MAR GREGORIOS COLLEGE
OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: COMPLEX ANALYSIS

SUBJECT CODE: SM26C

SEMESTER: VI

PREPARED BY: PROF.S.KAVITHA

UNIVERSITY OF MADRAS

B.Sc. DEGREE COURSE IN MATHEMATICS

SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC15

CORE-XV: COMPLEX ANALYSIS

(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 6 | **YEAR: III**
Credits : 4 | **SEMESTER: VI**

Learning outcomes:

Students will acquire knowledge about the basic ideas of analysis of Complex Functions in solving Complex Variables.

UNIT I

Analytic Functions:Functions of a Complex Variable – Limit- Theorems on Limits – Continuuous functions- Differentiability – Cauchy – Riemann equations – Analytic functions- Harmonic functions – Conformal mapping.
Chapter 1 – sec 2.1 to 2.9.

UNIT II

Bilinear Transformations:Elementary transformations – Bilinear transformations – Cross ratio- Fixed Points of Bilinear Transformations – Mapping by Elementary Functions - The Mapping $w = z^2$, z^n, n is a positive integer,$w = e^z$, sin z, cos z.
Chapter 3 – sec 3.1 to 3.4 , Chapter 5 – sec 5.1 to 5.5

UNIT III

Complex Integration – definite integral – Cauchy's Theorem – Cauchy's integral formula – Higher derivatives. Chapter 6 – sec 6.1 to 6.4

UNIT IV

Series expansions – Taylor's series – Laurent's Series – Zeroes of analytic functions- Singularities. Chapter 7 – 7.1 to 7.4

UNIT V

Residues – Cauchy's Residue Theorem – Evaluation of definite integrals.
Chapter 8 – 8.1 to 8.3.

Content and treatment as in

"Complex Analysis" byDr.S.Arumugam,Thangapandi Isaac, Dr.A.Somasundaram, SciTech publications(India) Pvt Ltd,2002.

Reference:

1. Complex variables and Applications (Sixth Edition) by James Ward Brown and RuelV.Churchill, Mc.Grawhill Inc.
2. Complex Analysis by P.Duraipandian,KayalakPachaiyappa,S.Chand& Co Pvt.Ltd.
3. Complex Analysis ,T.K.Manickavachagom Pillay, S.Viswanathan Publishers Pvt. Ltd.

e-Resources:

1. http://ebooks.lpude.in.complexanalysis.
2. https://nptel.ac.in.

UNIT-1
ANALYTIC FUNCTIONS

6.1 INTRODUCTION

We have learnt the complex numbers in the previous class. Here we will review the complex number. In this chapter we will learn how to add, subtract, multiply and divide complex numbers.

6.2 COMPLEX NUMBERS

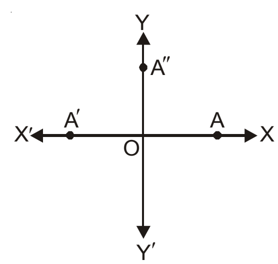
A number of the form $a + i\,b$ is called a complex number when a and b are real numbers and $i = \sqrt{-1}$. We call 'a' the real part and 'b' the imaginary part of the complex number $a + ib$. If $a = 0$ the number $i\,b$ is said to be purely imaginary, if $b = 0$ the number a is real.

A complex number $x + iy$ is denoted by z.

6.3 GEOMETRICAL REPRESENTATION OF IMAGINARY NUMBERS

Let OA be positive numbers which is represented by x and OA' by $-x$.

And $$-x = (i)^2\, x = i\,(ix) \text{ is on } OX'.$$



It means that the multiplication of the real number x by i twice amounts to the rotation of OA through two right angles to reach OA'.

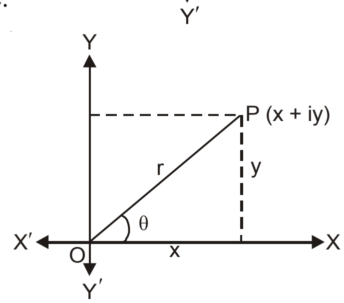
Thus, it means that multiplication of x by i is equivalent to the rotation of x through one right angle to reach OA''.

Hence, y-axis is known as imaginary axis.

Multiplication by i rotates its direction through a right angle.

6.4 ARGAND DIAGRAM

Mathematician Argand represented a complex number in a diagram known as Argand diagram. A complex number $x + iy$ can be represented by a point P whose co-ordinate are (x, y). The axis of x is called the real axis and the axis of y the imaginary axis. The distance OP is the **modulus** and the angle, OP makes with the x-axis, is the **argument** of $x + iy$.



6.5 EQUAL COMPLEX NUMBERS

If two complex numbers $a + ib$ and $c + id$ are equal, prove that

$$a = c \quad \text{and} \quad b = d$$

Solution. We have, $$a + ib = c + id \;\Rightarrow\; a - c = i(d - b)$$

$$(a - c)^2 = -(d - b)^2 \;\Rightarrow\; (a - c)^2 + (d - b)^2 = 0$$

Here sum of two positive numbers is zero. This is only possible if each number is zero.

i.e., $(a - c)^2 = 0 \;\Rightarrow\; a = c$ and $(d - b)^2 = 0 \;\Rightarrow\; b = d$ **Ans.**

6.6 ADDITION OF COMPLEX NUMBERS

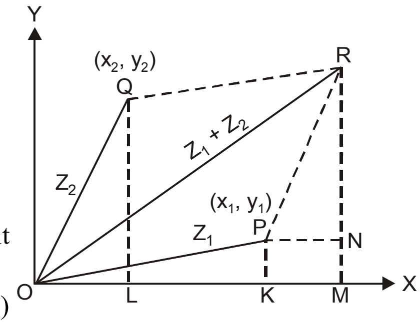
Let $a + ib$ and $c + id$ be two complex numbers, then

$$(a + ib) + (c + id) = (a + c) + i\,(b + d)$$

Procedure. In addition of complex numbers we add real parts with real parts and imaginary parts with imaginary parts.

6.7 ADDITION OF COMPLEX NUMBERS BY GEOMETRY

Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ be two complex numbers represented by the points P and Q on the Argand diagram.



Complete the parallelogram $OPRQ$.

Draw PK, RM, QL, perpendiculars on OX.

Also draw $PN \perp$ to RM.

$$OM = OK + KM = OK + \text{OL} = x_1 + x_2$$

and $$RM = MN + NR = KP + LQ = y_1 + y_2$$

$\therefore$ The co-ordinates of R are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number.

$$(x_1 + x_2) + i\,(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2)$$

Thus the sum of two complex numbers is represented by the extremity of the diagonal of the parallelogram formed by OP (z_1) and OQ (z_2) as adjacent sides.

$$|z_1 + z_2| = OR \quad \text{and} \quad \text{amp}\,(z_1 + z_2) = \angle ROM.$$

6.8 SUBTRACTION

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Procedure. In subtraction of complex numbers we subtract real parts from real parts and imaginary parts from imaginary parts.

SUBTRACTION OF COMPLEX NUMBERS BY GEOMETRY.

Let P and Q represent two complex numbers

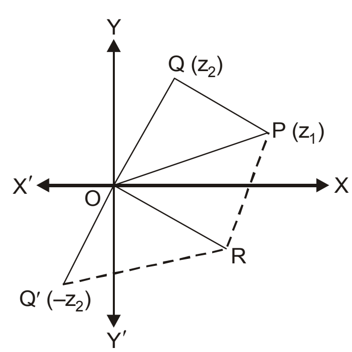
$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2.$$

Then $$z_1 - z_2 = z_1 + (-z_2)$$

$z_1 - z_2$ means the addition of z_1 and $-z_2$.

$-z_2$ is represented by OQ' formed by producing OQ to OQ' such that $OQ = OQ'$.

Complete the parallelogram $OPRQ'$, then the sum of z_1 and $-z_2$ represented by OR.



6.9 POWERS OF *i*

Some time we need various powers of i.

We know that $i = \sqrt{-1}$.

On squaring both sides, we get

$$i^2 = -1$$

Multiplying by i both sides, we get

$$i^3 = -i$$

Again, $$i^4 = (i^3)\,(i) = (-i)\,(i) = -(i^2) = -(-1) = 1$$

$$i^5 = (i^4)\,(i) = (1)\,(i) = i$$

$$i^6 = (i^4)(i^2) = (1)(-1) = -1$$
$$i^7 = (i^4)(i^3) = 1(-i) = -i$$
$$i^8 = (i^4)(i^4) = (1)(1) = 1.$$

Example 1. *Simplify the following:* (*a*) i^{49}, (*b*) i^{103}.

Solution. (*a*) We divide 49 by 4 and we get

$$49 = 4 \times 12 + 1$$
$$i^{49} = i^{4 \times 12 + 1} = (i^4)^{12}(i^1) = (1)^{12}(i) = i$$

(*b*) we divide 103 by 4, we get

$$103 = 4 \times 25 + 3$$
$$i^{103} = i^{4 \times 25 + 3} = (i^4)^{25}(i^3) = (1)^{25}(-i) = -i$$ **Ans.**

6.10 MULTIPLICATION

$$(a + ib) \times (c + id) = ac - bd + i(ad + bc)$$

Proof.
$$(a + ib) \times (c + id) = ac + iad + ibc + i^2bd$$
$$= ac + i(ad + bc) + (-1)bd \qquad [\because i^2 = -1]$$
$$= (ac - bd) + (ad + bc)i$$

Example 2. *Multiply 3 + 4i by 7 – 3i.*

Solution. Let $z_1 = 3 + 4i$ and $z_2 = 7 - 3i$

$$z_1.z_2 = (3 + 4i).(7 - 3i)$$
$$= 21 - 9i + 28i - 12i^2$$
$$= 21 - 9i + 28i - 12(-1) \qquad [\because i^2 = -1]$$
$$= 21 - 9i + 28i + 12$$
$$= 33 + 19i$$ **Ans.**

Multiplication of complex numbers (Polar form) :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

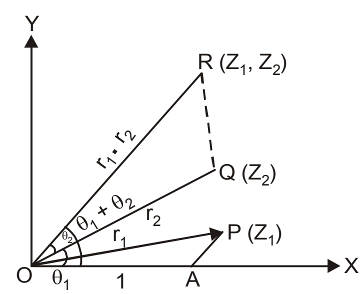
$$x_1 = r_1\cos\theta_1, \quad y_1 = r_1\sin\theta_1$$
$$x_2 = r_2\cos\theta_2, \quad y_2 = r_2\sin\theta_2$$
$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad |z_1| = r_1$$
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2) \qquad |z_2| = r_2$$
$$z_1 . z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1r_1[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \qquad |z_1z_2| = r_1r_2$$

The modulus of the product of two complex numbers is the product of their moduli and the argument of the product is the sum of their arguments.

Graphical method

Let *P*, *Q* represent the complex numbers.

$$z_1 = x_1 + iy_1$$
$$= r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + iy_2$$
$$= r_2(\cos\theta_2 + i\sin\theta_2)$$



Cut off $OA = 1$ along *x*-axis. Construct $\Delta\ ORQ$ on OQ similar to $\Delta\ OAP$.

So that $$\frac{OR}{OP} = \frac{OQ}{OA} \Rightarrow \frac{OR}{OP} = \frac{OQ}{1} \Rightarrow OR = OP \,.\, OQ = r_1 r_2$$

$$\angle XOR = \angle AOQ + \angle QOR = \theta_2 + \theta_1$$

Hence the product of two complex numbers z_1, z_2 is represented by the point R, such that

(*i*) $|z_1 \,.\, z_2| = |z_1|.|z_2|$ (*ii*) $\text{Arg}\,(z_1 \,.\, z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2)$

6.11 *i* (IOTA) AS AN OPERATOR

Multipliction of a complex number by i.

Let $$z = x + iy = r(\cos\theta + i\sin\theta)$$

$$i = 0 + i \,.\, 1 = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$i \,.\, z = r(\cos\theta + i\sin\theta) \,.\, \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

Hence a complex number multiplied by i results :

The rotation of the complex number by $\frac{\pi}{2}$ in anticlockwise direction without change in magnitude.

6.12 CONJUGATE OF A COMPLEX NUMBER

Two complex numbers which differ only in the sign of imaginary parts are called conjugate of each other.

A pair of complex number $a + ib$ and $a - ib$ are said to be conjugate of each other.

Theorem. Show that the sum and product of a complex number and its conjugate complex are both real.

Proof. Let $x + iy$ be a complex number and $x - iy$ its conjugate complex.

$$\text{Sum} = (x + iy) + (x - iy) = 2x \qquad \text{(Real)}$$

$$\text{Product} = (x + iy).(x - iy) = x^2 + y^2. \qquad \text{(Real)}$$

Proved.

Note. Let a complex number be z. Then the conjugate complex number is denoted by $\bar{z}$.

Example 3. *Find out the conjugate of a complex number 7 + 6i.*

Solution. Let $z = 7 + 6i$

To find conjugate complex number of $7 + 6i$ we change the sign of imaginary number.

Conjugate of $z = \bar{z} = 7 - 6i$ **Ans.**

6.13 DIVISION

To divide a complex number $a + ib$ by $c + id$, we write it as $\frac{a + ib}{c + id}$.

To simplify further, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a + ib}{c + id} = \frac{(a + ib)}{(c + id)} \times \frac{(c - id)}{(c - id)} = \frac{ac - iad + ibc - i^2 bd}{(c)^2 - (id)^2}$$

$$= \frac{ac - i(ad - bc) + bd}{c^2 - d^2 i^2} \qquad [\because i^2 = -1]$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$$

Example 4. *Divide 1 + i by 3 + 4i.*

Solution.

$$\frac{1+i}{3+4i} = \frac{1+i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{3 - 4i + 3i - 4i^2}{9 - 16i^2}$$

$$= \frac{3 - i + 4}{9 + 16} = \frac{7}{25} - \frac{1}{25} i \qquad \textbf{Ans.}$$

DIVISION (By Algebra)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

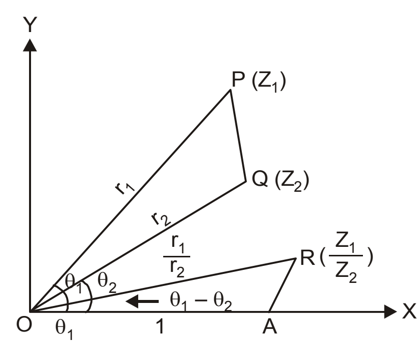
$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

The modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument of the quotient is the difference of their arguments.

6.14 DIVISION OF COMPLEX NUMBERS BY GEOMETRY

Let P and Q represent the complex numbers.

$$z_1 = x_1 + i\,y_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + i\,y_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$



Cut off $OA = 1$, construct $\Delta\, OAR$ on OA similar to $\Delta\, OQP$.

So that $\dfrac{OR}{OA} = \dfrac{OP}{OQ} \Rightarrow \dfrac{OR}{1} = \dfrac{OP}{OQ}$

$$OR = \frac{OP}{OQ} = \frac{r_1}{r_2}$$

$$\angle AOR = \angle QOP = \angle AOP - \angle AOQ = \theta_1 - \theta_2$$

$\therefore$ R represents the number $\dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Hence the complex number $\dfrac{z_1}{z_2}$ is represented by the point R.

(*i*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (*ii*) Arg. $\left(\dfrac{z_1}{z_2}\right)$ = Arg. (z_1) – Arg. (z_2).

Example 5. *If $a = \cos\theta + i\sin\theta$, prove that $1 + a + a^2 = (1 + 2\cos\theta)(\cos\theta + i\sin\theta)$.*

Solution. Here we have

$$\begin{aligned} a &= \cos\theta + i\sin\theta \\ 1 + a + a^2 &= 1 + (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2 \\ &= 1 + \cos\theta + i\sin\theta + \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\ &= (\cos\theta + i\sin\theta) + (1 - \sin^2\theta) + \cos^2\theta + 2i\sin\theta\cos\theta \\ &= (\cos\theta + i\sin\theta) + \cos^2\theta + \cos^2\theta + 2i\sin\theta\cos\theta \\ &= (\cos\theta + i\sin\theta) + 2\cos^2\theta + 2i\sin\theta\cos\theta \\ &= (\cos\theta + i\sin\theta) + 2\cos\theta(\cos\theta + i\sin\theta) \\ &= (\cos\theta + i\sin\theta)(1 + 2\cos\theta) \end{aligned}$$

Proved.

Example 6. *If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, prove that $\dfrac{a + ib}{1 + c} = \dfrac{1 + iz}{1 - iz}$.*

Solution. Here, we have $b + ic = (1 + a)z \quad \Rightarrow \quad z = \dfrac{b + ic}{1 + a}$

$$\frac{1 + iz}{1 - iz} = \frac{1 + i\dfrac{b + ic}{1 + a}}{1 - i\dfrac{b + ic}{1 + a}} = \frac{1 + a + ib - c}{1 + a - ib + c}$$

$$= \frac{[(1 + a + ib) - c]}{(1 + a + c - ib)} \times \frac{(1 + a + ib + c)}{(1 + a + c + ib)} = \frac{(1 + a + ib)^2 - c^2}{(1 + a + c)^2 + b^2}$$

$$= \frac{1 + a^2 - b^2 + 2a + 2ib + 2iab - c^2}{1 + a^2 + c^2 + 2a + 2c + 2ac + b^2} = \frac{1 + a^2 - b^2 - c^2 + 2a + 2ib + 2iab}{1 + (a^2 + b^2 + c^2) + 2a + 2c + 2ac}$$

Putting the value of $a^2 + b^2 + c^2 = 1$ in the above, we get

$$= \frac{1 + a^2 - (1 - a^2) + 2a + 2ib + 2iab}{1 + 1 + 2a + 2c + 2ac} = \frac{2(a^2 + a + ib + iab)}{2(1 + a + c + ac)} = \frac{2(1 + a)(a + ib)}{2(1 + a)(1 + c)} = \frac{a + ib}{1 + c}$$

Proved.

Example 7. *If $z = \cos\theta + i\sin\theta$, prove that*

$$\frac{2}{1 + z} = 1 - i\tan\frac{\theta}{2}$$

Solution. Here, we have $z = \cos\theta + i\sin\theta$

(*a*) $$\frac{2}{1 + z} = \frac{2}{1 + (\cos\theta + i\sin\theta)} = \frac{2}{(1 + \cos\theta) + i\sin\theta} \times \frac{(1 + \cos\theta) - i\sin\theta}{(1 + \cos\theta) - i\sin\theta}$$

$$= \frac{2[(1 + \cos\theta) - i\sin\theta]}{(1 + \cos\theta)^2 + \sin^2\theta}$$

$$= \frac{2[(1 + \cos\theta) - i\sin\theta]}{2(1 + \cos\theta)} = 1 - \frac{i\sin\theta}{1 + \cos\theta}$$

$$= 1 - i\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = 1 - i\tan\left(\frac{\theta}{2}\right)$$

$$\begin{aligned} &(1 + \cos\theta)^2 + \sin^2\theta \\ &= 1 + \cos^2\theta + 2\cos\theta + \sin^2\theta \\ &= 1 + (\sin^2\theta + \cos^2\theta) + 2\cos\theta \\ &= 1 + 1 + 2\cos\theta \\ &= 2 + 2\cos\theta \\ &= 2(1 + \cos\theta) \end{aligned}$$

Proved.

Example 8. *If $x = \cos\theta + i\sin\theta$, $y = \cos\phi + i\sin\phi$, prove that*

$$\frac{x - y}{x + y} = i\tan\left(\frac{\theta - \phi}{2}\right)$$ (*M.U. 2008*)

Solution. We have,

$$\frac{x - y}{x + y} = \frac{(\cos\theta + i\sin\theta) - (\cos\phi + i\sin\phi)}{(\cos\theta + i\sin\theta) + (\cos\phi + i\sin\phi)}$$

$$= \frac{(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)}{(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)}$$

$$= \frac{\left[-2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) + 2i\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]}{\left[2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + 2i\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)\right]}$$

$$= \frac{2i\sin\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]}{2\cos\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]} = i\tan\left(\frac{\theta-\phi}{2}\right)$$ **Proved.**

EXERCISE 6.1

1. If $z = 1 + i$, find (*i*) z^2 (*ii*) $\frac{1}{z}$ and plot them on the Argand diagram. **And.** (*i*) $2i$, (*ii*) $\frac{1}{2} - \frac{i}{2}$

Express the following in the form *a* + *ib*, where *a* and *b* are real (2 – 4):

2. $\frac{2-3i}{4-i}$ **Ans.** $\frac{11}{17} - \frac{10}{17}i$ **3.** $\frac{(3+4i)(2+i)}{1+i}$ **Ans.** $\frac{13}{2} + \frac{9}{2}i$

4. $\frac{(1+2i)^3}{(1+i)(2-i)}$ **Ans.** $-\frac{7}{2} + \frac{1}{2}i$

5. The points *A*, *B*, *C* represent the complex numbers z_1, z_2, z_3 respectively, and *G* is the centroid of the triangle *ABC*, if $4z_1 + z_2 + z_3 = 0$, show that the origin is the mid-point of AG.

6. *ABCD* is a parallelogram on the Argand plane. The affixes of *A*, *B*, *C* are $8 + 5i$, $-7 - 5i$, $-5 + 5i$, respectively. Find the affix of *D*. **Ans.** $10 + 15i$

7. If z_1, z_2, z_3 are three complex numbers and

$$a_1 = z_1 + z_2 + z_3$$
$$b_1 = z_1 + \omega z_2 + \omega^2 z_3$$
$$c_1 = z_1 + \omega^2 z_2 + \omega z_3$$

show that $|a_1|^2 + |b_1|^2 + |c_1|^2 = 3\{|z_1|^2 + |z_2|^2 + |z_3|^2\}$

where ω, ω^2 are cube roots of unity.

8. Find the complex conjugate of $\frac{2+3i}{1-i}$. **Ans.** $-\frac{1}{2} - \frac{5}{2}i$

9. If $x + iy = \frac{1}{a+ib}$, prove that $(x^2 + y^2)(a^2 + b^2) = 1$

10. Find the value of $x^2 - 6x + 13$, when $x = 3 + 2i$. **Ans**. 0

11. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (*M.U. 2008*)

12. If $\frac{1}{\alpha + i\beta} + \frac{1}{a + ib} = 1$, where α, β, a, b are real, express *b* in terms of α, β.

Ans. $\frac{-\beta}{\alpha^2 + \beta^2 - 2\alpha + 1}$

13. If $(x + iy)^{1/3} = a + ib$, then show that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

14. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.

15. Find the values of *x* and *y*, if $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$. **Ans.** $x = 3$ and $y = -1$

16. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

Functions of a Complex Variable

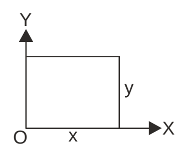
7.1 INTRODUCTION

The theory of functions of a complex variable is of atmost importance in solving a large number of problems in the field of engineering and science. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

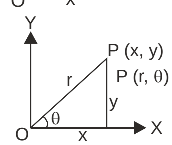
7.2 COMPLEX VARIABLE

$x + iy$ is a complex variable and it is denoted by z.

(1) $z = x + iy$ where $i = \sqrt{-1}$ (Cartesian form)



(2) $z = r(\cos\theta + i\sin\theta)$ (Polar form)



(3) $z = re^{i\theta}$ (Exponential form)

7.3 FUNCTIONS OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$
$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$.

7.4 NEIGHBOURHOOD OF Z_0

Let z_0 is a point in the complex plane and let z be any positive number, then the set of points z such that

$$|z - z_0| < \in$$

is called $\in$ -neighbourhood of z_0.

Closed set

A set S is said to be closed if it contains all of its limits point.

Interior Point

A point z_0 is called a interior point of a point set S if there exists a neighbourhood of z_0 lying wholly in S.

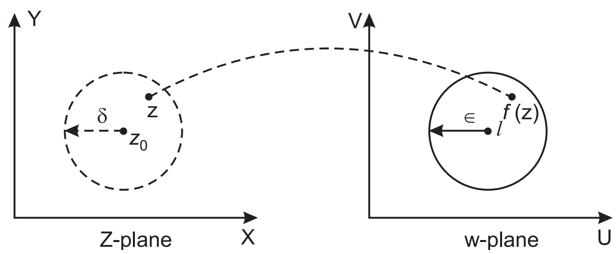
7.5 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\in > 0$, however small there exists a real number $\delta > 0$, such that

$| f(z) - l | < \in$ whenever $0 < | z - z_0 | < \delta$

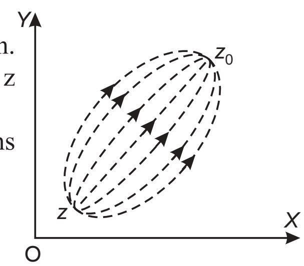
i.e. for every $z \neq z_0$ in δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane

In symbolic form, $\displaystyle\operatorname*{Lim}_{z \to z_0} f(z) = l$



Note: (I) δ usually depends upon $\in$.

(II) $z \to z_0$ implies that z approaches z_0 along any path. The limits must be independent of the manner in which z approaches z_0



If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.

Example 1. *Prove that* $\displaystyle\lim_{z \to 1-i} \frac{(z^2 + 4z + 3)}{z + 1} = 4 - i$

Solution. $\displaystyle\lim_{z \to 1-i} \frac{z^2 + 4z + 3}{z + 1} = \lim_{z \to 1-i} \frac{(z+1)(z+3)}{(z+1)} = \lim_{z \to 1-i} (z+3) = (1-i) + 3 = 4 - i$ **Proved.**

Example 2. *Show that* $\displaystyle\lim_{z \to 0} \frac{z}{|z|}$ *does not exist.*

Solution. $\displaystyle\lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x + iy}{\sqrt{x^2 + y^2}}$

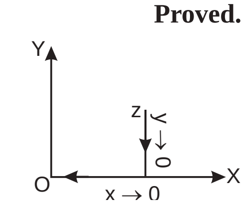
Let $y = mx$,

$$= \lim_{x \to 0} \frac{x + imx}{\sqrt{x^2 + m^2 x}} = \lim_{x \to 0} \frac{1 + im}{\sqrt{1 + m^2}} = \frac{1 + mi}{\sqrt{1 + m^2}}$$

The value of $\dfrac{1 + mi}{\sqrt{1 + m^2}}$ are different for different values of m.

Hence, limit of the function does not exist. **Proved.**

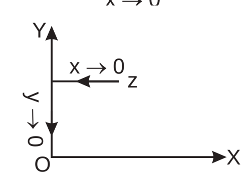
Example 3. *Prove that* $\displaystyle\lim_{z \to 0} \frac{z}{\bar{z}}$ *does not exist.*



Solution. Case I. $\displaystyle\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x + iy}{x - iy} = \lim_{x \to 0} \left[\lim_{y \to 0} \frac{x + iy}{x - iy} \right]$

$$= \lim_{x \to 0} \frac{x}{x} = 1$$

Here the path is $y \to 0$ and then $x \to 0$



Case II. Again $\displaystyle\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{y \to 0} \left[\lim_{x \to 0} \frac{x + iy}{x - iy} \right] = \lim_{y \to 0} \frac{iy}{-iy} = -1$

In this case, we have a different path first $x \to 0$, then $y \to 0$
As $z \to 0$ along two different paths we get different limits.
Hence the limit does not exist. **Proved.**

Example 4. *Find the limit of the following* $\lim\limits_{z\to\infty} \dfrac{iz^3 + iz - 1}{(2z+3i)(z-i)^2}$

Solution. On dividing numerator and denominator by z^3, we get

$$\lim_{x\to\infty} \frac{iz^3 + iz - 1}{(2z+3i)(z-i)^2} \quad \lim_{z\to\infty} \frac{\left(i + \frac{i}{z^2} - \frac{1}{z^3}\right)}{\left(2+\frac{3i}{z}\right)\left(1-\frac{i}{z}\right)^2} = \frac{i}{2} \qquad \textbf{Ans.}$$

Example 5. *Find the limit of the following* $\lim\limits_{z\to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

Solution.

$$\lim_{z\to 1+i} \frac{z^2 + 1 - z - i}{z^2 - 2z + 2} = \lim_{z\to 1+i} \frac{(z+i)(z-i) - 1(z+i)}{(z-1-i)(z+1+i)} = \lim_{z\to 1+i} \frac{(z+i)(z-i-1)}{(z-1-i)(z-1+i)} = \lim_{z\to 1+i} \frac{z+i}{z-1+i}$$

$$= \frac{1+i+i}{1+i-1+i} = \frac{1+2i}{2i} = \frac{(1+2i)(-i)}{2(i)(-i)} = \frac{-i+2}{2(1)} = \frac{2-i}{2} = 1 - \frac{i}{2} \qquad \textbf{Ans.}$$

EXERCISE 7.1

Show that the following limits do not exist:

1. $\lim\limits_{z\to 0} \dfrac{\text{Im}(z)^3}{\text{Re}(z)^3}$ **2.** $\lim\limits_{z\to -i} \dfrac{z^2}{z+i}$ **3.** $\lim\limits_{z\to 0} \dfrac{\text{Re}(z)^2}{\text{Im}\, z}$ **4.** $\lim\limits_{z\to 0} \dfrac{z}{(\bar{z})^2}$

Find the Limits of the following:

5. $\lim\limits_{z\to 0} \dfrac{\text{Re}(z)^2}{|z|}$ **Ans.** 0 **6.** $\lim\limits_{z\to 1+i} \dfrac{2z^3}{\text{Im}(z)^2}$ **Ans.** $2(-1+i)$ **7.** $\lim\limits_{z\to 0} \dfrac{z^2+6z+3}{z^2+2z+2}$ **Ans.** $\dfrac{3}{2}$

7.6 CONTINUITY

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number $\in$, we can find a number δ such that $|f(z) - f(z_0)| < \in$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

7.7 CONTINUITY IN TERMS OF REAL AND IMAGINARY PARTS

If $w = f(z) = u(x, y) + iv(x, y)$ is continuous function at $z = z_0$ then $u(x, y)$ and $v(x, y)$ are separately continuous functions of x, y at (x_0, y_0) where $z_0 = x_0 + i\,y_0$.

Conversely, if $u(x, y)$ and $v(x, y)$ are continuous functions of x, y at (x_0, y_0) then $f(z)$ is continuous at $z = z_0$.

