 0} \\
&= \lim_{t \to 0} t \sin \frac{1}{t} \\
&= 0 \ (\because \left| t \sin \frac{1}{t} \right| \leq 1) \\
\therefore f'(0) = 0
\end{aligned}$$

Mean Value Theorems:

Definition 3.34 ***Local Maximum, Local Minimum:*** *Let f be a real function defined on a metrics space X. We say that f has local maximum at a point p in X if there exists $\delta > 0$ such that $f(q) \leq f(p)$ $\forall q \in X$ with $d(p, q) < \delta$. f has a local minimum at p in X, if $f(p) \leq f(q)$ $\forall q \in X$ such that $d(p, q) < \delta$.*

Theorem 3.35 *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$ and if f' exists, then $f'(x)$=0. The analogous statement for local minimum is also true.*

Proof: Case(i) Assume that f has local maximum at x. To Prove: $f'(x) =$

0. Since f has local maximum at x, there exists $\delta > 0$ such that $(q, x) < \delta \Rightarrow f(q) \leq f(x)$

$$\begin{aligned}
\text{If } x - \delta < t < x \text{ then } & \frac{f(t) - f(x)}{t - x} \geq 0 \\
\Rightarrow \lim_{t \to x} & \frac{h(t) - h(x)}{t - x} \geq 0 \\
& (i.e.)\ f'(x) \geq 0 \text{ ....(1)} \\
\text{If } t^x < x^t < x + \delta \text{ then } & \frac{f(t) - f(x)}{t - x} \leq 0 \\
\Rightarrow \lim_{t \to x} & \frac{h(t) - h(x)}{t - x} \leq 0 \\
& \Rightarrow f'(x) \leq 0 \text{ ....(2)}
\end{aligned}$$

Since $f'(x)$ exists, (1),(2) $\Rightarrow f'(x) = 0$.
Case(ii) Assume that f has a local minimum at x. We show that $f'(x)$=0. Then there exists $\delta > 0$ such that $d(q, x) < \delta \Rightarrow f(q) \geq f(x)$

$$\begin{aligned}
\text{If } x - \delta < t < x \text{ then} & \frac{f(t) - f(x)}{t - x} \leq 0 \\
\Rightarrow \lim_{t \to x} & \frac{f(t) - f(x)}{t - x} \leq 0 \\
& (i.e.)\ f'(x) \leq 0 \text{ ....(3)} \\
\text{If } x < t < x + \delta \text{ then } & \frac{f(t) - f(x)}{t - x} \geq 0 \\
\Rightarrow \lim_{t \to x} & \frac{f(t) - f(x)}{t - x} \geq 0 \\
& \Rightarrow f'(x) \geq 0 \text{ ....(4)}
\end{aligned}$$

Since $f'(x)$ exists, and from (3) and (4) we get $f'(x)$=0.

Theorem 3.36 ***Generalised Mean Value Theorem:*** *If f and g are continuous real functions on $[a, b]$, which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.*
proof: Let $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t),\ t \in [a, b]$. Since f and g are differentiable in (a, b), $h(t)$ is also differentiable in (a, b). Now,

$$\begin{aligned}
h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\
&= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\
&= f(b)g(a) - g(b)f(a) \\
h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\
&= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) \\
&= g(a)f(b) - f(a)g(b)
\end{aligned}$$

Claim: $h'(x) = 0$ for some $x \in (a,b)$. If $h(t)$ is a constant then $h'(x) = 0 \; \forall x \in (a,b)$. If $h(t) < h(a), a < t < b$, then by Intermediate value theorem, there exists x in (a,b) at which h is minimum. $\therefore h'(x) = 0$ (by Theorem 3.35). If $h(t) > h(a)$ then h attains its maximum at some point $x \in (a,b)$. $\therefore$ $h'(x) = 0$ (by Theorem 3.35) (i.e.)

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x) = 0$$
$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$$

Theorem 3.37 ***Mean Value Theorem:*** *If f is a real continuous function on $[a,b]$ which is differentiable at (a,b) then there is a point $x \in (a,b)$ at which $f(b) - f(a) = (b-a)f'(x)$.*
Proof: Put $g(x) = x$ in theorem 3.36. $\therefore g'(x) = 1 \Rightarrow (f(b) - f(a)) = (b-a)f'(x)$.

Theorem 3.38 *Suppose f is differentiable in (a,b).*
(a) If $f'(x) \geq 0 \; \forall x \in (a,b)$, then f is monotonically increasing.
(b) If $f'(x) = 0 \; \forall x \in (a,b)$, then f is a constant.
(c) If $f'(x) \leq 0 \; \forall x \in (a,b)$, then f is monotonically decreasing.
Proof: (a) By theorem 3.37, If $x_1 < x_2$, then there exists $x_1 < x < x_2$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$...... (1)
If $f'(x) \geq 0$ then (1) $\Rightarrow f(x_2) - f(x_1) \geq 0$ ($\because (x_2 - x_1)f'(x) \geq 0) \Rightarrow f(x_1) \leq f(x_2)$ (i.e.) f is an increasing function
(b) If $f'(x)$=0 then (1) $\Rightarrow f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$. $\therefore f$ is constant.
(c) If $f'(x) \leq 0$ then (1)$\Rightarrow f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_1) \geq f(x_2)$. $\therefore f$ is an decreasing function.

The Continuity Of Derivatives

Theorem 3.39 *Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$, then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$. A similar result holds if $f'(a) > \lambda > f'(b)$.*
Proof: Let $g(t) = f(t) - \lambda t, t \in [a,b]$ then, $g'(t) = f'(t) - \lambda$; $g'(a) = f'(a) - \lambda < 0$. $\therefore$ there exists $a < t_1 < b$ such that $g(t_1) < g(a)$. Also, $g'(b) = f'(b) - \lambda > 0$. $\therefore$ there exists $a < t_2 < b$ such that $g(t_2) < g(b)$. $\therefore g$ attains minimum at $x \in (a,b)$. $\therefore g'(x)$=0 (by Theorem 3.35) (i.e.) $f'(x) - \lambda = 0 \Rightarrow f'(x) = \lambda$.

Corollary 3.40 *If f is differentiable on $[a,b]$, then f' is cannot have any simple discontinuity on $[a,b]$. But f' may have discontinuity of second kind.*
Proof: f' takes every value between $f(a)$ and $f(b)$. Let $a < x < b$. If f' is not continuous at x, then

1. $f'(x+), f'(x-)$ exists,

2. $f'(x+) \neq f'(x-)$,
3. $f'(x-) = f'(x+) \neq f'(x) \Rightarrow\Leftarrow$

$\therefore$ f' cannot have any simple discontinuity. In Example 3.33 f' has a discontinuity of second kind at $x \in [a, b]$.

Theorem 3.41 ***L'Hospital's Rule:*** *Suppose f and g are differentiable in (a, b) and $g'(x) \neq 0$ $\forall x \in (a, b)$ where $-\infty \leq a < b \leq \infty$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$........ (1).*
If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$........ (2) (or) if $g(x) \to \infty$ as $x \to a$....... (3), then $\frac{f(x)}{g(x)} \to A$ as $x \to a$....... (4). (The analogous statement is true if $x \to b$ (or) if $g(x) \to -\infty$ in (3)).
Proof: Case(i): Let $-\infty \leq A < \infty$. We choose r and q such that $A < r < q$. Given

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$$

Then there exists $c \in (a, b)$ such that $a < x < c \Rightarrow \frac{f'(x)}{g'(x)} < r$....... (i)
Now if $a < x < y < c$ then by generalised mean value theorem, there exists $t \in (a, b)$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$........ (ii)
Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. Then by taking limits as $x \to a$, then (ii) we get $\frac{f(y)}{g(y)} \leq r < q$........ (iii)
Suppose $g(x) \to \infty$ as $x \to a$, then by keeping y fixed in (ii) we can find $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ $\forall x \in (a, c_1)$. Multiply (ii) by $\frac{g(x)-g(y)}{g(x)}$, we get

$$\begin{aligned}
\frac{f(x) - f(y)}{g(x)} &< r\left(\frac{g(x) - g(y)}{g(x)}\right) \\
\Rightarrow \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} &< r\left(1 - \frac{g(y)}{g(x)}\right) \\
\Rightarrow \frac{f(x)}{g(x)} &< r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}
\end{aligned}$$

Since $g(x) \to \infty$ as $x \to a$, there exists $c_2 \in (a, c_1)$ such that $\frac{f(x)}{g(x)} < r$ $\forall x \in (a, c_2)$ (or) $\frac{f(x)}{g(x)} < q$ $\forall x \in (a, c_2)$.......(iv)
suppose $-\infty < A \leq \infty$. By choosing $p < A$ as above, we can show that there exists $c_3 \in (a, b)$ such that $p < \frac{f(x)}{g(x)}$ $\forall a < x < c_3$.....(v)
Thus in all cases $\frac{f(x)}{g(x)} \to A$ as $x \to a$. Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

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Derivatives Of Higher Order

Definition 3.42 *If f has a derivative f' on an interval and if f' is differentiable, we see the second derivative f'' exists. Similarly if $f^{n-1}(x)$ is differentiable we say $f^{(n)}$ exists.*

Theorem 3.43 ***Taylor's Theorem:*** *Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists $\forall t \in (a, b)$. Let α, β be distinct points of $[a, b]$ and define*

$$p(t) = \sum_{n=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k,$$

then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$.
Proof: If n=1, then $f(\beta) = f(\alpha) + f'(x)(\beta - \alpha)$; $\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(x)$. This is just the mean value theorem. Suppose $n > 1$. Define a number M such that $f(\beta) = p(\beta) + M(\beta - \alpha)^n$........(1)
Let $g(t) = f(t) - p(t) - M(t - \alpha)^n$........ (2)
Now,

$$\begin{aligned}
g(\alpha) &= f(\alpha) - p(\alpha) - M(\alpha - \alpha)^n \\
&= f(\alpha) - p(\alpha) \\
g(\alpha) &= f(\alpha) - f(\alpha) \ (\because p(\alpha) = f(\alpha)) \\
&= 0 \\
g(\beta) &= f(\beta) - p(\beta) - M(\beta - \alpha)^n \\
&= 0 \ (by\ (1)).......(4) \\
\text{Also } g^{(n)}(t) &= f^{(n)}(t) - 0 - Mn!.......(5) \\
g^{(k)}(\alpha) &= f^{(k)}(\alpha) - p^{(k)}(\alpha) \\
&= f^{(k)}(\alpha) - f^{(k)}(\alpha) \\
&= 0......(6)
\end{aligned}$$

(i.e.) $g(\alpha) = g'(\alpha) = \cdots = g^{n-1}(\alpha) = 0$. Since $g(\alpha) = 0$ and $g(\beta) = 0$, there exists $x_1 \in (\alpha, \beta)$, by mean value theorem, such that $g'(x_1)$=0. Now since $g'(\alpha) = 0$; $g'(x_1) = 0$ again by mean value theorem there exists $x_2 \in (\alpha, x_1)$ such that $g"(x_2) = 0$. Proceeding this way we get $\alpha < x_n < x_{n-1}$, such that $g^{(n)}(x_n) = 0$ (i.e.) $f^{(n)}(x_n) - Mn! = 0$ (by (5)). $\therefore M = \frac{f^n(x_n)}{n!}$, sub M in (1) $\Rightarrow f(\beta) = p(\beta) + \frac{f^{(n)}(x_n)}{n!}(\beta - \alpha)^n$, $\forall x \in (\alpha, x_{n-1})$

UNIT V

RIEMANN INTEGRAL AND POINTWISE CONVERGENCE

The Riemann-Steiltjes integral and Sequences and series of functions

Definition 4.1 *Let $[a, b]$ be an interval. By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, ..., xn$, where $a = x_0 \leq x_1 \leq, ..., \leq x_{i-1} \leq x_i \leq , ..., \leq x_n = b$.*

Remark 4.2 1. $\Delta x_i = x_i - x_{i-1}$ $\forall i = 1, 2, ..., n$.

2. *Let f be a bounded real function on $[a, b]$ then $m_i = \inf f(x), M_i = \sup f(x) \quad \forall x_{i-1} \leq x \leq x_i$.*

3.
$$\begin{aligned}
L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\
U(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\
L(P, f) &\leq \int_a^b f(x)dx \leq U(P, f) \\
L(P, f) &\leq U(P, f).
\end{aligned}$$

4. $\int_{\underline{a}}^{b} f(x)dx = \sup L(P, f)$

5. $\int_a^{\bar{b}} f(x)dx = \inf U(P, f)$ *(The* $\inf$ *and* $\sup$ *are taken over all partition P of $[a, b]$).*

6. *If the upper and lower reimann interval over is same then f is said to be Reimann integrable over $[a, b].f \in \mathcal{R}(\mathcal{R}$ is the set of all Reimann integrable functions)*

7.
$$\int_{\underline{a}}^{b} f(x)dx = \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx$$

Result 4.3 *For every partition P of $[a, b]$ and every bounded function f there exists 2 real numbers m, M such that $m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$.*
Solution: Let $m = \inf f(x)$ and $M = \sup f(x), a \leq x \leq b$. Let $P =$

$\{x_0, x_1, ..., x_n\}$ be the given partition of $[a, b]$,

$$\begin{aligned}
m &\leq m_i \leq M_i \leq M \\
m\Delta x_i &\leq m_i\Delta x_i \leq M_i\Delta x_i \leq M\Delta x_i \ (\Delta x_i \geq 0) \\
\sum_{i=1}^{n} m\Delta x_i &\leq \sum_{i=1}^{n} m_i\Delta x_i \leq \sum_{i=1}^{n} M_i\Delta x_i \leq \sum_{i=1}^{n} M\Delta x_i \\
m(\sum_{i=1}^{n} \Delta x_i) &\leq L(P, f) \leq U(P, f) \leq M \sum_{i=1}^{n} \Delta x_i ....(1) \\
\text{Now, } \sum_{i=1}^{n} \Delta x_i &= \Delta x_1 + \Delta x_2 + ... + \Delta x_n \\
&= (x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1}) \\
&= x_n - x_0 \\
&= b - a........(2)
\end{aligned}$$

sub (2) in (1) we get, $m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$.

Definition 4.4 *Let α be a monotonically increasing function on $[a, b]$. Corresponding to each partition P of $[a, b]$*
we define $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Clearly, $\Delta\alpha_i \geq 0$

$$\begin{aligned}
L(P, f, \alpha) &= \sum_{i=1}^{n} m_i\Delta\alpha_i \\
U(P, f, \alpha) &= \sum_{i=1}^{n} M_i\Delta\alpha_i \\
\sup L(P, f, \alpha) &= \int_{\underline{a}}^{b} f d\alpha \\
U(P, f, \alpha) &= \int_{a}^{\bar{b}} f d\alpha
\end{aligned}$$

where infimum and suprimum are taken over all partitions. If

$$\int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\bar{b}} f d\alpha,$$

then f is Reimann Stieljes integrable with respect to,

$$\int_{a}^{b} f d\alpha = \int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\bar{b}} f d\alpha,$$

we also write $f \in \mathcal{R}(\alpha)$.

Note 4.5 *By taking $\alpha(x) = x$, we see that the Reimann integral is the special case of Riemann's Stieltjes integral.*

Definition 4.6 *The partition P^* of $[a,b]$ is called a refinement of P if $P \subset P^*$. Given two partition P_1 and P_2, we say that $P = P_1 \cup P_2$ is the common refinement of P_1 and P_2.*

Theorem 4.7 *If P^* is an refinement of P, then $L(P,f,\alpha) \leq L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \leq U(P,f,\alpha)$.*

Proof: Let $P = \{x_0, x_1, ..., x_{i-1}, x_i, ..., x_n\}$ be a partition of $[a,b]$ and let $P^* = \{x_0, x_1, x_2, ..., x_{i-1}, x^*, x_i, ..., x_n\}$ be an refinement of P. Let

$$\begin{aligned} m_i &= \inf f(x),\ x_{i-1} \leq x \leq x_i \\ w_1 &= \inf f(x),\ x_{i-1} \leq x \leq x^* \\ w_2 &= \inf f(x),\ x^* \leq x \leq x_i \end{aligned}$$

$\therefore w_1 \geq m_i$ and $w_2 \geq m_i$. Now,

$$\begin{aligned} L(P^*,f,\alpha) &= m_1\Delta\alpha_1 + m_2\Delta\alpha_2 + ... + m_{i-1}\Delta\alpha_{i-1} + w_1(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + w_2(\alpha(x_i) - \alpha(x^*)) + m_{i+1}\Delta\alpha_{i+1} ... + m_n\Delta\alpha_n ......(1) \\ L(P,f,\alpha) &= m_1\Delta\alpha_1 + m_2\Delta\alpha_2 + ... + m_{i-1}\Delta\alpha_{i-1} + m_i\Delta\alpha_i \\ &\quad + m_{i+1}(\Delta\alpha_{i+1}) + ... + m_n\Delta\alpha_n ......(2) \end{aligned}$$

(1)-(2) $\Rightarrow$

$$\begin{aligned} L(P^*,f,\alpha) - L(P,f,\alpha) &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) - m_i\Delta\alpha_i \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i(\alpha(x_i) - \alpha(x_{i-1})) \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i(\alpha(x_i) - \alpha(x^*)) - m_i(\alpha(x^*) - \alpha(x_{i-1})) \\ &= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \\ &\geq 0 (\because w_1 \text{ and } w_2 \geq m_i) \\ L(P^*,f,\alpha) - L(P,f,\alpha) &\geq 0 \\ \Rightarrow L(P,f,\alpha) &\leq L(P^*,f,\alpha) \\ \therefore L(P,f,\alpha) &\leq L(P^*,f,\alpha) \end{aligned}$$

Let $P^* = \{x_0, x_1, ..., x_{i-1}, x^*, x_i, ..., x_n\}$ be refinement of P. Let

$$\begin{aligned} M_i &= \sup f(x), x_{i-1} \leq x \leq x_i \\ w_1 &= \sup f(x), x_{i-1} \leq x \leq x^* \\ w_2 &= \sup f(x), x^* \leq x \leq x_i \\ \therefore w_1 &\geq M_i \text{ and } w_2 \geq M_i \end{aligned}$$

Now

$$\begin{aligned}U(P^*,f,\alpha) &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + ... + M_{i-1}\Delta\alpha_{i-1} + w_1(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + w_2(\alpha(x_i) - \alpha(x^*)) + M_{i+1}\Delta\alpha_{i+1} + ... + M_n\Delta\alpha_n ....(1) \\ U(P,f,\alpha) &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + ... + M_{i-1}\Delta\alpha_{i-1} + M_i\Delta\alpha_i \\ &\quad + M_{i+1}(\Delta\alpha_{i+1}) + ... + M_n\Delta\alpha_n ......(2)\end{aligned}$$

(1)-(2) $\Rightarrow$

$$\begin{aligned}U(P^*,f,\alpha) - U(P,f,\alpha) &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) \\ &\quad - \alpha(x^*)) - M_i\Delta\alpha_i \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - M_i(\alpha(x_i) - \alpha(x_{i-1})) \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - M_i(\alpha(x_i) - \alpha(x^*)) - M_i(\alpha(x^*) - \alpha(x_{i-1})) \\ &= (w_1 - M_i)(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + (w_2 - M_i)(\alpha(x_i - \alpha(x^*))) \\ &\leq 0(\because w_1 \text{ and } w_2 \leq M) \\ (i.e.)\ U(P^*,f,\alpha) - U(P,f,\alpha) &\leq 0 \\ \Rightarrow U(P^*,f,\alpha) &\leq U(P,f,\alpha) \\ \therefore U(P^*,f,\alpha) &\leq U(P,f,\alpha)\end{aligned}$$

If P^* contains k-points more than P, we repeat this reasoning k-times and get the result.

Theorem 4.8

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$$

Proof: Let P_1 and P_2 be two partition of $[a,b]$ and let $P^* = P_1 U P_2$. (i.e.) P^* is a common refinement of P_1 and P_2. $L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha) \Rightarrow L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$. Keeping P_1 fixed and taking infimum over all partition P_2, we get

$$L(P,f,\alpha) \leq \overline{\int_a^b} f d\alpha.$$

Now, by taking suprimum over all partition P_1 we get

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$$

Theorem 4.9 ***Criterion for Riemann Integrability:*** *Let $f \in \mathcal{R}(\alpha)$ iff $\forall \in > 0$, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$.*

Proof: Let $\in > 0$, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$
Claim: $f \in \mathcal{R}(\alpha)$. We know that

$$U(P,f,\alpha) \geq \int_a^{\bar{b}} f d\alpha .....(1)$$

$$L(P,f,\alpha) \leq \int_{\underline{a}}^{b} f d\alpha .....(2)$$

$$(2) \times -1 \Rightarrow -L(P,f,\alpha) \geq -\int_{\underline{a}}^{b} f d\alpha .....(3)$$

$$(1)+(3)\ U(P,f,\alpha) - L(P,f,\alpha) \geq \int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^{b} f d\alpha$$

$$(or)\ \int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^{b} f d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha)$$
$$< \epsilon$$

Since ϵ is arbitrary,

$$\int_{\underline{a}}^{b} f d\alpha = \int_a^{\bar{b}} f d\alpha . (i.e.)\ f \in \mathcal{R}(\alpha).$$

Conversely: Assume $f \in \mathcal{R}(\alpha)$. To Prove: let $\epsilon > 0$, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$
let $\epsilon > 0$ be given
Then there exists two partition P_1 and P_2 such that
$U(P_1,f,\alpha) < \int_a^b f d\alpha + \frac{\epsilon}{2}.....(4)$ and $\int_a^b f d\alpha - \frac{\epsilon}{2} < L(P_2,f,\alpha).......(5)$
Let $P = P_1 U P_2$ (i.e.) P is the common refinement of P_1 and P_2
Now

$$\begin{aligned} U(P,f,\alpha) &\leq U(P_1,f,\alpha) \\ &\leq \int_a^b f d\alpha + \frac{\epsilon}{2} \text{ (by (4))} \\ &< L(P_2,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (by (5))} \\ &= L(P_2,f,\alpha) + \epsilon \\ &\leq L(P,f,\alpha) + \epsilon \\ \therefore U(P,f,\alpha) - L(P,f,\alpha) &< \epsilon \end{aligned}$$

Theorem 4.10 *Let P be a partition $\in$: $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$...(1)*
(a) if (1) holds for some P and ϵ then (1) holds for every refinement of P.
(b) if (1) holds for $P = \{x_0, x_1, ..., x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

(c) if $f \in \mathcal{R}(\alpha)$ and the hypothesis of (b) holds then

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon.$$

Proof: (a) Let P^* be a refinement of P. We know that

$$\begin{aligned} U(P^*, f, \alpha) &\leq U(P, f, \alpha)......(2) \\ L(P^*, f, \alpha) &\leq L(P, f, \alpha) \text{ (by Theorem 4.7)} \\ -L(P^*, f, \alpha) &\leq -L(P, f, \alpha)......(3) \end{aligned}$$

(2)+(3) gives

$$\begin{aligned} U(P^*, f, \alpha) - L(P^*, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \\ &< \epsilon \text{ (by (1))} \\ (i.e.) U(P^*, f, \alpha) - L(P^*, f, \alpha) &< \epsilon \end{aligned}$$

(b) $s_i, t_i \in [x_{i-1}, x_i]$; $f(s_i), f(t_i) \in f[x_{i-1}, x_i]$; $m_i \leq f(s_i), f(t_i) \leq M_i$

$$\begin{aligned} \therefore |f(s_i) - f(t_i)| &\leq M_i - m_i \ (\because M_i - m_i \geq 0) \\ \Rightarrow |f(s_i) - f(t_i)|\Delta\alpha_i &\leq (M_i - m_i)\Delta\alpha_i \\ \Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\ &= \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \\ &= U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1))} \\ \therefore \sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i &< \epsilon. \end{aligned}$$

(c) We have

$$\begin{aligned} m_i &\leq f(t_i) \leq M_i \\ \Rightarrow m_i\Delta\alpha_i &\leq f(t_i)\Delta\alpha_i \leq M_i\Delta\alpha_i \\ \Rightarrow \sum_{i=1}^{n} m_i\Delta\alpha_i &\leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq \sum_{i=1}^{n} M_i\Delta\alpha_i \\ \Rightarrow L(P, f, \alpha) &\leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha)......(4) \\ L(P, f, \alpha) &\leq \int_a^b f d\alpha \leq U(P, f, \alpha)......(5) \end{aligned}$$

(4) and (5) $\Rightarrow$

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| \leq U(P,f,\alpha) - L(P,f,\alpha)$$
$$= \epsilon \text{ (by (1))}$$
$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon.$$

Theorem 4.11 *If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$.*
Proof: Let $\epsilon > 0$ be given. Choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$...(1)
Since f is continuous on $[a,b]$ and $[a,b]$ is compact, f is uniformly continuous. Then there exists $\delta > 0$ such that $|x - \epsilon| < \delta \Rightarrow |f(x) - f(\epsilon)| < \eta$..... (2)
Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta$ $\therefore$ (2) guarantees that $|M_i - m_i| < \eta$ (i.e.) $M_i - m_i < \eta$......(3)
Now,

$$\begin{aligned}
U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \\
&= \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \\
&< \eta(\sum_{i=1}^{n}\Delta\alpha_i) \text{ (by (3))} \\
&= \eta[\Delta\alpha_1 + \Delta\alpha_2 + ... + \Delta\alpha_n] \\
&= \eta[(\alpha(x_1) - \alpha(x_0)) + (\alpha(x_2) - \alpha(x_1)) + ... + (\alpha(x_n) - \alpha(x_{n-1}))] \\
&= \eta(\alpha(x_n) - \alpha(x_0)) \\
&= \eta[\alpha(b) - \alpha(a)] \\
&< \epsilon
\end{aligned}$$

$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ (by Theorem 4.9)

By Theorem 4.9, $f \in \mathcal{R}(\alpha)$.