Example 6. *Examine the continuity of the following*

$$f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, & z \neq i \\ 0, & z = i \end{cases} \qquad \text{at } z = i$$

Solution. $\lim\limits_{z \to 0} \dfrac{z^3 - iz^2 + z - i}{z - i} = \lim\limits_{z \to i} \dfrac{z^2(z-i) + 1(z-i)}{z - i}$

$$= \lim_{z \to i} \frac{(z-i)(z^2+1)}{z-i} = \lim_{z \to i}(z^2+1) = -1+1 = 0$$

$f(i) = 0$

$$\boxed{\lim_{z \to i} f(z) = f(i)}$$

Hence $f(z)$ is continuous at $z = i$ **Ans.**

Example 7. *Show that the function f(z) defined by*

$$f(z) = \begin{cases} \dfrac{\text{Re}(z)}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

is not continuous at $z = 0$

Solution. Here $f(z) = \dfrac{\text{Re}(z)}{z}$ when $z \neq 0$

$$\lim_{z \to 0} \frac{\text{Re}(z)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x+iy} = \lim_{x \to 0}\left[\lim_{y \to 0} \frac{x}{x+iy}\right] = \lim_{x \to 0} \frac{x}{x} = 1$$

Again $$\lim_{z \to 0} \frac{\text{Re}(z)}{z} = \lim_{y \to 0}\left[\lim_{x \to 0} \frac{x}{x+iy}\right] = 0$$

As $z \to 0$, for two different paths limit have two different values. So, limit does not exist.

Hence $f(z)$ is not continuous at $z = 0$ **Proved.**

EXERCISE 7.2

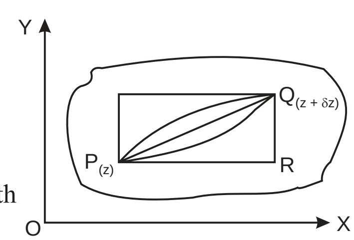
Examine the continuity of the following functions.

1. $f(z) = \begin{cases} \dfrac{\text{Im}(z)}{|z|} & , z \neq 0 \\ 0 & z = 0 \end{cases}$ at $z = 0$ **Ans.** Not Continuous

2. $f(z) = \dfrac{z^2 + 3z + 4}{z^2 + i}$ at $z = 1 - i$ **Ans.** Continuous

3. Show that the following functions are continuous for z
   (*i*) $\cos z$ (*ii*) e^{2z}

7.8 DIFFERENTIABILITY



Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z \to 0$.

Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.

Example 8. *Consider the function*

$$f(z) = 4x + y + i(-x + 4y)$$

and discuss $\dfrac{df}{dz}$.

Solution. Here, $f(z) = 4x + y + i(-x + 4y) = u + iv$

so $u = 4x + y$ and $v = -x + 4y$

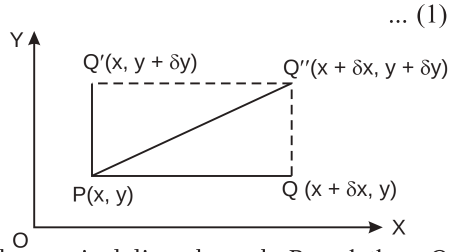
$$f(z+\delta z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

$$f(z+\delta z) - f(z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y) - 4x - y + ix - 4iy$$

$$= 4\delta x + \delta y - i\,\delta x + 4i\delta y$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\,\delta x + 4\,i\delta y}{\delta x + i\delta y}$$

$$\Rightarrow \quad \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \quad \text{... (1)}$$



(*a*) **Along real axis:** If Q is taken on the horizontal line through $P\,(x, y)$ and Q then approaches P along this line, we shall have $\delta y = 0$ and $\delta z = \delta x$.

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

(*b*) **Along imaginary axis:** If Q is taken on the vertical line through P and then Q approaches P along this line, we have

$$z = x + iy = 0 + iy,\ \delta z = i\delta y,\ \delta x = 0.$$

Putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1+4i) = 4 - i$$

(*c*) **Along a line $y = x$:** If Q is taken on a line $y = x$.

$$z = x + iy = x + ix = (1 + i)x$$

$$\delta z = (1 + i)\delta x \quad \text{and} \quad \delta y = \delta x$$

On putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x} = \frac{4 + 1 - i + 4i}{1+i} = \frac{5+3i}{1+i} = \frac{(5+3i)(1-i)}{(1+i)(1-i)} = 4 - i$$

In all the three different paths approaching Q from P, we get the same values of $\dfrac{\delta f}{\delta z} = 4 - i.$ In such a case, the function is said to be differentiable at the point z in the given region.

Example 9. *If* $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0, \\ 0, & z = 0 \end{cases}$ *then discuss* $\dfrac{df}{dz}$ *at* $z = 0$.

Solution. If $z \to 0$ along radius vector $y = mx$

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \left[\frac{\dfrac{x^3 y(y-ix)}{x^6+y^2} - 0}{x+iy} \right] = \lim_{z\to 0} \left[\frac{-ix^3 y\,(x+iy)}{(x^6+y^2)(x+iy)} \right]$$

$$= \lim_{z\to 0} \left[\frac{-ix^3 y}{x^6+y^2} \right] = \lim_{x\to 0} \left[\frac{-ix^3 (mx)}{x^6 + m^2 x^2} \right] \quad [\because\ y = mx]$$

$$= \lim_{x\to 0} \left[\frac{-imx^2}{x^4+m^2} \right] = 0$$

But along $y = x^3$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{-ix^3 y}{x^6+y^2}\right] = \lim_{x\to 0}\frac{-ix^3(x^3)}{x^6+(x^3)^2} = -\frac{i}{2}$$

In different paths we get different values of $\frac{df}{dz}$ *i.e.* 0 and $\frac{-i}{2}$. In such a case, the function is not differentiable at $z = 0$.

Theorem: Continuity is a necessary condition but not sufficient condition for the existence of a finite derivative.

Proof. We have, $f(z_0+\delta z) - f(z_0) = \delta z \left\{\frac{f(z_0+\delta z) - f(z_0)}{\delta z}\right\}$ ... (1)

Taking lim of both sides of (1), as $\delta z \to 0$, we get

$$\lim_{\delta z\to 0}[f(z_0+\delta z) - f(z_0)] = 0.f'(z_0) \Rightarrow \lim_{\delta z\to 0}[f(z_0+\delta z) - f(z_0)] = 0$$

$\Rightarrow$ $\lim_{z\to z_0}[f(z) - f(z_0)] = 0$ $\Rightarrow$ $\lim_{z\to z_0} f(z) = f(z_0)$

$\Rightarrow$ $f(z)$ is continuous at $z = z_0$. **Proved.**

The converse of the above theorem is not true.

This can be shown by the following example.

Example 10. *Prove that the function* $f(z) = |z|^2$ *is continuous everywhere but no where differentiable except at the origin.*

Solution. Here, $f(z) = |z|^2$.

$\therefore$ But $|z| = \sqrt{(x^2+y^2)}$ $\Rightarrow$ $|z|^2 = x^2 + y^2$

Since x^2 and y^2 are polynomial so $x^2 + y^2$ is continuous everywhere, therefore, $|z|^2$ is continuous everywhere.

Now, we have $f'(z) = \lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$

$$f'(z) = \lim_{\delta z\to 0}\frac{|z+\delta z|^2 - |z|^2}{\delta z} \qquad (z\bar{z} = |z|^2)$$

$$= \lim_{\delta z\to 0}\frac{(z+\delta z)(\bar{z}+\delta\bar{z}) - z\bar{z}}{\delta z} = \lim_{\delta z\to 0}\frac{z\bar{z} + z\delta\bar{z} + \delta z.\bar{z} + \delta z.\delta\bar{z} - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z\to 0}\frac{z.\delta\bar{z} + \delta z.\bar{z} + \delta z\delta\bar{z}}{\delta z} = \lim_{\delta z\to 0}\left\{\bar{z} + \delta\bar{z} + z\frac{\delta\bar{z}}{\delta z}\right\} = \lim_{\delta z\to 0}\left\{\bar{z} + z\frac{\delta\bar{z}}{\delta z}\right\} \quad ...(1)$$

[Since, $\delta z \to 0$ so $\delta\bar{z} \to 0$]

Let $\delta z = r(\cos\theta + i\sin\theta)$ and $\delta\bar{z} = r(\cos\theta - i\sin\theta)$

$\Rightarrow$ $\frac{\delta\bar{z}}{\delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$ $\Rightarrow$ $\frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)^{-1}$

$\Rightarrow$ $\frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)(\cos\theta - i\sin\theta)$ $\Rightarrow$ $\frac{\delta\bar{z}}{\delta z} = (\cos\theta - i\sin\theta)^2$

$\Rightarrow$ $\frac{\delta\bar{z}}{\delta z} = \cos 2\theta - i\sin 2\theta$

since $\frac{\delta\bar{z}}{\delta z}$ depends on θ. It means for different values of θ, $\frac{\delta\bar{z}}{\delta z}$ has different values.

It means $\frac{\delta\bar{z}}{\delta z}$ has different values for different z. $z = r(\cos\theta + i\sin\theta)$

Therefore $\lim_{\delta z \to 0} \frac{\delta \bar{z}}{\delta z}$ does not tend to a unique limit when $z \neq 0$.

Thus, from (1), it follows that $f'(z)$ is not unique and hence $f(z)$ is not differentiable when $z \neq 0$.

But when $z = 0$ then $f'(z) = 0$ *i.e.,* $f'(0) = 0$ and is unique.

Hence, the function is differentiable at $z = 0$. **Proved.**

By different method, the above example 10 is again solved as example 11 on page 143.

7.9 ANALYTIC FUNCTION

A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is **analytic** at every point of the domain.

The point at which the function is not differentiable is called a **singular point** of the function.

An analytic function is also known as "holomorphic", "regular", "monogenic".

Entire Function. A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

For Example 1. Polynomials rational functions are entire.

2. $|\bar{z}|^2$ is differentiable only at $z = 0$. So it is no where analytic.

Note: (*i*) An entire is always analytic, differentiable and continuous function. But converse is not true.

(*ii*) Analytic function is always differentiable and continuous. But converse is not true.

(*iii*) A differentiable function is always continuous. But converse is not true

7.10 THE NECESSARY CONDITION FOR F (Z) TO BE ANALYTIC

Theorem. *The necessary conditions for a function f (z) = u + iv to be analytic at all the points in a region R are*

(*i*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (*ii*) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ provided $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist.

Proof: Let $f(z)$ be an analytic function in a region R,

$$f(z) = u + iv,$$

where u and v are the functions of x and y.

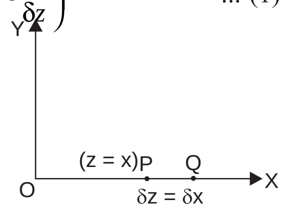
Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

$$\therefore \quad f(z + \delta z) = (u + \delta u) + i(v + \delta v)$$

Now $$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{(u+\delta u)+i(v+\delta v)-(u+iv)}{\delta z} = \frac{\delta u + i\delta v}{\delta z} = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \text{ or } f'(z) = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \dots (1)$$

since δz can approach zero along any path.



(*a*) **Along real axis (*x*-axis)**

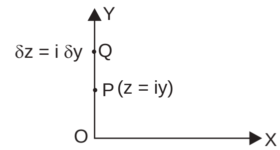
$z = x + iy$ but on x-axis, $y = 0$

$\therefore$ $z = x$, $\delta z = \delta x$, $\delta y = 0$

Putting these values in (1), we have

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \dots (2)$$

(*b*) **Along imaginary axis (*y*-axis)**



$z = x + iy$ but on y-axis, $x = 0$

$z = 0 + iy$ $\delta x = 0,\ \delta z = i\delta y.$

Putting these values in (1), we get

$$f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right) = \lim_{\delta y \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{... (3)}$$

If $f(z)$ is differentiable, then two values of $f'(z)$ must be the same.

Equating (2) and (3), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ **are known as Cauchy Riemann equations.**

7.11 SUFFICIENT CONDITION FOR F (Z) TO BE ANALYTIC

Theorem. *The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(*ii*) $\dfrac{\partial u}{\partial x}, \ \dfrac{\partial u}{\partial y}, \ \dfrac{\partial v}{\partial x}, \ \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in region R.*

Proof. Let $f(z)$ be a single-valued function having

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y}$$

at each point in the region R. Then the $C - R$ equations are satisfied.

By Taylor's Theorem:

$$f(z + \delta z) = u(x + \delta x,\ y + \delta y) + iv(x + \delta x,\ y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \ldots + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\right]$$

$$= [u(x, y) + iv(x, y)] + \left[\frac{\partial u}{\partial x}\cdot\delta x + i\frac{\partial v}{\partial x}\cdot\delta x\right] + \left[\frac{\partial u}{\partial y}\delta y + i\frac{\partial v}{\partial y}\cdot\delta y\right] + \ldots$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y + \ldots$$

(Ignoring the terms of second power and higher powers)

$$\Rightarrow \quad f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \quad \text{... (1)}$$

We know $C - R$ equations *i.e.*,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Replacing $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively in (1), we get

$$f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right).\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right).\delta y \qquad \text{(taking } i \text{ common)}$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right).\delta x+\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right).i\delta y=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right).(\delta x+i\delta y)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right).\delta z$$

$$\Rightarrow \quad \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \boxed{\boldsymbol{f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}}}$$

$$\Rightarrow \quad \boxed{\boldsymbol{f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}}}$$

Proved.

Remember: 1. If a function is analytic in a domain D, then u, v satisfy $C-R$ conditions at all points in D.

2. $C-R$ conditions are necessary but not sufficient for analytic function.

3. $C-R$ conditions are sufficient if the partial derivative are continuous.

Example 11. *Determine whether $\frac{1}{z}$ is analytic or not?* (*R.G.P.V. Bhopal, III Sem., June 2003*)

Solution. Let $w=f(z)=u+iv=\frac{1}{z} \quad \Rightarrow \quad u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$

Equating real and imaginary parts, we get

$$u=\frac{x}{x^2+y^2}, \qquad v=\frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Thus C – R equations are satisfied. Also partial derivatives are continuous except at (0, 0). Therefore $\frac{1}{z}$ is analytic everywhere except at $z=0$.

Also $$\frac{dw}{dz}=-\frac{1}{z^2}$$

This again shows that $\frac{dw}{dz}$ exists everywhere except at $z=0$. Hence $\frac{1}{z}$ is analytic everywhere except at $z=0$. **Ans.**

Example 12. *Show that the function $e^x(\cos y+i\sin y)$ is an analytic function, find its derivative.* (*R.G.P.V., Bhopal, III Semester, June 2008*)

Solution. Let $e^x(\cos y + i \sin y) = u + iv$

So, $e^x \cos y = u$ and $e^x \sin y = v$ then $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial y} = e^x \cos y$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Here we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

These are $C - R$ equations and are satisfied and the partial derivatives are continuous. Hence, $e^x(\cos y + i \sin y)$ is analytic.

$$f(z) = u + iv = e^x(\cos y + y \sin y) \text{ and } \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^x.e^{iy} = e^{x+iy} = e^z.$$

Which is the required derivative. **Ans.**

Example 13. *Test the analyticity of the function w = sin z and hence derive that:*

$$\frac{d}{dz}(\sin z) = \cos z$$

Solution. $w = \sin z = \sin(x + iy)$
$= \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$, $v = \cos x \sinh y$ $\left[\begin{matrix}\cos iy = \cosh y \\ \sin iy = i \sinh y\end{matrix}\right]$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \frac{\partial v}{\partial y} = \cos x \cosh y$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So $C - R$ equations are satisfied and partial derivatives are continuous.

Hence, $\sin z$ is an analytic function.

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}[\sin x \cosh y + i \cos x \sinh y]$$

$$= \frac{\partial}{\partial x}(\sin x \cosh y + i \cos x \sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y = \cos x \cos iy - \sin x \sin iy$$

$$= \cos(x + iy) = \cos z$$

Ans.

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{... (1)}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{... (2)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{... (3)}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \text{... (4)}$$

From (1) $\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$

From (3) $\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2}$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

From (4) $\sin i x = \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$

$$= i\,\frac{e^x - e^{-x}}{2} = i \sinh x$$

From (2) $\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = i \sin x$

Example 14. *Show that the real and imaginary parts of the function w = log z satisfy the Cauchy-Riemann equations when z is not zero. Find its derivative.*

Solution. To separate the real and imaginary parts of log *z*, we put $x = r\cos\theta;\ y = r\sin\theta$

$$w = \log z = \log(x + iy)$$

$\Rightarrow$ $u + iv = \log(r\cos\theta + ir\sin\theta) = \log r(\cos\theta + i\sin\theta) = \log_e r.e^{i\theta}$

$$= \log_e r + \log_e e^{i\theta} = \log r + i\theta = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \qquad \left[\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x}\end{aligned}\right]$$

So $$u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2),\ v = \tan^{-1}\frac{y}{x}$$

On differentiating *u*, *v*, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}.(2x) = \frac{x}{x^2+y^2} \qquad \text{... (1)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \qquad \text{... (2)}$$

From (1) and (2), $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ ... (A)

Again differentiating *u*, *v*, we have

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2} \qquad \text{... (3)}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \qquad \text{... (4)}$$

From (3) and (4), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{... (B)}$$

Equations (A) and (B) are *C* – *R* equations and partial derivatives are continuous.
Hence, *w* = log *z* is an analytic function except
when $x^2 + y^2 = 0 \Rightarrow x = y = 0 \Rightarrow x + iy = 0 \Rightarrow z = 0$
Now $w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

Which is the required derivative. **Ans.**

Example 15. *Discuss the analyticity of the function* $f(z) = z\bar{z}$.

Solution. $f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2$

$$f(z) = x^2+y^2 = u+iv.$$

$$u = x^2+y^2,\ v = 0$$

At origin, $$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,\ 0)-u(0,\ 0)}{h} = \lim_{h\to 0}\frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0,0)}{k} = \lim_{k \to 0} \frac{k^2}{k} = 0$$

Also,
$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0,0)}{k} = 0$$

Thus,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, C – R equations are satisfied at the origin.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^2 + y^2) - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{(x^2 + m^2 x^2)}{(x + imx)} = \lim_{x \to 0} \frac{(1 + m^2) x}{1 + im} = 0$$

Therefore, $f'(0)$ is unique. Hence the function $f(z)$ is analytic at $z = 0$. **Ans.**

Example 16. *Show that the function f (z) = u + iv, where*

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ = 0, & z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at z = 0. Is the function analytic at z = 0 ? Justify your answer. *(MDU Dec 2009)*

Solution.
$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = u + iv$$

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v = \frac{x^3 + y^3}{x^2 + y^2}$$

[By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at (0, 0) we get $\frac{0}{0}$, so we apply first principle method]

At the origin

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,\ 0+k) - u(0,\ 0)}{k} = \lim_{k \to 0} \frac{\frac{-k^3}{k^2}}{k} = -1 \qquad \text{(Along } y\text{- axis)}$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,\ 0) - v(0,\ 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,\ 0+k) - v(0,\ 0)}{k} = \lim_{k \to 0} \frac{\frac{k^3}{k^2}}{k} = 1 \qquad \text{(Along } y\text{-axis)}$$

Thus we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again $$f'(0) = \lim_{z\to 0} \frac{f(0+z) - f(0)}{z} = \lim_{z\to 0} \left[\frac{\frac{x^3 - y^3 + i\,(x^3 + y^3)}{x^2 + y^2} - (0)}{x + i\,y} \right]$$

$$= \lim_{z\to 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} \right]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{x^2 + x^2} \left(\frac{1}{x + ix} \right)$$

$$= \frac{2i}{2(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i+1}{1+1} = \frac{1}{2}(1+i) \qquad \text{... (1)}$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = (1+i) \qquad \text{[Increment = } z\text{]} \qquad \text{... (2)}$$

From (1) and (2), we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

Example 17. *Show that the function*

$$f(z) = e^{-z^{-4}}, \qquad (z \neq 0 \quad \text{and}$$
$$f(0) = 0$$

is not analytic at $z = 0$,
although, Cauchy-Riemann equations are satisfied at the point. How would you explain this.

Solution. $$f(z) = u + iv = e^{-z^{-4}} = e^{-(x+iy)^{-4}} = e^{-\frac{1}{(x+iy)^4}}$$

$$\Rightarrow \qquad u + iv = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2)-i4xy(x^2-y^2)]}$$

$$\Rightarrow \qquad u + iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} . e^{\frac{-i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\Rightarrow \qquad u + iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} - i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]}$$

Equating real and imaginary parts, we get

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}, \quad v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

At $z = 0$ $$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h\to 0} \frac{e^{-h^{-4}}}{h} = \operatorname{Lim}_{h\to 0} \frac{1}{h\,e^{\frac{1}{h^4}}}$$

$$= \operatorname{Lim}_{h\to 0} \left[\frac{1}{h\left[1 + \frac{1}{h^4} + \frac{1}{2!h^8} + \frac{1}{3!h^{12}} + ...\right]} \right], \quad \left(e^x = 1 + x + \frac{x^2}{2!} + ... \right)$$

$$= \lim_{h \to 0} \left[\frac{1}{\left[h + \frac{1}{h^3} + \frac{1}{2\,h^7} + \frac{1}{6h^{11}} \ldots\ldots \right]} \right] = \frac{1}{0 + \infty} = \frac{1}{\infty} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,\, 0+k) - u(0,\, 0)}{k} = \lim_{k \to 0} \frac{e^{-k^{-4}}}{k} = \lim_{k \to 0} \frac{1}{k\, e^{\frac{1}{k^4}}} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,\; 0) - v(0,\, 0)}{h} = \lim_{h \to 0} \frac{e^{-h^{-4}}}{h} = \lim_{h \to 0} \frac{1}{h . e^{\frac{1}{h^4}}} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,\, 0+k) - v(0,\, 0)}{k} = \lim_{k \to 0} \frac{e^{-k^{-4}}}{k} = \lim_{k \to 0} \frac{1}{k . e^{\frac{1}{k^4}}} = 0$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($C - R$ equations are satisfied at $z = 0$)

But $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{e^{-z^{-4}}}{z}$

Along $z = re^{i\frac{\pi}{4}}$
$$f'(0) = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e^{-\left(e^{i\frac{\pi}{4}}\right)^{-4}}}{r\,e^{i\frac{\pi}{4}}} = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e^{-\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-4}}}{r\,e^{i\frac{\pi}{4}}}$$

$$= \lim_{r \to 0} \frac{e^{-r^{-4}} e^{-\cos\pi}}{r\,e^{i\frac{\pi}{4}}} = \lim_{r \to 0} \frac{e^{-r^{-4}} \cdot e}{r\,e^{i\frac{\pi}{4}}} = \infty$$

Showing that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$. **Proved.**

Example 18. *Examine the nature of the function*

$$f(z) = \frac{x^2\, y^5\, (x + iy)}{x^4 + y^{10}}\, ; z \neq 0$$

$$f(0) = 0$$

in the region including the origin.

Solution. Here $f(z) = u + iv = \frac{x^2 y^5\, (x + iy)}{x^4 + y^{10}}\, ; z \neq 0$

Equating real and imaginary parts, we get

$$u = \frac{x^3 y^5}{x^4 + y^{10}}, \quad v = \frac{x^2 y^6}{x^4 + y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0 + k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} - 0\right] \cdot \frac{1}{x+iy} \quad \text{(Increment} = z)$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = \frac{0}{1} = 0 \qquad \text{... (1)}$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \qquad \text{... (2)}$$

(1) and (2) shows that $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there. **Ans.**

7.12 C–R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

(*MDU, Dec. 2010, RGPV., K.U. 2009, Bhopal, III Sem. Dec. 2007*)

Proof. We know that $x = r \cos \theta$, and u is a function of x and y.

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t., "r", we get

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(r e^{i\theta}).e^{i\theta} \qquad \text{... (2)}$$

Differentiating (1) w.r.t. "θ", we get

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = f'(r e^{i\theta})\, r e^{i\theta} i \qquad \text{... (3)}$$

Substituting the value of $f'(r e^{i\theta}) e^{i\theta}$ from (2) in (3), we obtain

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}\right) i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = ir \frac{\partial u}{\partial r} - r \frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad \Rightarrow \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

And $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ **Proved.**

7.13 DERIVATIVE OF W OR F (Z) IN POLAR FORM

We know that $w = u + iv$, $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

But
$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + i\cdot r\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial}{\partial r}(u + iv)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta \qquad [\because w = u + iv]$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} \qquad \text{... (1)}$$

Second form of $\frac{dw}{dz}$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial (u + iv)}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \qquad [w = u + iv]$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right)$$

$$= -\frac{i}{r}\frac{\partial}{\partial \theta}(u + iv)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right) = -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \qquad [w = u + iv]$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta} \qquad \text{... (2)}$$

$$\boxed{\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}} \qquad \left[-\frac{i}{r}\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial r}\right]$$

$$\boxed{\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}}$$

These are the two forms for $\frac{dw}{dz}$.

EXERCISE 7.3

Determine which of the following functions are analytic:

1. $x^2 + iy^2$ **Ans.** Analytic at all points $y = x$

2. $2xy + i(x^2 - y^2)$ **Ans.** Not analytic

3. $\dfrac{x-iy}{x^2+y^2}$ **Ans.** Not analytic

4. $\dfrac{1}{(z-1)(z+1)}$ **Ans.** Analytic at all points, except $z = \pm 1$

5. $\dfrac{x-iy}{x-iy+a}$ **Ans.** Not analytic

6. $\sin x \cosh y + i \cos x \sinh y$ **Ans.** Yes, analytic

7. $xy + iy^2$ **Ans.** Yes, analytic at origin

8. Discuss the analyticity of the function $f(z) = z\bar{z} + \bar{z}^2$ in the complex plane, where $\bar{z}$ is the complex conjugate of z. Also find the points where it is differentiable but not analytic.

Ans. Differentiable only at $z = 0$, No where analytic.

9. Show the function of $\bar{z}$ is not analytic any where.

10. If $f(z) = \begin{cases} \dfrac{x^2 y\,(y-ix)}{x^4+y^2}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases}$

prove that $\dfrac{f(z) - f(0)}{z} \to 0$, as $z \to 0$, along any radius vector but not as $z \to 0$ in any manner. (*AMIETE, Dec. 2010*)

11. If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant. (*AMIETE, Dec. 2009*)

Choose the correct answer :

12. The Cauchy-Riemann equations for $f(z) = u(x, y) + iv(x, y)$ to be analytic are :

(*a*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

(*b*) $\dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(*c*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(*d*) $\dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$ **Ans.** (*b*)

(*R.G.P.V., Bhopal, III Semester, Dec. 2006*)

13. Polar form of *C-R* equations are :

(*a*) $\dfrac{\partial u}{\partial \theta} = \dfrac{1}{r}\dfrac{\partial v}{\partial r},\ \dfrac{\partial u}{\partial r} = r\dfrac{\partial v}{\partial \theta}$

(*b*) $\dfrac{\partial u}{\partial \theta} = r\dfrac{\partial v}{\partial r},\ \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$

(*c*) $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta},\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$

(*d*) $\dfrac{\partial u}{\partial r} = r\dfrac{\partial v}{\partial \theta},\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$ **Ans.** (*c*)

(*R.G.P.V., Bhopal, III Semester, June, 2007*)

14. The curve $u(x, y) = C$ and $v(x, y) = C'$ are orthogonal if

(*a*) *u* and *v* are complex functions (*b*) $u + iv$ is an analytic function.

(*c*) $u - v$ is an analytic function. (*d*) u + v is an analytic function

Ans. (*b*) $\left[\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r},\ \dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r}\right]$

15. If $f(z) = \dfrac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\left(\dfrac{\alpha x}{y}\right)$ be an analytic function α is equal to α, if

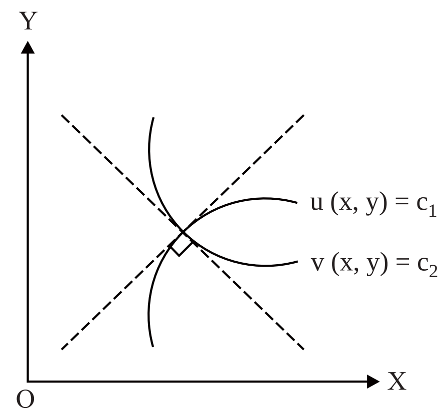
(*a*) + 1 (*b*) – 1 (*c*) + 2 (*d*) – 2 (*AMIETE, Dec. 2009*)

7.14 ORTHOGONAL CURVES (U.P. III SEMESTER, JUNE 2009)

Two curves are said to be orthogonal to each other, when they intersect at right angle at each of their points of intersection.

The analytic function $f(z) = u + iv$ consists of two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ which form an orthogonal system.

$$u(x, y) = c_1 \quad ...(1)$$
$$v(x, y) = c_2 \quad ...(2)$$



Differentiating (1), $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}$$

Similarly from (2), $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2 \ (say)$

The product of two slopes

$$m_1 m_2 = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right) \qquad (C-R \text{ equations})$$
$$= -1$$

Since $m_1 m_2 = -1$, two curves $u = c_1$ and $v = c_2$ are orthogonal, and c_1, c_2 are parameters, $u = c_1$ and $v = c_2$ form an orthogonal system.

7.15 HARMONIC FUNCTION

(*U.P., III Semester 2009-2010*)

Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem. If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$, be an analytic function, then we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad ...(1) \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \quad ...(2) \end{aligned}\right] \quad C-R \text{ equations.}$$

Differentiating (1) with respect to x, we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ... (3)

Differentiating (2) w.r.t. 'y' we have $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ ... (4)

Adding (3) and (4) we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore both u and v are harmonic functions.

Such functions u, v are called **Conjugate harmonic functions if $u + iv$ is also analytic function.**

Example 19. *Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate function of the function $U(x, y) = 2x(1 - y)$.* (*U.P., III Semester Dec. 2009*)

Solution. See Art. 4.15

Here, we have $U(x, y) = 2x(1 - y)$. Let V be the harmonic conjugate of U.