Theorem 4.12 *If f is monotonic on $[a,b]$ and if α is continuous in $[a,b]$, then $f \in \mathcal{R}(\alpha)$.*
Proof: Let
epsilon > 0 be given. For every positive integer n, we choose a partition P such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$. This is possible since α is continuous.
Case(i): f is monotonic increasing. $\therefore$ $M_i = f(x_i)$; $m_i = f(x_{i-1})$ $\forall i =$

$1, 2, ..., n$. Now,

$$\begin{aligned}
&U(P,f,\alpha) - L(P,f,\alpha)\\
&\quad = \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i\\
&\quad = \sum_{i=1}^{n}(M_i\Delta\alpha_i - m_i\Delta\alpha_i)\\
&\quad = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i\\
&\quad = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))(\frac{\alpha(b)-\alpha(a)}{n})\\
&\quad = \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]\\
&\quad = \frac{\alpha(b)-\alpha(a)}{n}\{(f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + ...\\
&\qquad + (f(x_n) - f(x_{n-1}))\}\\
&\quad = \frac{\alpha(b)-\alpha(a)}{n}[f(x_n) - f(x_0)]\\
&\quad = \frac{\alpha(b)-\alpha(a)}{n}(f(b) - f(a))\\
&\quad < \epsilon \text{ as } n \to \infty.\\
&\therefore f \in \mathcal{R}(\alpha).
\end{aligned}$$

Case(ii): f is monotonic decreasing. $\therefore M_i = f(x_i);\ m_i = f(x_{i-1})\ \forall i = 1, 2, ..., n$. Now,

$$\begin{aligned}
&U(P,f,\alpha) - L(P,f,\alpha)\\
&\quad = \sum_{i=1}^{n}(M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i)\Delta\alpha_i\\
&\quad = \sum_{i=1}^{n}(M_i\Delta\alpha_i - m_i\Delta\alpha_i)\\
&\quad = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i\\
&\quad = \sum_{i=1}^{n}(f(x_{i-1}) - f(x_i))(\frac{\alpha(b)-\alpha(a)}{n})\\
&\quad = \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}[f(x_{i-1}) - f(x_i)]
\end{aligned}$$

$$= \frac{\alpha(b) - \alpha(a)}{n}\{(f(x_0) - f(x_1)) + (f(x_1) - f(x_2)) + ...$$
$$+ (f(x_{n-1}) - f(x_n))\}$$
$$= \frac{\alpha(b) - \alpha(a)}{n}[f(x_0) - f(x_n)]$$
$$= \frac{\alpha(b) - \alpha(a)}{n}(f(a) - f(b))$$
$$< \epsilon \text{ as } n \to \infty.$$
$$\therefore f \in \mathcal{R}(\alpha).$$

Hence the proof.

Theorem 4.13 *Suppose f is bounded on $[a,b], f$ has only finitely many point of discontinuity on $[a,b]$ and α is continuous at every point at which f is discontinuous, then $f \in \mathcal{R}(\alpha)$.*
Proof: Let $\epsilon > 0$ be given. Put $M = sup|f(x)|$. Let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint $[u_j, v_j] \subset [a,b]$ such that the sum of the corresponding differences

$$\sum_j [\alpha(v_j) - \alpha(u_j)] < \epsilon.$$

Also we place these intervals in such a way that every point of $E \cap (a,b)$ lies in the interval of some $[u_j, v_j]$. Remove the segments (u_j, v_j) from $[a,b]$. The remaining set K is compact. hence f is uniformly continuous on K. $\therefore$ there exists $\delta > 0$ such that $|s-t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon \quad \forall s,t \in K$. We form a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ as follows. Each u_j occurs in P, each v_j occurs in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j's then $\Delta x_i < \delta$. we observe that $M_i - m_i \leq 2\mu$, $\forall i$ and $M_i - m_i \leq \epsilon$ unless x_{i-1} is one of the u_j's. $\therefore U(P,f,\alpha) - L(P,f,\alpha) \leq [\alpha(b) - \alpha(a)]\epsilon + 2M\epsilon$. (By Theorem 4.11) Since ϵ is arbitrary, Theorem 4.9 guarantees that $f \in \mathcal{R}(\alpha)$.

Theorem 4.14 *Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b], m \leq f \leq M, \phi$ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$, then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.*
Proof: Let $\epsilon > 0$ be given. Since $\phi : [m,M] \to R$ is continuous and $[m,M]$ is compact, ϕ is uniformly continuous. $\therefore$ There exists $\delta > 0$ such that $\delta < \epsilon, |s-t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \epsilon$ for $s,t \in [m,M]$...... (1)
Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$...... (2)
To Prove: $h \in \mathcal{R}(\alpha)$. Let $M_i^* = \sup h(x), x_{i-1} \leq x \leq x_i$ and $m_i^* = \inf h(x), x_{i-1} \leq x \leq x_i$. Let $A = \{i | 1 \leq i \leq n, M_i - m_i < \delta\}$; $B =$

$\{i|1 \leq i \leq n, M_i - m_i \geq \delta\}$

$$\begin{aligned}
\text{for } i \in A, |M_i - m_i| &< \delta \Rightarrow |\phi(M_i) - \phi(m_i)| < \epsilon \text{ (by (1))} \\
\Rightarrow |M_i^* - m_i^*| &< \epsilon ......(3) \\
\text{For } i \in B, |M_i^* - m_i^*| &\leq |M_i^*| + |m_i^*| \\
&\leq k + k \quad \text{where } k = sup|\phi(t)|, t \in [m, M] \\
|M_i^* - m_i^*| &\leq 2k.....(4) \\
\text{Also } \delta \sum_{i \in B} \Delta\alpha_i &\leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \\
&\leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i \\
&= U(P, f, \alpha) - L(P, f, \alpha) \\
&< \delta^2 \text{ (by (2))} \\
(i.e.)\ \delta \sum_{i \in B} \Delta\alpha_i &< \delta^2 \\
\Rightarrow \sum_{i \in B} \Delta\alpha_i &< \delta.....(5)
\end{aligned}$$

$$\begin{aligned}
\text{Now } U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i=1}^{n} M_i^* \Delta\alpha_i - \sum_{i=1}^{n} m_i^* \Delta\alpha_i \\
&= \sum_{i=1}^{n} (M_i^* - m_i^*)\Delta\alpha_i \\
&= \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i \\
&< \epsilon \sum_{i \in A} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i \text{ (by (3) and (4))} \\
&< \epsilon \sum_{i=1}^{n} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i \\
&< \epsilon[\alpha(b) - \alpha(a)] + 2k\delta \\
&< \epsilon[\alpha(b) - \alpha(a)] + 2k\epsilon \ (\because \delta < \epsilon) \\
&= \epsilon[\alpha(b) - \alpha(a) + 2k]
\end{aligned}$$

(i.e.) $U(P, h, \alpha) - L(P, h, \alpha) < \epsilon[\alpha(b) - \alpha(a) + 2k]$
since ϵ is arbitrary, Theorem 4.9, implies that $h \in \mathcal{R}(\alpha)$.

Lemma 4.15 *If $f \in \mathcal{R}(\alpha)$ and $f \geq 0$ on $[a, b]$ then $\int_a^b f d\alpha \geq 0$.*

Proof: Since $f \geq 0,\ M_i \geq 0 \forall_i$.

$$\therefore \sum_{i=1}^{n} M_i \Delta\alpha_i \geq 0$$
$$\Rightarrow U(P, h, \alpha) \geq 0$$
$$\Rightarrow \inf U(P, h, \alpha) \geq 0$$
$$\Rightarrow \int_a^b f d\alpha \geq 0.$$

Properties of Integral

Theorem 4.16 *(a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha), cf_1 \in \mathcal{R}(\alpha)$ for every constant c and $\int_a^b (f_1+f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha, \int_a^b cf_1 d\alpha = c\int_a^b f_1 d\alpha$.*
(b) If $f_1(x) \leq f_2(x)$ on $[a, b]$ then $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$.
(c) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and on $[a, b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$
(d) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M$ then $|\int_a^b f d\alpha| \leq [\alpha(b) - \alpha(a)]$.
(e) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ then $f \in R(\alpha_1+\alpha_2)$ and $\int_a^b f d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$. If $f \in \mathcal{R}(\alpha)$ and c is positive constant then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f d\alpha = c\int_a^b f d\alpha$.

Proof: **(a)** Let $\epsilon > 0$ be given. Since $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in [a, b]$, there exists two partitions P_1 and P_2 of $[a, b]$ such that $U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon$..... (1) and $U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon$.....(2)
Let $P = P_1 \cup P_2$ be the common refinement of $[a, b]$.

$$\therefore U(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha)$$
$$L(P_1, f_1, \alpha) \leq L(P_1, f_1, \alpha)$$
$$\Rightarrow U(P, f_1, \alpha) + L(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha) + L(P, f_1, \alpha)$$
$$\Rightarrow U(P, f_1, \alpha) - L(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha)$$
$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon \text{ (by (1))}.......(3)$$
$$\text{Similarly } U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon \text{ (by (2))}.......(4)$$

(3)+(4)$\Rightarrow$

$$\begin{aligned} U(P, f_1, \alpha) + U(P, f_2, \alpha) - (L(P, f_1, \alpha)) + L(P, f_2, \alpha) & \\ < 2\epsilon......(5) & \\ \text{Now } L(P, f_1, \alpha) + L(P, f_2, \alpha) &\leq L(P, f_1 + f_2, \alpha) \\ &\leq U(P, f_1 + f_2, \alpha) \\ &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha)......(6) \end{aligned}$$

(5),(6)$\Rightarrow U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) < 2\epsilon$. $\therefore f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$.
To prove:

$$\int_a^b (f_1 + f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Since $f_1,\ f_2 \in \mathcal{R}(\alpha)$, there exists partition P_1 and P_2 of $[a,b]$

$$U(P_1,f_1,\alpha) < \int_a^b f_1 d\alpha + \epsilon \text{ (by Theorem 4.9)}.......(1*)$$
$$U(P_2,f_2,\alpha) < \int_a^b f_2 d\alpha + \epsilon........(2*)$$

(1)+(2)⇒

$$U(P_1,f_1,\alpha) + U(P_2,f_2,\alpha) < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon......(3*)$$

Let $P = P_1 \cup P_2$

$$U(P,f_1,\alpha) \leq U(P_1,f_1,\alpha).......(4*)$$
$$U(P,f_2,\alpha) \leq U(P_2,f_2,\alpha).......(5*)$$

(4*)+(5*)⇒

$$\begin{aligned} U(P,f_1,\alpha) + U(P,f_2,\alpha) &\leq U(P_1,f_1,\alpha) + \leq U(P_2,f_2,\alpha) \\ &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon......(6*) \text{ (by (3*))} \\ U(P,f_1+f_2,\alpha) &\leq U(P,f_1,\alpha) + U(P,f_2,\alpha) \\ &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \text{ (by (6*))} \end{aligned}$$

Taking infimum over all partition P,

$$\int_a^b (f_1+f_2) d\alpha < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon$$

Since ϵ is arbitrary,

$$\int_a^b (f_1+f_2) d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.....(7*)$$

Replacing f_1 and f_2 in (7*) by $-f_1$ and $-f_2$ respectively we get,

$$\int_a^b (-f_1-f_2) d\alpha \leq \int_a^b (-f_1) d\alpha + \int_a^b (-f_2) d\alpha$$
$$\Rightarrow \int_a^b (f_1+f_2) d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.....(8*)$$

From (7*)and(8*) we get,

$$\int_a^b (f_1+f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

To Prove: $cf_1 \in \mathcal{R}(\alpha)$ where c is a constant.
For any partition P, of $[a,b]$

$$U(P,cf_1,\alpha) = \begin{cases} cU(P,f_1,\alpha) & c \geq 0 \\ cL(P,f_1,\alpha) & c \leq 0 \end{cases}$$

and

$$L(P,cf_1,\alpha) = \begin{cases} cL(P,f_1,\alpha) & c \geq 0 \\ cU(P,f_1,\alpha) & c \leq 0 \end{cases}$$

$$U(P,cf_1,\alpha) - L(P,cf_1,\alpha) = \begin{cases} c(U(P,f_1,\alpha) - L(P,f_1,\alpha)) & c \geq 0 \\ -c(U(P,f_1,\alpha) - L(P,f_1,\alpha)) & c \leq 0 \end{cases}$$

$$U(P,cf_1,\alpha) - L(P,cf_1,\alpha) = |c|(U(P,f_1,\alpha) - L(P,f_1,\alpha)).....(1A)$$

Since $f_1 \in \mathcal{R}(\alpha)$ there exists a partition P of $[a,b]$ such that

$$U(P,f_1,\alpha) - L(P,cf_1,\alpha) < \frac{\epsilon}{|c|}......(2A)$$

Sub (2A) in (1A), we get

$$\begin{aligned} U(P,cf_1,\alpha) - L(P,cf_1,\alpha) &< |c|\frac{\epsilon}{|c|} \\ U(P,cf_1,\alpha) - L(P,cf_1,\alpha) &< \epsilon \\ &\therefore cf_1 \in \mathcal{R}(\alpha). \end{aligned}$$

To Prove:

$$\overline{\int_a^b} cf_1 d\alpha = \underline{\int_a^b} cf_1 d\alpha$$

$$\begin{aligned} \text{If } c \geq 0, \text{ then } U(P,cf_1,\alpha) &= cU(P,f_1,\alpha) \\ \Rightarrow \inf U(P,cf_1,\alpha) &= \inf(cU(P,f_1,\alpha)) \\ \Rightarrow \inf U(P,cf_1,\alpha) &= c \inf U(P,cf_1,\alpha) \\ \Rightarrow \int_a^b cf_1 d\alpha &= \int_a^b cf_1 d\alpha \\ \text{If } c \leq 0, \text{ then } L(P,cf_1,\alpha) &= cU(P,f_1,\alpha) \\ &= -|c|U(P,f_1,\alpha) \ (\because c \leq 0) \\ \Rightarrow \sup L(P,cf_1,\alpha) &= \sup(-|c|U(P,f_1,\alpha)) \\ &= |c| \sup(-U(P,f_1,\alpha)) \\ &= -|c| \inf(U(P,f_1,\alpha)) \\ \Rightarrow \int_a^b cf_1 d\alpha &= -|c| \int_a^b f_1 d\alpha \\ &= c \int_a^b f_1 d\alpha \\ \text{When } c = 0, \int_a^b cf_1 d\alpha &= \int_a^b f_1 d\alpha \ (= 0) \end{aligned}$$

To Prove:

$$f_1 \leq f_2 \Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

Proof of b: Given $f_1 \leq f_2 \Rightarrow f_2 - f_1 \geq 0$

$$\begin{aligned}
&\Rightarrow \int_a^b (f_2 - f_1) d\alpha \geq 0\\
&\Rightarrow \int_a^b f_2 + \int_a^b (-f_1) d\alpha \geq 0\\
&\Rightarrow \int_a^b f_2 d\alpha + \int_a^b (-f_1) d\alpha \geq 0 \text{ (by (a))}\\
&\Rightarrow \int_a^b f_2 d\alpha - \int_a^b f_1 d\alpha \geq 0\\
&\Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha
\end{aligned}$$

Proof of (c): Given $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$ for $\epsilon < 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon ......(1B)$$

Let $P^* = P \cup \{c\}$. Now P^* is a refinement of P and induces two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively. Now,

$$\begin{aligned}
U(P, f, \alpha) &\geq U(P^*, f, \alpha)\\
&= U(P_1, f, \alpha) + U(P_2, f, \alpha).....(2B)\\
\Rightarrow U(P_1, f, \alpha) &\leq U(P, f, \alpha)......(3B)\\
\text{and } \ U(P_2, f, \alpha) &\leq U(P, f, \alpha).......(4B)\\
L(P, f, \alpha) &\leq L(P^*, f, \alpha)\\
&= L(P_1, f, \alpha) + L(P_2, f, \alpha).......(5B)\\
-L(P, f, \alpha) &\geq -L(P_1, f, \alpha) - L(P_2, f, \alpha)\\
-L(P_1, f, \alpha) &\leq -L(P, f, \alpha)......(6B)\\
\text{and } - L(P_2, f, \alpha) &\leq -L(P, f, \alpha).......(7B)\\
(3B) + (6B) \Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1B))}\\
&< \epsilon\\
&\therefore f \in \mathcal{R}(\alpha) \text{ on } [a, c].\\
(4B) + (7B) \Rightarrow U(P_2, f, \alpha) - L(P_2, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1B))}\\
&< \epsilon\\
&\therefore f \in \mathcal{R}(\alpha) \text{ on } [c, b].
\end{aligned}$$

To Prove:

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

$$
\begin{aligned}
(2B) \Rightarrow U(P,f,\alpha) &\geq U(P_1,f,\alpha)+U(P_2,f,\alpha)\\
&\geq \int_a^c f d\alpha + \int_c^b f d\alpha\\
\Rightarrow \inf U(P,f,\alpha) &\geq \int_a^c f d\alpha + \int_c^b f d\alpha\\
\int_a^b f d\alpha &\geq \int_a^c f d\alpha + \int_c^b f d\alpha ......(8B)\\
(5B) \Rightarrow L(P,f,\alpha) &\leq L(P_1,f,\alpha)+L(P_2,f,\alpha)\\
&\leq \int_a^c f d\alpha + \int_c^b f d\alpha\\
\Rightarrow \sup U(P,f,\alpha) &\leq \int_a^c f d\alpha + \int_c^b f d\alpha\\
\int_a^b f d\alpha &\leq \int_a^c f d\alpha + \int_c^b f d\alpha ....(9B)
\end{aligned}
$$

$\therefore$ (8B) and (9B), we get

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

Proof of (d): Given $f \in \mathcal{R}(\alpha)$ and $|f(x)| \leq M$
To Prove: $|\int_a^b f d\alpha| \leq [\alpha(b) - \alpha(a)]$
we have, for any partition P of $[a,b]$,

$$
\begin{aligned}
\int_a^b f d\alpha &\leq U(P,f,\alpha)\\
\left|\int_a^b f d\alpha\right| &\leq |U(P,f,\alpha)|\\
&= \left|\sum_{i=1}^n M_i \Delta\alpha_i\right|\\
&< \sum_{i=1}^n |M_i \Delta\alpha_i|\\
&= \sum_{i=1}^n |M_i| \Delta\alpha_i \ (\because \Delta\alpha_i \geq 0)\\
&\leq \sum_{i=1}^n M \Delta\alpha_i \ (\because |f(x)| \leq M)\\
&= M\sum_{i=1}^n \Delta\alpha_i\\
\left|\int_a^b f d\alpha\right| &\leq M[\alpha(b) - \alpha(a)]
\end{aligned}
$$

Proof of (e): Given $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$. To Prove: $f \in \mathcal{R}(\alpha_1 + \alpha_2)$.

Let $\alpha = \alpha_1 + \alpha_2$. For any partition p of $[a, b]$,

$$\begin{aligned}
U(P, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} M_i[(\alpha_1 + \alpha_2)(x_i) - (\alpha_1 + \alpha_2)(x_{i-1})] \\
&= \sum_{i=1}^{n} M_i[\alpha_1(x_i) + \alpha_2(x_i)] - [\alpha_1(x_{i-1}) + \alpha_2(x_{i-1})] \\
&= \sum_{i=1}^{n} M_i[\alpha_1(x_i) - \alpha_1(x_{i-1})] + \sum_{i=1}^{n} M_i[\alpha_2(x_i) - \alpha_2(x_{i-1})] \\
U(P, f, \alpha) &= U(P, f, \alpha_1) + U(P, f, \alpha_2) ....(1C)
\end{aligned}$$

Similarly $L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2) ....(2C)$

since $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, there exists partitions P_1 and P_2 of $[a, b]$ such that

$$\begin{aligned}
U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) &< \epsilon \\
\text{and } U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) &< \epsilon
\end{aligned}$$

Let P^* be the common refinement of P_1 and P_2 of $[a, b]$. $P^* = P_1 \cup P_2$

$$\begin{aligned}
&U(P^*, f, \alpha_1) - L(P^*, f, \alpha_1) < \epsilon ....(3C) \\
&U(P^*, f, \alpha_2) - L(P^*, f, \alpha_2) < \epsilon ....(4C) \text{ (by Theorem 4.10)}
\end{aligned}$$

Now,

$$\begin{aligned}
U(P^*, f, \alpha) - L(P^*, f, \alpha) &= U(P^*, f, \alpha_1) + U(P^*, f, \alpha_2) \\
&\quad - [L(P^*, f, \alpha_1) + L(P^*, f, \alpha_2)] \text{ (by (1C) and (2C))} \\
&= [U(P^*, f, \alpha_1) - L(P^*, f, \alpha_1)] \\
&\quad + [U(P^*, f, \alpha_2) - L(P^*, f, \alpha_2)] \\
&< \epsilon + \epsilon \text{ (by (3C) and (4C))} \\
U(P^*, f, \alpha) - L(P^*, f, \alpha) &< 2\epsilon.
\end{aligned}$$

Since ϵ arbitrary, we get $f \in \mathcal{R}(\alpha)$ (i.e.) $f \in \mathcal{R}(\alpha_1 + \alpha_2)$.
To Prove:

$$\int_a^b d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\begin{aligned}
(1C) \Rightarrow U(P,f,\alpha) &= U(P,f,\alpha_1) + U(P,f,\alpha_2) \\
&\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\Rightarrow \inf U(P,f,\alpha) &\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\int_a^b f d\alpha &\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 ......(5C) \\
(2C) \Rightarrow L(P,f,\alpha) &= L(P,f,\alpha_1) + L(P,f,\alpha_2) \\
&\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\sup U(P,f,\alpha) &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\int_a^b f d\alpha &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 ......(6C)
\end{aligned}$$

from (5C) and (6C) we get,

$$\begin{aligned}
\int_a^b f d\alpha &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
(i.e.) \int_a^b d(\alpha_1 + \alpha_2) &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.
\end{aligned}$$

To Prove: Given $f \in \mathcal{R}(\alpha)$ and $c > 0$
To Prove: $f \in \mathcal{R}(\alpha)$, for any partition P,

$$\begin{aligned}
U(P,f,c\alpha) &= \sum_{i=1}^{n} M_i \Delta(c\alpha_i) \\
&= \sum_{i=1}^{n} M_i (c\alpha(x_i) - c\alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} M_i c[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \sum_{i=1}^{n} c M_i \Delta\alpha_i \\
&= cU(P,f,\alpha) ....(7C) \\
\text{Similarly } L(P,f,c\alpha) &= cL(P,f,\alpha) \\
U(P,f,c\alpha) - L(P,f,c\alpha) &= cU(P,f,\alpha) - cL(P,f,\alpha) \\
&= c[U(P,f,\alpha) - L(P,f,\alpha)] ......(8C)
\end{aligned}$$

Since $f \in \mathcal{R}(\alpha)$, given $\epsilon > 0$, there exists partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{c} ....(9C)$$

sub (9C)in (8C) we get

$$U(P, f, c\alpha) - L(P, f, c\alpha) < c \cdot \frac{\epsilon}{c} = \epsilon$$

$\therefore f \in \mathcal{R}(c\alpha)$. To Prove:

$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha$$

$$\begin{aligned}
(7C) \Rightarrow U(P, f, c\alpha) &= cU(P, f, \alpha)\\
\Rightarrow \inf U(P, f, c\alpha) &= \inf cU(P, f, \alpha)\\
&= c \inf U(P, f, \alpha)\\
\Rightarrow \int_a^b f d(c\alpha) &= c \int_a^b f d\alpha
\end{aligned}$$

Theorem 4.17 *If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$,then*
(a) $f \cdot g \in \mathcal{R}(\alpha)$
(b) $|f| \in \mathcal{R}(\alpha)$ and

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha.$$

Proof: **(a)** Let $\phi(t) = t^2$, clearly ϕ is continuous

$$\begin{aligned}
h(x) &= \phi(f(x)) \text{ (by Theorem 4.14)}\\
&= f(x)^2\\
&= f^2(x)\\
\therefore f^2 &\in \mathcal{R}(\alpha) ....(1) \ (\because f \in \mathcal{R}(\alpha))\\
\text{Now,} f, g &\in \mathcal{R}(\alpha)\\
\Rightarrow f + g, f - g &\in \mathcal{R}(\alpha) \text{ (by Theorem 4.16)}\\
\Rightarrow (f + g)^2, (f - g)^2 &\in \mathcal{R}(\alpha)\\
\Rightarrow (f + g)^2 - (f - g)^2 &\in \mathcal{R}(\alpha)\\
\Rightarrow 4fg &\in \mathcal{R}(\alpha)\\
\Rightarrow fg &\in \mathcal{R}(\alpha) \text{ (by Theorem 4.16)}
\end{aligned}$$

(b) $|f| \in \mathcal{R}(\alpha)$ and $| \int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$.
To Prove: $|f| \in \mathcal{R}(\alpha)$. Let $\phi(t) = |t|$; $h(x) = \phi(f(x)) = |f(x)|$. $\therefore$ By Theorem 4.14, $|f| \in \mathcal{R}(\alpha)$
To prove:

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha.$$

Choose $c = \pm 1$ so that $c\int_a^b f d\alpha \geq 0$

$$\begin{aligned}
\therefore |\int_a^b f d\alpha| &= c\int_a^b f d\alpha \\
&= \int_a^b cf d\alpha \text{ (by Theorem 4.16(a))} \\
&\leq \int_a^b |f| d\alpha \ (\because cf \leq |f|) \text{ by Theorem 4.16(b)}
\end{aligned}$$

Hence the proof.