By total differentiation

$$\left[\begin{array}{l} U = 2x - 2xy \\ \dfrac{\partial U}{\partial x} = 2 - 2y \\ \dfrac{\partial U}{\partial y} = -2x \end{array}\right]$$

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \\ &= -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \\ &= -(-2x)\, dx + (2 - 2y)\, dy + C \\ &= 2x\, dx + (2\, dy - 2y\, dy) + C \\ V &= x^2 + 2y - y^2 + C \end{aligned}$$

Hence, the harmonic conjugate of U is $x^2 + 2y - y^2 + C$ **Ans.**

Example 20. *Prove that* $u = x^2 - y^2$ *and* $v = \dfrac{y}{x^2 + y^2}$ *are harmonic functions of* (x, y), *but are not harmonic conjugates.*

Solution. We have, $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic

$$v = \frac{y}{x^2 + y^2}, \qquad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2(-2y) - (-2xy)\,2(x^2 + y^2)\,2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(-2y) - (-2xy)\,4x}{(x^2 + y^2)^3} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2 + y^2)\cdot 1 - y\,(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \text{... (1)}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - (x^2 - y^2)\,2(x^2 + y^2)(2y)}{(x^2 + y^2)^4} = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)(4y)}{(x^2 + y^2)^3}$$

$$= \frac{-2x^2 y - 2y^3 - 4x^2 y + 4y^3}{(x^2 + y^2)^3} = \frac{-6x^2 y + 2y^3}{(x^2 + y^2)^3} \qquad \text{... (2)}$$

On adding (1) and (2), we get $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

$v(x, y)$ also satisfies Laplace equations, hence $v(x, y)$ is also harmonic function.

But $\qquad \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y} \qquad$ and $\qquad \dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

Therefore u and v are not harmonic conjugates. **Proved.**

Example 21. *If u(x, y) and v(x, y) are harmonic functions in a region R, prove that the function*

$$\left[\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$

is an analytic function of z = x + iy. (*R.G.P.V., Bhopal, III Semester, Dec. 2004*)

Solution. Since $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, therefore

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1) \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \quad \ldots (2)$$

Let $$F(z) = R+iS = \left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Equating real and imaginary parts, we get

$$R = \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x},$$

$$\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2} \quad \ldots (3) \qquad \frac{\partial R}{\partial y} = \frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\,\partial y} \quad \ldots (4)$$

$$S = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

$$\frac{\partial S}{\partial x} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y} \quad \ldots (5) \qquad \frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y^2} \quad \ldots (6)$$

Putting the value of $\frac{\partial^2 u}{\partial x^2}$ from (1) in (5), we get

$$\frac{\partial S}{\partial x} = -\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y} \quad \ldots (7)$$

Putting the value of $\frac{\partial^2 v}{\partial y^2}$ from (2) in (6), we get

$$\frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2} \quad \ldots (8)$$

From (3) and (8), $$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y}$$

From (4) and (7), $$\frac{\partial R}{\partial y} = -\frac{\partial S}{\partial x}$$

Therefore, C-R equations are satisfied and hence the given function is analytic. **Proved.**

7.16 APPLICATION TO FLOW PROBLEMS

Consider two dimensional irrotational motion in a plane parallel to xy-plane.

The velocity v of fluid can be expressed as

$$v = v_x\,\hat{i}+v_y\,\hat{j} \quad \ldots (1)$$

Since the motion is irrotational, a scalar function $\phi((x, y)$ gives the velocity components.

$$V = \nabla\,\phi(x, y) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}\right)\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y} \quad \ldots (2)$$

On comparing (1) and (2), we get

$$= \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} \qquad \text{... (3)}$$

7.17 VELOCITY POTENTIAL FUNCTION

The scalar function $\phi(x, y)$ which gives the velocity component is called the velocity potential function.

As the fluid is incompressible

$$div\ v = 0$$

$$\nabla v = 0 \quad \Rightarrow \quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot(\hat{i}v_x + \hat{j}v_y) = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{... (4)}$$

Putting the values of v_x and v_y from (3) in (4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

This is Laplace equation. The function ϕ is harmonic and is a real part of analytic function

$$f(z) = \phi(x, y) + i\phi(x, y)$$

We know that

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}} \qquad f(z) = \psi + \phi \quad [CR\text{-equations}]$$

$$= \frac{v_y}{v_x} \qquad \text{[Using (3)]}$$

Here the resultant velocity $\sqrt{v_x^2 + v_y^2}$ of the fluid is along the tangent to the curve

$$\psi(x, y) = C'$$

Such curves are known as *stream lines* and $\psi(x, y)$ is known as *stream function*.

The curves represented by $\phi(x, y) = c$ are called *equipotential lines*.

As $\phi(x, y)$ and $\psi(x, y)$ are conjugates of analytic function $f(z)$. The equipotential lines $\phi(x, y) = C$ and the stream potential line $\psi(x, y) = C'$ intersect each other orthogonally.

$$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial y} = v_x - iv_y$$

The magnitide of the resultant velocity $= \left|\frac{df}{dz}\right| = \sqrt{v_x^2 + v_y^2}$

The function $f(z)$ which represents the flow pattern is called the *complex potential.*

7.18 METHOD TO FIND THE CONJUGATE FUNCTION

Case I. Given. If $f(z) = u + iv$, and u is known.

To find. v, conjugate function.

Method. We know that $dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$ ... (1)

Replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ in (1), we get [C-R equations]

$$dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$$

$$v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}\cdot dy$$

$\Rightarrow$ $$v = \int M\,dx + \int N\,dy \quad \text{... (2)}$$

where $$M = -\frac{\partial u}{\partial y} \text{ and } N = \frac{\partial u}{\partial x}$$

so that $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

since u is a conjugate function, so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow$ $$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{... (3)}$$

Equation (3) satisfies the condition of an exact differential equation.
So equation (2) can be integrated and thus v is determined.
Case II. Similarly, if $v = v(x, y)$ is given
To find out u.

We know that $$du = \frac{\partial u}{\partial x}dx + i\,\frac{\partial u}{\partial y}\,dy \quad \text{... (4)}$$

On substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in (4) , we get

$$du = \frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy$$

On integrating, we get

$$u = \int \frac{\partial v}{\partial y}\,dx - \int \frac{\partial v}{\partial x}\,dy \quad \text{...(5)}$$

(since v is already known so $\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ on R.H.S. are also known)

Equation (5) is an exact differential equation. On solving (5), u can be determined.
Consequently $f(z) = u + iv$ can also be determined.

Example 22. *Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.*

(*R.G.P.V., Bhopal, III Semester, June 2005*)

Solution. We have,

$$u = x^2 - y^2 - 2xy - 2x + 3y$$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Since Laplace equation is satisfied, therefore u is harmonic. **Proved.**

We know that $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad ...(1) \qquad \left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right]$$

Putting the values of $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ in (1), we get

$$dv = -(-2y - 2x + 3)\, dx + (2x - 2y - 2)dy$$

$$\Rightarrow \quad v = \int (2y + 2x - 3)dx + \int (-2y - 2)dy + C \qquad \text{(Ignoring } 2x\text{)}$$

Hence, $v = 2xy + x^2 - 3x - y^2 - 2y + C$ **Ans.**

Now,
$$\begin{aligned} f(z) &= u + iv \\ &= (x^2 - y^2 - 2xy - 2x + 3y) + i\,(2xy + x^2 - 3x - y^2 - 2y) + iC \\ &= (x^2 - y^2 + 2ixy) + (ix^2 - iy^2 - 2xy) - (2 + 3i)\,x - i\,(2 + 3i)\,y + iC \\ &= (x^2 - y^2 + 2ixy) + i\,(x^2 - y^2 + 2ixy) - (2 + 3i)\,x - i\,(2 + 3i)y + iC \\ &= (x + iy)^2 + i\,(x + iy)^2 - (2 + 3i)\,(x + iy) + iC \\ &= z^2 + iz^2 - (2 + 3i)\,z + iC \\ &= (1 + i)\,z^2 - (2 + 3i)\,z + iC \end{aligned}$$

Which is the required expression of $f(z)$ in terms of z. **Ans.**

Example 23. *If $w = \phi + i\psi$ represents the complex potential for an electric field and*

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2},$$

determine the function ϕ.

Solution. $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2 + y^2)\cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{x(2y)}{(x^2 + y^2)^2} = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

We know that, $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$

$$= \left(-2y - \frac{2xy}{(x^2 + y^2)^2}\right)dx - \left(2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right)dy$$

$$\phi = \int \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] dx + c$$

This is an exact differential equation.

$$\phi = -2xy + \frac{y}{x^2 + y^2} + C$$ **Ans.**

Which is the required function.

Example 24. *An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.* (*R.G.P.V., Bhopal, III Semester, Dec. 2001*)

Solution. Let $\psi(x, y)$ be a stream function

We know that $$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = \left(-\frac{\partial \phi}{\partial y}\right)dx + \left(\frac{\partial \phi}{\partial x}\right)dy \qquad \text{[C-R equations]}$$

$$= \{-(3x^2 - 3y^2)\}\, dx + 6xy\, dy$$
$$= -3x^2\, dx + (3y^2\, dx + 6xy\, dy)$$
$$= -d\,(x^3) + 3d\,(xy^2)$$
$$\psi = \int -d\,(x^3) + 3d\,(xy^2) + c$$
$$\psi = -x^3 + 3xy^2 + c$$

ψ is the required stream function. **Ans.**

Example 25. *Find the imaginary part of the analytic function whose real part is* $x^3 - 3xy^2 + 3x^2 - 3y^2$. (*R.G.P.V., Bhopal, III Semester, Dec. 2008, 2005*)

Solution. Let $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$
$$\frac{\partial u}{\partial y} = -6xy - 6y$$

We know that

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \Rightarrow \quad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\Rightarrow \quad dv = (6xy + 6y)\, dx + (3x^2 - 3y^2 + 6x)\, dy$$

This is an exact differential equation.

$$v = \int (6x\,y + 6y)\, dx + \int -3y^2 dy + C$$
$$= 3x^2 y + 6xy - y^3 + C$$

Which is the required imaginary part. **Ans.**

Example 26. *If* $u - v = (x - y)(x^2 + 4\,xy + y^2)$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of z.*

Solution. $u + iv = f(z) \quad \Rightarrow \quad iu - v = i f(z)$

Adding these, $(u - v) + i\,(u + v) = (1 + i)\, f(z)$

Let $U + iV = (1 + i)\, f(z)$ where $U = u - v$ and $V = u + v$

$$F(z) = (1 + i)\, f(z)$$
$$U = u - v = (x - y)(x^2 + 4\,xy + y^2)$$
$$= x^3 + 3\,x^2 y - 3\,xy^2 - y^3$$
$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$$
$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

We know that $dV = \frac{\partial V}{\partial x} \cdot dx + \frac{\partial V}{\partial y} dy = -\frac{\partial U}{\partial y} \cdot dx + \frac{\partial U}{\partial x} \cdot dy$ [C-R equations]

On putting the values of $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, we get

$$= (-3x^2 + 6\,xy + 3\,y^2)\, dx + (3\,x^2 + 6xy - 3\,y^2) \,.\, dy$$

Integrating, we get

$$V = \int (-3x^2 + 6xy + 3y^2) dx + \int (-3y^2) dy$$

(y as constant) (Ignoring terms of x)

$$= -x^3 + 3\,x^2 y + 3xy^2 - y^3 + c$$
$$F(z) = U + iV$$
$$= (x^3 + 3\,x^2 y - 3\,xy^2 - y^3) + i\,(-x^3 + 3\,x^2 y + 3xy^2 - y^3) + ic$$
$$= (1 - i)\, x^3 + (1 + i)\, 3\,x^2 y - (1 - i)\, 3\,xy^2 - (1 + i)\, y^3 + ic$$
$$= (1 - i)\, x^3 + i\,(1 - i)\, 3\,x^2 y - (1 - i)\, 3\,xy^2 - i\,(1 - i)\, y^3 + ic$$

$$= (1 - i)\,[x^3 + 3\,ix^2y - 3\,xy^2 - iy^3] + ic$$
$$= (1 - i)\,(x + iy)^3 + iC = (1 - i)\,z^3 + ic$$
$$(1 + i)\,f(z) = (1 - i)\,z^3 + ic, \qquad [F(z) = (1 + i)\,f(z)]$$
$$f(z) = \frac{1-i}{1+i} z^3 + \frac{ic}{1+i} = -\frac{i\,(1+i)}{(1+i)} z^3 + \frac{i\,(1-i)}{(1+i)\,(1-i)} c = -iz^3 + \frac{1+i}{2} c \qquad \textbf{Ans.}$$

Example 27. *If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}\,[(x - y)\sin y - (x + y)\cos y]$ find $f(z)$.* (*U.P. III Semester, 2009-2010*)

Solution. We know that,

$$f(z) = u + iv \qquad \dots (1)$$
$$i\,f(z) = i\,u - v \qquad \dots (2)$$
$$F(z) = U + iV$$
$$U = u - v = e^{-x}\,[(x - y)\sin y - (x + y)\cos y]$$
$$\frac{\partial U}{\partial x} = -\,e^{-x}\,[(x - y)\sin y - (x + y)\cos y] + e^{-x}\,[\sin y - \cos y]$$
$$\frac{\partial U}{\partial y} = e^{-x}\,[(x - y)\cos y - \sin y - (x + y)(-\sin y) - \cos y]$$

We know that,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \qquad \text{[C – R equations]}$$
$$= -\,e^{-x}\,[(x - y)\cos y - \sin y + (x + y)\sin y - \cos y]\,dx$$
$$-\,e^{-x}\,[(x - y)\sin y - (x + y)\cos y - \sin y + \cos y]\,dy$$
$$= -\,e^{-x}\,x\,\{(\cos y + \sin y)\,dx - e^{-x}\,(-y\cos y - \sin y + y\sin y - \cos y)\,dx$$
$$-\,e^{-x}\,[(x - y)\sin y - (x + y)\cos y - \sin y + \cos y]\,dy$$
$$V = (\cos y + \sin y)\,(x\,e^{-x} + e^{-x}) + e^{-x}\,(-y\cos y - \sin y + y\sin y - \cos y) + C$$
$$F(z) = U + iV$$
$$F(z) = e^{-x}\,[(x - y)\sin y - (x + y)\cos y] + i\,e^{-x}\,[x\cos y + \cos y + x\sin y + \sin y$$
$$- y\cos y - \sin y + y\sin y - \cos y] + iC$$
$$= e^{-x}\,[\{x\sin y - y\sin y - x\cos y - y\cos y\} + i\{x\cos y + x\sin y - y\cos y + y\sin y\}] + iC$$
$$= e^{-x}\,[(x + iy)\sin y - (x + i\,y)\cos y + (-y + i\,x)\sin y + (-\,y + i\,x)\cos y] + iC$$
$$= e^{-x}\,[(x + iy)\sin y - (x + i\,y)\cos y + i\,(x + i\,y)\sin y + i\,(x + i\,y)\cos y] + iC$$
$$= e^{-x}\,(x + i\,y)\,[\sin y - \cos y + i\sin y + i\cos y] + i\,C$$
$$= e^{-x}\,(x + i\,y)\,[(1 + i)\sin y + i\,(1 + i)\cos y] + i\,C$$
$$(1 + i)\,f(z) = e^{-x}\,(x + iy)\,(1 + i)\,(\sin y + i\cos y) + i\,C$$
$$f(z) = e^{-x}\,(x + i\,y)\,(\sin y + i\cos y) + \frac{iC}{1+i}$$
$$= i\,z\,e^{-x}\,(\cos y - i\sin y) + \frac{iC}{1+i}$$
$$= i\,z\,e^{-x}\,e^{-iy} = i\,z\,e^{-(x + iy)} = i\,z\,e^{-z} + \frac{iC}{1+i} \qquad \textbf{Ans.}$$

Let $\phi_1(x, y) = -\,e^{-x}\,[(x - y)\sin y - (x + y)\cos y] + e^{-x}\,[\sin y - \cos y]$

$\phi_1(z, 0) = -\,e^{-z}\,[z\sin 0 - z\cos 0] + e^{-z}\,[\sin 0 + \cos 0]$

$= -\,e^{-z}\,[z - 1]$

Let $\phi_2(x, y) = e^{-x}\,[(x - y)\cos y - \sin y + (x + y)\sin y - \cos y]$

$\phi_2(z, 0) = e^{-z}\,[(z)\cos 0 - \sin 0 + z\sin 0 - \cos 0]$

$$= e^{-z}[z-1]$$
$$F(z) = U + iV$$
$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = f_1(z, 0) - i f_2(z, 0)$$
$$= e^{-z}(z-1) - i e^{-z}(z-1) = (1-i) e^{-z}(z-1) = (1-i) e^{-z}(z-1)$$
$$F(z) = (1-i)\left[z\frac{e^{-z}}{-1} - \int\frac{e^{-z}}{-1}dz\right] + C = (1-i)[-z e^{-z} - e^{-z}] + C$$
$$(1+i) f(z) = (-1+i)(z+1) e^{-z} + C$$
$$f(z) = \frac{(-1+i)}{1+i}(z+1)e^{-z} + C = \frac{(-1+i)(1-i)}{(1+i)(1-i)}(z+1)e^{-z} + C$$
$$= i(z+1)e^{-z} + C$$ **Ans.**

Example 28. *Let f(z) = u (r, θ) + iv (r, θ) be an analytic function and u = – r^3 sin 3θ. then construct the corresponding analytic function f (z) in terms of z.*

Solution. $u = -r^3 \sin 3\theta$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \qquad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

We know that $dv = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial \theta}d\theta$

$$= \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right)dr + \left(r\frac{\partial u}{\partial r}\right)d\theta \qquad \left(\begin{array}{l} C-R \text{ equations} \\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}\end{array}\right)$$

$$= -\frac{1}{r}(-3r^3 \cos 3\theta)\, dr + r(-3r^2 \sin 3\theta)\, d\theta$$
$$= 3r^2 \cos 3\theta \cdot dr - 3r^3 \sin 3\theta\, d\theta$$
$$v = \int (3r^2 \cos 3\theta) dr - c = r^3 \cos 3\theta + c$$
$$f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic = ir^3(\cos 3\theta + i \sin 3\theta) + ic$$
$$= i r^3 e^{i3\theta} + ic = i(r e^{i\theta})^3 + ic = i z^3 + ic$$ **Ans.**

This is the required analytic function.

Example 29. *Find analytic function f (z) = u(r, θ) + iv (r, θ) such that*
$$v(r, \theta) = r^2 \cos 2\theta - r\cos\theta + 2.$$

Solution. We have, $v = r^2 \cos 2\theta - r \cos\theta + 2$ ... (1)

Differentiating (1), we get

$$\frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin\theta \qquad \text{... (2)}$$
$$\frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos\theta \qquad \text{... (3)}$$

Using $C-R$ equations in polar coordinates, we get

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r\sin\theta \qquad \text{[From (2)]}$$
$$\Rightarrow \quad \frac{\partial u}{\partial r} = -2r \sin 2\theta + \sin\theta \qquad \text{... (4)}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \qquad \text{[From (3)]}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta \qquad \text{... (5)}$$

By total differentiation formula

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta = (-2r\sin 2\theta + \sin\theta)dr + (-2r^2\cos 2\theta + r\cos\theta)d\theta$$

$$= -[(2r\,dr)\sin 2\theta + r^2(2\cos 2\theta\, d\theta)] + [\sin\theta\cdot dr + r(\cos\theta\, d\theta)]$$

$$= -[(2r\,dr)\sin 2\theta - \sin\theta dr] + [-r^2 2\cos 2\theta\, d\theta + r\cos\theta\, d\theta]$$

$$= -d(r^2\sin 2\theta) + d(r\sin\theta) \qquad \text{(Exact differential equation)}$$

Integrating, we get

$$u = -r^2\sin 2\theta + r\sin\theta + c$$

Hence,

$$f(z) = u + iv$$

$$= (-r^2\sin 2\theta + r\sin\theta + c) + i(r^2\cos 2\theta - r\cos\theta + 2)$$

$$= ir^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + 2i + c$$

$$= ir^2e^{2i\theta} - ir\,e^{i\theta} + 2i + c = i(r^2e^{2i\theta} - r\,e^{i\theta}) + 2i + c. \qquad \textbf{Ans.}$$

This is the required analytic function.

Example 30. *Deduce the following with the polar form of Cauchy-Riemann equations :*

(*a*) $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ (*MDU, Dec. 2010, K.U. 2009*) (*b*) $f'(z) = \dfrac{r}{z}\left[\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right]$

Solution. We know that polar form of C-R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{... (1)}$$

$$\frac{\partial u}{\partial \theta} = -r\,\frac{\partial v}{\partial r} \qquad \text{... (2)}$$

(*a*) Differentiating (1) partially w.r.t. r., we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta} \qquad \text{... (3)}$$

Differentiating (2) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta\partial r} \qquad \text{...(4)}$$

Thus using (1), (3) and (4), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial \theta\partial r}\right) = 0 \qquad \left[\frac{\partial^2 v}{\partial r\partial \theta} = \frac{\partial^2 v}{\partial \theta\partial r}\right]$$

Proved.

(*b*) Now,

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+r\sin\theta\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+iy\left(\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\right) \qquad \text{(By C-R equations)}$$

$$= x\,f'(z) + iy\,f'(z) = (x+iy)\,f'(z) = z\,f'(z).$$

$$\therefore \qquad f'(z)=\frac{r}{z}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \qquad \textbf{Proved.}$$

7.19 MILNE THOMSON METHOD (TO CONSTRUCT AN ANALYTIC FUNCTION)

By this method $f(z)$ is directly constructed without finding v and the method is given below:

Since $\qquad z = x + iy$ and $\bar{z} = x - iy$

$$\therefore \qquad x=\frac{z+\bar{z}}{2},\; y=\frac{z-\bar{z}}{2i}$$

$$f(z) \equiv u(x,y)+iv(x,y) \qquad \text{... (1)}$$

$$f(z) \equiv u\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)+iv\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)$$

This relation can be regarded as a formal identity in two independent variables z and $\bar{z}$. Replacing $\bar{z}$ by z, we get

$$f(z) \equiv u(z,0)+iv(z,0)$$

Which can be obtained by replacing x by z and y by 0 in (1)

Case I. If *u* is given

We have $\qquad f(z) = u + iv$

$$\therefore \qquad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}, \qquad f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad (C-R \text{ equations})$$

If we write $\qquad \dfrac{\partial u}{\partial x}=\phi_1(x,y), \qquad \dfrac{\partial u}{\partial y}=\phi_2(x,y)$

$$f'(z)=\phi_1(x,y)-i\phi_2(x,y) \quad \text{or} \quad f'(z)=\phi_1(z,0)-i\phi_2(z,0)$$

On integrating $\qquad f(z)=\int\phi_1(z,0)\,dz-i\int\phi_2(z,0)dz+c$

Case II. If *v* is given

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x} = \psi_1(x,y)+i\,\psi_2(x,y)$$

when $\qquad \psi_1(x,y)=\dfrac{\partial v}{\partial y}, \qquad \psi_2(x,y)=\dfrac{\partial v}{\partial x}.$

$$f(z)=\int\psi_1(z,0)\,dz+i\int\psi_2(z,0)\,dz+c$$

7.20 WORKING RULE: TO CONSTRUCT AN ANALYTIC FUNCTION BY MILNE THOMSON METHOD

Case I. When u is given

Step 1. Find $\dfrac{\partial u}{\partial x}$ and equate it to $\phi_1(x, y)$.

Step 2. Find $\dfrac{\partial u}{\partial y}$ and equate it to $\phi_2(x, y)$.

Step 3. Replace x by z and y by 0 in $\phi_1(x, y)$ to get $\phi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\phi_2(x, y)$ to get $\phi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula $f(z) = \int\{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c$

Case II. When v is given

Step 1. Find $\frac{\partial v}{\partial x}$ and equate it to $\psi_2(x, y)$.

Step 2. Find $\frac{\partial v}{\partial y}$ and equate it to $\psi_1(x, y)$.

Step 3. Replace x by z and y by 0 in $\psi_1(x, y)$ to get $\psi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\psi_2(x, y)$ to get $\psi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula

$$f(z) = \int\{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c$$

Case III. When $u - v$ is given.

We know that
$$f(z) = u + iv \quad \text{...(1)}$$
$$if(z) = iu - v \quad \text{...(2) [Multiplying by } i]$$

Adding (1) and (2), we get
$$(1 + i) f(z) = (u - v) + i(u + v)$$
$$\Rightarrow \quad F(z) = U + iV$$

where
$$F(z) = (1 + i) f(z) \quad \text{...(3)} \quad \begin{bmatrix} U = u - v \\ V = u + v \end{bmatrix}$$

Here, $U = (u - v)$ is given

Find out F(z) by the method described in case I, then substitute the value of F(z) in (3), we get
$$f(z) = \frac{F(z)}{1+i}$$

Case IV. When $u + v$ is given.

We know that
$$f(z) = u + iv \quad \text{...(1)}$$
$$if(z) = iu - v \quad \text{[Multiplying by } i]\text{...(2)}$$

Adding (1) and (2), we get
$$(1 + i) f(z) = (u - v) + i(u + v)$$
$$\Rightarrow \quad F(z) = U + iV$$

where
$$F(z) = (1 + i) f(z) \quad \text{...(3)} \quad \begin{bmatrix} U = u - v \\ V = u + v \end{bmatrix}$$

Here, $V = (u + v)$ is given

Find out F(z) by the method described in case II, then substitute the value of F(z) in (3), we get
$$f(z) = \frac{F(z)}{1+i}$$

Example 31. *If $u = x^2 - y^2$, find a corresponding analytic function.*

Solution.
$$\frac{\partial u}{\partial x} = 2x = \phi_1(x, y), \qquad \frac{\partial u}{\partial y} = -2y = \phi_2(x, y)$$

On replacing x by z and y by 0, we have
$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + C$$
$$= \int [2z - i(0)]dz + c = \int 2z\,dz + c = z^2 + C \qquad \textbf{Ans.}$$

This is the required analytic function.

Example 32. *Show that e^x $(x \cos y - y \sin y)$ is a harmonic function. Find the analytic function for which e^x $(x \cos y - y \sin y)$ is imaginary part.*

(*U.P., III Semester, June 2009, R.G.P.V., Bhopal, III Semester, June 2004*)

Solution. Here $v = e^x\,(x \cos y - y \sin y)$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial v}{\partial x} = e^x\ (x\cos y - y\sin y) + e^x \cos y = \psi_2\,(x, y), \qquad \text{(say)} \quad \dots (1)$$

$$\frac{\partial v}{\partial y} = e^x\ (-x\sin y - y\cos y - \sin y) = \psi_1\ (x,\ y) \qquad \text{(say)} \quad \dots (2)$$

$$\frac{\partial^2 v}{\partial x^2} = e^x\ (x\cos y - y\sin y) + e^x \cos y + e^x \cos y$$

$$= e^x\,(x\cos y - y\sin y + 2\cos y) \quad \dots (3)$$

and $$\frac{\partial^2 v}{\partial y^2} = e^x\,(-x\cos y + y\sin y - 2\cos y) \quad \dots (4)$$

Adding equations (3) and (4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \Rightarrow v \text{ is a harmonic function.}$$

Now putting $x = z$, $y = 0$ in (1) and (2), we get

$$\psi_2\,(z, 0) = ze^z + e^z \qquad \psi_1\ (z, 0) = 0$$

Hence by Milne-Thomson method, we have

$$f\,(z) = \int [\psi_1\,(z,\ 0) + i\,\psi_2\ (z,\ 0)]\ dz + C$$

$$= \int [0 + i\,(ze^z + e^z)]\ dz + C = i\,(ze^z - e^z + e^z) + C = i\,z\,e^z + C.$$

This is the required analytic function. **Ans.**

Example 33. *If* $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, *find* f (*z*).

(*R.G.P.V., Bhopal, III Semester, Dec. 2003*)

Solution. $$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2} = \phi_1(x, y)$$

$$\psi_1\ (z,\ 0) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x, y)$$

$$\psi_2\ (z,\ 0) = 0$$

$$f\,(z) = \int\ [\psi_1(z,0) - i\psi_2(z,0)]\ dz + C = \int \frac{(2\cos 2z + 2)}{(1 + \cos 2z)^2}\,dz + C = 2\int \frac{1}{1 + \cos 2z}\,dz + C$$

$$= 2\int \frac{1}{2\cos^2 z}\,dz + C = \int \sec^2 z dz + C = \tan z + C \qquad \textbf{Ans.}$$

which is the required function.

Example 34. *Show that the function* $u = e^{-2xy}\ \sin(x^2 - y^2)$ *is harmonic. Find the conjugate function v and express u + iv as an analytic function of z.*

(*R.G.P.V. Bhopal, III Semester, June, 2007, Dec. 2006*)

Solution. We have,

$$u = e^{-2xy}\ \sin(x^2 - y^2) \quad \dots (1)$$

Differentiating (1), w.r.t. x, we get

$$\frac{\partial u}{\partial x} = 2x\ e^{-2xy}\ \cos(x^2 - y^2) - 2y\,e^{-2xy}\ \sin\ (x^2\ -\ y^2)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = e^{-2xy}\,[2x\cos(x^2-y^2) - 2y\sin(x^2-y^2)] = \psi_1(x,y) \quad \text{... (2)}$$

$\psi_1(z, 0) = 2z \cos z^2$

Differentiating (1), w.r.t. y, we get

$$\frac{\partial u}{\partial y} = -\,2y\,e^{-2xy}\cos(x^2-y^2) - 2x\,e^{-2xy}\sin(x^2-y^2)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = e^{-2xy}\,[-2y\cos(x^2-y^2) - 2x\sin(x^2-y^2)] = \phi_2(x,y) \quad \text{... (3)}$$

$\psi_2(z, 0) = -\,2z \sin z^2$

Differentiating (2), w.r.t, 'x', we get

$$\frac{\partial^2 u}{\partial x^2} = -2y\,e^{-2xy}\,[2x\cos(x^2-y^2) - 2y\sin(x^2-y^2)]$$
$$+\,e^{-2xy}\,[2\cos(x^2-y^2) + 2x\,(2x)\,\{-\sin(x^2-y^2)\} - 2y(2x)\cos(x^2-y^2)]$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^{-2xy}\,[-4xy\cos(x^2-y^2) + 4y^2\sin(x^2-y^2) + 2\cos(x^2-y^2)$$
$$-4x^2\sin(x^2-y^2) - 4xy\cos(x^2-y^2)\,]$$

$$= e^{-2xy}\,[-8xy\cos(x^2-y^2) + 4y^2\sin(x^2-y^2) + 2\cos(x^2-y^2) - 4x^2\sin(x^2-y^2)] \quad \text{...(4)}$$

Differentiating (3), w.r.t. 'y', we get

$$\frac{\partial^2 u}{\partial y^2} = -2x\,e^{-2xy}\,[-2y\cos(x^2-y^2) - 2x\sin(x^2-y^2)]$$
$$+\,e^{-2xy}\Big[-2\cos(x^2-y^2) + 2y\,(-2y)\,\sin(x^2-y^2) - 2x(-2y)\cos(x^2-y^2)\Big]$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = e^{-2xy}\,[4xy\cos(x^2-y^2) + 4x^2\sin(x^2-y^2) - 2\cos(x^2-y^2)$$
$$-4y^2\sin(x^2-y^2) + 4xy\cos(x^2-y^2)]$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-2xy}\,[8xy\cos(x^2-y^2) + 4x^2\sin(x^2-y^2) - 2\cos(x^2-y^2) - 4y^2\sin(x^2-y^2)] \quad \text{... (5)}$$

Adding (4) and (5), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Which proves that u is harmonic.