Definition 4.18 ***Unit Step Function:***

$$I(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x > o \end{cases}$$

Theorem 4.19 *If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x - s)$, then*

$$\int_a^b f d\alpha = f(s).$$

Proof: Consider partitions $P = \{x_0, x_1, x_2, x_b\}$ of $[a,b]$ where $x_0 x_1 = s, s < x_2 < b, x_2 = b$. Now,

$$\begin{aligned}
U(P,f,\alpha) &= \sum_{i=1}^{3} M_i \Delta\alpha_i \\
&= M_i\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3 \\
&= M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)] + M_3[\alpha(x_3) - \alpha(x_2)] \\
&= M_1[I(x_1 - s) - I(x_0 - s)] + M_2[I(x_2 - s) - I(x_1 - s)] \\
&\quad + M_3[I(x_3 - s) - I(x_2 - s)] \\
&= M_1[I(s - s) - I(a - s)] + M_2[I(x_2 - s) - I(s - s)] \\
&\quad + M_3[I(b - s) - I(x_2 - s)] \\
&= M_1[I(0) - I(a - s)] + M_2[I(x_2 - s) - I(0)] \\
&\quad + M_3[I(b - s) - I(x_2 - s)] \\
&= M_1[0 - 0] + M_2[1 - 0] + M_3[1 - 1] \text{ (by definition of } i) \\
&= M_2
\end{aligned}$$

In a similar fashion we can get $L(P,f,\alpha) = m_2$.

$$\begin{aligned}
\int_a^b f d\alpha &= \inf U(P,f,\alpha) = \sup L(P,f,\alpha) \\
&= \inf M_2 = \sup m_2 \\
&= f(s) \ (\because x_2 \to s, f(x_2) \to f(x) \text{ as } f \text{ is continuous at } s)
\end{aligned}$$

Theorem 4.20 *Suppose* $c_n \geq 0$ *for* $1, 2, 3...$, $\sum c_n$ *converges,* $\{s_n\}$ *is a sequence of distinct point in* (a, b) *and* $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. *Let* f *be continuous on* $[a, b]$, *then*

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof: We have $|I(x - s_n)| \leq 1$. $\therefore |c_n I(x - s_n)| \leq c_n$. Since

$$\sum_{n=1}^{\infty} c_n$$

is convergent, by comparison test,

$$\sum_{n=1}^{\infty} c_n I(x - s_n)$$

also converges. Now,

$$\begin{aligned}
\alpha(a) &= \sum_{n=1}^{\infty} c_n I(a - s_n) \\
&= 0 ....(1) \ (\because I(a - s_n) = 0) \\
\text{and } \alpha(b) &= \sum_{n=1}^{\infty} c_n I(b - s_n) \\
&= \sum_{n=1}^{\infty} c_n .....(2) \ (\because I(b - s_n) = 0)
\end{aligned}$$

Claim: α is monotonically increasing. Let $x < y$ and let $x < s_k < y$

$$\begin{aligned}
\alpha(x) &= \sum_{n=1}^{\infty} c_n I(x - s_n) \\
&= c_1 + c_2 + ... + c_{k-1} \\
\alpha(y) &= \sum_{n=1}^{\infty} c_n I(y - s_n) \\
&= c_1 + c_2 + ... + c_{k-1} + c_k \\
\therefore \alpha(x) &\leq \alpha(y)
\end{aligned}$$

Hence the claim. Since

$$\sum_{n=1}^{\infty} c_n$$

is convergent, given $\epsilon > 0$, there exists $N >$ such that

$$\sum_{n=N+1}^{\infty} c_n < \epsilon ......(3)$$

Let

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$$
$$\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$$

Clearly $\alpha(x) = \alpha_1(x) + \alpha_2(x)$. Let $\alpha_{1i} = I(x - s_i), i = 1, 2, ..., N$.

$$\begin{aligned} \therefore \alpha_1(x) &= \sum_{n=1}^{N} c_n \alpha_{1n}(x) \\ &= (c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N})x \\ \text{(or) } \alpha_1 &= c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N} \end{aligned}$$

Now,

$$\begin{aligned} \int_a^b f d\alpha_1 &= \int_a^b f d(c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N}) \\ &= c_1 \int_a^b f d\alpha_{11} + c_2 \int_a^b f d\alpha_{12} + ...c_N \int_a^b f d\alpha_{1N} \text{ (by Theorem 4.16(e))} \\ &= c_1 f(s_1) + c_2 f(s_2) + ... + c_N f(s_N) \text{ (by Theorem 4.19)} \\ &= \sum_{n=1}^{N} c_n f(s_n) ....(4) \end{aligned}$$

Now,

$$\begin{aligned} \alpha_2(a) &= \sum_{n=N+1}^{\infty} c_n I(a - s_n) \\ &= 0........(5) \\ \alpha_2(b) &= \sum_{n=N+1}^{\infty} c_n I(b - s_n) \\ &= \sum_{n=N+1}^{\infty} c_n \\ &< \epsilon \text{ (by (3))......(6)} \end{aligned}$$

Let $M = |f(x)|, x \in [a,b]$. By Theorem 4.16(d),

$$\begin{aligned} \left|\int_a^b f d\alpha_2\right| &\leq [\alpha_2(b) - \alpha_2(a)] \\ &\leq M\epsilon \text{ (by (5)and(6))}, \\ (i.e.) \left|\int_a^b f d\alpha_2\right| &\leq M\epsilon \\ \Rightarrow \left|\int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 - \int_a^b f d\alpha_1\right| &\leq M\epsilon \end{aligned}$$

$$\begin{aligned} \Rightarrow \left|\int_a^b f d(\alpha_1 + \alpha_2) - \int_a^b f d\alpha_1\right| &\leq M\epsilon \text{ (by theorem 4.16(d))} \\ \Rightarrow \left|\int_a^b f d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right| &\leq M\epsilon \text{ (by (4))} \end{aligned}$$

Taking limits as $N \to \infty$,

$$\begin{aligned} \left|\int_a^b f d\alpha - \sum_{n=1}^{\infty} c_n f(s_n)\right| &\leq M\epsilon \\ \therefore \left|\int_a^b f d\alpha\epsilon\right| &= \sum_{n=1}^{\infty} c_n f(s_n) \end{aligned}$$

Theorem 4.21 *Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a,b]$, Let f be a bounded real function on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}$. In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.*

Proof: Let $\epsilon > 0$ be given. Since $\alpha' \in R$, there exists a partition $P = \{x_1, x_2, ..., x_n\}$ of $[a,b]$ such that $U(P,\alpha') - L(P,\alpha') < \epsilon$........ (1)

By mean value theorem , there exists $t :\in [x_{i-1}, x_i]$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$ (i.e.) $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$..... (2)

By Theorem 4.10(b), $\forall s_i, t_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon......(3)$$

Now,

$$\begin{aligned}
&\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \\
&\quad = \left|\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \\
&\quad = \left|\sum_{i=1}^{n} f(s_i)[\alpha'(t_i) - \alpha'(s_i)]\Delta x_i\right| \\
&\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \\
&\quad \leq \sum_{i=1}^{n} |f(s_i)||\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \\
&\quad \leq \sum_{i=1}^{n} M|\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \quad \text{where } M = \sup|f(x)| \\
&\quad = M\sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \\
&\quad \leq M\epsilon \text{ (by (3))} \\
&(i.e.)\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \leq M\epsilon \\
&\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(\alpha')(s_i)\Delta x_i\right| \leq M\epsilon.....(4)
\end{aligned}$$

Since inequality (4) is true for any s_i in $[x_{i-1}, x_i]$, we can replace $(f\alpha')(s_i)$ by M_i' and m_i', where $m_i' = \inf(f\alpha')s_i,\ M_i' = \sup(f\alpha')(s_i), s_i \in [x_{i-1}, x_i]$

$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} M_i'\Delta x_i\right| \leq M\epsilon.......(5)$$

$$\text{and } \left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} m_i'\Delta x_i\right| \leq M\epsilon......(6)$$

Again by replacing $f(s_i)$ by M_i in (5) and by m_i in (6)
we get

$$\begin{aligned}
&\left|\sum_{i=1}^{n} M_i'\Delta\alpha_i - \sum_{i=1}^{n} M_i'\Delta x_i\right| \leq M\epsilon \text{ and} \\
&\left|\sum_{i=1}^{n} m_i'\Delta\alpha_i - \sum_{i=1}^{n} m_i'\Delta x_i\right| \leq M\epsilon \\
\Rightarrow\ &|U(P, f, \alpha) - U(P, f, \alpha')| \leq M\epsilon......(7) \text{ and} \\
&|L(P, f, \alpha) - L(P, f, \alpha')| \leq M\epsilon.......(8)
\end{aligned}$$

Since ϵ is arbitrary, (7) and (8)

$$\begin{aligned}
\Rightarrow U(P, f, \alpha) &= U(P, f, \alpha') \text{ and} \\
L(P, f, \alpha) &= L(P, f, \alpha') \\
\Rightarrow \inf U(P, f, \alpha) &= \inf U(P, f, \alpha') \text{ and} \\
\sup L(P, f, \alpha) &= \sup L(P, f, \alpha') \\
\Rightarrow \int_a^{\bar{b}} f d\alpha &= \int_a^{\bar{b}} (f\alpha') d\alpha ....(9) \text{ and} \\
\int_{\underline{a}}^{b} f d\alpha &= \int_{\underline{a}}^{b} (f\alpha') d\alpha ....(10) \\
\therefore f \in \mathcal{R}(\alpha) \Leftrightarrow \int_{\underline{a}}^{b} f d\alpha &= \int_a^{\bar{b}} f d\alpha \\
\Leftrightarrow \int_{\underline{a}}^{b} (f\alpha') d\alpha &= \int_a^{\bar{b}} (f\alpha') d\alpha \text{ (by (9) and (10))} \\
\Leftrightarrow f(\alpha') &\in \mathcal{R}. \\
\text{Now, } \int_a^b f d\alpha &= \int_a^{\bar{b}} f d\alpha \\
&= \int_a^{\bar{b}} (f\alpha') dx \ (by (9)) \\
&= \int_a^b (f\alpha') dx \\
&= \int_a^b f(x)\alpha'(x) dx \\
\therefore \int_a^b f d\alpha &= \int_a^b f(x)\alpha'(x) dx
\end{aligned}$$

Remark 4.22 *The above theorem gives the relation of $\mathcal{R}$ integral and $\mathcal{R}(\alpha)$ integral.*

Theorem 4.23 ***Change of Variable:*** *Suppose ϕ is a strictly increasing function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\phi(y)), g(y) = f(\phi(y))$, then $g \in \mathcal{R}(\beta)$ and $\int_A^B gd(\beta) = \int_a^b f d\alpha$.*
Proof: $g(y) = (f \cdot \phi)x = f(\phi(y)) = f(x)$

$$[A, B] \xrightarrow{\phi} [a, b] \xrightarrow{f} \mathcal{R}$$

$$[A, B] \xrightarrow{\phi} [a, b] \xrightarrow{\alpha} \mathcal{R}$$

$$\begin{aligned}
\beta(y) &= (\alpha \cdot \phi)y \\
&= \alpha(\phi(y)) \\
&= \alpha(x)
\end{aligned}$$

Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be any partition of $[a, b]$. Since ϕ is onto for each i, there exists $y_i \in [A, B]$ such that $\phi(y_i) = x_i,\ i = 0, 1, 2, ..., n$. $\therefore$ $\{y_0, y_1, y_2, ..., y_n\}$ is a partition of $[A, B]$ every partition of $[A, B]$ can be obtained in this way (since ϕ is monotonically increasing)

$$\begin{aligned}
\text{For } y &\in [y_{i-1}, y_i] \\
g(y) &= (f \cdot \phi)y \\
g(y) &= f(\phi(y)) \\
&= f(x) \text{ where } x = \phi(y),\ x \in [x_{i-1}, x_i] \\
\Rightarrow \sup g(y) &= \sup f(x) \\
\Rightarrow M_{i'} &= M_i ....(1) \\
\text{Similarly } \inf g(y) &= \inf f(x) \\
m_{i'} &= m_i ....(2) \\
\text{Now } \Delta\beta_i &= \beta(y_i) - \beta(y_{i-1}) \\
&= (\alpha \circ \phi)y_i - (\alpha \circ \phi)y_{i-1} \\
&= \alpha(\phi(y_i)) - \alpha(\phi(y_{i-1})) \\
&= \alpha(x_i) - \alpha(x_{i-1}) \\
&= \Delta\alpha_i ......(3) \\
\therefore U(Q, g, \beta) &= \sum_{i=1}^{n} M_i' \Delta\beta_i \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i \text{ (by (1) and (3))} \\
&= U(P, f, \alpha) ......(4) \\
\text{Similarly } L(Q, g, \beta) &= L(P, f, \alpha) ......(5)
\end{aligned}$$

Since $f \in \mathcal{R}(\alpha)$, given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\begin{aligned}
U(P, f, \alpha) - L(P, f, \alpha) &< \epsilon \\
\Rightarrow U(Q, g, \beta) - L(Q, g, \beta) &< \epsilon \text{ (by (4) and (5))} \\
\therefore g &\in \mathcal{R}(\beta)
\end{aligned}$$

$$\begin{aligned}
\text{Also } \int_A^B g d\beta &= \inf U(Q, g, \beta) \\
&= \inf U(P, f, \alpha) \text{ (by (4))} \\
&= \int_a^b f d\alpha.
\end{aligned}$$

Note 4.24 *Let $\alpha(x) = x$ and $\phi' \in \mathcal{R}$ on $[A, B]$.*

$$\begin{aligned}
\therefore \beta(y) &= (\alpha \circ \phi)y, \\
&= \alpha(\phi(y)) \\
&= \phi(y)\ \forall y \in [A, B] \\
\therefore \beta &= \phi \\
\int_A^B g d\beta &= \int_a^b f d\alpha \text{ (by previous theorem)} \\
\int_a^b f(x)dx &= \int_A^B g d\beta \\
&= \int_A^B g d\phi \\
&= \int_A^B g(y)\phi'(y)dy \text{ (by theorem 4.21)}
\end{aligned}$$

Integrations and Differentiations:

Theorem 4.25 *Let $f \in R$ on $[a, b]$, for $a \leq x \leq b$, put $F(x) = \int_a^x f(t)dt$, then F is continuous on $[a, b]$, further more if f is continuous at some point x_0 of $[a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof: Given $F(x) = \int_a^x f(t)dt$. To Prove: $F(x)$ is continuous on $[a, b]$. Let $a \leq x \leq y \leq b$. Now,

$$\begin{aligned}
F(y) - F(x) &= \int_a^y f(t)dt - \int_a^x f(t)dt \\
&= \int_a^x f(t)dt + \int_x^y f(t)dt - \int_a^x f(t)dt \\
&= \int_x^y f(t)dt \\
\Rightarrow |F(y) - F(x)| &= |\int_x^y f(t)dt| \\
&\leq \int_x^y |f(t)|dt \\
&\leq \int_x^y M dt \text{ where } M = \sup |f(t)|,\ t \in [a, b] \\
&= M(y - x) \\
(i.e.)\ |F(y) - F(x)| &\leq M|y - x|\ (\because (y - x) = 0)
\end{aligned}$$

Given $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{M}$ such that $|y - x| < \delta \Rightarrow |F(y) - F(x)| < \epsilon$ (i.e.) F is continuous on $[a, b]$. (infact F is uniformly continuous on $[a, b]$). Suppose f is continuous at $x_0 \in [a, b]$. To Prove: $F'(x_0) = f(x_0)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \epsilon$ for $t \in [a, b]$...... (1)

Let $x_0 - \delta < s \leq x_0 \leq t \leq x_0 + \delta$. Now,

$$\begin{aligned}
F(t) - F(s) &= \int_a^t f(t)dt - \int_a^s f(t)dt \\
&= \int_a^s f(t)dt + \int_s^t f(t)dt - \int_a^s f(t)dt \\
F(t) - F(s) &= \int_s^t f(t)dt \\
\Rightarrow \frac{F(t) - F(s)}{t - s} &= \frac{1}{t - s}\int_s^t f(t)dt \\
\Rightarrow \frac{F(t) - F(s)}{t - s} - f(x_0) &= \frac{1}{t - s}\int_s^t f(t)dt - f(x_0) \\
\frac{F(t) - F(s)}{t - s} - f(x_0) &= \frac{1}{t - s}\{\int_s^t f(t)dt - (t - s)f(x_0)\} \\
&= \frac{1}{t - s}\{\int_s^t f(t)dt - \int_s^t f(x_0)dt\} \\
&= \frac{1}{t - s}\int_s^t (f(t) - f(x_0))dt \\
\left|\frac{F(t) - F(s)}{t - s} - f(x_0)\right| &= \left|\frac{1}{t - s}\int_s^t (f(t) - f(x_0))dt\right| \\
&\leq \frac{1}{t - s}\int_s^t |f(t) - f(x_0)|dt \\
&< \frac{\in}{t - s}\int_s^t dt \text{ (by (1))} \\
\left|\frac{F(t) - F(s)}{t - s} - f(x_0)\right| &< \epsilon
\end{aligned}$$

It follows that $F'(x_0) = f(x_0)$.

Theorem 4.26 ***The Fundamental Theorem of Calculus:*** *If $f \in R$ on $[a, b]$ and if there is a differentiable function F such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof: Since $f \in R$ on $[a, b]$, given $\in 0$, there exists a partition $P = \{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$...... (1)

Since F is differentiable we can apply the mean value theorem to it on $[x_{i-1}, x_i]$. There exists $t_i \in [x_{i-1}, x_i]$ such that

$$\begin{aligned}
F(x_i) - F(x_{i-1}) &= (x_{i-1} - x_i)F'(t_i) \\
&= \Delta x_i f(t_i) \ (\because F' = f)
\end{aligned}$$

Summing over i, we get,

$$\sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n}\Delta x_i f(t_i)$$

$$F(b) - F(a) = \sum_{i=1}^{n} f(t_i)\Delta x_i ....(2)$$

By Theorem 4.10(c), (1) implies that

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < \epsilon.......(3)$$

Using (2) and (3) we get, $|(F(b)-F(a))-\int_a^b f(x)dx| < \epsilon$. Since ϵ is arbitrary, $\int_a^b f(x)dx = F(b) - F(a)$. Hence the proof.

Theorem 4.27 ***Integration by parts:*** *Suppose F and G are differentiable functions on $[a,b], F' = f \in \mathcal{R}, G' = g \in \mathcal{R}$, then*

$$\int_a^b f(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Proof: Let $H(x) = F(x)G(x)$. $\therefore H'(x) = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + f(x)G(x)$....... (1)
Given f and $g \in \mathcal{R}$. Since F and G are differentiable, they are continuous. $\therefore$ By Theorem 4.11, F and G are integrable ($\in \mathcal{R}$). $\therefore$ By Theorem 4.16 $F(x)g(x) + f(x)G(x) \in \mathcal{R}$ (i.e.) $H'(x) \in R$. By fundamental theorem of calculus,

$$\int_a^b H'(x)dx = H(b) - H(a)$$

$$(i.e.) \int_a^b (F(x)g(x) + f(x)G(x))dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Hence the proof.

Definition 4.28 ***Integration of vector valued functions:*** *Let $f_1, f_2, ..., f_k$ be real functions on $[a,b]$ and let $\bar{f} = (f_1, f_2, ..., f_k)$ be a mapping of $[a,b] \to \mathbb{R}^k$. Suppose α increases monotonically on $[a,b]$, then $\bar{f} \in \mathcal{R}(\alpha) \Leftrightarrow$ for each $f_i \in \mathcal{R}(\alpha)$, and in this case*

$$\int_a^b \bar{f}d\alpha = (\int_a^b f_1 d\alpha, \int_a^b f_2 d\alpha, ..., \int_a^b f_k d\alpha)$$

Theorem 4.29 ***Fundamental Theorem of calculus for vector valued functions:*** *If $\bar{F}, \bar{f}$ map $[a,b]$ into $\mathbb{R}^k$ and if $\bar{f} \in \mathcal{R}$ on $[a,b]$ and if $\bar{F}' = \bar{f}$ then $\int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$.*
Proof: Let

$$\bar{f} = (f_1, f_2, ..., f_k)$$
$$\bar{F} = (F_1, F_2, ..., F_k)$$
$$\bar{F}' = (F_1', F_2', ..., F_k')$$

Given $\bar{F}' = \bar{f}$. $\therefore (F_1', F_2', ..., F_k') = (f_1, f_2, ..., f_k) \Rightarrow F_i' = f_i \quad \forall i = 1, 2, ..., k$. Since $\bar{f} \in \mathcal{R}$, each $f_i \in \mathcal{R}$. $\therefore$ By fundamental theorem of calculus, for any i.

$$\int_a^b F_i'(t)dt = F_i(b) - F_i(a)$$

$$(i.e.) \int_a^b f_i(t)dt = F_i(b) - F_i(a)........(1)$$

Now,

$$\begin{aligned}
\int_a^b \bar{f}(t)dt &= \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, ..., \int_a^b f_k(t)dt\right) \text{ (by definition)} \\
(1) \Rightarrow &= (F_1(b) - F_1(a), F_2(b) - F_2(a), ..., F_k(b) - F_k(a)) \\
&= (F_1(b), F_2(b), ..., F_k(b)) - (F_1(a), F_2(a), ..., F_k(a)) \\
&= \bar{F}(b) - \bar{F}(a)
\end{aligned}$$

$$\therefore \int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$$

Note 4.30 ***Schwartz inequality:***

$$\left|\sum_{j=1}^n a_j \bar{b_j}\right|^2 \leq \left(\sum_{j=1}^n |a_j|^2\right)\left(\sum_{j=1}^n |b_j|^2\right) \quad \textit{(or)}$$

$$\left|\sum_{j=1}^n a_j \bar{b_j}\right| \leq \left(\sum_{j=1}^n |a_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^n |b_j|^2\right)^{\frac{1}{2}}$$

Theorem 4.31 *If $\bar{f}$ maps $[a, b]$ into $\mathbb{R}^k$ and if $\bar{f} \in \mathcal{R}(\alpha)$ for some monotonically increasing function $[a, b]$, then $|\bar{f}| \in \mathcal{R}(\alpha)$ and $|\int_a^b \bar{f}(t)d\alpha| \leq \int_a^b |\bar{f}(t)|d\alpha$.*
Proof:

$$\begin{aligned}
\bar{f} &= (f_1, f_2, ..., f_k) \\
|\bar{f}| &= (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2)^{1/2} \\
\text{Since } \bar{f} &\in \mathcal{R}(\alpha) \\
\Rightarrow f_i &\in \mathcal{R}(\alpha) \ \forall i = 1, 2, ..., k \\
\Rightarrow f_i^2 &\in \mathcal{R}(\alpha) \\
\Rightarrow (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2) &\in \mathcal{R}(\alpha) \\
\Rightarrow (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2)^2 &\in \mathcal{R}(\alpha) \text{(by Theorem 4.17}, \phi(t) = t^{1/2}) \\
\Rightarrow |\bar{f}| &\in \mathcal{R}(\alpha)
\end{aligned}$$