Now we have to express $u + iv$ as a function of z

$$f(z) = \int [\psi_1(z, 0) - i\,\psi_2(z, 0)]\,dz = \int [2z\cos z^2 - i\,(-\,2z\sin z^2)]\,dz$$

$$= \sin z^2 - i\cos z^2 + C = -\,i\,(\cos z^2 + i\sin z^2) + C = -\,i\,e^{iz^2} + C \qquad \textbf{Ans.}$$

Example 35. *If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z by Milne Thomson method.*

Solution. We know that

$$f(z) = u + iv \quad \text{... (1)}$$
$$i\,f(z) = i\,u - v \quad \text{... (2)}$$

Adding (1) and (2), we get

$$(1 + i)\,f(z) = (u - v) + i\,(u + v)$$

$$F(z) = U + iV$$
$$U = u - v = (x - y)(x^2 + 4xy + y^2)$$
$$\frac{\partial U}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$$
$$= x^2 + 4xy + y^2 + 2x^2 + 4xy - 2xy - 4y^2 = 3x^2 + 6xy - 3y^2$$
$$\phi_1(x, y) = 3x^2 + 6xy - 3y^2$$
$$\phi_1(z, 0) = 3z^2$$
$$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$
$$= -x^2 - 4xy - y^2 + 4x^2 + 2xy - 4xy - 2y^2 = 3x^2 - 6xy - 3y^2$$
$$\phi_2(x, y) = 3x^2 - 6xy - 3y^2$$
$$\phi_2(z, 0) = 3z^2$$
$$F(z) = \text{U} + i\text{V}$$
$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \phi,(`, 0) - i\,\phi_2(z, 0) = 3z^2 - i\,3z^2$$
$$= 3(1 - i)z^2$$
$$F(z) = (1 - i)z^3 + C$$
$$(1 + i)f(z) = (1 - i)z^3 + C$$
$$f(z) = \frac{1-i}{1+i}z^3 + \frac{C}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)}z^3 + C_1$$
$$= \frac{1-2i+(-i)^2}{1+1}z^3 + C_1 = \frac{1-2i-1}{2}z^3 + C_1 = -i\,z^3 + C_1 \qquad \textbf{Ans.}$$

Note: This example has already been solved on page 162 as Example 33.

Example 36. *If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$, prove that*

$$f(z) = \frac{1}{2}\left[1 - \cot\frac{z}{2}\right] \text{ when } f\left(\frac{\pi}{2}\right) = 0. \qquad \text{(R.G.P.V. Bhopal, III Semester, Dec. 2007)}$$

Solution. We know that $f(z) = u + iv$

$\therefore$ $i\,f(z) = iu - v$ [Multiplying by i]

On adding, we get $(1 + i)f(z) = (u - v) + i(u + v)$

$\Rightarrow$ $\text{F}(z) = \text{U} + i\text{V}$

$\text{U} = u - v$

$\text{V} = u + v$

$\Rightarrow$ $(1 + i)f(z) = \text{F}(z)$

We have, $U = u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

$$\Rightarrow \quad U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y} \qquad [\because e^{-y} = \cosh y - \sinh y]$$

$$= \frac{\cos x - \cosh y}{2(\cos x - \cosh y)} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} = \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \qquad ...(1)$$

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cos x - (\sin x + \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}\right]$$
$$= \frac{1}{2}\left[\frac{(\cos^2 x + \sin^2 x - \cosh y\cos x + \sinh y\sin x)}{(\cos x - \cosh y)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1-\cosh y\cos x+\sinh y\sin x}{(\cos x-\cosh y)^2}\right]=\frac{1-\cos iy\cos x+\sin iy\sin x}{(\cos x-\cosh y)^2}$$

$$\phi_1(x, y) = \frac{1-\cos(x+iy)}{(\cos x-\cosh y)^2} \qquad ...(2)$$

Replacing x by z and y by 0 in (2), we get

$$\phi_1(z, 0) = \frac{1}{2}\left[\frac{1-\cos z}{(\cos z-1)^2}\right]=\frac{1}{2(1-\cos z)}$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{2}\left[\frac{(\cos x-\cosh y)\,.\,\cosh y-(\sin x+\sinh y)(-\sinh y)}{(\cos x-\cosh y)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos x\cosh y)+\sin x\sinh y-(\cosh^2 y-\sinh^2 y)}{(\cos x-\cosh y)^2}\right]$$

$$\phi_2(x, y) = \frac{1}{2}\left[\frac{\cos x\cosh y+\sin x\sinh y-1}{(\cos x-\cosh y)^2}\right] \qquad ...(3)$$

Replacing x by z and y by 0 in (3), we have

$$\phi_2(z, 0) = \frac{1}{2}\left[\frac{\cos z-1}{(\cos z-1)^2}\right]=\frac{1}{2}\cdot\left(\frac{-1}{1-\cos z}\right)$$

$$F'(z) = \frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial U}{\partial x}-i\frac{\partial U}{\partial y} \qquad \text{[C–R equations]}$$

$$= \phi_1(z, 0) - i\,\phi_2(z, 0)$$

By Milne Thomson Method,

$$F(z) = \int\left[\phi_1(z, 0)-i\phi_2(z, 0)\right]dz+C$$

$$= \int\left[\frac{1}{2}\cdot\frac{1}{(1-\cos z)}+\frac{i}{2}\cdot\frac{1}{1-\cos z}\right]dz+C$$

$$= \frac{1+i}{2}\int\frac{1}{2\sin^2 z/2}\,dz+C=\frac{1+i}{4}\int\operatorname{cosec}^2(z/2)\,dz+C$$

$$= \left(\frac{1+i}{4}\right)\cdot\frac{(-\cot z/2)}{\left(\frac{1}{2}\right)}+C=-\left(\frac{1+i}{2}\right)\cot\frac{z}{2}+C$$

$$\Rightarrow \quad (1+i)\,f(z) = -\left(\frac{1+i}{2}\right)\cot\frac{z}{2}+C \quad \Rightarrow \quad f(z) = -\frac{1}{2}\cot\frac{z}{2}+\frac{C}{1+i} \qquad ...(4)$$

On putting $z = \dfrac{\pi}{2}$ in (4), we get

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2}\cot\frac{\pi}{4}+\frac{C}{1+i}$$

$$0 = -\frac{1}{2}+\frac{C}{1+i} \quad \Rightarrow \quad \frac{C}{1+i}=\frac{1}{2} \qquad \left[\,f\left(\frac{\pi}{2}\right) = 0, \text{ given}\right]$$

On putting the value of $\dfrac{C}{1+i}$ in (4), we get

$$f(z) = -\frac{1}{2}\cot\frac{z}{2}+\frac{1}{2}$$

Hence, $f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$, when $f\left(\frac{\pi}{2}\right) = 0.$ **Proved.**

EXERCISE 7.4

Show that the following functions are harmonic and determine the conjugate functions.

1. $u = 2x(1-y)$ **Ans.** $v = x^2 - y^2 + 2y + C$ **2.** $u = 2x - x^3 + 3xy$ **Ans.** $v = 2y - 3x^2y + y^3 + C$

Determine the analytic function, whose real part is

3. $\log\sqrt{x^2 + y^2}$ (*K.U., 2009*) **Ans.** $\log z + C$ **4.** $\cos x \cosh y$ **Ans.** $\cos z + c$

5. $e^{-x}(\cos y + \sin y)$ (*AMIETE, June 2010*)

6. $e^{2x}(x\cos 2y - y\sin 2y)$ **Ans.** $ze^{2z} + iC$ **7.** $e^{-x}(x\cos y + y\sin y)$ and $f(0) = i.$ **Ans.** $ze^{-z} + i$

Determine the analytic function, whose imaginary part is

8. $v = \log(x^2 + y^2) + x - 2y$ **Ans.** $2i\log z - (2-i)z + C$ **9.** $v = \sinh x \cos y$ **Ans.** $\sin iz + C$

10. $v = \left(r - \frac{1}{r}\right)\sin\theta$ **Ans.** $z + \frac{1}{z} + C$

11. Find the analytic function whose real part is $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$ (*MDU Dec. 2010*)

[**Hint:** See sloved Example 41 on page 168] **Ans.** $f(x) = \cot z + c$

12. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition that $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}.$ **Ans.** $f(z) = \cot\frac{z}{2} + \frac{1-i}{2}$

13. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $V(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2.$ **Ans.** $i[z^2 - z + 2]$

14. Show that the function $u = x^2 - y^2 - 2xy - 2x - y - 1$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as a function of z where $z = x + iy$. **Ans.** $(1 + i)z^2 + (-2 + i)z - 1$

15. Construct an analytic function of the form $f(z) = u + iv$, where v is $\tan^{-1}(y/x)$, $x \neq 0$, $y \neq 0$ **Ans.** $\log cz$

16. Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. **Ans.** $v = e^{-2xy}\cos(x^2 - y^2) + C$, $f(z) = -ie^{iz^2} + C_1$

17. Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$. (*AMIETE, June 2009*)

Choose the correct answer:

18. The harmonic conjugate of $u = x^3 - 3xy^2$ is

(*a*) $y^3 - 3xy^2$ (*b*) $3x^2y - y^3$ (*c*) $3xy^2 - y^3$ (*d*) $3xy^2 - x^3$ (*AMIETE, June 2010*)

7.21 PARTIAL DIFFERENTIATION OF FUNCTION OF COMPLEX VARIABLE

Example 37. *Prove that*

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

Solution. We know that

$x + iy = z$ ... (1), $x - iy = \bar{z}$ ... (2)

From (1) and (2), we get

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{-i}{2}(z - \bar{z})$$

$\Rightarrow$ $$\frac{\partial x}{\partial z} = \frac{1}{2}, \qquad \frac{\partial y}{\partial z} = -\frac{i}{2}$$

and $$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \qquad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

We know that,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(\frac{-i}{2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad ...(3)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial \bar{z}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial \bar{z}}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(\frac{i}{2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad ... (4)$$

From (3) and (4), we get

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + i\frac{\partial^2}{\partial x \partial y} - i\frac{\partial^2}{\partial x \partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$\Rightarrow$ $$4\frac{\partial^2}{\partial z\, \partial \bar{z}} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$ **Proved.**

Example 38. *If f (z) is a harmonic function of z, show that*

$$\left\{\frac{\partial}{\partial x}|\, f(z)\,|\right\}^2 + \left\{\frac{\partial}{\partial y}|\, f(z)\,|\right\}^2 = |\, f'(z)\,|^2$$ (*K.U., 2009, U.P. III Semester, June 2009*)

Solution. Since $f(z) = u(x, y) + i\, v(x, y)$

so $$|\, f(z)\,| = \sqrt{u^2 + v^2} \qquad ... (1)$$

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial}{\partial x}|\, f(z)\,| = \frac{\partial}{\partial x}(\sqrt{u^2 + v^2})$$

$$= \frac{1}{2}(u^2 + v^2)^{\frac{-1}{2}}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{\sqrt{u^2 + v^2}} = \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{|\, f(z)|} \qquad ... (2)$$

Similarly $$\frac{\partial}{\partial y}|\, f(z)| = \frac{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}}{|\, f(z)|} \qquad ... (3)$$

Squaring (2) and (3) adding, we get

$$\left[\frac{\partial}{\partial x}|\, f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|\, f(z)|\right]^2 = \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2}{|\, f(z)|^2}$$

$$= \frac{\left(u\frac{\partial u}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial u}{\partial y}.v\frac{\partial v}{\partial y} + \left(v\frac{\partial v}{\partial y}\right)^2}{|\, f(z)|^2}$$

By C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$2uv\ \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 2uv\left(\frac{\partial v}{\partial y}\right)\left(-\frac{\partial u}{\partial y}\right)$$

Putting the value of $2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} = -\,2uv\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y}$ in (4), we get

$$\left[\frac{\partial}{\partial x}|\,f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|\,f(z)|\right]^2 = \frac{\left(u\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y} + \left(v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \left(v\frac{\partial v}{\partial y}\right)^2}{|\,f(z)|^2}$$

$$= \frac{u^2\left(\frac{\partial u}{\partial x}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2}{|\,f(z)|^2} = \frac{u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]}{|\,f(z)|^2}$$

$$= \frac{u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{-\partial v}{\partial x}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]}{|\,f(z)|^2} \qquad \text{[C – R equations]}$$

$$= \frac{(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}{|\,f(z)|^2} = \frac{|\,f(z)\,|^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}{|\,f(z)|^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$[\ |\,f(x)\,|^2 = u^2 + v^2\ ]$

$= |\,f'(z)\,|^2$ **Proved.**

Example 39. *Prove that* $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\,u\,|^P = P(P-1)|\,u\,|^{P-2}|\,f'(z)\,|^2$

Solution. We know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}$ [Example 46, page 173]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\,u\,|^P = \frac{1}{2^P}\frac{4\partial^2}{\partial z\partial\bar{z}}[\,f(z) + f(\bar{z})]^P \qquad \left[\because\ u = \frac{1}{2}[\,f(z) + f(\bar{z})]\right]$$

$$= \frac{4}{2^P}\frac{\partial}{\partial\bar{z}}P[\,f(z) + f(\bar{z})]^{P-1}f'(z) = \frac{1}{2^{P-2}}P(P-1)[\,f(z) + f(\bar{z})]^{P-2}f'(z)f'(\bar{z})$$

$$= P(P-1)\left[\frac{1}{2}\{\,f(z) + f(\bar{z})\}\right]^{P-2}[\,f'(z)f'(\bar{z})] = P(P-1)|\,u\,|^{P-2}|\,f'(z)\,|^2$$ **Proved.**

Example 40. *Prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|\,f'(z)\,| = 0$$

Solution. We have, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}$ (*Example 46 on page 173*)

Hence $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\{\log|\,f'(z)\,|\} = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\{\log|\,f'(z)\,|\} = 4\frac{\partial^2}{\partial z\partial\bar{z}}\frac{1}{2}\log|\,f'(z)\,|^2$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\log\{\,f'(z)f'(\bar{z})\} = 2\frac{\partial^2}{\partial z\partial\bar{z}}[\log f'(z) + \log f'(\bar{z})] = 2\frac{\partial}{\partial z}\left(0 + \frac{1}{f'(\bar{z})}\,f''(\bar{z})\right)$$

$$= 2\frac{\partial}{\partial z}\,\frac{f''(\bar{z})}{f'(\bar{z})} \qquad \left[\begin{array}{l}\bar{z} \text{ is constant in regards to} \\ \text{differentiation } w.r.t.\ z\end{array}\right]$$

$$= 2 \times 0$$

$$= 0$$

Proved.

Example 41. *Prove that* $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$

Solution. $f(z) = u + iv$ or $R\,f(z) = u \Rightarrow$ Real part of $f(z) = u$

$$\frac{\partial}{\partial x}u^2 = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}u^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} \qquad \dots (1)$$

Similarly,
$$\frac{\partial^2}{\partial y^2}u^2 = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} \qquad \dots (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] + 0 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2\right]\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right) = 2|f'(z)|^2$$

$$\Rightarrow \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2 = 2\,|f'(z)|^2$$

Proved.

Example 42. *If f (z) is regular function of z, show that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$ (*R.G.P.V., Bhopal, III Semester, June 2004*)

Solution.
$$f(z) = u + iv$$

$$|f(z)|^2 = u^2 + v^2 \qquad \dots (1)$$

Let
$$\phi = u^2 + v^2$$

Differentiating (1) *w.r.t. x*, we get

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \qquad \dots (2)$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \qquad \dots (3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) + \left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}\right] \qquad \dots (4)$$

By C – R equations
$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 \qquad \dots (5)$$

By Laplace equations $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

On putting the values of $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ from (5) in (4), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right], \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\left|\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right|^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\,f(z)|^2 = 4\,|\,f'(z)|^2$$ **Proved.**

Example 43. *If |f(z)| is constant, prove that f(z) is also constant.*

Solution. $f(z) = u + iv$

$|\,f(z)\,|^2 = u^2 + v^2$

$|\,f(z)\,| = \text{constant} = c$ (given)

$u^2 + v^2 = c^2$ ... (1)

Differentiating (1) w.r.t. x, we get $2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ ... (2)

Differentiating (1) w.r.t. 'y', we get $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \qquad \text{... (3)}$$

Squaring (2) and (3) and then adding, we get

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

As $f(z) = u + iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$f'(\bar{z}) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$$

$$|\,f'(z)\,|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$|\,f'(z)\,|^2 = 0 \quad \Rightarrow \quad f(z)$ is constant. **Proved.**

UNIT-2

MAPPINGS

7.28 GEOMETRICAL REPRESENTATION

To draw a curve of complex variable (x, y) on z-plane we take two axes *i.e.*, one real axis and the other imaginary axis. A number of points (x, y) are plotted on z-plane, by taking different value of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called *Argand diagram* in z-plane.

But a complex function $w = f(z)$ *i.e.*, $(u + iv) = f(x + iy)$ involves four variables x, y and u, v.

A figure of only three dimensions (x, y, z) is possible in a plane. A figure of four dimensional region for x, y, u, v is not possible.

So, we choose two complex planes z-plane and w-plane. In the w-plane we plot the corresponding points $w = u + iv$. By joining these points we have a corresponding curve C' in w-plane.

7.29 TRANSFORMATION

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this "transformation or mapping of z-plane into w-plane". If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along a curve C in z-plane, the point $P'(u, v)$ will move along a corresponding curve C' in w-plane, then we say that a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the relation $w = f(z)$.

Example 64. *Transform the rectangular region ABCD in z-plane bounded by $x = 1$, $x = 3$; $y = 0$ and $y = 3$. Under the transformation $w = z + (2 + i)$.*

Solution. Here,

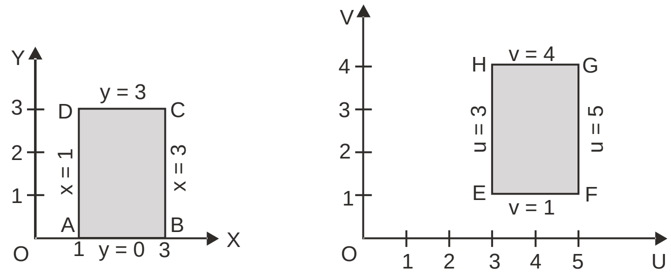
$$w = z + (2 + i)$$

$$\Rightarrow \quad u + iv = x + iy + (2 + i) = (x + 2) + i(y + 1)$$

By equating real and imaginary quantities, we have $u = x + 2$ and $v = y + 1$.

z-plane	w-plane	z-plane	w-plane
x	$u = x + 2$	y	$v = y + 1$
1	$= 1 + 2 = 3$	0	$= 0 + 1 = 1$
3	$= 3 + 2 = 5$	3	$= 3 + 1 = 4$

Here the lines $x = 1$, $x = 3$; $y = 0$ and $y = 1$ in the z-plane are transformed onto the line $u = 3$, $u = 5$; $v = 1$ and $v = 4$ in the w-plane. The region $ABCD$ in z-plane is transformed into the region $EFGH$ in w-plane.



Ans.

Example 65. *Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$.*

Solution. $w = z^2 = (x + y)^2$, $u + iv = x^2 - y^2 + 2ixy$

This gives $u = x^2 - y^2$ and $v = 2xy$

Table of (x, y) and (u, v)

x	2	2.5	3	3.5	4	4.5	5
$y = \pm\sqrt{x^2 - 4}$	0	± 1.5	± 2.2	± 2.9	± 3.5	± 4.1	± 4.6
$u = x^2 - y^2$	4	4	4	4	4	4	4
$v = 2xy$	0	± 7.5	± 13.2	± 20.3	± 28	± 36.9	± 46

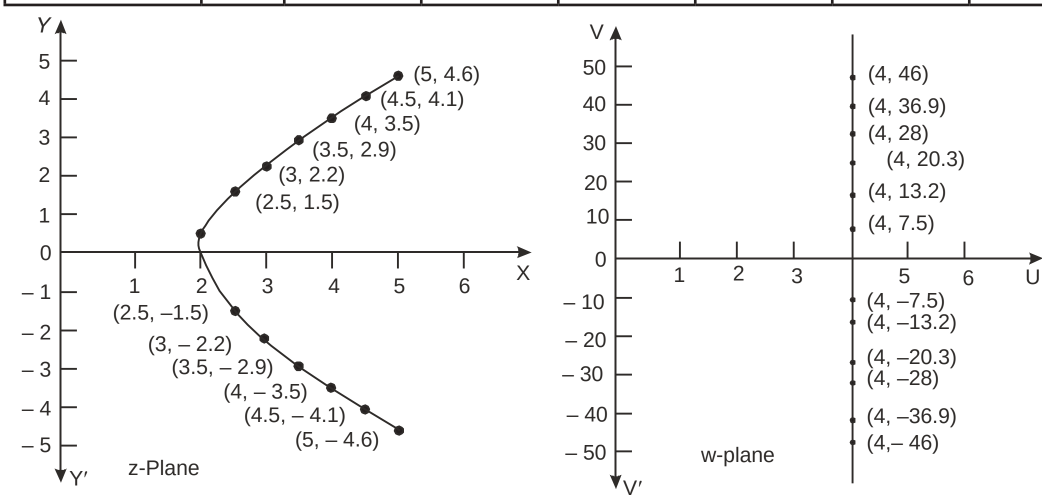


Image of the curve $x^2 - y^2 = 4$ is a straight line, $u = 4$ parallel to the v-axis in w-plane. **Ans.**

7.30 CONFORMAL TRANSFORMATION

(*U.P. III Semester Dec., 2006, 2005*)

Let two curves C, C_1 in the z-plane intersect at the point P and the corresponding curve C', C'_1 in the w-plane intersect at P'. **If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves of w-plane at P' in magnitude and sense, then the transformation is called conformal:**

conditions: (*i*) $f(z)$ is analytic. (*ii*) $f(z) \neq 0$ Or

If the sense of the rotation as well as the magnitude of the angle is preserved, the transformation is said to be **conformal**.

If only the magnitude of the angle is preserved, transformation is **Isogonal**.

7.31 THEOREM. If *f*(*z*) is analytic, mapping is conformal (*U.P. III Semester Dec. 2005*)

Proof. Let C_1 and C_2 be the two curves in the z-plane intersecting at the point z_0 and let the tangents at this point make angles α_1 and α_2 with the real axis. Let z_1 and z_2 be the points on the curves C_1 and C_2 near to z_0 at the same distance r from z_0, so that we have

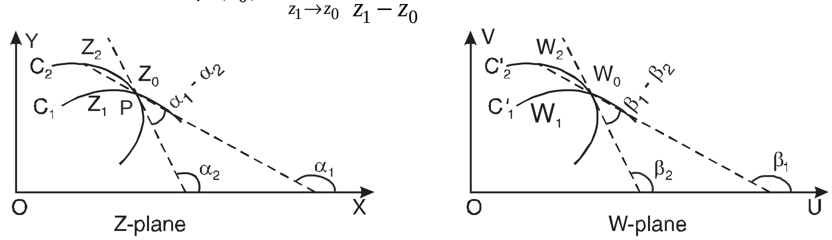
$$z_1 - z_0 = re^{i\theta_1}, \qquad z_2 - z_0 = re^{i\theta_2}$$

As $\quad r \to 0,\ \theta_1 \to \alpha_1$ and $\theta_2 \to \alpha_2$

Let the image of the curves C_1, C_2 be C_1' and C_2' in w-plane and images of z_0, z_1 and z_2 be w_0, w_1 and w_2.

Let $\quad w_1 - w_0 = r.e^{i\phi_1},\ w_2 - w_0 = r.e^{i\phi_2}$

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0}$$



$$Re^{i\lambda} = \lim_{r \to 0} \frac{r_1 e^{i\phi_1}}{re^{i\theta_1}} \qquad \text{(since } f'(z_0) = Re^{i\lambda}\text{)}$$

$$Re^{i\lambda} = \frac{r_1}{r} e^{i\phi_1 - i\theta_1} = \frac{r_1}{r} e^{i(\phi_1 - \theta_1)}$$

Hence $$\lim_{r \to 0}\left[\frac{r_1}{r}\right] = R = |f'(z_0)| \text{ and } \lim\ (\phi_1 - \theta_1) = \lambda$$

$\Rightarrow \qquad \lim\ \phi_1 - \lim \theta_1 = \lambda$ or $\beta_1 - \alpha_1 = \lambda$ *i.e.*, $\beta_1 = \alpha_1 + \lambda$

Similarly it can be proved $\beta_2 = \alpha_2 + \lambda$ curve C_1' has a definite tangent at w_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ so curve C_2'.

Angle between two curves C_1' and C_2'

$$= \beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 + \lambda) = (\alpha_1 - \alpha_2)$$

so the transformation is conformal at each point where $f'(z) = 0$

Note 1. The point at which $f'(z) = 0$ is called a **critical point** of the transformation. Also the points where $\frac{dw}{dz} \neq 0$ are called **ordinary points**.

2. Let $\phi = \alpha_1 - \alpha_2$ or $\alpha_1 = \alpha_2 + \phi$ shows that the tangent at P to the curve is rotated through an $\angle\phi$ = amp $\{f'(z)\}$ under the given transformation.

Angle of rotation = $\tan^{-1} \frac{v}{u}$.

3. In formal transformation, element of arc passing through P is magnified by the factor $|f'(z)|$. The area element is also magnified by the factor $|f'(z)|$ or $J = \frac{\partial(u, v)}{\partial(x, y)}$ in a conformal transformation.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2 = |f'(x+iy)|^2$$

$|f'(z)|$ is called the **coefficient of magnification**.

4. Conjugate functions remain conjugate functions after conformal transformation. A function which is the solution of Laplace's equation, its transformed function again remains the solution of Laplace's equation after conformal transformation.

7.32 THEOREM

Prove that an analytic function f (z) ceases to be conformal at the points where f´ (z) = 0. (*U.P. III Semester, Dec. 2006*)

Proof. Let $f'(z) = 0$ and $f'(z_0) = 0$ at $z = z_0$

Suppose that $f'(z_0)$ has a zero of order $(n - 1)$ at the point z_0, then first $(n - 1)$ derivatives of $f(z)$ vanish at z_0 but $f^n(z_0) \neq 0$, hence at any point z in the neighbourhood of z_0, we have by Taylor's Theorem.

$$f(z) = f(z_0) + a_n (z - z_0)^n + \ldots\ldots$$

where $a_n = \dfrac{f^n(z_0)}{n!}$, so that $a_n \neq 0$.

Hence, $f(z_1) - f(z_0) = a_n (z_1 - z_0)^n + \ldots\ldots$

i.e. $w_1 - w_0 = a_n (z_1 - z_0)^n + \ldots.$

or $\rho_1 e^{i\phi_1} = |a_n| \,.\, r^n e^{i(n\theta_1 + \lambda)} + \ldots\ldots.$ where $\lambda = \text{amp } a_n$

Hence, $\text{Lim } \phi_1 = \text{Lim } (n\theta_1 + \lambda) = n\alpha_1 + \lambda$

Similary, $\text{Lim } \phi_2 = n\alpha_2 + \lambda.$

Thus the curves γ_1 and γ_2 still have definite tangents at w_0.

But the angle between the tangents

$$= \text{Lim } \phi_2 - \text{Lim } \phi_1 = n(d_2 - d_1)$$

So magnitude of the angle is not preserved.

Also the linear magnification $R = \text{Lim } (\rho_1 / r) = 0.$

Hence, the conformal property does not hold good at a point where $f'(z) = 0$.

Example 66. *If $u = 2x^2 + y^2$ and $v = \dfrac{y^2}{x}$, show that the curves u = constant and v= constant cut orthogonally at all intersections but that the transformation w = u + iv is not conformal.* (*Q. Bank U.P. III Semester 2002*)

Solution. For the curve, $2x^2 + y^2 = u$

$$2x^2 + y^2 = \text{constant} = k_1 \text{ (say)} \qquad \ldots(1)$$

Differentiating (1), we get

$$4x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{y} = m_1 \text{ (say)} \qquad \ldots(2)$$

$$\frac{y^2}{x} = v$$

For the curve, $\dfrac{y^2}{x} = \text{constant} = k_2$ (say),

$$\Rightarrow \qquad y^2 = k_2 x. \qquad \ldots(3)$$

Differentiating (3), we get

$$2y\frac{dy}{dx} = k_2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{k_2}{2y} = \frac{y^2}{x} \times \frac{1}{2y} = \frac{y}{2x} = m_2 \text{ (say)} \qquad \ldots(4)$$

From (2) and (4), we see that

$$m_1 m_2 = \left(\frac{-2x}{y}\right)\left(\frac{y}{2x}\right) = -1$$

Hence, two curves cut orthogonally.

However, since $\quad \frac{\partial u}{\partial x} = 4x, \qquad \frac{\partial u}{\partial y} = 2y$

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}, \qquad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

The Cauchy-Riemann equations are not satisfied by u and v.

Hence, the function $u + iv$ is not analytic. So, the transformation is not conformal. **Proved**

Example 67. *For the conformal transformation $w = z^2$, show that*

(*a*) *The coefficient of magnification at $z = 2 + i$ is* $2\sqrt{5}$

(*b*) *The angle of rotation at $z = 2 + i$ is $\tan^{-1} 0.5$.*

(*c*) *The co-efficient of magnification at $z = 1 + i$ is* $2\sqrt{2}$.

(*d*) *The angle of rotation at $z = 1 + i$ is* $\frac{\pi}{4}$.