To Prove:

$$\left|\int_a^b \bar{f}(t)d\alpha\right| \leq \int_a^b |\bar{f}(t)|d\alpha$$

Let $\bar{y} = \int_a^b \bar{f}(t)d\alpha$. If $\bar{y} = 0$, then the inequality is trivial (for, $\bar{y} = 0 \Rightarrow$ L.H.S=0 and $|\bar{f}| \geq 0 \Rightarrow \int_a^b |\bar{f}(t)|d\alpha \geq 0$ (i.e.) R.H.S ≥ 0)
Let $\bar{y} \neq 0$

$$\therefore \bar{y} = \int_a^b \bar{f}d\alpha = \left(\int_a^b f_1 d\alpha, \int_a^b f_2 d\alpha, ..., \int_a^b f_k d\alpha\right)$$

$$= (y_1, y_2, ..., y_k) \text{ where } y_i = \int_a^b f_i d\alpha$$

$$\text{Now } |\bar{y}|^2 = y_1^2 + y_2^2 + ... + y_k^2$$

$$(i.e.)\ |\bar{y}|^2 = \sum_{i=1}^k y_i^2$$

$$= \sum_{i=1}^k y_i y_i$$

$$= \sum_{i=1}^k y_i (\int_a^b f_i d\alpha)$$

$$= \sum_{i=1}^k \int_a^b (y_i f_i) d\alpha$$

$$= \int_a^b (\sum_{i=1}^k y_i f_i) d\alpha$$

$$\leq \int_a^b \left(\sum_{i=1}^k |y_i|^2\right)^{1/2} \left(\sum_{i=1}^k |f_i|^2\right)^{1/2} d\alpha \text{ (by schwartz inequality)}$$

$$(i.e.)\ |\bar{y}|^2 \leq \int_a^b \left(\sum_{i=1}^k y_i^2\right)^{1/2} \left(\sum_{i=1}^k f_i^2\right)^{1/2} d\alpha$$

$$= \int_a^b |\bar{y}||\bar{f}|d\alpha$$

$$= |\bar{y}| \int_a^b |\bar{f}|d\alpha$$

$$(i.e.)\ |\bar{y}|^2 \leq |\bar{y}| \int_a^b |\bar{f}|d\alpha$$

$$\Rightarrow |\bar{y}| \leq \int_a^b |\bar{f}|d\alpha$$

$$\left|\int_a^b \bar{f}d\alpha\right| \leq \int_a^b |\bar{f}|d\alpha$$

Uniform Convergence:

Definition 4.32 ***Uniform Convergence:*** *We say that $\{f_n\}$ of function $n = 1, 2, ...$ converges uniformly on E to a function f is every $\epsilon > 0$ there is an integer N such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$.*

Note 4.33 *If* $\{f_n\}$ *converges pointwise on* E*, then there exists a function* f *such that for every* $\epsilon > 0$ *and for every* x *in* E *there is an integer* N *depending on* ϵ *and* x *such that* $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$. *If* $\{f_n\}$ *converges uniformly on* E*, it is possible for each* $\epsilon > 0$*, to find one integer* N *which will do for all* x *in* E. *We say that the series* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* E *if the* $\{s_n\}$ *of partial sums defined by* $s_n(x) = \sum_{i=1}^{n} f_i(x)$ *converges uniformly on* E.

Theorem 4.34 ***Cauchy's Criterian for Uniform Convergence:*** *The sequence of functions* $\{f_n\}$*, defined on* E*, converges uniformly on* E *iff for every* $\epsilon > 0$ *there exists an integer* N *such that* $n, m \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| < \epsilon$.

Proof: For the 'only if' part we assume that $\{f_n\} \to f$ uniformly. To Prove: There exists N such that $x \in E \;\; n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \epsilon$. Let $\epsilon > 0$ such that $|f_n(x) - f(x)| \leq \epsilon/2$...... (1) $\quad \forall n \geq N \;\; \forall x \in E$

Now, for $n, m \geq N$

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\
&\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\
&\leq \epsilon/2 + \epsilon/2 \text{ (by (1))} \\
(i.e.)\ |f_n(x) - f_m(x)| &\leq \epsilon
\end{aligned}$$

For the $'if'$ part we assume that there exists $N > 0$ such that $n, m \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| \leq \epsilon$........ (2)

For fixed x, (2) implies that $\{f_n(x)\}$ is a cauchy sequence $\therefore \{f_n(x)\} \to f(x)(|f_n(x) - f(x)| \to 0)$. To Prove: $\{f_n\} \to f$ uniformly. In (2), keeping n fixed and taking limit as $m \to \infty$ we get $|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N$ $\forall x \in E$. $\therefore \{f_n\} \to f$ uniformly.

Theorem 4.35 *Suppose*

$$\lim_{n \to \infty} f_n = f(x), \;\; (x \in E).$$

Put $M_n = \sup_{x \in E} |f_n(x) - f(x)|$*, then* $\{f_n\} \to f$ *uniformly on* E *iff* $M_n \to 0$ *as* $n \to \infty$.

Proof: For the 'only if' part, we assume that $\{f_n\} \to f$. To Prove: $M_n \to 0$ as $n \to \infty$. By hypothesis, given $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N \;\; \forall x \in E \Rightarrow \sup x \in E |f_n(x) - f(x)| \leq \epsilon$ $\forall n \geq N \Rightarrow M_n \leq \epsilon \; \forall n \geq N$ (i.e.) $M_n \to 0$ as $n \to \infty$. For the 'if' part, let $M_n \to 0$ as $n \to \infty$. Then there exists $N > 0$ such that $M_n \leq \epsilon$ $\forall n \geq N \Rightarrow \sup_{x \in E} |f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N \Rightarrow |f_n(x) - f(x)| \leq \epsilon$ $\forall n \geq N, x \in E \Rightarrow \{f_n\} \to f$ uniformly.

Theorem 4.36 ***Weristress*** M ***test for uniform convergence:*** *Suppose* $\{f_n\}$ *is a sequence of function defined on* E *and suppose that* $|f_1(x)| \leq M_n$

$(x \in E, n = 1, 2...)$ then $\sum f_n$ converges uniformly on E its $\sum M_n$ converges.
Proof: Assume that $\sum M_n$ converges. To Prove: $\sum f_n$ converges uniformly. Let $\epsilon > 0$ be given. Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums of $\sum f_n$ and $\sum M_n$ respectively. Since $\sum M_n$ converges, $\{t_n\}$ also converges. Since any convergence sequence is a Cauchy sequence $\{t_n\}$ is also a Cauchy sequence. Then there exists $N > 0$ such that $|t_n - t_m| \leq \epsilon \quad \forall n, m \geq N$. Let $m > n(\geq N)$

$$|t_n - t_m| = \left|\sum_{n+1}^{m} M_k\right| \leq \epsilon.......(1)$$

Now, for $x \in E$,

$$\begin{aligned}
|s_n(x) - s_m(x)| &= \left|\sum_{n+1}^{m} f_k(x)\right| \\
&\leq \sum_{n+1}^{m} |f_k(x)| \\
&\leq \sum_{n+1}^{m} M_k \leq \epsilon \text{ (by (1))} \\
\therefore |s_n(x) - s_m(x)| &< \epsilon
\end{aligned}$$

$\therefore$ By Cauchy's criteria 4.34 the $\{s_n\}$ converges uniformly on E. $\therefore \sum f_n$ converges uniformly.

Theorem 4.37 ***[Uniform Convergence and Continuity]*** *Suppose $\{f_n\}$ converges to f uniformly on a set E, in a metric space. Let x be a limit point of E and suppose that $\lim_{t\to x} f_n(t) = A_n (n = 1, 2, 3...)$, then $\{A_n\}$ converges $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. In other words $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$.*
Proof: Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges to f uniformly on E, by Theorem 4.34, there exists an integer $N > 0$ such that $|f_n(t) - f_m(t)| \leq \epsilon$ $\forall n, m \geq N, t \in E......$ (1)
Letting $t \to x$ in (1) we get $|A_n - A_m| \leq \epsilon \quad \forall n, m \geq N (\because \lim_{t\to x} = A_n)$ (i.e.) $\{A_n\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\{A_n\}$ converges to some $A($ in $\mathbb{R})$ (i.e.) $\{A_n\} \to A$. $\therefore$ there exists $N_1 > 0$ such that $|A_n - A| \leq \epsilon/3, \ \forall n \geq N_1......$ (2)
Now,

$$\begin{aligned}
|f(t) - A| &= |f(t) - f_n(t)| + (f_n(t) - A_n) + |(A_n - A)| \\
&\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + (A_n - A)|.......(3)
\end{aligned}$$

Since $\{f_n\} \to f$ uniformly, there exists $N_2 > 0$ such that $|f_n(t) - f(t)| \leq \epsilon/3$ $\forall n \geq N_2, t \in E.......$ (4)

Since x is a limit point of E and $\because \lim_{t\to x} f_n(t) = A_n$, there exists a neighbourhood V of x such that $|f_n(t) - A_n| \leq \epsilon/3 \quad \forall t \in V \cap E$....... (5)
Let $N_3 = max\{N_1, N_2\}$. Now using (2),(4) and (5) in (3) we get

$$|f(t) - A| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 \ \forall n \geq N_3 \ \forall t \in V \cap E.$$
$$(i.e.) \ |f(t) - A| \leq \epsilon$$
$$(i.e.) \ \lim_{t\to x} f(t) = A \text{ (or)}$$
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\lim_{t\to x} f_n(t))$$
$$\therefore \lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

Theorem 4.38 *If $\{f_n\}$ is a sequence of continuous functions on E, and if $\{f_n\}$ converges to f uniformly on E then f is continuous on E.*
Proof: Enough To Prove: $\lim_{t\to x} f(t) = f(x)$

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t)) \ (\because f_n \to f \ \text{ uniformly})$$
$$\lim_{t\to x} f(t) = \lim_{n\to\infty}(\lim_{t\to x} f_n(t)) \ (\text{by Theorem } 4.37)$$
$$= \lim_{n\to\infty} f_n(x) \ (\because f_n \text{ is continuous})$$
$$= f(x) \ (\because f_n \to f \text{ uniformly})$$

Remark 4.39 *The converse of the above theorem need not be true. (i.e.) a sequence of continuous function may converse to a continuous function, although the convergence is not uniform.*

Example 4.40 $f_n(x) = n^2x(1-x^2)^n, \ 0 \leq x \leq 1, \ n = 1, 2, 3, ...$ *Clearly, each f_n is continuous. Also f is continuous. But the convergence is not uniform. By Theorem 4.35, for let*

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$$
$$= \sup_{x\in[0,1]} |n^2x(1-x^2)^n - 0|$$
$$= n^2 \sup_{x\in[0,1]} \{x(1-x^2)^n\}$$
$$\nrightarrow 0 \ as \ n \to \infty.$$

By Theorem 4.35, the convergence is not uniform.

Theorem 4.41 ***[Dini's Theorem]*** *Suppose K is compact and*
(a) $\{f_n\}$ is a sequence of continuous functions on K.
(b) $\{f_n\}$ converges pointwise to a continuous functions f on K.
(c) $f_n(x) \geq f_{n+1}(x) \quad \forall x \in K, \ n = 1, 2, 3...$

then $f_n \to f$ uniformly on K.

Proof: Given K is compact. Let $g_n = f_n - f$. Since each f_n is continuous and f is continuous, g_n is continuous for all n. Since $\{f_n\}$ converges pointwise to f, $\{g_n\}$ converges pointwise to 0. Since $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K,\ n = 1, 2...$ $f_n(x) - f(x) \geq f_{n+1}(x) - f(x)$. (i.e.) $g_n(x) \geq g_{n+1}(x)$ $\forall x,\ n = 1, 2...$ (i.e.) $\{g_n\}$ is also a monotonic decreasing sequence. To prove that $\{f_n\}$ converges to f uniformly. It is enough to prove that $\{g_n\}$ converges to 0 uniformly. Let $\epsilon > 0$ be given. For each n, let $K_n = \{x \in K | g_n(x) \geq \epsilon\}$. Now,

$$\begin{aligned} K_n &= \{x \in K | g_n(x) \geq \in [\epsilon, \infty)\} \\ &= \{x \in K | x \in g_n^{-1}[\epsilon, \infty)\} \\ &= g_n^{-1}[\epsilon, \infty). \end{aligned}$$

Since $[\epsilon, \infty)$ is closed in R and g_n is continuous, $g_n^{-1}[\epsilon, \infty)$ is closed in K. (i.e.) K_n is a closed subspace of the compact space K. $\therefore K_n$ is compact ($\because$ every closed subspace of a compact space is compact). Claim: $K_n \supset K_{n+1},\ n = 1, 2, 3...$ Let $x \in K_{n+1} \Rightarrow g_{n+1}(x) \geq \epsilon$. But $g_n(x) \geq g_{n+1}(x)$ (by (1)). $\therefore g_n(x) \geq g_{n+1}(x) \geq \epsilon \Rightarrow g_n(x) \geq \epsilon \Rightarrow x \in K_n \therefore K_{n+1} \subset K_n$. Fix $x \in K$. Since $\{g_n\}$ converges pointwise to 0. $\{g_n(x)\} \to 0$. Then there exists $N(x) > 0$ such that $|g_n(x) - 0| < \epsilon \quad \forall n \geq N(x) \Rightarrow g_n(x) < \epsilon \quad \forall n \geq N(x) \Rightarrow x \notin K_n \quad \forall n \geq N(x) \Rightarrow x \notin \bigcap_{n=1}^{\infty} K_n$. Since x is arbitrary,$\bigcap_{n=1}^{\infty} K_n = \phi \Rightarrow K_N = \phi$ for some N. $\therefore g_N(x) < \epsilon \quad \forall x \in K$. But

$$\begin{aligned} 0 \leq g_n(x) \leq g_N(x) < \epsilon\ \forall x \in K,\ \forall n \geq N \\ g_n(x) < \epsilon\ \forall x \in K,\ \forall n \geq N \\ (i.e.)\ |g_n(x) - 0| < \epsilon\ \forall x \in K,\ \forall n \geq N \end{aligned}$$

Hence $\{g_n\} \to 0$ uniformly.