(*Q. Bank U.P. III Semester 2002*)

Solution. (*i*) Let $\quad w = f(z) = z^2$

$\therefore \quad f'(z) = 2z$

$$f'(2 + i) = 2(2 + i) = 4 + 2i.$$

(*a*) Coefficient of magnification at $z = 2 + i$ is $= |f'(2+i)| = |4 + 2i| = \sqrt{16+4} = 2\sqrt{5}.$

(*b*) Angle of rotation at $z = 2 + i$ is = amp. $f'(2 + i) = (4 + 2i) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(0.5).$

and $\quad f'(1 + i) = 2(1 + i) = 2 + 2i$

$\therefore$ (*c*) The co-efficient of magnification at $z = 1 + i$ is $|f'(1+i)| = |2 + 2i| = \sqrt{4+4} = 2\sqrt{2}$

(*d*) The angle of rotation at $z = 1 + i$ is amp. $|f'(1+i)| = 2 + 2i = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$ **Ans.**

Standard transformations

7.33 TRANSLATION $\quad \boldsymbol{w = z + C,}$

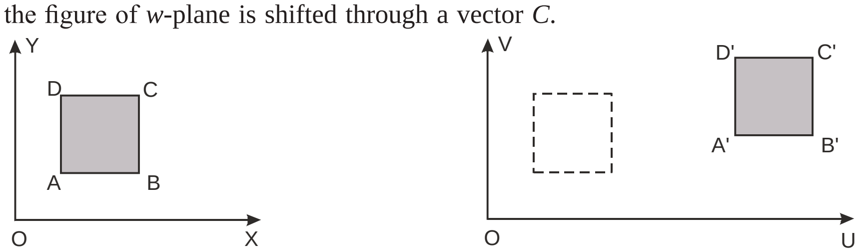
where $\quad C = a + ib$

$$u + iv = a + iy + a + ib$$

$$u = x + a \text{ and } v = y + b$$

On substituting the values of x and y in the equation of the curve to be transformed, we get the equation of the image in the w-plane.

The point P (x, y) in the z-plane is mapped onto the point $P'(x + a, y + b)$ in the w-plane. Similarly other points of z-plane are mapped onto w-plane. Thus if w-plane is superimposed on the z-plane, the figure of w-plane is shifted through a vector C.



In other words the transformation is mere translation of the axes.

7.34 ROTATION $\quad w = ze^{i\theta}$

The figure in z-plane rotates through an angle θ in anticlockwise in w-plane.

Example 68. *Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w-plane corresponding to the triangular region R bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in z-plane.*

Solution.

$$w = ze^{i\pi/4}$$

$$w = (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \quad u + iv = (x+iy)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[x - y + i(x+y)]$$

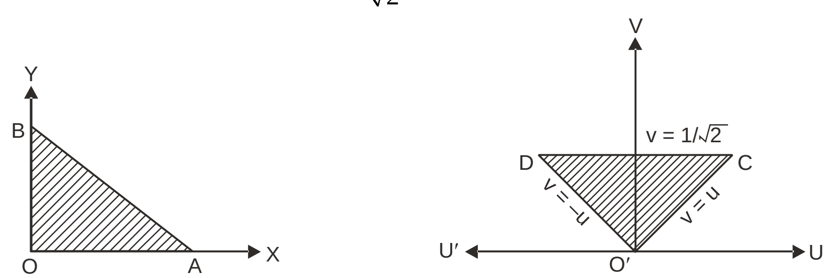
Equating real and imaginay parts, we get

$$\Rightarrow \quad u = \frac{1}{\sqrt{2}}(x-y), \qquad v = \frac{1}{\sqrt{2}}(x+y) \qquad \text{... (1)}$$

(i) Put $x = 0$, $\quad u = -\frac{1}{\sqrt{2}}y, \quad v = \frac{1}{\sqrt{2}}y$ or $v = -u$

(ii) Put $y = 0$, $\quad u = \frac{1}{\sqrt{2}}x, \quad v = \frac{1}{\sqrt{2}}x$ or $v = u$

(iii) Putting $x + y = 1$ in (1), we get $v = \frac{1}{\sqrt{2}}$



Hence the triangular region OAB in z-plane is mapped on a triangular region O′CD of w-plane bounded by the lines $v = u$, $v = -u$, $v = \frac{1}{\sqrt{2}}$.

$$f'(z) = \frac{1}{\sqrt{2}}(1+i)$$

$$f(z) = \frac{1}{\sqrt{2}}[(x-y) + i(x+y)]$$

Amp. $f'(z) = \tan^{-1}(1) = \frac{\pi}{4}$

The mapping $w = ze^{i\pi/4}$ performs a rotation of R through an angle $\pi/4$. **Ans.**

7.35 MAGNIFICATION

$$\boxed{w = cz}$$

where c is a real quantity.

(i) The figure in w-plane is magnified c-times the size of the figure in z-plane.

(ii) Both figures in z-plane and w-plane are singular.

Example 69. *Determine the region in w-plane on the transformation of rectangular region enclosed by $x = 1$, $y = 1$, $x = 2$ and $y = 2$ in the z-plane. The transformation is $w = 3z$.*

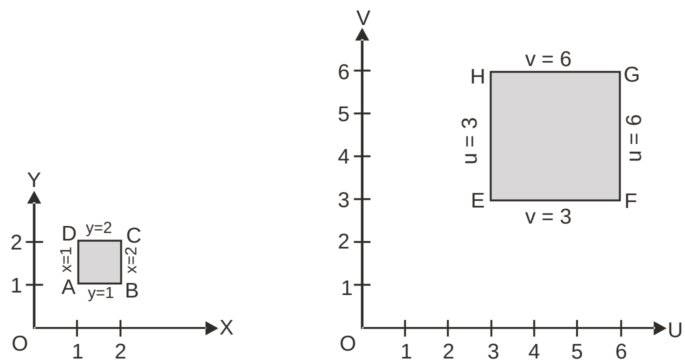
Solution. We have, $w = 3z$

$$u + iv = 3(x + iy)$$

Equating the real and imaginary parts, we get

$$u = 3x \quad \text{and} \quad v = 3y$$

z-plane		*w*-plane	
x	y	$u = 3x$	$v = 3y$
1	1	3	3
2	2	6	6



7.36 MAGNIFICATION AND ROTATION

$$\boldsymbol{w = c\, z} \qquad \text{... (1)}$$

where c, z, w all are complex numbers.

$$c = ae^{i\alpha}, \quad z = re^{i\theta}, \quad w = Re^{i\phi}$$

Putting these values in (1), we have

$$Re^{i\phi} = (ae^{i\alpha})(re^{i\theta}) = are^{i(\theta+\alpha)}$$

i.e. $R = ar$ and $\phi = \theta + \alpha$

Thus we see that the transform $w = cz$ corresponding to a rotation, together with magnification.

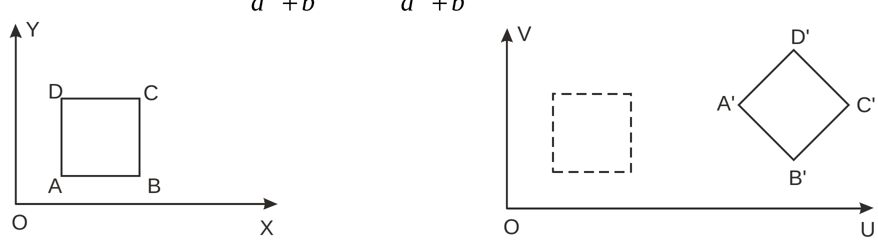
Algebraically $w = c\,z$ or $u + iv = (a + ib)(x + iy)$

$\Rightarrow$ $u + iv = ax - by + i(ay + bx)$

$$u = ax - by \text{ and } v = ay + bx$$

On solving these equations, we can get the values of x and y.

$$x = \frac{au + bv}{a^2 + b^2}, \; y = \frac{-bu + av}{a^2 + b^2}$$



On putting values of x and y in the equation of the curve to be transformed we get the equation of the image.

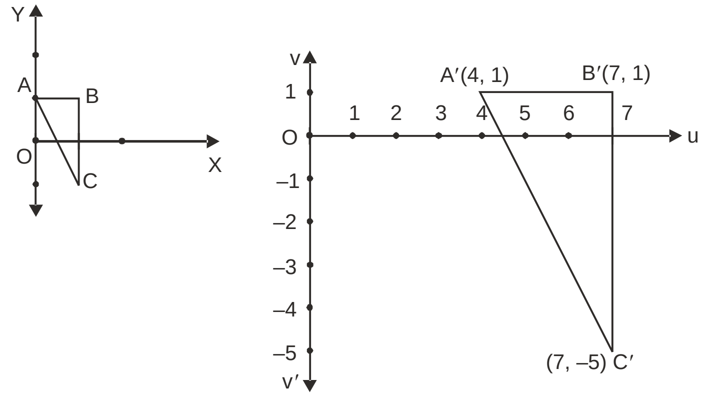
Example 70. *Find the image of the triangle with vertices at* i, $1 + i$, $1 - i$ *in the z-plane, under the transformation*

(i) $w = 3z + 4 - 2i$, (ii) $w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$

Solution. (i) $w = 3z + 4 - 2i$

$\Rightarrow$ $u + iv = 3(x + iy) + 4 - 2i \Rightarrow u = 3x + 4,\ v = 3y - 2$

S. No.	x	y	$u = 3x + 4$	$v = 3y - 2$
1.	0	1	4	1
2.	1	1	7	1
3.	1	–1	7	– 5



(ii)

$$w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$$

$$\Rightarrow \quad u + iv = \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)(x + iy) - 2 + 4i$$

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(x + iy) - 2 + 4i$$

$$= \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y + i\left(-\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4\right)$$

$$\Rightarrow \quad u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y \quad \text{and} \quad v = -\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4$$

S.No.	*z-Plane*		*w-plane*	
	x	y	$u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y$	$v = -\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4$
1.	0	1	$-2 + \frac{\sqrt{3}}{2}$	$\frac{9}{2}$
2.	1	1	$-\frac{3}{2} + \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} + \frac{9}{2}$
3.	1	–1	$-\frac{3}{2} - \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} + \frac{7}{2}$

7.37 INVERSION AND REFLECTION

$$w = \frac{1}{z} \quad \ldots (1)$$

If $z = r e^{i\theta}$ and $w = R e^{i\phi}$

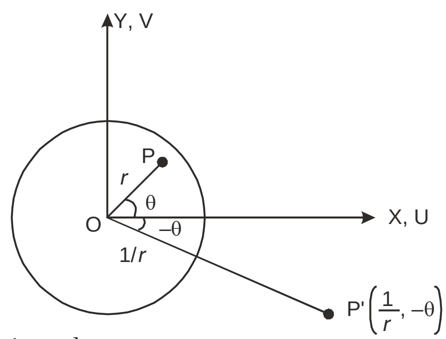
Putting these values in (1), we get

$$\text{Re}^{i\phi} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

On equating, $R = \frac{1}{r}$ and $\phi = -\theta$

Thus the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'\left(\frac{1}{r}, -\theta\right)$ in the w-plane.

Hence the transformation is an inversion of z and followed by reflection into the real axis. The points inside the unit circle ($|z| = 1$) map onto points outside it, and points outside the unit circle into points inside it.



Algebraically $w = \frac{1}{z}$ or $z = \frac{1}{w}$

$$x + iy = \frac{1}{u + iv}$$

$$\Rightarrow \quad x + iy = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}, \quad y = -\frac{v}{u^2 + v^2}$$

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1) be in z-plane.

On substituting the values of x and y in (1), we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g \frac{u}{u^2 + v^2} + 2f \frac{(-v)}{u^2 + v^2} + c = 0$$

This is the equation of circle in w-plane. This shows that a circle in z-plane transforms to another circle in w-plane.

But a circle through origin transforms into a straight line.

Example 71. *Under the transformation* $w = \frac{1}{z}$, *find the image of* $y - x + 1 = 0$ *(PTU May 2007)*

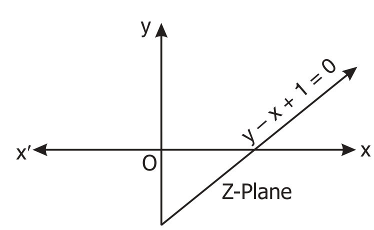
Solution. Here the equation of straight line may be given

$$y - x + 1 = 0$$

$$w = \frac{1}{z} \quad \Rightarrow \quad z = \frac{1}{w}$$

$$\Rightarrow \quad x + iy = \frac{1}{u + iv} = \frac{u + iv}{u^2 + v^2}$$

so that $x = \frac{u}{u^2 + v^2}$ and $y = -\frac{v}{u^2 + v^2}$

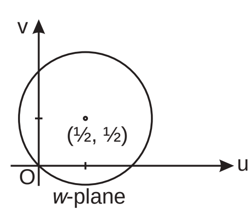


Putting the values of x, y in terms of u, v, we get

$$-\frac{v}{u^2 + v^2} - \frac{u}{u^2 + v^2} + 1 = 0$$

$$\Rightarrow \quad -u - v + u^2 + v^2 = 0$$

$$\Rightarrow \quad u^2 + v^2 - u - v = 0$$



This is the equation of a circle with centre at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $= \frac{1}{\sqrt{2}}$ **Ans.**

Example 72. *Find the image of* $|z - 3i| = 3$ *under the mapping* $w = \frac{1}{z}$.

(*Uttarakhand, III Semester 2008*)

Solution. $w = \frac{1}{z} \quad \Rightarrow \quad z = \frac{1}{w}$

$$\Rightarrow \quad x + iy = \frac{1}{u + iv} = \frac{u - iv}{(u + iv)(u - iv)} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow \quad x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2} \qquad \text{... (1)}$$

The given curve is $|z - 3i| = 3$

$$\Rightarrow \quad |x + iy - 3i| = 3 \quad \Rightarrow \quad x^2 + (y - 3)^2 = 9 \qquad \text{... (2)}$$

On substituting the values of x and y from (1) into (2), we get

$$\frac{u^2}{(u^2 + v^2)^2} + \left(-\frac{v}{u^2 + v^2} - 3\right)^2 = 9$$

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{(-v - 3u^2 - 3v^2)^2}{(u^2 + v^2)^2} = 9$$

$$\Rightarrow \quad u^2 + (-v - 3u^2 - 3v^2)^2 = 9(u^2 + v^2)^2$$

$$\Rightarrow \quad u^2 + v^2 + 9u^4 + 9v^4 + 6u^2v + 6v^3 + 18u^2v^2 = 9u^4 + 18u^2v^2 + 9v^4$$

$$\Rightarrow \quad u^2 + v^2 + 6u^2v + 6v^3 = 0$$

$$\Rightarrow \quad u^2 + v^2 + 6v(u^2 + v^2) = 0$$

$$\Rightarrow \quad (u^2 + v^2)(6v + 1) = 0$$

$\Rightarrow \quad 6v + 1 = 0$ is the equation of the image. **Ans.**

Second Method. $|z - 3i| = 3, \quad z = \frac{1}{w}$

$$\left|\frac{1}{w} - 3i\right| = 3 \quad \Rightarrow \quad |1 - 3iw| = 3|w|$$

$$\Rightarrow \quad |1 - 3i(u + iv)| = 3|u + iv| \quad \Rightarrow \quad |1 + 3v - 3iu| = 3|u + iv|$$

$$\Rightarrow \quad (1 + 3v)^2 + 9u^2 + 9(u^2 + v^2) \quad \Rightarrow \quad 1 + 6v + 9v^2 + 9u^2 = 9(u^2 + v^2)$$

$\Rightarrow \quad 1 + 6v = 0$ **Ans.**

Third Method. $|z - 3i| = 3 \quad \Rightarrow \quad z - 3i = 3e^{i\theta} \quad \Rightarrow \quad z = 3i + 3e^{i\theta}$

$$w = \frac{1}{z} = \frac{1}{3i + 3e^{i\theta}} \quad \Rightarrow \quad 3w = \frac{1}{i + e^{i\theta}}$$

$$3(u + iv) = \frac{1}{i + \cos\theta + i\sin\theta}$$

$$\Rightarrow \quad (3u + 3iv) = \frac{\cos\theta - i(1 + \sin\theta)}{\cos^2\theta + (1 + \sin\theta)^2} \quad \Rightarrow \quad 3v = -\frac{1 + \sin\theta}{2 + 2\sin\theta} = -\frac{1}{2}$$

$6v + 1 = 0$ **Ans.**

Example 73. *Image of* $|z + 1| = 1$ *under the mapping* $w = \frac{1}{z}$ is

(*a*) $2v + 1 = 1$ (*b*) $2v - 1 = 1$ (*c*) $2u + 1 = 0$ (*d*) $2u - 1 = 0$ (*AMIETE, June 2009*)

Solution. $w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}$

$$\Rightarrow \quad u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}$$

Given $|z + 1| = 1 \Rightarrow |x + iy + 1| = 1 \Rightarrow (x + 1)^2 + y^2 = 1$

$$\Rightarrow \quad x^2 + y^2 + 2x = 0 \Rightarrow x^2 + y^2 = -2x \Rightarrow \frac{1}{2} = -\frac{x}{x^2 + y^2} = -u$$

$$\Rightarrow \quad \frac{1}{2} = -u \quad \Rightarrow \quad 2u + 1 = 0$$

Hence (*c*) is correct answer. **Ans.**

Example 74. *Show that under the transformation* $w = \frac{1}{z}$, *the image of the hyperbola* $x^2 - y^2 = 1$ *is the lemniscate* $R^2 = \cos 2\phi$.

Solution. $x^2 - y^2 = 1$

Putting $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 1 \quad \Rightarrow \quad r^2(\cos^2\theta - \sin^2\theta) = 1$

$$\Rightarrow \quad r^2 \cos 2\theta = 1 \qquad \ldots (1)$$

And $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow r\,e^{i\theta} = \frac{1}{R\,e^{i\phi}} \Rightarrow r\,e^{i\theta} = \frac{1}{R}e^{-i\phi}$

Equating real and imaginary parts, we get

$$\therefore \quad r = \frac{1}{R} \quad \text{and} \quad \theta = -\phi$$

Putting the values of r and θ in (1), we get

$$\frac{1}{R^2}\cos 2(-\phi) = 1 \quad \Rightarrow \quad R^2 = \cos 2\phi$$ **Proved.**

EXERCISE 7.8

1. Find the image of the semi infinite, strip $x > 0$, $0 < y < 2$ under the transformation $w = iz + 1$.

   Ans. Strip $-1 < u < 1$, $v > 0$

2. Determine the region in the w-plane in which the rectangle bounded by the lines $x = 0$, $y = 0$, $x = 2$ and $y = 1$ is mapped under the transformation $w = \sqrt{2}\, e^{i\pi/4} z$.

   (*Q. Bank U.P. III Semester 2002*)

   Ans. Region bounded by the lines $v = -u$, $v = u$, $u + v = 4$ and $v - u = 2$.

3. Show that the condition for transformation $w = a^2 + blcz + d$ to make the circle $|w| = 1$ correspond to a straight line in the z-plane is $(a) = (c)$.

4. What is the region of the w-plane in two ways the rectangular region in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - i)$?

   Ans. Region bounded by $u = 2$, $v = -1$, $u = 3$ and $v = 1$.

5. Find the image of $|z - 2i|$ under the mapping $w = \frac{1}{z}$ **Ans.** $v + \frac{1}{4} = 0$

6. For the mapping $w(z) = 1/z$, find the image of the family of circles $x^2 + y^2 = ax$, where a is real.

Ans. $u = \frac{1}{a}$ is a straight line || to v-axis.

7. Show that the function $w = \frac{4}{z}$ transforms the straight line $x = c$ in the z-plane into a circle in the w-plane.

8. If $(w+1)^2 = \frac{4}{z}$, then prove that the unit circle in the w-plane corresponds to a parabola in the z-plane, and the inside of the circle to the outside of the parabola.

9. Find the image of $|z - 2i| = 2$ under the mapping $w = \frac{1}{z}$

(*Q. Bank U.P. 2002*) **Ans.** $4v + 1 = 0$

10. The image of the circle $|z - 1| = 1$ in the complex plane, under the mapping $w = u + iv = \frac{1}{z}$ is

(*i*) $|w - 1| = 1$ (*ii*) $u^2 + v^2 = 1$ (*iii*) $u = \frac{1}{2}$ (*iv*) $v = \frac{1}{2}$ **Ans.** (*iii*)

11. Inverse transformation $w = \frac{1}{z}$ transforms the straight line $ay + bx = 0$ into

(*i*) Circle (*ii*) straight line through the origin
(*iii*) straight line (*iv*) none of these

12. The analytic function $f(z)$, which maps the angular region $0 \le \theta \le \pi/4$ onto the region $\pi/4 \le \phi \le \pi/2$ is

(*i*) $ze^{i\pi/4}$ (*ii*) $z + \pi/4$ (*iii*) iz (*iv*) $e^{z+i\pi/4}$ **Ans.**

7.38 BILINEAR TRANSFORMATION (Mobius Transformation)

$$\boxed{w = \frac{az+b}{cz+d}} \qquad ad - bc \ne 0 \qquad \ldots (1)$$

(1) is known as bilinear transformation.

If $ad - bc \ne 0$ then $\frac{dw}{dz} \ne 0$ *i.e.* transformation is conformal.

From (1), $$z = \frac{-dw+b}{cw-a} \qquad \ldots(2)$$

This is also bilinear except $w = \frac{a}{c}$.

Note. From (1), every point of z-plane is mapped into unique point in w-plane except $z = -\frac{d}{c}$

.

From (2), every point of w-plane is mapped into unique point in z-plane except $w = \frac{a}{c}$.

7.39 INVARIANT POINTS OF BILINEAR TRANSFORMATION

We know that $$w = \frac{az+b}{cz+d} \qquad \ldots (1)$$

If z maps into itself, then $w = z$

(1) becomes $$z = \frac{az+b}{cz+d} \qquad \ldots (2)$$

Roots of (2) are the invariants or fixed points of the bilinear transformation.
If the roots are equal, the bilinear transformation is said to be parabolic.

7.40 CROSS-RATIO

If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ is called the cross-ratio of z_1, z_2, z_3, z_4.

7.41 THEOREM

A bilinear transformation preserves cross-ratio of four points

Proof. We know that $\quad \boldsymbol{w = \frac{az + b}{cz + d}}$.

As w_1, w_2, w_3, w_4 are images of z_1, z_2, z_3, z_4 respectively, so

$$w_1 = \frac{az_1 + b}{cz_1 + d}, \quad w_2 = \frac{az_2 + b}{cz_2 + d}$$

$$\therefore \qquad w_1 - w_2 = \frac{(ad - bc)}{(cz_1 + d)(cz_2 + d)}(z_1 - z_2) \qquad \text{...(1)}$$

Similarly

$$w_2 - w_3 = \frac{ad - bc}{(cz_2 + d)(cz_3 + d)}(z_2 - z_3) \qquad \text{...(2)}$$

$$w_3 - w_4 = \frac{ad - bc}{(cz_3 + d)(cz_4 + d)}(z_3 - z_4) \qquad \text{...(3)}$$

$$w_4 - w_1 = \frac{ad - bc}{(cz_4 + d)(cz_1 + d)}(z_4 - z_1) \qquad \text{...(4)}$$

From (1), (2), (3) and (4), we have

$$\boxed{\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}}$$

$$\Rightarrow \qquad (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

7.42 PROPERTIES OF BILINEAR TRANSFORMATION

1. A bilinear transformation maps circles into circles.

2. A bilinear transformation preserves cross ratio of four points.

If four points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively.

$$\Rightarrow \qquad \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Hence, under the **bilinear** transform of four points cross-ratio is preserved.

7.43 METHODS TO FIND BILINEAR TRANSFORMATION

1. By finding a, b, c, d for $\frac{az+b}{cz+d}$ with the given conditions.

2. With the help of cross-ratio

$$\boxed{\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}}$$

Example 75. *Find the bilinear transformation which maps the points* $z = 1, i, -1$ *into the points* $w = i, 0, -i$.

Hence find the image of $|z| < 1$. (*U.P., III Semester, 2008, Summer 2002*)
(*U.P. (Agri. Engg.) 2002*)

Solution. Let the required transformation be $w = \frac{az+b}{cz+d}$

or $$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z + 1} = \frac{pz+q}{rz+1} \quad \ldots (1) \qquad \left[p = \frac{a}{d},\ q = \frac{b}{d},\ r = \frac{c}{d}\right]$$

z	w
1	i
i	0
-1	$-i$

On substituting the values of z and corresponding values of w in (1), we get

$$i = \frac{p+q}{r+1} \quad \Rightarrow \quad p + q = ir + i \quad \ldots (2)$$

$$0 = \frac{pi+q}{ri+1} \quad \Rightarrow \quad pi + q = 0 \quad \ldots (3)$$

$$-i = \frac{-p+q}{-r+1} \quad \Rightarrow \quad -p + q = ir - i \quad \ldots (4)$$

On subtracting (4) from (2), we get $2p = 2i$ or $p = i$

On putting the value of p in (3) , we have $i\,(i) + q = 0$ or $q = 1$

On substituting the values of p and q in (2), we obtain

$$i + 1 = i\,r + i \quad \text{or} \quad 1 = i\,r \quad \text{or} \quad r = -i$$

Putting the values of p, q, r in (1), we have

$$w = \frac{iz+1}{-iz+1}$$

$$u + iv = \frac{i(x+iy)+1}{-i(x+iy)+1} = \frac{(ix-y+1)(ix+y+1)}{(-ix+y+1)(ix+y+1)} = \frac{-x^2-y^2+1+2ix}{x^2+(y+1)^2}$$

Equating real parts, we get

$$u = \frac{-x^2-y^2+1}{x^2+(y+1)^2} \quad \ldots (5)$$

But $\quad |z|<1 \Rightarrow x^2+y^2<1 \quad \Rightarrow \quad 1-x^2-y^2>0$

From (5) $\quad u > 0$ As denominator is positive. **Ans.**

Example 76. *Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find the image of the unit circle $|z| = 1$.*

[*Uttarakhand, III Semester 2008, U.P. III semester (C.O.) 2003*]

Solution. On putting $z = 0, -1, i$ into $w = i, 0, \infty$ respectively in

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \ldots (1)$$

z	w
0	i
-1	0
i	∞

$$\Rightarrow \quad \frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(-1)}{(-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)} \quad \Rightarrow \quad \left(\frac{w-i}{-i}\right) = \frac{z(1+i)}{z-i}$$

$$\Rightarrow \quad w - i = \frac{(-i+1)z}{z-i} \quad \Rightarrow \quad w = \frac{(1-i)z}{z-i} + i = \frac{(1-i)z+iz+1}{z-i}$$

$$\Rightarrow \quad w = \frac{z+1}{z-i} \quad \ldots (2)$$ **Ans.**

From (2) $$z = \frac{iw+1}{w-1} \quad \ldots (3) \qquad \left[\text{Inverse transformation is } z = \frac{-dw+b}{cw-a}\right]$$

And $|z| = 1$

$$\Rightarrow \quad \left|\frac{iw+1}{w-1}\right| = 1 \quad \Rightarrow \quad |1 + iw| = |w - 1|$$

$$\Rightarrow \quad |1 + i(u + iv)| = |u + iv - 1| \quad \Rightarrow \quad |1 - v + iu| = |u - 1 + iv|$$

$$\Rightarrow \quad (1 - v)^2 + u^2 = (u - 1)^2 + v^2 \quad \Rightarrow \quad 1 + v^2 - 2v + u^2 = u^2 + 1 - 2u + v^2$$

$$\Rightarrow \quad u - v = 0 \quad \Rightarrow \quad v = u$$ **Ans.**

Example 77. *Find the fixed points and the normal form of the following bilinear transformations.*

(*a*) $w = \frac{3z-4}{z-1}$ and (*b*) $w = \frac{z-1}{z+1}$

Discuss the nature of these transformations.

Solution. (*a*) The fixed points are obtained by

$$z = \frac{3z-4}{z-1} \quad \text{or} \quad z^2 - 4z + 4 = 0 \quad \text{or} \quad (z-2)^2 = 0 \Rightarrow z = 2$$

$z = 2$ is the only fixed point. This transformation is parabolic.

Normal Form

$$w = \frac{3z-4}{z-1} \quad \Rightarrow \quad \frac{1}{w-2} = \frac{1}{\frac{3z-4}{z-1} - 2} = \frac{z-1}{3z-4-2z+2} = \frac{z-1}{z-2}$$

and $$\frac{1}{w-2} = \frac{1}{z-2} + 1$$

(*b*) The fixed points are obtained by

$$z = \frac{z-1}{z+1} \quad \Rightarrow \quad z^2 + z = z - 1 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z = \pm i$$

Hence $\pm\, i$ are the two fixed points.

Normal Form

$$w = \frac{z-1}{z+1}$$

$$w - i = \frac{z-1}{z+1} - i = \frac{z-1-i(z+1)}{z+1} \quad \text{... (1)}$$

and $$w + i = \frac{z-1}{z+1} + i = \frac{z-1+i(z+1)}{z+1} \quad \text{...(2)}$$

On dividing (1) by (2), we get

$$\frac{w-i}{w+i} = \frac{z-1-iz-i}{z-1+iz+i} = \frac{(1-i)(z-i)}{(1+i)(z+i)} = \frac{(-i^2-i)(z-i)}{(1+i)(z+i)}$$

$$\frac{w-1}{w+1} = -i\left(\frac{z-i}{z+i}\right) = k\left(\frac{z-i}{z+i}\right) \text{ where } k = -i$$

The transformation is elliptic. **Ans.**

Example 78. *The fixed points of the transformation* $w = \frac{2z-5}{z+4}$ *are given by*:

(*a*) $\left(\frac{5}{2}, 0\right)$ (*b*) $(-4, 0)$ (*c*) $(-1 + 2i, -1 - 2i)$ (*d*) $(-1+\sqrt{6}, -1-\sqrt{6})$

(*AMIETE, Dec. 2010*)

Solution. Here $f(z) = \frac{2z-5}{z+4}$

In the case of fixed point $z = \frac{2z-5}{z+4}$

$\Rightarrow$ $z^2 + 4z = 2z - 5$ $\Rightarrow$ $z^2 + 2z + 5 = 0$

$\Rightarrow$ $z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

Thus, $z = -1 \pm 2i$ are the only fixed points.