Note 4.42 *Compactness is really needed in the above theorem.*

Example 4.43 $f_n(x) = \frac{1}{nx+1},\ 0 < x < 1,\ n = 1, 2, 3...$ $\{f_n\} \to f$ *pointwise where* $f(x) = 0 \forall x \in (0, 1)$ *and* $(0, 1)$ *is not compact. Clearly, each* f_n *is continuous. Also* f *is continuous. Now,*

$$\begin{aligned} n + 1 &> n \\ \Rightarrow (n + 1)x &> nx \\ \Rightarrow (n + 1)x + 1 &> nx + 1 \\ \Rightarrow \frac{1}{(n + 1)x + 1} &< \frac{1}{nx + 1} \\ \Rightarrow f_{n+1}(x) &< f_n(x) \end{aligned}$$

$\Rightarrow \{f_n\}$ is a decreasing sequence. But $\{f_n\} \to f$ uniformly. For, if $\{f_n\} \to f$ uniformly then, given $\epsilon > 0$, there exists $N > 0$ such that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq \epsilon \ \forall n \geq N,\ \forall x \in (0,1) \\
(i.e.)\ \left|\frac{1}{nx+1} - 0\right| &\leq \epsilon \ \forall x \in (0,1) \\
\left|\frac{1}{nx+1}\right| &\leq \epsilon \ \forall x \in (0,1) \\
\text{Put } x = \frac{1}{n}.\ \text{Then } \frac{1}{2} &\leq \epsilon \\
&\Rightarrow\Leftarrow
\end{aligned}$$

$\therefore$ The convergence is not uniform.

Definition 4.44 *If X is a metric space $\mathscr{C}(x)$ denotes the set of all complex valued continuous bounded functions with domain X. $\mathscr{C}(X) = \{f/f : X \to c, f$ is continuous and bounded$\}$. If X is compact, $\mathscr{C}(X) = \{f/f : X \to c, f$ is continuous$\}$ ($\because$ any continuous function on a compact space is bounded). For any f in $\mathscr{C}(f)$, $\sup \|f\| = \sup_{x\in X} |f(x)|$, since f is bounded $\|f\| < \infty$.*

Result 4.45 *$\mathscr{C}(X)$ is a metric space. Given $f, g \in \mathscr{C}(X)$ define*

$$\begin{aligned}
(i)\ d(f,g) &= \|f - g\| \\
&= \sup_{x\in E} |f(x) - g(x)| \\
&\geq 0 \\
\therefore d(f,g) &\geq 0 \\
(ii)\ d(f,g) &= \sup_{x\in E} |f(x) - g(x)| \\
&= \sup_{x\in E} |g(x) - f(x)| \\
&= \|g - f\| \\
&= d(f,g) \\
(iii)\ d(f,g) = 0 &\Leftrightarrow \|f - g\| = 0 \\
&\Leftrightarrow \sup_{x\in E} |f(x) - g(x)| \\
&\Leftrightarrow |f(x) - g(x)| = 0 \forall x \in E \\
&\Leftrightarrow f(x) = g(x) \\
&\Leftrightarrow f = g
\end{aligned}$$

$$\begin{aligned}
(iv)\ d(f,g) &= \|f-g\| \\
&= \sup_{x\in E}|f(x)-g(x)| \\
&= \sup_{x\in E}|(f(x)-h(x))+(h(x)-g(x))| \\
&\le \sup_{x\in E}\{|(f(x)-h(x))|+|(h(x)-g(x))|\} \\
&\le \sup_{x\in E}|(f(x)-h(x))|+\sup_{x\in E}|(f(x)-g(x))| \\
&= \|f-h\|+\|h-g\| \\
&= d(f,h)+d(h,g) \\
(i.e.)\ d(f,g) &\le d(f,h)+d(h,g)
\end{aligned}$$

$\therefore$ *$(\mathscr{C}(X), d)$ is a metric space.*

Result 4.46 ***(Analogue of Theorem 4.35)*** *A sequence $\{f_n\} \to f$ with respect to the metric space $\mathscr{C}(X)$ iff $\{f_n\} \to f$ uniformly on X.*

Proof: 'only if' part:

Assume that $\{f_n\} \to f$ in $\mathscr{C}(X)$. $\|f_n - f\| \to 0$ as $n \to \infty$ (i.e.) $\sup_{x\in E}|f_n(x) - f(x)| \to 0$ as $n \to \infty$ (i.e.) $M_n \to 0$ as $n \to \infty$ (Theorem 4.35). $\{f_n\} \to f$ uniformly (by Theorem 4.35)

'if' part:

Suppose $\{f_n\} \to f$ uniformly. Then $M_n \to 0$ as $n \to \infty$ (Theorem 4.35) (i.e.) $\sup x \in E|f_n(x) - f(x)| \to 0$ as $n \to \infty$ (i.e.)$\|f_n - f\| \to 0$ as $n \to \infty$. $\therefore \{f_n\} \to f$ in $\mathscr{C}(X)$

Note 4.47 *(i) Closed subsets of $\mathscr{C}(X)$ are called uniformly closed subsets.*
(ii) If $A \subset \mathscr{C}(X)$ then the closure of A is called the uniform closure of A.

Theorem 4.48 *$\mathscr{C}(X)$ is a complete metric space.*

Proof: Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. Let $\epsilon > 0$ be given. Then there exists $N > 0$ such that $\|f_n - f_m\| < \epsilon \quad \forall n, m \ge N$...... (1)
(i.e.) $\sup_{x\in E}|f_n(x) - f_m(x)| \le \epsilon \quad \forall n, m \ge N. \ \Rightarrow |f_n(x) - f_m(x)| \le \epsilon$ $\forall n, m \ge N, x \in X$. By Theorem 4.34, guarantees that $\{f_n\}$ converges uniformly, say f. (i.e.) $\lim_{n\to\infty} f_n(x) = f(x), x \in X$. Claim: $f \in \mathscr{C}(X)$. Since each f_n is continuous and $\{f_n\} \to f$ uniformly (Theorem 4.38). Theorem 4.38 demands that f is also continuous. Again, since $\{f_n\} \to f$ uniformly, there exists $N_1 > 0$ such that $|f_n(x) - f(x)| < 1\ \forall n \ge N_1, x \in X$. In particular, $|f_{N_1}(x) - f(x)| < 1$....... (2) $\forall x \in X$
Since $f_{N_1}(x) \in \mathscr{C}(X), |f_{N_1}(x)| \le K$......... (3) $\forall x \in X$
Now,

$$\begin{aligned}
|f(x)| &= |(f(x) - f_{N_1}(x)) + f_{N_1}(x)| \\
|f(x)| &\le |f(x) - f_{N_1}(x)| + |f_{N_1}(x)| \\
&< 1 + K \text{ (by (2) and (3)) } \forall x \in X \\
(i.e.)\ |f(x)| &< 1 + K\ \forall x \in K.
\end{aligned}$$

$\therefore f$ is bounded. Hence $f \in \mathscr{C}(X)$. It remains to prove that $\{f_n\} \to f$ in $\mathscr{C}(X)$. For, $\{f_n\} \to f$ uniformly $\Rightarrow M_n \to 0 \Rightarrow \sup_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$ (by Theorem 4.35) $\Rightarrow \|f_n - f\| \to 0$ as $n \to \infty$. So $\{f_n\} \to f$ in the metric space $\mathscr{C}(X)$. $\therefore \mathscr{C}(X)$ is a complete metric space.

Uniform Convergence and Integration

Theorem 4.49 *Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ for $n = 1,2,3....$ and suppose $f_n \to f$ uniformly on $[a,b]$ then $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f d\alpha$.*

Proof: Let $\epsilon_n = \sup_{a \le x \le b} |f(x) - f_n(x)|$....... (1) (Theorem 4.35)

$$
\begin{aligned}
\therefore |f - f_n| &\le \epsilon_n \ \forall n = 1,2,3... \\
-\epsilon &\le f - f_n \le \epsilon_n \\
\Rightarrow f_n - \epsilon_n &\le f \le f_n + \epsilon_n \\
\Rightarrow \int_a^b (f_n - \epsilon_n) d\alpha &\le \int_{\underline{a}}^b f d\alpha \le \int_a^{\bar{b}} f d\alpha \le \int_a^b (f_n + \epsilon_n) d\alpha........(2) \\
\Rightarrow \int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha &\le \int_{\underline{a}}^b f d\alpha \le \int_a^{\bar{b}} f d\alpha \le \int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha \\
\Rightarrow \int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^b f d\alpha &\le (\int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha) - (\int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha) \\
&= 2\int_a^b \epsilon_n d\alpha \\
&= 2\epsilon_n \int_a^b d\alpha \\
&= 2\epsilon_n[\alpha(b) - \alpha(a)]
\end{aligned}
$$

$$
(i.e.) \int_a^{\bar{b}} f d\alpha - \int_{\underline{a}}^b f d\alpha \le 2\epsilon_n(\alpha(b) - \alpha(a))
$$

$$
\to 0 \ (\because \epsilon_n \to 0 \ \text{ as } \ f_n \to f \ \text{ uniformly by theorem } 4.35)
$$

$$
\therefore \int_a^{\bar{b}} f d\alpha = \int_{\underline{a}}^b f d\alpha
$$

Hence $f \in \mathcal{R}(\alpha)$. **II part:** To prove:

$$
\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha
$$

Now, (2)⇒

$$\begin{aligned}
\int_a^b (f_n - \epsilon_n) d\alpha &\le \int_a^b f d\alpha \le \int_a^b (f_n + \epsilon_n) d\alpha \\
\int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha &\le \int_a^b f d\alpha \le \int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha \\
\Rightarrow \int_a^b f_n d\alpha - \epsilon_n \int_a^b d\alpha &\le \int_a^b f d\alpha \le \int_a^b f_n d\alpha + \epsilon_n \int_a^b d\alpha \\
\Rightarrow -\epsilon_n \int_a^b d\alpha &\le \int_a^b f d\alpha - \int_a^b f_n d\alpha \le \epsilon_n \int_a^b d\alpha \\
\Rightarrow \left| \int_a^b f d\alpha - \int_a^b f_n d\alpha \right| &\le \epsilon_n \int_a^b d\alpha \\
&= \epsilon_n (\alpha(b) - \alpha(a)) \\
&\to 0 \text{ as } n \to \infty \ (\because \epsilon_n \to 0) \\
\lim_{n\to\infty} \int_a^b f_n d\alpha &= \int_a^b f d\alpha.
\end{aligned}$$

Corollary 4.50 *If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x) (a \le x \le b)$, the series converges uniformly on $[a, b]$, then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$.(the series may be integrated term by term)*

Proof: Given $\sum f_n = f$ (uniformly). Let $s_n = \sum_{k=1}^{n} f_k$. By hypothesis $\{s_n\} \to f$ uniformly. By Theorem 4.49,

$$\begin{aligned}
\int_a^b f d\alpha &= \lim_{n\to\infty} \int_a^b s_n d\alpha \\
&= \lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} f_k \right) d\alpha \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_a^b f_k d\alpha \right) \\
&= \sum_{k=1}^{\infty} \int_a^b f_k d\alpha
\end{aligned}$$