Hence (c) is correct answer. **Ans.**

Example 79. *Show that the transformation* $w = i\,\frac{1-z}{1+z}$ *transforms the circle* $|z| = 1$ *onto the real axis of the w-plane and the interior of the circle into the upper half of the w-plane.*

(*U.P., III Semester, Dec. 2003*)

Solution. $w = i\left(\frac{1-z}{1+z}\right)$

$$(u+iv) = i\left(\frac{1-(x+iy)}{1+(x+iy)}\right) = \frac{(i-ix+y)}{[1+(x+iy)]}\,\frac{[(1+x)-iy]}{[(1+x)-iy]} \quad \text{(Rationalizing)}$$

$$= \frac{i+ix+y-ix-ix^2-xy+y+xy-iy^2}{(1+x)^2+y^2} = \frac{y-xy+y+xy+i+ix-ix-ix^2-iy^2}{(1+x)^2+y^2}$$

$$= \frac{2y+i(1-x^2-y^2)}{(1+x)^2+y^2}$$

Equating the real and imaginary parts, we get

$$u = \frac{2y}{(1+x)^2+y^2} \quad \text{... (1)}$$

and $$v = \frac{1-(x^2+y^2)}{(1+x)^2+y^2} \quad \text{... (2)}$$

when $x^2 + y^2 = 1$, then $v = \frac{1-1}{(1+x)^2+y^2} = 0$

$v = 0$ is the equation of the real axis in the w – plane. **Proved.**

(*b*) Now the equation of the interior of the circle is $x^2 + y^2 < 1$.

Dividing (1) by (2), we get

$$\frac{u}{v} = \frac{2y}{1-(x^2+y^2)}, \qquad u - u(x^2+y^2) = 2vy, \qquad u(x^2+y^2) = u - 2vy$$

$$x^2+y^2 = 1-\frac{2vy}{u}, \quad 1-\frac{2vy}{u} < 1 \qquad [\text{as } x^2+y^2 < 1]$$

$$-\frac{2vy}{u} < 0, \qquad 2vy > 0$$

$v > 0$

$v > 0$ is the equation of the upper half of w-plane. **Proved.**

Example 80. *Show that* $\omega = \frac{i-z}{i+z}$ *maps the real axis of the z-plane into the circle* $|w| = 1$ *and (ii) the half-plane* $y > 0$ *into the interior of the unit circle* $|\omega| < 1$ *in the w-plane.*

(*U.P., III Semester, Dec. 2005, 2002*)

Solution. We have $\omega = \frac{i-z}{i+z}$

$$|\omega| = \left|\frac{i-z}{i+z}\right| = \frac{|i-z|}{|i+z|} = \frac{|i-x-iy|}{|i+x+iy|}$$

$$|\omega| = \left|\frac{-x+i(1-y)}{x+i(1+y)}\right|, \quad |\omega| = \frac{\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+(1+y)^2}}$$

Now the real axis in z-plane *i.e.i* $y = 0$, transform into

$$|\omega| = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1, \qquad |\omega| = 1 \qquad |z| = 1$$

Hence the real axis in the z-plane is mapped into the circle $|\omega| = 1$.

(*ii*) The interior of the circle *i.e.* $|w| < 1$ gives.

$$\frac{\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+(1+y)^2}} < 1 \quad \Rightarrow \quad \frac{x^2+(1-y)^2}{x^2+(1+y)^2} < 1 \qquad \Rightarrow \quad x^2+(1-y)^2 < x^2+(1+y)^2$$

$$\Rightarrow \qquad 1+y^2-2y < 1+y^2+2y \quad \Rightarrow \quad -4y < 0 \quad \Rightarrow \quad y > 0.$$

Thus the upper half of the z-plane corresponds to the interior of the circle $|w| = 0$. **Proved.**.

Example 81. *Show that the transformation* $w = \dfrac{3-z}{z-2}$ *transforms the circle with centre* $\left(\dfrac{5}{2}, 0\right)$ *and radius* $\dfrac{1}{2}$ *in the z-plane into the imaginary axis in the w-plane and the interior of the circle into the right half of the plane.* *(A.M.I.E.T.E. Summer 2000)*

Solution. $w = \dfrac{3-z}{z-2} \quad \Rightarrow \quad u+iv = \dfrac{3-x-iy}{x+iy-2} \quad \Rightarrow \quad (u+iv)(x+iy-2) = 3-x-iy$

$$\Rightarrow \qquad ux + iuy - 2u + ivx - vy - 2iv = 3 - x - iy$$

$$\Rightarrow \qquad ux - 2u - vy + i(uy + vx - 2v) = 3 - x - iy$$

Equating real and imaginary quantities, we have

$$ux - vy - 2u = 3 - x \quad \text{and} \quad vx - 2v + uy = -y$$

$$\Rightarrow \qquad (u+1)x - vy = 2u + 3 \text{ and } vx + (u+1)y = 2v$$

On solving the equations for x and y, we have

$$x = \frac{2u^2+2v^2+5u+3}{u^2+v^2+2u+1}, \quad y = \frac{-v}{u^2+v^2+2u+1}$$

Here, the equation of the given circle is $\left(x-\dfrac{5}{2}\right)^2 + y^2 = \dfrac{1}{4}$ ... (1)

Putting the values of x and y in (1), we have

$$\left(\frac{2u^2+2v^2+5u+3}{u^2+v^2+2u+1} - \frac{5}{2}\right)^2 + \left(\frac{-v}{u^2+v^2+2u+1}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \qquad \left(\frac{-u^2-v^2+1}{2(u^2+v^2+2u+1)}\right)^2 + \left(\frac{-v}{u^2+v^2+2u+1}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \qquad (-u^2-v^2+1)^2 + 4v^2 = (u^2+v^2+2u+1)^2$$

$$\Rightarrow \qquad (u^2+v^2-1)^2 + 4v^2 = [(u^2+v^2-1)+(2u+2)]^2$$

$$\Rightarrow \qquad (u^2+v^2-1)^2 + 4v^2 = (u^2+v^2-1)^2 + (2u+2)^2 + 2(u^2+v^2-1)(2u+2)$$

$$\Rightarrow \qquad v^2 = (u+1)^2 + (u^2+v^2-1)(u+1)$$

$$\Rightarrow \qquad v^2 = u^2+2u+1+u^3+uv^2-u+u^2+v^2-1$$

$$\Rightarrow \qquad 0 = u^3+2u^2+u+uv^2$$

$\Rightarrow \qquad u(u^2+2u+1+v^2) = 0 \quad \Rightarrow \quad u = 0$ *i.e.*, equation of imaginary axis.

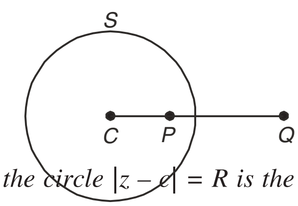
Equation of the interior of the circle is $\left(x-\frac{5}{2}\right)^2 + y^2 < \frac{1}{4}$.
Then corresponding equation in u, v is

$$u(u^2+2u+1+v^2) > 0 \quad \text{or} \quad u[(u+1)^2+v^2] > 0$$

As $(u+1)^2+v^2>0$ so $u = 0$ *i.e.*, equation of the right half plane. **Ans.**

7.44 INVERSE POINT WITH RESPECT TO A CIRCLE

Two points P and Q are said to be the inverse points with respect to a circle S if they are collinear with the centre C on the same side of it, and if the product of their distances from the centre is equal to r^2 where r is the radius of the circle.



Thus when P and Q are the inverse points of the circle, then the three points C, P, Q are collinear, and also $CP.CQ = r^2$

Example 82. *Show that the inverse of a point a, with respect to the circle* $|z-c| = R$ *is the point* $c+\dfrac{R^2}{\bar{a}-\bar{c}}$

Solution. Let b be the inverse point of the point a' with respect to the circle $|z-c| = R$.

Condition I. The points a, b, c are collinear. Hence

$$arg\ (\bar{b}-\bar{c}) = arg\ (\bar{a}-\bar{c}) = -arg\ (\bar{a}-\bar{c}) \qquad (\text{since } \arg z = -\arg \bar{z})$$

$$\Rightarrow \quad arg\ (\bar{b}-\bar{c}) + arg\ (\bar{a}-\bar{c}) = 0 \quad \text{or} \quad arg\ (\bar{b}-\bar{c})\ (\bar{a}-\bar{c}) = 0$$

$\therefore \quad (\bar{b}-\bar{c})(\bar{a}-\bar{c})$ is real, so that

$$(\bar{b}-\bar{c})(\bar{a}-\bar{c}) = |(\bar{b}-\bar{c})(\bar{a}-\bar{c})|$$

Condition II. $|\bar{b}-\bar{c}|\,|\bar{a}-\bar{c}| = R^2 \quad \Rightarrow \quad |\bar{b}-\bar{c}|\,|\bar{a}-\bar{c}| = R^2 \qquad \{|z| = |\bar{z}|\}$

$$|(\bar{b}-\bar{c})\ (\bar{a}-\bar{c})| = R^2 \quad \Rightarrow \quad (\bar{b}-\bar{c})\ (\bar{a}-\bar{c}) = R^2 \quad \Rightarrow \quad \bar{b}-\bar{c} = \frac{R^2}{\bar{a}-\bar{c}}$$

$$\Rightarrow \qquad \bar{b} = \bar{c} + \frac{R^2}{\bar{a}-\bar{c}}.$$ **Proved.**

Example 83. *Find a Mobius transformation which maps the circle* $|w| \le 1$ *into the circle* $|z-1| < 1$ *and maps* $w = 0$, $w = 1$ *respectively into* $z = \frac{1}{2}$, $z = 0$.

Solution. Let the transformation be,

$$w = \frac{az+b}{cz+d} \qquad \text{... (1)}$$

Since, $w = 0$ maps into $z = \frac{1}{2}$,
From (1), we get

z	w
½	0
0	1

$$0 = \frac{\frac{a}{2}+b}{\frac{c}{2}+d} \quad \Rightarrow \quad \frac{a}{2}+b = 0 \quad \Rightarrow \quad b = \frac{-a}{2} \qquad \text{... (2)}$$

Since $w = 1$ maps into $z = 0$, from (1), we get

$$1 = \frac{0+b}{0+d} \quad \Rightarrow \quad b = d \qquad \text{... (3)}$$

Here $\qquad |w| = 1$ corresponding to $|z-1| = 1$

Therefore points $w, \dfrac{1}{w}$ inverse with respect to the circle $|w| = 1$ correspond to the points

$z,\ 1+\dfrac{1}{z-1}$ inverse with respect to the circle $|z-1| = 1$

[z and $a+\dfrac{R^2}{z-a}$ are inverse points on the circle $|z-a| = R$]

Particular $w = 0$ and $\propto$ correspond to

$$z=\frac{1}{2},\ 1+\frac{1}{\frac{1}{2}-1} \Rightarrow z=\frac{1}{2},\ -1$$

Since $w = 0$ maps into $z = -1$, from (1), we get

$$\propto = \frac{-a+b}{-c+d} \Rightarrow -c+d = 0 \quad \Rightarrow \quad c = d \qquad \text{... (4)}$$

From (2), (3) and (4), $\qquad b = -\dfrac{a}{2}, \qquad b = c = d$

From (1) $$w=\frac{az+b}{cz+d}=\frac{-2bz+b}{bz+b}=\frac{-2z+1}{z+1}$$ **Ans.**

Example 84. *Show that bilinear transformation of a circle of z-plane into a circle of w-plane and inverse points are transformed into inverse points.*

In particular case in which the circle in the z-plane transform into a straight line in the w-plane, the inverse points transform into points symmetrical about this line.

Solution. The equation of a circle is

$$\frac{|z-p|}{|z-q|}=k \qquad \text{... (1) with inverse points } p, q, k \neq 1.$$

Let the bilinear transformation is $w=\dfrac{az+b}{cz+d}$ ... (2)

Under this transformation points p, q in the z-plane map into $\dfrac{aq+b}{cq+d}$ and $\dfrac{aq+b}{cq+d}$ in the w-plane.

From (2), we get $$z=\frac{dw-b}{-cw+a} \qquad \text{... (3)}$$

Putting the value of z from (3) into (1), we get

$$\frac{\left|\dfrac{dw-b}{-cw+a}-p\right|}{\left|\dfrac{dw-b}{-cw+a}-q\right|}=k \quad \Rightarrow \quad \frac{\left|w-\dfrac{ap+b}{cp+d}\right|}{\left|w-\dfrac{aq+b}{cq+d}\right|}=k\frac{|cq+d|}{|cp+d|} \qquad \text{... (4)}$$

This is the equation of circle in w-plane. Its inverse points are

$$\frac{c\,p+b}{c\,p+d} \text{ and } \frac{aq+b}{cq+d}.$$

Particular case. If $\qquad k\dfrac{|cp+d|}{|cq+d|}=1$

then equation (4), becdomes

$$\frac{\left|w-\dfrac{ap+b}{cp+d}\right|}{\left|w-\dfrac{aq+b}{cp+d}\right|}=1 \qquad \text{... (5)}$$

(5) is the equation of a line bisecting at right angles to the join of the points $\dfrac{ap+b}{cp+d}$ and $\dfrac{aq+b}{cq+d}$.

Example 85. *Find two bilinear transformations whose fixed points are 1 and 2.*

(*Q. Bank U.P.T.U. 2002*)

Solution. We have, $$w = \frac{az+b}{cz+d} \quad \text{... (1)}$$

Fixed points are given by

$$z = \frac{az+b}{cz+d}$$

$$\Rightarrow \quad cz^2 - (a-d)\,z - b = 0 \quad \Rightarrow \quad z^2 - \frac{(a-d)}{c}z - \frac{b}{c} = 0 \quad \text{... (2)}$$

Fixed points are 1 and 2, so

$$(z-1)(z-2) = 0$$

$$\Rightarrow \quad z^2 - 3z + 2 = 0 \quad \text{... (3)}$$

Equating the coefficeints of z and constants in (2) and (3), we get

$$\therefore \quad \frac{a-d}{c} = 3 \quad \text{and} \quad -\frac{b}{c} = 2$$

$$\Rightarrow \quad b = -2c \quad \text{and} \quad d = a - 3c$$

Putting the values of b and d in (1), we get

$$w = \frac{az-2c}{cz+a-3c}$$ has its fixed points at $z = 1$ and $z = 2$.

Taking $a = 1$, $c = -1$ and $a = 2$, $c = -1$, we have

$$w = \frac{z+2}{4-z} \quad \text{and} \quad w = \frac{2(z+1)}{5-z}$$ **Ans.**

Example 86. *Show that the transformation* $w = \dfrac{2z+3}{z-4}$ *maps the circle* $x^2 + y^2 - 4x = 0$ *onto the straight line* $4u + 3 = 0$.

Solution. We have, $$w = \frac{2z+3}{z-4}$$

The inverse transformation is $$z = \frac{4w+3}{w-2} \quad \text{... (1)}$$

Now the circle $x^2 + y^2 - 4x = 0$ can be written as $z\bar{z} - 2(z+\bar{z}) = 0$ $\quad \begin{bmatrix} z = x+iy \\ \bar{z} = x-iy \end{bmatrix}$

Substituting for z and $\bar{z}$ from (1), we get

$$\frac{4w+3}{w-2} \cdot \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$\Rightarrow \quad 16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} + 3\bar{w} - 8w - 6 + 4w\bar{w} + 3w - 8\bar{w} - 6) = 0$$

$$\Rightarrow \quad 22(w+\bar{w}) + 33 = 0 \quad \Rightarrow \quad 22(2u) + 33 = 0 \Rightarrow 4u + 3 = 0 \quad \begin{bmatrix} w = u+iv \\ \bar{w} = u-iv \end{bmatrix}$$

Thus, circle is transformed into a straight line. **Ans.**

Example 87. *If a is any real positive number, show that the transformation* $w = \dfrac{z-a}{z+a}$ *transforms conformally the plane* $x > 0$ *to the unit circle* $|w| < 1$. *What are the transforms of* $|w|$ = *constant and* arg w = *constant in z-plane?*

(*Q. Bank U.P. III Semester 2002*)

Solution. We have, $$w = \frac{z-a}{z+a}$$

(*i*) $$|w| < 1$$

$$\Rightarrow \quad \left|\frac{z-a}{z+a}\right| < 1$$
$$\Rightarrow \quad |z - a| < |z + a|$$
$$\Rightarrow \quad |x - a + iy| < |x + a + iy|$$
$$\Rightarrow \quad (x - a)^2 + y^2 < (x + a)^2 + y^2$$
$$\Rightarrow \quad -2ax < 2ax$$
$$\Rightarrow \quad 4ax > 0$$
$$\Rightarrow \quad x > 0$$

(*ii*) Hence the transformation (1) transforms conformally the plane $x > 0$ to the unit circle $|w| < 1$.

The circle $|w| = k$ transform into

$$\left|\frac{z-a}{z+a}\right| = k \quad \Rightarrow \quad \left|\frac{x-a+iy}{x+a+iy}\right| = k \quad \Rightarrow \quad \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = k^2$$
$$(x-a)^2 + y^2 = k^2[(x+a)^2 + y^2]$$
$$\Rightarrow \quad x^2 + y^2 + a^2 - 2ax\left(\frac{1+k^2}{1-k^2}\right) = 0 \qquad \dots (2)$$

There is a series of coaxal circles is z-plane.

(*iii*) From (1), we have

$$\text{Re}^{i\phi} = \frac{z-a}{z+a}$$
$$\Rightarrow \quad \text{Re}^{i\phi} = \frac{(x-a)+iy}{(x+a)+iy} \qquad \dots (3)$$

Take logarithm of both sides of (3), we get

$$\log R + i\phi = \log\{(x-a) + iy\} - \log\{(x+a) + iy\}$$
$$\therefore \quad \phi = \tan^{-1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x+a}\right)$$
$$\Rightarrow \quad \tan\phi = \frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{x^2 - a^2}} = \frac{y(x+a) - y(x-a)}{x^2 - a^a + y^2}$$

So, the lines $\phi = \alpha$ transform into

$$\tan\alpha = \frac{2ay}{x^2 + y^2 - a^2}$$

$\Rightarrow$ $x^2 + y^2 - a^2 - 2ay \cot\alpha = 0$ which are coaxal circles orthgonal to (2). **Ans.**

EXERCISE 7.9

1. Find the bilinear transformation that maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$ respectively. **Ans.** $w = \frac{3z + 2i}{iz + 6}$

2. Determine the bilinear transformation which maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ onto $w_1 = i$, $w_2 = -1$, $w_3 = -i$ respectively. **Ans.** $w = \frac{z-i}{iz-1}$

3. Verify that the equation $w = \frac{1+iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half plane $v > 0$.

4. Find the bilinear transformation which maps 1, i, – 1 to 2, i, – 2 respectively. Find the fixed and critical points of the transformation. **Ans.** i, $2i$

5. Show that the transformation $w = \frac{i(1-z)}{1+z}$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane.

6. Show that the transformation $w = \frac{iz+2}{4z+i}$ transforms the real axis in the z-plane into circle in the w-plane. Find the centre and the radius of this circle. (*A.M.I.E.T.E., Winter 2000*)

Ans. $\left(0, \frac{7}{8}\right), \frac{9}{8}$

7. Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$

8. If z_0 is the upper half of the z-plane show that the bilinear transformation

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z}_0}\right)$$

maps the upper half of the z-plane into the interior of the unit circle at the origin in the w-plane.

9. Find the condition that the transformation $w = \frac{az+b}{cz+d}$ transforms the unit circle in the w-plane into straight line in the z-plane. **Ans.** If $\left|\frac{c}{a}\right| = 1$ or $|a| = |c|$

10. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of the circle $|z| = 1$ under this transformation ?

Ans. Straight line $2u + 1 = 0$

11. Show that the map of the real axis of the z-plane on the w-plane by the transformation is a circle and find its centre and radius. **Ans.** Centre $\left(0, -\frac{1}{2}\right)$, Radius $= \frac{1}{2}$

12. Find the invariant points of the transformation $w = -\left(\frac{2z+4i}{iz+1}\right)$. Prove also that these two points together with any point z and its image w, form a set of four points having a constant cross ratio. **Ans.** $4i$ and $-i$

13. Show that under the transformation $w = \frac{z-i}{z+i}$, the real axis in z-plane is mapped into the circle$|w| = 1$. What portion of the z-plane corresponds to the interior of the circle ?

Ans. The half z-plane above the real axis corresponds to the interior of the circle $|w| = 1$.

14. Discuss the application of the transformation $w = \frac{iz+1}{z+i}$ to the areas in the z-plane which are respectively inside and outside the unit circle with its centre at the origin.

15. What is the form of a bilinear transformation which has one fixed point α and the other fixed point ∞?

16. Prove that, in general, in the bilinear transformation $w = \frac{az+b}{cz+d}$, there are two values of z (invariant points) for which $w = z$ but there is only one value if $(a - d)^2 + 4bc = 0$.

Choose the correct alternative:

17. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are

(*i*) $-4/5, -5$ (*ii*) $2, 2$ (*iii*) $-2, -2$ (*iv*) $2, -2$ **Ans.** (*iv*)

18. The fixed points of the mapping $f(z) = \frac{3iz+13}{z-3i}$ are

(*i*) $3i \pm 2$ (*ii*) $3 \pm 2i$ (*iii*) $2 \pm 3i$ (*iv*) $-2 \pm 3i$ **Ans.** (*i*)

7.45 TRANSFORMATION: $\boldsymbol{w = z^2}$

Solution. $w = z^2$

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

Equating real and imaginary parts, we get $u = x^2 - y^2$, $v = 2xy$

(*i*) (*a*) Any line parallel to *x*-axis, *i.e.*, $y = c$, maps into

$$u = x^2 - c^2, \quad v = 2cx$$

Eliminating *x*, we get $v^2 = 4c^2\,(u + c^2)$ ... (1) which is a parabola.

(*b*) Any line parallel to *y*-axis, *i.e.*, $x = b$, maps into a curve

$$u = b^2 - y^2,\ v = 2by$$

Eliminating *y*, we get $v^2 = -4b^2(u - b^2)$, ... (2) which is a parabola.

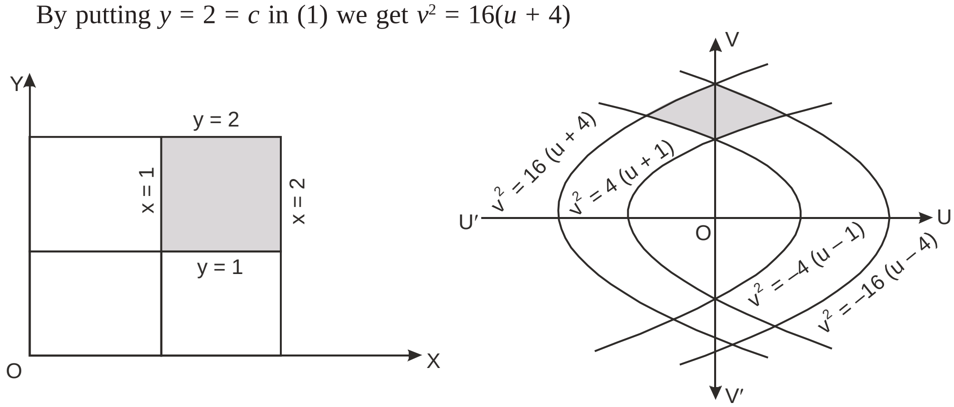
(*c*) The rectangular region bounded by the lines $x = 1$, $x = 2$, and $y = 1$, $y = 2$ maps into the region bounded by the parabolas.

By putting $x = 1 = b$ in (2) we get $v^2 = -4(u - 1)$,,

By putting $x = 2 = b$ in (2) we get $v^2 = -16(u - 4)$

By putting $y = 1 = c$ in (1) we get $v^2 = 4(u + 1)$,

By putting $y = 2 = c$ in (1) we get $v^2 = 16(u + 4)$



(*ii*) (*a*) In polar co-ordinates: $z = re^{i\theta}$, $w = Re^{i\phi}$

$$w = z^2$$

$$Re^{i\phi} = r^2\, e^{2i\theta}$$

Then $R = r^2,\ \phi = 2\theta$

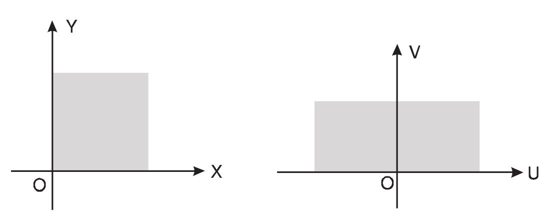
In *z*-plane, a circle $r = a$ maps into $R = a^2$ in *w*-plane.

Thus, circles with centre at the origin map into circles with centre at the origin.

(*b*) If $\theta = 0$, $\phi = 0$ *i.e.*, real axis in *z*-plane maps into real axis in *w*-plane

If , $\theta = \dfrac{\pi}{2}, \phi = \pi$ *i.e.*, the positive imaginary axis in *z*-plane maps into negative real axis in *w*-plane.

Thus, the first quadrant in *z*-plane $0 \le \theta \le \dfrac{\pi}{2}$, maps into upper half of *w*-plane $0 \le \phi \le \pi$.



The angles in *z*-plane at origin maps into double angle in *w*-plane at origin.

Hence, the mapping $w = z^2$ is not conformal at the origin.

It is conformal in the entire *z*-plane except origin. Since $\dfrac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is critical point of mapping.

Example 88. *For the conformal transformation $w = z^2$, show that*
(*a*) *the coefficient of magnification at $z = 2 + i$ is* $2\sqrt{5}$
(*b*) *the angle of rotation at $z = 2 + i$ is* $\tan^{-1}(0.5)$.

Solution.
$$z = 2 + i$$
$$f(z) = w = z^2$$
$$= (2 + i)^2 = 4 - 1 + 4i = 3 + 4i$$
$$f'(z) = 2z = 2(2 + i) = 4 + 2i$$

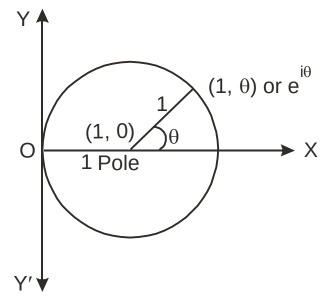
(*a*) Coefficient of magnification $= |f'(z)| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$ **Proved.**

(*b*) The angle of rotation $= \tan^{-1}\dfrac{v}{u} = \tan^{-1}\dfrac{2}{4} = \tan^{-1}(0.5)$ **Proved.**

Example 89. *For the conformal transformation $w = z^2$, show that the circle $|z - 1| = 1$ transforms into the cardioid $R = 2(2 + \cos\phi)$ where $Re^{i\phi}$ in the w-plane.*

Solution. $|z - 1| = 1$ ... (1)

Equation (1) represents a circle with centre at (1, 0) and radius 1.
Shifting the pole to the point (1, 0), any point on (1) is $1 + e^{i\theta}$
Transformation is under $w = z^2$.



$$Re^{i\phi} = (1 + e^{i\theta})^2$$
$$= e^{i\theta}\left(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}\right)^2$$
$$= e^{i\theta}\left(2\cos\frac{\theta}{2}\right)^2 = 4e^{i\theta}\cos^2\frac{\theta}{2}$$

This gives $R = 4\cos^2\dfrac{\theta}{2}$,

$\Rightarrow$ $R = 2\left(2\cos^2\dfrac{\phi}{2}\right)$ $[\phi = \theta]$

$\Rightarrow$ $R = 2(\cos\phi + 1)$ **Proved.**

7.46 TRANSFORMATION: $w = z^n$ $(n \in N)$

$$Re^{i\phi} = (re^{i\theta})^n = r^n e^{in\theta}$$

Hence, $R = r^n,\ \phi = n\theta$

Mapping of simple figures

z-plane	*w-plane*
Circle, $r = a$	Circle, $R = a^n$
The initial line, $\theta = 0$	The initial line, $\phi = 0$
The straight line, $\theta = \theta_0$	The straight line, $\phi = n\,\theta_0$

7.47 TRANSFORMATION: $w = z + \dfrac{1}{z}$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

At $z = \pm 1$, $\dfrac{dw}{dz} = 0$, so transformation is not conformal at $z = \pm 1$.

$$w = z + \frac{1}{z} = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\frac{u}{r + \frac{1}{r}} = \cos\theta \quad \text{and} \quad \frac{v}{r - \frac{1}{r}} = \sin\theta$$

$$\sin^2\theta + \cos^2\theta = \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} \quad \Rightarrow \quad 1 = \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2}$$

z-plane	*w*-plane
Circle, $r = r$	Ellipses
Circle, $r = 1$	Lines $u = 2$
Lines, $\theta = \theta_0$	Hyperbola : $\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$

7.48 TRANSFORMATION: $w = e^z$

$$u + iv = e^{x + iy} = e^x.e^{iy} = e^x(\cos y + i \sin y)$$

Equating real and imaginary parts, we have

$$u = e^x \cos y, \qquad v = e^x \sin y$$

Again $\quad w = e^z$

$$Re^{i\phi} = e^{x+iy} = e^x.e^{iy}$$

Hence $\quad R = e^x$ or $x = \log_e R$ and $y = \phi$

Mapping of simple figures

z-plane	*w-plane*
The straight line $x = c$	Circle $R = e^c$
y-axis ($x = 0$)	Unit Circle $R = e^0 = 1$
Region between $y = 0$, $y = \pi$	Upper half plane
Region between $y = 0$, $y = -\pi$	Lower half plane
Region between the lines $y = c$ and $y = c + 2\pi$	Whole plane

Example 90. *Find the image and draw a rough sketch of the mapping of the region $1 \le x \le 2$ and $2 \le y \le 3$ under the mapping $w = e^z$.*

Solution. $\quad z = x + iy$

Let $\quad w = Re^{i\phi}$ ... (1)

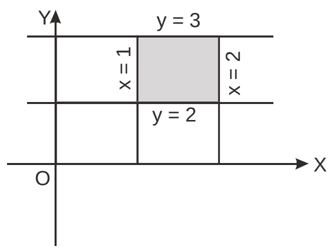
But $\quad w = e^z + e^{x+iy}$ ... (2)

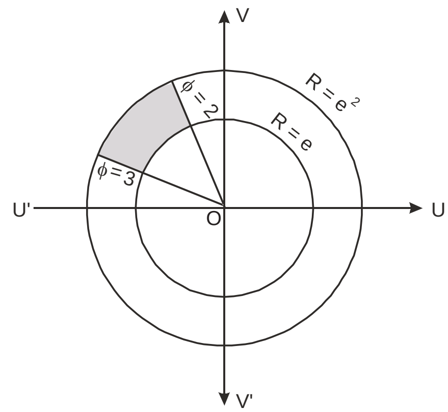
From (1) and (2); $\quad Re^{i\phi} = e^{x+iy} = e^x.e^{iy}$

Equating real and imaginary parts, we get $R = e^x$ ... (3) and $\phi = y$

(i) Here $1 \le x$, then $R = e^1$ is circle of radius $e^1 = 2.7$

$x = 2$, then $R = e^2$ represents a circle of radius $e^2 = 7.4$

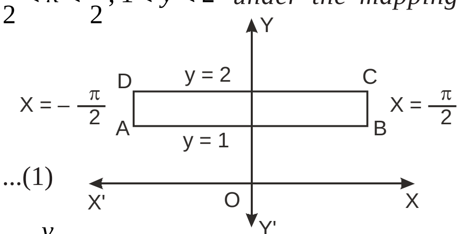




(ii) $y = 2$ and $\phi = 2$ represents radial line making an angle of 2 radians with the x-axis. $y = 3$, then $\phi = 3$ represents radial line making an angle 3 radians with x-axis.

Hence, the mapping of the region $1 \le x \le 2$ and $2 \le y \le 3$ maps the shaded sectors in the figure. **Ans.**

Example 91. *Find the image of the strip* $-\frac{\pi}{2} < x < \frac{\pi}{2}, 1 < y < 2$ *under the mapping* $w(z) = \sin z$.



Solution. $w(z) = \sin z = \sin(x + iy)$

$$= \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

$$u = \sin x \cosh y \quad \Rightarrow \quad \sin x = \frac{u}{\cosh y} \quad \text{...(1)}$$

$$v = \cos x \sinh y \quad \Rightarrow \quad \cos x = \frac{v}{\sinh y} \quad \text{...(2)}$$

Eliminating x from (1) and (2), we get

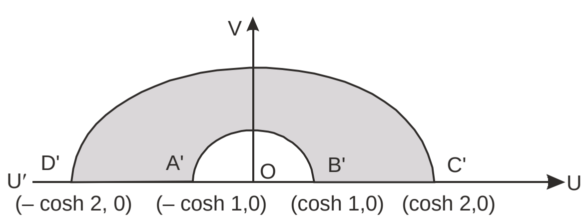
$$\sin^2 x + \cos^2 x = \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} \quad \Rightarrow \quad 1 = \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y}$$

Hence $y = 2$, maps into the ellipse

$$\frac{u^2}{\cosh^2 2} + \frac{v^2}{\sinh^2 2} = 1 \quad \Rightarrow \quad \frac{u^2}{14.15} + \frac{v^2}{13.15} = 1$$

Also $y = 1$, maps into the ellipse.

$$\frac{u}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1 \quad \Rightarrow \quad \frac{u^2}{2.38} + \frac{v^2}{1.38} = 1$$



The image of $A\left(\frac{-\pi}{2}, 1\right)$ in z-plane is $(-\cosh 1, 0)$ *i.e.* $(-1.543, 0)$ in w-plane

The image of the point $D\left(-\frac{\pi}{2},\ 2\right)$ in z-plane is (– cosh 2, 0) *i.e.*, (– 3.762, 0).

Hence, AD line in z-plane maps into $A'D'$ line in w-plane.

The image of $B\left(\frac{\pi}{2}, 1\right)$ is (cosh 1, 0) *i.e.*, (1.543, 0) in w-plane.

The image of $C\left(\frac{\pi}{2}, 2\right)$ is (cosh 2, 0) *i.e.*, (3.762, 0) in w-plane.

Hence, BC line maps into $B'C'$ line in w-plane.

Hence, the strip $\frac{-\pi}{2} < x < \frac{\pi}{2},\ 1 < y < 2$ maps into the shaded region of w-plane bounded by the ellipses and u-axis. **Ans.**

7.49 TRANSFORMATION:

w = cosh z

$$u + iv = \cosh(x + iy) = \cos i(x + iy) = \cos(ix - y)$$

$$= \cos ix \cos y + \sin ix \sin y = \cosh x \cos y + i \sinh x \sin y$$

So $\qquad u = \cosh x \cos y, \quad v = \sinh x \sin y$

$$\Rightarrow \qquad \cosh x = \frac{u}{\cos y} \quad \text{and} \quad \sinh x = \frac{v}{\sin y}$$

On eliminating x, we get $\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$ ... (1) $(\cosh^2 x - \sinh^2 x = 1)$

On eliminating y, we get $\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$ ... (2) $(\cos^2 y + \sin^2 y = 1)$

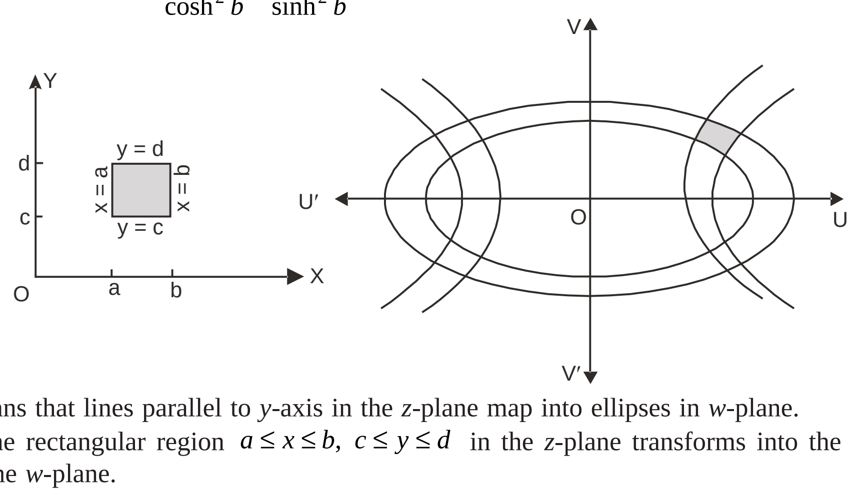
(*a*) On putting $y = a$ (constant) in (1), we get

$$\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1 \quad \textit{i.e.}, \text{ Hyperbola.}$$

It shows that the lines parallel to x-axis in the z-plane map into hyperbola in the w-plane.

(*b*) On substituting $x = b$ (constant) in (2), we obtain

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$



It means that lines parallel to y-axis in the z-plane map into ellipses in w-plane.

(*c*) The rectangular region $a \le x \le b,\ c \le y \le d$ in the z-plane transforms into the shaded portion in the w-plane.

UNIT-3

COMPLEX INTEGRATION

In case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the x-axis from $x = a$ to $x = b$. But in case of a complex function $f(z)$ the path of the definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z = a$ to $z = b$.

$z = x + iy \quad \Rightarrow \quad dz = dx + idy$ ... (1) $\quad dz = dx$ if $y = 0$ ... (2) $dz = idy$ if $x = 0$ ... (3)

In (1), (2), (3) the direction of dz are different. Its value depends upon the path (curve) of integration. But the value of integral from a to b remains the same along any regular curve from a to b.

In case the initial point and final point coincide so that c is a closed curve, then this integral is called *contour integral* and is denoted by $\oint_C f(z)\,dz.$

If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + idy$,
we have

$$\oint_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Let us consider a few examples:

Complex Integral (Line Integral)

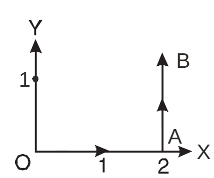
Example 48. *Evaluate* $\int_0^{2+i} (\bar{z})^2 dz$ *along the real axis from z = 0 to z = 2 and then along a line parallel to y-axis from z = 2 to z = 2 + i.*

(*R.G.P.V., Bhopal, III Semester, June 2005*)

Solution. $\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$

$$= \int_{OA} (x^2)\,dx + \int_{AB} (2-iy)^2\, idy$$

[Along OA, $y = 0$, $dy = 0$, x varies 0 to 2.
Along AB, $x = 2$, $dx = 0$ and y varies 0 to 1]



$$= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2\, i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\int_0^1 (4-4iy-y^2)\,dy = \frac{8}{3} + i\left[4y - 2iy^2 - \frac{y^3}{3}\right]_0^1$$

$$= \frac{8}{3} + i\left[4 - 2i - \frac{1}{3}\right] = \frac{8}{3} + \frac{i}{3}(11-6i) = \frac{1}{3}(8+11i+6) = \frac{1}{3}(14+11i)$$

Which is the required value of the given integral. **Ans.**

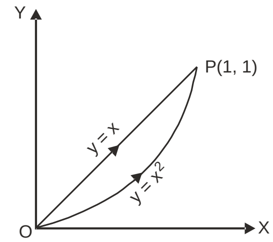
Example 49. *Evaluate* $\int_0^{1+i}(x^2 - iy)\, dz$, *along the path*

(a) $y = x$ *(R.G.P.V., Bhopal, III Semester, Dec. 2007)* (b) $y = x^2$.

Solution. (a) Along the line $y = x$,

$dy = dx$ so that $dz = dx + idy$

$\Rightarrow$ $dz = dx + idx = (1 + i)\, dx$



$\therefore$ $\int_0^{1+i}(x^2 - iy)\, dz$

[On putting $y = x$ and $dz = (1 + i)dx$]

$$= \int_0^1 (x^2 - ix)(1+i)\, dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\,\frac{x^2}{2}\right]_0^1 = (1+i)\left(\frac{1}{3} - \frac{1}{2}i\right) = \frac{(1+i)(2-3i)}{6} = \frac{5}{6} - \frac{1}{6}\,i.$$

Which is the required value of the given integral. **Ans.**

(b) Along the parabola $y = x^2$, $dy = 2x\, dx$ so that

$dz = dx + idy \Rightarrow dz = dx + 2ix\, dx = (1 + 2ix)\, dx$

and x varies from 0 to 1.

$$\therefore \int_0^{1+i}(x^2 - iy)\, d\, x = \int_0^1 (x^2 - ix^2)(1 + 2ix)dx = \int_0^1 x^2\,(1-i)(1+2ix)\, dx$$

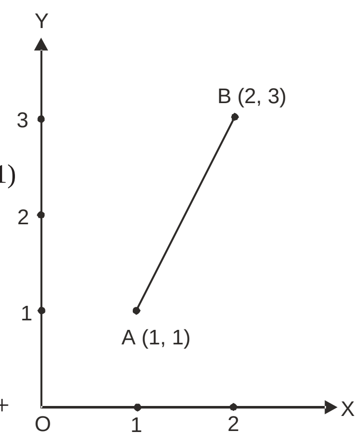
$$= (1-i)\int_0^1 x^2\,(1+2ix)\, dx = (1-i)\left[\frac{x^3}{3} + i\,\frac{x^4}{2}\right]_0^1$$

$$= (1-i)\left[\frac{1}{3} + \frac{1}{2}\,i\right] = \frac{(1-i)(2+3i)}{6} = \frac{1}{6}\,(2 + 3i - 2i + 3) = \frac{5}{6} + \frac{1}{6}i$$

Which is the required value of the given integral. **Ans.**

Example 50. *Evaluate* $\int_C (12z^2 - 4iz)\, dz$ *along the curve C joining the points (1, 1) and (2, 3)* (*U.P., III Semester, Dec. 2009*)

Solution. Here, we hae



$$\int_C (12z^2 - 4iz)\;\; dz = \int_C [12\,(x+iy)^2 - 4i\,(x+iy)]\,(dx + i\,dy)$$

$$= \int_C [12\,(x^2 - y^2 + 2i\,xy) - 4ix + 4y]\,(dx + i\,dy)$$

$$= \int_C (12\,x^2 - 12y^2 + 24\,ixy - 4ix + 4y)\,(dx + i\,dy) \;\;...\; (1)$$

Equation of the time AB passing through (1, 1) and (2, 3) is

$$y - 1 = \frac{3-1}{2-1}\,(x-1)$$

$y - 1 = 2\,(x - 1) \Rightarrow y = 2x - 1 \Rightarrow dy = 2\, dx$

Putting the values of y and dy in (1), we get

$= \int_1^2 (12x^2 - 12\,(2x - 1)^2 + 24\,ix\,(2x - 1) - 4ix +$ $4\,(2x - 1)]\,(dx + 2i\,dx)$

$$= \int_1^2 [12x^2 - 48x^2 + 48x - 12 + 48\,ix^2 - 24\,ix - 4ix + 8x - 4]\,(1 + 2i)\,dx$$

$$= (1 + 2i)\int_1^2 [-36 + 48i)\,x^2 + (56 - 28i)\,x - 16]\,dx$$

$$= (1 + 2i)\left[(-36+48i)\frac{x^3}{3} + (56 - 28i)\,\frac{x^2}{2} - 16x\right]_1^2$$

$$= (1+2i)\left[(-36+48i)\frac{8}{3}+(56-28i)\,2-16\times 2-(36+48i)\frac{1}{3}-(56-28i)\frac{1}{2}+16\right]$$

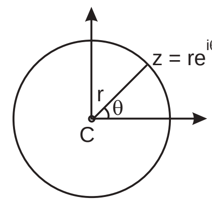
$$= (1+2i)\,(-96+128i+112-56i-32+12-16i-28+14i+16$$

$$= (1+2i)\,(-16+70i) = -16+70i-32i-140 = -156+38i \qquad \textbf{Ans.}$$

Example 51. *Evaluate* $\int (z-a)^n\, dz$ *where c is the circle with centre a and r. Discuss the case when n = –1.*

Solution. The equation of circle C is $|z-a| = r$ or $z-a = re^{i\theta}$ where θ varies from 0 to 2π

$$dz = ire^{i\theta}d\theta$$



$$\oint_C (z-a)^n\, dz = \int_0^{2\pi} r^n e^{in\theta}.ire^{i\theta}\, d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta = ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \qquad [\because\ n \neq -1]$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i2(n+1)\pi}-1\right] = \frac{r^{n+1}}{n+1}[\cos 2(n+1)\,\pi + i\sin 2(n+1)\pi - 1] = \frac{r^{n+1}}{n+1}[1+0i-1]$$

$$= 0. \qquad [\text{When } n \neq -1]$$

Which is the required value of the given integral.

When $n = -1$,

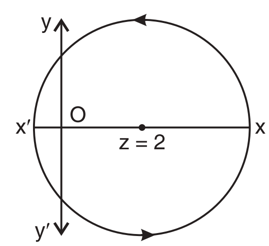
$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi}\frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i. \qquad \textbf{Ans.}$$

Example 52. *Evaluate* $\int_C (z-z^2)dz,$ *where C is the upper half of the circle* $|z-2| = 3$. *What is the value of the integral if C is the lower half of the above given circle?*

(*MDU, Dec 2009*)

Solution. Put $z = re^{i\theta} = 3e^{i\theta} \Rightarrow dz = 3i.e^{i\theta}\, d\theta$

Upper Circle



$$\int_C (z-z^2)dz = \int_0^{\pi}(3e^{i\theta}-9e^{2i\theta})3ie^{i\theta}d\theta$$

$$= \int_0^{\pi}[9ie^{2i\theta}-27ie^{3i\theta}]d\theta = \left[\frac{9}{2}e^{2i\theta}-\frac{27}{3}e^{3i\theta}\right]_0^{\pi}$$

$$= \left(\frac{9}{2}e^{2i\pi}-9e^{3i\pi}\right)-\left(\frac{9}{2}-9\right)$$

$$= \frac{9}{2}\left[e^{2i\pi}-2e^{3i\pi}+1\right]$$

$$= \frac{9}{2}[1+2+1] = 18$$

$$\left(\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ \therefore\quad e^{i2\pi} &= 1, \quad e^{i3\pi} = -1 \end{aligned}\right)$$

Ans.

Lower Circle

$$\int_C (z-z^2)dz = \int_{\pi}^{2\pi}(9ie^{2i\theta}-27ie^{3i\theta})\,d\theta$$

$$= \left[\frac{9}{2}e^{2i\theta}-9e^{3i\theta}\right]_{\pi}^{2\pi} = \left(\frac{9}{2}e^{4\pi i}-9e^{6\pi i}\right)-\left(\frac{9}{2}e^{2\pi i}-9e^{3\pi i}\right)$$

$$= \left(\frac{9}{2}-9\right)-\left(\frac{9}{2}+9\right) = -18 \qquad \textbf{Ans.}$$

EXERCISE 7.6

1. Integrate $f(z) = x^2 + ixy$ from A (1, 1) to B (2, 8) along
(*i*) the straight line AB; (*ii*) the curve C, $x = t$, $y = t^3$. **Ans.** (*i*) $-\frac{1}{3}(147-71)i$ (*ii*) $-\left(\frac{1094}{21}-\frac{124i}{5}\right)$

2. Evaluate $\int_{1-i}^{2+i}(2x+iy+1)\,dz$ along
(*i*) $x = t+1,\ y = 2t^2-1$; (*ii*) the straight line joining $1 - i$ and $2 + i$. **Ans.** (*i*) $4+\frac{25}{3}i$ (*ii*) $4 + 8i$
(*R.G.P.V., Bhopal, Dec. 2008*)

3. Evalute the line integral $\int_C z^2 dz$ where C is the boundary of a triangle with vertices 0, $1 + i$. $-1 + i$ clockwise. **Ans.** 0

4. Evaluate $\int_C (z+1)^2 dz$ where C is the boundary of the rectangle with vertices at the points $a + ib$, $-a + ib$, $-a - ib$, $a - ib$. **Ans.** 0

5. Evaluate the integral $\int_c |z|\,dz$, where C is the straight line from $z = -i$ to $z = i$. **Ans.** i

6. Evaluate the integral $\int_c |z|\,dz$, where C is the left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$. **Ans.** $2i$

7. Evaluate the integral $\int_c \log z\,dz$, where C is the unit circle $|z| = 1$. **Ans.** $2\pi i$

8. Integrate xz along the straight line from A (1, 1) to B (2, 4) in the complex plane. Is the value the same if the path of integration from A to B is along the curve $x = t$, $y = t^2$?
Ans. $-\frac{151}{15}+\frac{45i}{4}$

9. Evaluate $\int_0^{2+i}(\bar{z})^2 dz$, along
(*i*) the real axis to 2 and then vertically to $2 + i$, (*ii*) the line $y = x/2$.
(*U.P., III Semester, June 2009*) **Ans.** (*i*) $\frac{1}{3}(14+i)$, (*ii*) $\frac{5}{3}(2-i)$

Choose the correct answer:

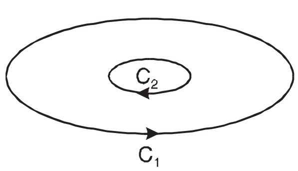
10. The value of $\int_C \frac{4z^2+z+5}{z-4}\,dz$, where C : $9x^2 + 4y^2 = 36$
(*i*) – 1 (*ii*) 1 (*iii*) 2 (*iv*) 0 (*AMIETE, June 2009*) **Ans.** (*iv*)

7.23 IMPORTANT DEFINITIONS

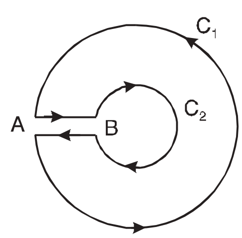
(*i*) **Simply connected Region.** A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region D are the points of the region D.

(*ii*) **Multi-Connected Region.** Multi-connected region is bounded by more than one curve. We can convert a multi-connected region into a simply connected one, by giving it one or more cuts.

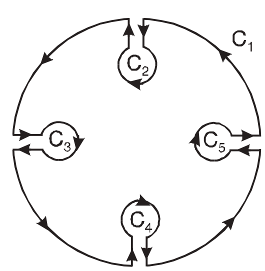
Note. A function $f(z)$ is said to be **meromorphic** in a region R if it is analytic in the region R except at a finite number of poles.



Multi-Connected Region



Simply Connected Region



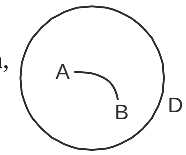
Simply Connected Region

(*iii*) **Single-valued and Multi-valued function**

If a function has only one value for a given value of z, then it is a single valued function.

For example $f(z) = z^2$

If a function has more than one value, it is known as multi-valued function,



For example $f(z) = z^{\frac{1}{2}}$

(*iv*) **Limit of a function**

A function $f(z)$ is said to have a limit l at a point $z = z_0$, if for a given an arbitrary chosen positive number ε, there exists a positive number δ, such that

$$|f(z) - l| < \varepsilon \text{ for } |z - z_0| < \delta$$

It may be written as $\lim_{z \to z_0} f(z) = l$

(*v*) **Continuity**

A function $f(z)$ is said to be continuous at a point $z = z_0$ if for a given an arbitrary positive number ε, there exists a positive number δ, such that

$$|f(z) - l| < \varepsilon \text{ for } |z - z_0| < \delta$$

In other words, a function $f(z)$ is continuous at a point $z = z_0$ if

(*a*) $f(z_0)$ exists (*b*) $\lim_{z \to z_0} f(z) = f(z)_{z=0}$

(*vi*) **Multiple point.** If an equation is satisfied by more than one value of the variable in the given range, then the point is called a multiple point of the arc.

(*vii*) **Jordan arc.** A continuous arc without multiple points is called a Jordan arc.

Regular arc. If the derivatives of the given function are also continuous in the given range, then the arc is called a regular arc.

(*viii*) **Contour.** A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

The contour is said to be closed if the starting point A of the arc coincides with the end point B of the last arc.

(*ix*) **Zeros of an Analytic function.**

The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of f(z). **For example,** (1) Zeros of $z^2 - 3z + 2$ are $z = 1$ and $z = 2$.

(2) Zeros of cos z is $\pm (2n-1)\frac{\pi}{2}$, where $n = 1, 2, 3$..........

7.24 CAUCHY'S INTEGRAL THEOREM

(*AMIETE, Dec. 2009, U.P. III Semester, 2009-2010, R.G.P.V., Bhopal, III Semester, Dec. 2002*)

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)\,dz = 0$.

Proof. Let the region enclosed by the curve c be R and let

$$f(z) = u + iv, \quad z = x + iy,\ dz = dx + idy$$

$$\int_c f(z)\,dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{(By Green's theorem)}$$

Replacing $-\frac{\partial v}{\partial x}$ by $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ we get

$$\int_c f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy = 0 + i0 = 0$$

$\Rightarrow \quad \int_C f(z)\,dz = 0$ **Proved.**

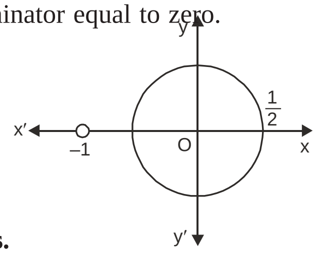
Note. If there is no pole inside and on the contour then the value of the integral of the function is zero.

Example 53. *Find the integral* $\int_c \frac{3z^2+7z+1}{z+1}\,dz,$ where C is the circle $|z| = \frac{1}{2}.$

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$z + 1 = 0 \quad \Rightarrow \quad z = -1$

The given circle $|z| = \frac{1}{2}$ with centre at $z = 0$ *and radius* $\frac{1}{2}$ does not enclose any singularity of the given function.



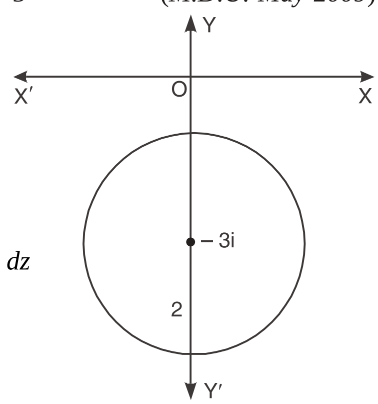
$$\int_C \frac{3z^2+7z+1}{z+1}\,dz = 0$$ (*By Cauchy Integral theorem*) **Ans.**

Example 54. *Evaluate* $\oint_C \frac{dz}{z^2+9}$, *where C is*

(*i*) $|z + 3i| = 2$ (*ii*) $|z| = 5$ (*M.D.U. May 2009*)

Solution. Here $f(z) = \frac{1}{z^2+9}$

The poles of $f(z)$ can be determined by equating the denominator equal to zero.



(*i*) $\therefore \quad z^2 + 9 = 0 \quad \Rightarrow \quad z = \pm 3i$

Pole at $z = -3i$ lies in the given circle C.

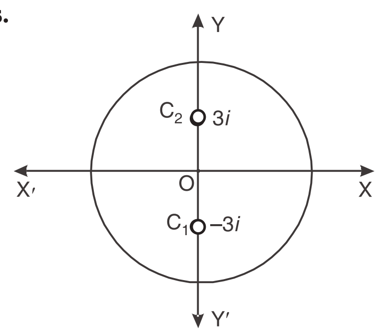
$$\int_C f(z)\,dz = \int_C \frac{1}{z^2+9}\,dz = \int_C \frac{1}{(z+3i)(z-3i)}\,dz$$

$$= \int_C \frac{\frac{1}{z-3i}}{z+3i}\,dz$$

$$= 2\pi i \left[\frac{1}{z-3i}\right]_{z=-3i}$$

$$= 2\pi i \left[\frac{1}{-3i-3i}\right] = \frac{-2\pi i}{6i} = -\frac{\pi}{3}$$ **Ans.**

(*ii*) Both the poles $z = 3i$ and $z = -3i$ lie inside the given contour



$$\int_C f(z)\,dz = \int_C \frac{1}{z^2+9}\,dz = \int_C \frac{1}{(z+3i)(z-3i)}\,dz$$

$$= \int_{C_1} \frac{\frac{1}{z-3i}}{z+3i}\,dz + \int_{C_2} \frac{\frac{1}{z+3i}}{z-3i}\,dz$$

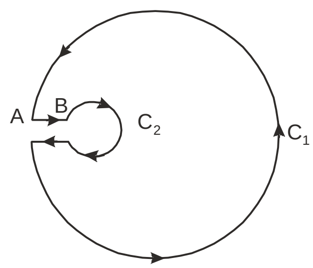
$$= 2\pi i \left[\frac{1}{z-3i}\right]_{z=-3i} + 2\pi i \left[\frac{1}{z+3i}\right]_{z=3i}$$

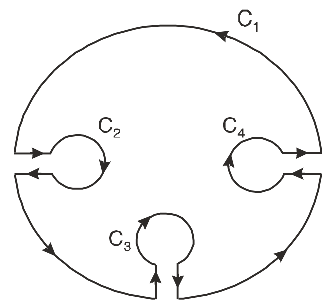
$$= 2\pi i \left[\frac{1}{-3i-3i}\right] + 2\pi i \left[\frac{1}{3i+3i}\right] = -\frac{\pi}{3} + \frac{\pi}{3} = 0$$ **Ans.**

7.25 EXTENSION OF CAUCHY'S THEOREM TO MULTIPLE CONNECTED REGION

If $f(z)$ is analytic in the region R between two simple closed curves c_1 and c_2 then

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$





Proof. $\int f(z)\,dz = 0$
where the path of integration is along AB, and curves c_2 in clockwise direction and along BA and along c_1 in anticlockwise direction.

$$\int_{AB} f(z)\,dz - \int_{c_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{c_1} f(z)dz = 0$$

$$\Rightarrow \qquad -\int_{c_2} f(z)\,dz + \int_{c_1} f(z)\,dz = 0 \qquad \text{as } \int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$$

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$ **Proved.**

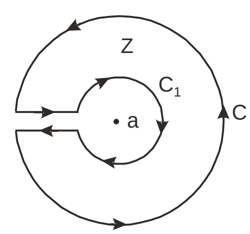
Corollary. $\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$

7.26 CAUCHY INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_z \frac{f(z)}{z-a}\,dz$$

(*AMIETE June 2010, U.P., III Semester Dec. 2009 R.G.P.V., Bhopal, III Semester, June 2008*)



Proof. Consider the function $\dfrac{f(z)}{z-a}$, which is analytic at all points within C, except $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $\dfrac{f(z)}{z-a}$ is analytic in the region between C and C_1; hence by Cauchy's Integral Theorem for multiple connected region, we have

$$\int_c \frac{f(z).dz}{z-a} = \int_{c_1} \frac{f(z)}{z-a}\,dz = \int_{c_1} \frac{f(z) - f(a) + f(a)}{z-a}\,.\,dz$$

$$= \int_{c_1} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_{c_1} \frac{dz}{z-a} \qquad \ldots (1)$$

For any point on C_1

$$\text{Now, } \int_{c_1} \frac{f(z)-f(a)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta}) - f(a)}{re^{i\theta}}\, ire^{i\theta}d\theta \qquad [\, z-a = r\,e^{i\theta} \text{ and } dz = ir\,e^{i\theta}d\theta \,]$$

$$= \int_0^{2\pi} [\, f(a+re^{i\theta}) - f(a)]\; id\theta = 0 \qquad \text{(where } r \text{ tends to zero).}$$

$$\int_{c_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ir\,e^{\,i\theta}d\theta}{r\,e^{i\theta}} = \int_0^{2\pi} id\theta = i\left[\theta\right]_0^{2\pi} = 2\pi i$$

Putting the values of the integrals in R.H.S. of (1), we have

$$\int_c \frac{f(z)\,dz}{z-a} = 0 + f(a)\,(2\pi i) \;\Rightarrow\; f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$ **Proved.**

7.27 CAUCHY INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION *(R.G.P.V., Bhopal, III Semester, Dec. 2007)*

If a function $f(z)$ is analytic in a region R, then its derivative at any point $z = a$ of R is also analytic in R, and is given by

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\,dz$$

where c is any closed curve in R surrounding the point $z = a$.

Proof. We know Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}\,dz \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'a', we get

$$f'(a) = \frac{1}{2\pi i}\int_c f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)dz$$

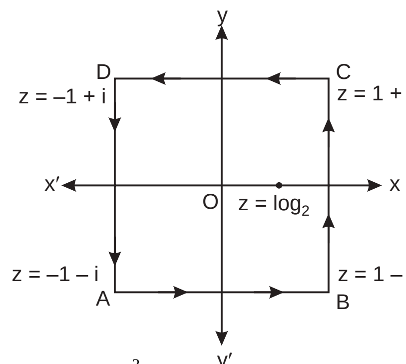
$$\boxed{\boldsymbol{f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\,dz}}$$

Similarly, $\boxed{\boldsymbol{f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^3}}} \quad \Rightarrow \quad \boxed{\boldsymbol{f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}}}$

Example 55. *Evaluate* $\int_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$, *where* C *is the square with vertices at* $\pm 1, \pm i$

(M.D.U. Dec. 2009)

Solution. Here we have $\int_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$



The pole is found by putting the denominator equal to zero

$(z - \log 2)^4 = 0 \;\Rightarrow\; z = \log 2$

The integral has a pole of fourth order.

$$\int_C \frac{e^{3z}}{(z-\log_2)^4}\,dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}(e^{3z})_{z=\log 2}$$

[By Cauchy formula]

$$= \frac{2\pi i}{3!}\;3.3.3(e^{3z})_{z=\log 2} = 9\pi i\, e^{3\log 2} = 9\pi i\, e^{\log 2^3} = 9\pi i(2)^3 = 72\pi i \qquad \textbf{Ans.}$$

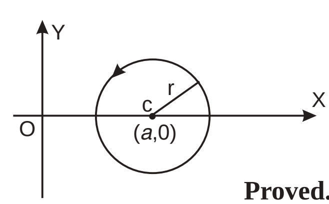
Example 56. *Prove that* $\int_C \frac{dz}{z-a} = 2\pi i$, *where C is the circle* $|z - a| = r$

(R.G.P.V., Bhopal, III Semester, Dec. 2006)

Solution. We have,

$\int_C \frac{dz}{z-a}$, where C is the circle with centre $(a, 0)$ and radius r.

By Cauchy Integral Formula



$$\left[\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\; f(a)\right]$$

$$\int_C \frac{dz}{z-a} = 2\pi i\;(1)$$

$$\Rightarrow \quad \int_C \frac{dz}{z-a} = 2\pi i$$

Proved.

Example 57. *Use Cauchy's integral formula to evaluate* $\int_c \frac{z}{(z^2-3z+2)}\, dz$ *where c is the circle* $|z-2| = \frac{1}{2}$ (*U.P. III Semester, June 2009*)

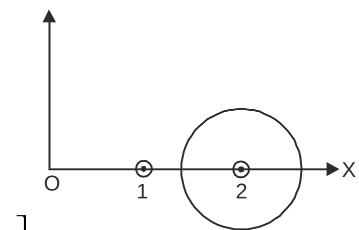
Solution. Here, we have

$$\int_c \frac{z}{(z^2-3z+2)}\, dz$$

The poles are determined by putting the denominator equal to zero

i.e.; $z^2 - 3z + 2 = 0 \quad \Rightarrow \quad (z-1)\quad(z-2) = 0$

$\Rightarrow \qquad z = 1, 2$



So, there are two poles $z = 1$ and $z = 2$.

There is only one pole at $z = 2$ inside the given circle.

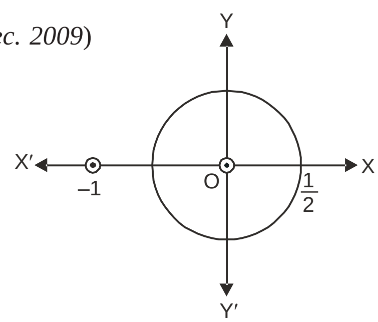
$$\int_c \frac{z}{(z^2-3z+2)}\, dz = \int_c \frac{z}{(z-1)\,(z-2)}\, dz$$

$$= \int_c \frac{\frac{z}{z-1}}{z-2}\, dz \qquad \left[\int_c \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)\right]$$

$$= 2\pi i \left[\frac{z}{z-1}\right]_{z=2} = 2\pi i \left(\frac{2}{2-1}\right) = 4\pi i$$ **Ans.**

Example 58. *Use Cauchy's integral formula to calculate*

$\int_C \frac{2z+1}{z^2+z}\, dz$ *where C is* $|z| = \frac{1}{2}$. (*AMIETE, Dec. 2009*)



Solution. Poles are given by

$$z^2 + z = 0$$

$\Rightarrow \qquad z(z+1) = 0 \qquad \Rightarrow \qquad z = 0, -1$

$|z| = \frac{1}{2}$ is a circle with centre at origin and radius $\frac{1}{2}$.

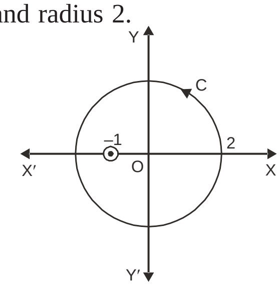
Therefore it encloses only one pole $z = 0$.

$$\therefore \int_C \frac{2z+1}{z\,(z+1)}\, dz = \int_C \frac{\frac{2z+1}{z+1}}{z}\, dz = 2\pi i \left[\frac{2z+1}{z+1}\right]_{z=0} = 2\pi i$$ **Ans.**

Example 59. *Evaluate:* $\int_C \frac{e^z}{(z-1)\,(z-4)}\, dz$ *where C is the circle* $|z| = 2$ *by using Cauchy's Integral Formula.* (*R.G.P.V., Bhopal, III Semester, June 2006*)

Solution. We have,

$\int_C \frac{e^z}{(z-1)\,(z-4)}\, dz$ where C is the circle with centre at origin and radius 2.



Poles are given by putting the denominator equal to zero.

$$(z-1)(z-4) = 0$$

$\Rightarrow \qquad z = 1, 4$

Here there are two simple poles at z = 1 and z = 4.

There is only one pole at z = 1 inside the contour. Therefore

$$\int_C \frac{e^z}{(z-1)\,(z-4)}\, dz = \int \frac{\frac{e^z}{(z-4)}}{(z-1)}\, dz = 2\pi i \left[\frac{e^z}{z-4}\right]_{z=1}$$ (By Cauchy Integral Theorem)

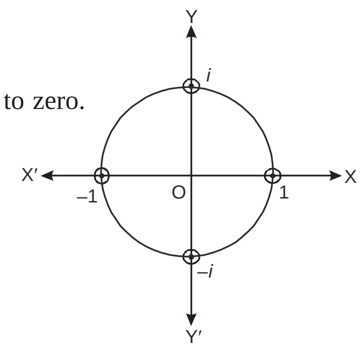
$$= 2\pi i\left(\frac{e}{1-4}\right) = -\frac{2\pi i\, e}{3}$$

Which is the required value of the given integral. **Ans.**

Example 60. *State the Cauchy's integral formula. Show that* $\int_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$

if t > 0 and C is the circle | z | = 3 (*U.P., III Semester, Dec. 2009*)

Solution. Here, we have $\int_C \frac{e^{zt}}{z^2+1}\,dz$



The poles are determined by putting the denominator equal to zero.

i.e., $z^2 + 1 = 0$

$\Rightarrow \quad z^2 = -1$

$\Rightarrow \quad z = \pm\sqrt{-1} = \pm i$

$\Rightarrow \quad z = i, -i$

The integrand has two simple poles at $z = i$ and at $z = -i$. Both poles are inside the given circle with centre at origin and radius 3.

Now, $\int_C \frac{e^{zt}}{z^2+1}\,dz = \frac{1}{2i}\int_C\left(\frac{e^{zt}}{z-i} - \frac{e^{zt}}{z+i}\right)dz$ [By partial fraction]

$$= \frac{1}{2i}\left[\int_{C_1}\frac{e^{zt}}{z-i}\,dz - \int_{C_2}\frac{e^{zt}}{z+i}\,dz\right] = \frac{1}{2i}\left[2\pi i\,(e^{zt})_{z=i} - 2\pi i\,(e^{zt})_{z=-i}\right]$$

$$= \frac{2\pi i}{2i}\left[e^{ti} - e^{-ti}\right] = 2\pi i \sin t$$

Example 61. *Evaluate the following integral using Cauchy integral formula*

$$\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz \quad \textit{where c is the circle } |z| = \frac{3}{2}.$$

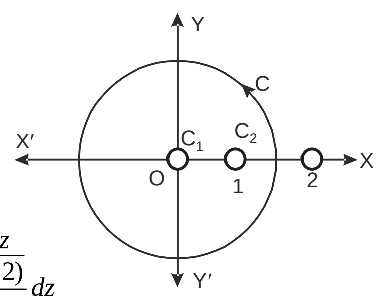
(*AMIETE, Dec. 2009, R.G.P.V., Bhopal, III Semester, June 2008*)

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$z(z-1)(z-2)$ or $z = 0, 1, 2$

The integrand has three simple poles at $z = 0, 1, 2$.

The given circle $|z| = \frac{3}{2}$ with centre at $z = 0$ and radius $= \frac{3}{2}$ encloses two poles $z = 0$, and $z = 1$.



$$\int_C \frac{4-3z}{z\,(z-1)(z-2)}\,dz = \int_{C_1}\frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2}\frac{\frac{4-3z}{z\,(z-2)}}{z-1}\,dz$$

$$= 2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i.\frac{4}{(-1)(-2)} + 2\pi i\,\frac{4-3}{1(1-2)} = 2\pi i(2-1) = 2\pi i$$

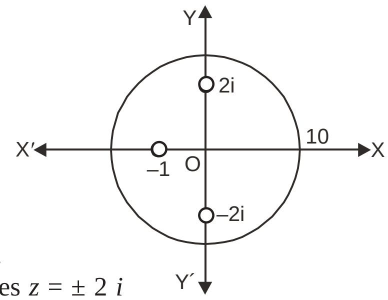
Which is the required value of the given integral. **Ans.**

Example 62. *Evaluate* $\int_c \frac{z^2-2z}{(z+1)^2\,(z^2+4)}\,dz$

where c is the circle | *z* | = *10.* *(U.P. III Semester, June 2009)*

Solution. Here, we have

$$\int_c \frac{z^2-2z}{(z+1)^2\,(z^2+4)}\,dz$$



The poles are determined by putting the denominator equal to zero.

i.e.; $(z+1)^2\,(z^2+4)=0$

$\Rightarrow$ $z=-1,-1$ and $z=\pm 2\,i$

The circle | z | = 10 with centre at origin and radius = 10.

encloses a pole at z = – 1 of second order and simple poles $z=\pm 2\,i$

The given integral = $I_1+I_2+I_3$

$$I_1=\int_{c_1}\frac{z^2-2z}{(z+1)^2\,(z^2+4)}\,dz=\int_{c_1}\frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2}=2\pi i\left[\frac{d}{d\,z}\,\frac{z^2-2z}{z^2+4}\right]_{z=-1}$$

$$=2\pi i\left[\frac{(z^2+4)\,(2z-2)-(z^2-2z)\,2z}{(z^2+4)^2}\right]_{z=-1}=2\pi i\left[\frac{(1+4)\,(-2-2)-(1+2)\,2\,(-1)}{(1+4)^2}\right]$$

$$=2\pi i\left(-\frac{14}{25}\right)=\frac{-28\pi i}{25}$$

$$I_2=\int_{c_2}\frac{\frac{z^2-2z}{(z+1)^2\,(z+2i)}}{(z-2i)}=2\pi i\left[\frac{z^2-2z}{(z+1)^2\,(z+2i)}\right]_{z=2i}=2\pi i\left[\frac{-4-4i}{(2i+1)^2\,(2i+2i)}\right]=2\pi i\,\frac{(1+i)}{4+3i}$$

$$I_3=\int_{c_3}\frac{\frac{z^2-2z}{(z+1)^2\,(z-2i)}}{(z+2i)}\,dz=2\pi i\left[\frac{z^2-2z}{(z+1)^2\,(z-2i)}\right]_{z=-2i}$$

$$=2\pi i\left[\frac{-4+4i}{(-2i+1)^2\,(-2i-2i)}\right]=2\pi i\,\frac{(i-1)}{(3i-4)}$$

$$\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz=I_1+I_2+I_3$$

$$=\frac{-28\pi i}{25}+2\pi i\left(\frac{1+i}{4+3i}\right)+2\pi i\left(\frac{i-1}{3i-4}\right)$$

$$=2\pi i\left[\frac{-14}{25}+\frac{1+i}{(4+3i)}+\frac{(i-1)}{(3i-4)}\right]$$

$$=2\pi i\left[\frac{-14}{25}+\frac{(1+i)\,(3i-4)+(i-1)\,(4+3i)}{(-9-16)}\right]$$

$$=\frac{2\pi i}{-25}\,[14+(3i-4-3-4i)+(4i-3-4-3i)]$$

$$=0$$ **Ans.**

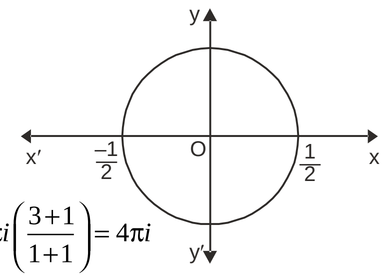
Example 63. *Find the value of* $\int_C \frac{3z^2+z}{z^2-1}\,dz$.

If c is circle $|z-1| = 1$ (*R.G.P.V., Bhopal, III Semester, June 2007*)

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$$z^2 - 1 = 0,\ z^2 = 1,\ z = \pm 1$$

The circle with centre $z = 1$ and radius unity encloses a simple pole at $z = 1$.



By Cauchy Integral formula

$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{\frac{3z^2+z}{z+1}}{z-1}\,dz = 2\pi i\left[\frac{3z^2+z}{z+1}\right]_{z=1} = 2\pi i\left(\frac{3+1}{1+1}\right) = 4\pi i$$

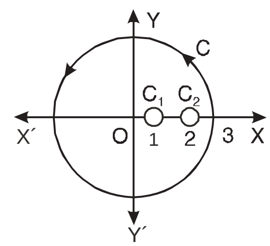
Which is the required value of the given integral. **Ans.**

Example 64. *Use Cauchy integral formula to evaluate.*

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$$

where c is the circle $|z| = 3$.

(*AMIETE, Dec. 2010, R.G.P.V., Bhopal, III Semester, June 2003*)



Solution. $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$

Poles of the integrand are given by putting the denominator equal to zero.

$$(z-1)(z-2) = 0 \quad \Rightarrow \quad z = 1, 2$$

The integrand has two poles at $z = 1, 2$.

The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the poles $z = 1$, and $z = 2$.

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)}\,dz + \int_{C_2} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)}\,dz$$

$$= 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}$$

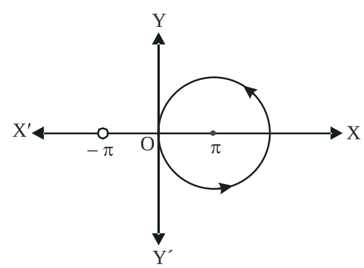
$$= 2\pi i\left(\frac{\sin \pi + \cos \pi}{1-2}\right) + 2\pi i\left(\frac{\sin 4\pi + \cos 4\pi}{2-1}\right) = 2\pi i\left(\frac{-1}{-1}\right) + 2\pi i\left(\frac{1}{1}\right) = 4\pi i$$

Which is the required value of the given integral. **Ans.**

Example 65. *Derive Cauchy Integral Formula.*

Evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$

where C is the circle $|z-\pi| = 3.2$



Solution. Here, $I = \int_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$

Where C is a circle $\{|z-\pi| = 3.2\}$ with centre (π, 0) and radius 3.2.

Poles are determined by putting the denominator equal to zero.

$$(z+\pi)^3 = 0 \qquad \Rightarrow \qquad z = -\pi, -\pi, -\pi$$

There is a pole at $z - \pi$ of order 3. But there is no pole within C.

By Cauchy Integral Formula $\int_C \frac{e^{3iz}}{(z+\pi)^3}\, dz = 0$ **Ans.**

Example 66. *Evaluate, using Cauchy's integral formula,*

$$\int_C \frac{\log z}{(z-1)^3}\, dz \quad \text{where } C \text{ is } |z-1| = \frac{1}{2}.$$ *(MDU. Dec. 2010)*

Solution. Using Cauchy's Integral formula,

$$\int_C \frac{\log z}{(z-1)^3} dz \qquad C: |z-1| = \frac{1}{2}$$

Poles are determined by putting denominater equal to zero.

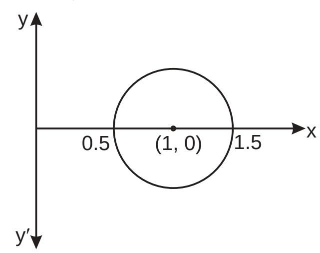
$$(z-1)^3 = 0 \qquad \Rightarrow \qquad z = 1, 1, 1$$

There is one pole of order three at $z = 1$ which is inside the circle C.

$$\int \frac{f(z)}{(z-a)^3}\, dz = 2\pi i f^2(a)$$

$$= 2\pi i\left[\frac{d^2}{dz^2}\log z\right]_{z=1} = 2\pi i\left[\frac{d}{dz}\left(\frac{1}{z}\right)\right]_{z=1}$$

$$= 2\pi i\left(-\frac{1}{z^2}\right)_{z=1} = -2\pi i$$

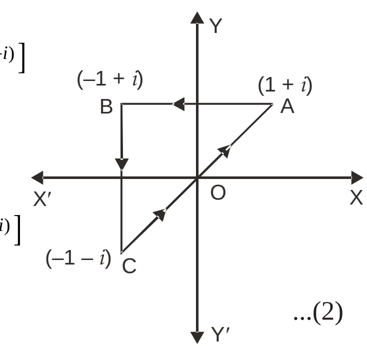


Example 67. *Verify, Cauchy theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1 + i$, $-1 + i$ and $-1 - i$.*

Solution.

$$\int_{AB} e^{iz} dz = \left(\frac{e^{iz}}{i}\right)_{1+i}^{-1+i} = \frac{1}{i}\left[e^{i(-1+i)} - e^{i(1+i)}\right]$$

$$= \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right] \qquad ...(1)$$



$$\int_{BC} e^{iz} dz = \left[\frac{e^{iz}}{i}\right]_{-1+i}^{-1-i} = \frac{1}{i}\left[e^{i(-1-i)} - e^{i(-1+i)}\right]$$

$$= \frac{1}{i}\left[e^{-i+1} - e^{-i-1}\right] \qquad ...(2)$$

$$\int_{CA} e^{iz} dz = \left[\frac{e^{iz}}{i}\right]_{-1-i}^{1+i} = \frac{1}{i}\left[e^{i(1+i)} - e^{i(-1-i)}\right] = \frac{1}{i}\left[e^{i-1} - e^{-i+1}\right] \qquad ...(3)$$

On adding (1), (2) and (3), we get

$$\int_{AB} e^{iz} dz + \int_{BC} e^{iz} dz + \int_{CA} e^{iz} dz = \frac{1}{i}\left[\left(e^{-i-1} - e^{i-1}\right) + \left(e^{-i+1} - e^{-i-1}\right) + \left(e^{i-1} - e^{-i+1}\right)\right]$$

$$\Rightarrow \qquad \int_{\Delta ABC} e^{iz}\, dz = 0 \qquad ...(4)$$

The given function has no pole. So there is no pole in Δ ABC.
The given function e^{iz} is analytic inside and on the triangle ABC.

By Cauchy Theorem, we have $\int_C e^{iz} dz = 0$ ...(5)

From (4) and (5) theorem is verified.

EXERCISE 7.7

Evaluate the following:

1. $\int_C \frac{1}{z-a}\,dz,$ where c is a simple closed curve and the point $z = a$ is
(*i*) outside c; (*ii*) inside c. **Ans.** (*i*) 0 (*ii*) $2\pi i$

2. $\int_c \frac{e^z}{z-1}\,dz$, where c is the circle $|\,z\,| = 2$. **Ans.** $2\pi ie$

3. $\int_C \frac{\cos \pi z}{z-1}\,dz$, where c is the circle $|\,z\,| = 3$. **Ans.** $-2\pi i$

4. $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$, where c is the circle $|z| = 3$. **Ans.** $4\pi i$

5. $\int_C \frac{e^{-2z}}{(z+2)^5}\,dz$, where c is the circle $|z| = 3$. **Ans.** $\frac{\pi ie^2}{12}$

6. $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where c is the circle $|z| = 2$. **Ans.** $\frac{8\pi}{3} ie^{-2}$

7. $\int_C \frac{2z^2+z}{z^2-1}\,dz$ where c is the circle $|z - 1| = 1$ **Ans.** $3\pi i$

8. $\int_C \frac{e^z}{z^2(z+1)^3}\,dz,$ C : $|\,z\,| = 2$. (*AMIETE, June 2009*) **Ans.** ???

9. Evaluate $\oint_C \frac{z^3+z+1}{z^2-3z+2}\,dz$, where C is the ellipse $4x^2 + 9y^2 = 1$.
(*M.D.U. Dec. 2005, May 2008*) **Ans.** 0

10. Evaluate $\int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz,$ where C is $|z| = 1$. (*M.D.U. May 2006, Dec. 2006*) **Ans.** πi

11. Evaluate $\oint \frac{\sin 6z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$, where C is the circle $|z| = 1$. (*M.D.U. May 2005*) **Ans.** $\frac{21}{16}\pi i$

12. If $f(\xi) = \oint_C \frac{4z^2+z+5}{z-\xi}\,dz,$ where C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1,$ find $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$.
(*J.N.T.U. 2005; M.D.U., Dec. 2007*) **Ans.** $20\pi i$, $2\pi(i - 1)$, $-14\pi i$, $16\pi i$

13. $\int_C \frac{z}{z^2-3z+2}\,dz,$ where C is $|z - 2| = \frac{1}{2}$. (*U.P.T.U. 2009; M.D.U. 2007*) **Ans.** $4\pi i$

14. $\int_C \frac{e^z\,dz}{(z+1)^2},$ where C is $|z - 1| = 3$. (*M.D.U. Day 2006*) **Ans.** $\frac{2\pi i}{e}$

Choose the correct alternative:

15. The value of the integral $\int_C \frac{z^2+1}{(z+1)(z+2)}\,dz,$ where C is $|\,z\,| = \frac{3}{2}$ is
(*i*) πi (*ii*) 0 (*iii*) $2\pi i$ (*iv*) $4\pi i$ **Ans.** (*iv*)
(*AMIETE, June 2010*)

16. Cauchy's Integral formula states that if $f(z)$ is analytic within a and on a closed curve C and if a is any point within C then $f(a) =$: (*R.G.P.V., Bhopal, III Semester, June 2007*)
(*i*) $\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-a}$ (*ii*) $\frac{1}{2\pi i}\oint f(z)\,dz$ (*iii*) $\frac{1}{2\pi i}\oint \frac{dz}{z-a}$ (*iv*) none of these. **Ans.** (*i*)

UNIT-4
POWER SERIES

7 Taylor and Laurent series

7.1 Introduction

We originally defined an analytic function as one where the derivative, defined as a limit of ratios, existed. We went on to prove Cauchy's theorem and Cauchy's integral formula. These revealed some deep properties of analytic functions, e.g. the existence of derivatives of all orders.

Our goal in this topic is to express analytic functions as infinite power series. This will lead us to Taylor series. When a complex function has an isolated singularity at a point we will replace Taylor series by Laurent series. Not surprisingly we will derive these series from Cauchy's integral formula.

Although we come to power series representations after exploring other properties of analytic functions, they will be one of our main tools in understanding and computing with analytic functions.

7.2 Geometric series

Having a detailed understanding of geometric series will enable us to use Cauchy's integral formula to understand power series representations of analytic functions. We start with the definition:

Definition. A finite geometric series has one of the following (all equivalent) forms.

$$\begin{aligned} S_n &= a(1 + r + r^2 + r^3 + \ldots + r^n) \\ &= a + ar + ar^2 + ar^3 + \ldots + ar^n \\ &= \sum_{j=0}^{n} ar^j \\ &= a \sum_{j=0}^{n} r^j \end{aligned}$$

The number r is called the ratio of the geometric series because it is the ratio of consecutive terms of the series.

Theorem. The sum of a finite geometric series is given by

$$S_n = a(1 + r + r^2 + r^3 + \ldots + r^n) = \frac{a(1 - r^{n+1})}{1 - r}. \quad (1)$$

Proof. This is a standard trick that you've probably seen before.

$$\begin{array}{lllll} S_n & = a + & ar + ar^2 + \ldots + ar^n & \\ rS_n & = & ar + ar^2 + \ldots + ar^n & + ar^{n+1} \end{array}$$

When we subtract these two equations most terms cancel and we get

$$S_n - rS_n = a - ar^{n+1}$$

Some simple algebra now gives us the formula in Equation 1.

Definition. An infinite geometric series has the same form as the finite geometric series except there is no last term:

$$S = a + ar + ar^2 + \ldots = a\sum_{j=0}^{\infty} r^j.$$

Note. We will usually simply say 'geometric series' instead of 'infinite geometric series'.

Theorem. If $|r| < 1$ then the infinite geometric series converges to

$$S = a\sum_{j=0}^{\infty} r^j = \frac{a}{1-r} \tag{2}$$

If $|r| \geq 1$ then the series does not converge.

Proof. This is an easy consequence of the formula for the sum of a finite geometric series. Simply let $n \to \infty$ in Equation 1.

Note. We have assumed a familiarity with convergence of infinite series. We will go over this in more detail in the appendix to this topic.

7.2.1 Connection to Cauchy's integral formula

Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$

Inside the integral we have the expression

$$\frac{1}{w-z}$$

which looks a lot like the sum of a geometric series. We will make frequent use of the following manipulations of this expression.

$$\frac{1}{w-z} = \frac{1}{w}\cdot\frac{1}{1-z/w} = \frac{1}{w}\left(1 + (z/w) + (z/w)^2 + \ldots\right) \tag{3}$$

The geometric series in this equation has ratio z/w. Therefore, the series converges, i.e. the formula is valid, whenever $|z/w| < 1$, or equivalently when

$$|z| < |w|.$$

Similarly,

$$\frac{1}{w-z} = -\frac{1}{z}\cdot\frac{1}{1-w/z} = -\frac{1}{z}\left(1 + (w/z) + (w/z)^2 + \ldots\right) \tag{4}$$

The series converges, i.e. the formula is valid, whenever $|w/z| < 1$, or equivalently when

$$|z| > |w|.$$

7.3 Convergence of power series

When we include powers of the variable z in the series we will call it a power series. In this section we'll state the main theorem we need about the convergence of power series. Technical details will be pushed to the appendix for the interested reader.

Theorem 7.1. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There is a number $R \geq 0$ such that:

1. If $R > 0$ then the series converges absolutely to an analytic function for $|z - z_0| < R$.
2. The series diverges for $|z - z_0| > R$. R is called the radius of convergence. The disk $|z - z_0| < R$ is called the disk of convergence.
3. The derivative is given by term-by-term differentiation

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

   The series for f' also has radius of convergence R.

4. If γ is a bounded curve inside the disk of convergence then the integral is given by term-by-term integration

$$\int_\gamma f(z)\, dz = \sum_{n=0}^{\infty} \int_\gamma a_n (z - z_0)^n$$

Notes.

- The theorem doesn't say what happens when $|z - z_0| = R$.
- If $R = \infty$ the function $f(z)$ is entire.
- If $R = 0$ the series only converges at the point $z = z_0$. In this case, the series does not represent an analytic function on any disk around z_0.
- Often (not always) we can find R using the ratio test.

The proof of this theorem is in the appendix.

7.3.1 Ratio test and root test

Here are two standard tests from calculus on the convergence of infinite series.

Ratio test. Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_{n+1}/c_n|$ exists, then:

1. If $L < 1$ then the series converges absolutely.

2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test gives no information.

Note. In words, L is the limit of the absolute ratios of consecutive terms.

Again the proof will be in the appendix. (It boils down to comparison with a geometric series.)

Example 7.2. Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the absolute ratios of consecutive terms is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|}{|z^n|} = |z|$$

Thus, the ratio test agrees that the geometric series converges when $|z| < 1$. We know this converges to $1/(1-z)$. Note, the disk of convergence ends exactly at the singularity $z = 1$.

Example 7.3. Consider the series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The limit from the ratio test is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|/(n+1)!}{|z^n|/n!} = \lim \frac{|z|}{n+1} = 0.$$

Since $L < 1$ this series converges for every z. Thus, by Theorem 7.1, the radius of convergence for this series is ∞. That is, $f(z)$ is entire. Of course we know that $f(z) = \mathrm{e}^z$.

Root test. Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_n|^{1/n}$ exists, then:

1. If $L < 1$ then the series converges absolutely.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test gives no information .

Note. In words, L is the limit of the nth roots of the (absolute value) of the terms.

The geometric series is so fundamental that we should check the root test on it.

Example 7.4. Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the nth roots of the terms is

$$L = \lim_{n\to\infty} |z^n|^{1/n} = \lim |z| = |z|$$

Happily, the root test agrees that the geometric series converges when $|z| < 1$.

7.4 Taylor series

The previous section showed that a power series converges to an analytic function inside its disk of convergence. Taylor's theorem completes the story by giving the converse: around each point of analyticity an analytic function equals a convergent power series.

Theorem 7.5. (Taylor's theorem) Suppose $f(z)$ is an analytic function in a region A. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in A. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{5}$$

(Where γ is any simple closed curve in A around z_0, with its interior entirely in A.)

We call the series the power series representing f around z_0.

The proof will be given below. First we look at some consequences of Taylor's theorem.

Corollary. The power series representing an analytic function around a point z_0 is unique. That is, the coefficients are uniquely determined by the function $f(z)$.

Proof. Taylor's theorem gives a formula for the coefficients.

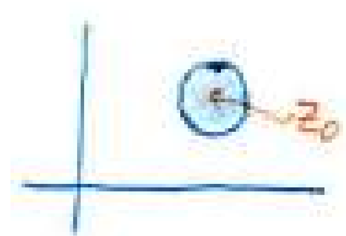
7.4.1 Order of a zero

Theorem. Suppose $f(z)$ is analytic on the disk $|z - z_0| < r$ and f is not identically 0. Then there is an integer $k \geq 0$ such that $a_k \neq 0$ and f has Taylor series around z_0 given by

$$f(z) = (z-z_0)^k(a_k + a_{k+1}(z-z_0) + \ldots) = (z-z_0)^k \sum_{n=k}^{\infty} a_n(z-z_0)^{n-k}. \tag{6}$$

Proof. Since $f(z)$ is not identically 0, not all the Taylor coefficients are zero. So, we take k to be the index of the first nonzero coefficient.

Theorem 7.6. Zeros are isolated. If $f(z)$ is analytic and not identically zero then the zeros of f are isolated. (By isolated we mean that we can draw a small disk around any zeros that doesn't contain any other zeros.)



Isolated zero at z_0: $f(z_0) = 0$, $f(z) \neq 0$ elsewhere in the disk.

Proof. Suppose $f(z_0) = 0$. Write f as in Equation 6. There are two factors:

$$(z - z_0)^k$$

and

$$g(z) = a_k + a_{k+1}(z - z_0) + \ldots$$

Clearly $(z - z_0)^k \neq 0$ if $z \neq z_0$. We have $g(z_0) = a_k \neq 0$, so $g(z)$ is not 0 on some small neighborhood of z_0. We conclude that on this neighborhood the product is only zero when $z = z_0$, i.e. z_0 is an isolated 0.

Definition. The integer k in Theorem 7.6 is called the order of the zero of f at z_0.

Note, if $f(z_0) \neq 0$ then z_0 is a zero of order 0.

7.5 Taylor series examples

The uniqueness of Taylor series along with the fact that they converge on any disk around z_0 where the function is analytic allows us to use lots of computational tricks to find the series and be sure that it converges.

Example 7.7. Use the formula for the coefficients in terms of derivatives to give the Taylor series of $f(z) = \mathrm{e}^z$ around $z = 0$.

Solution: Since $f'(z) = \mathrm{e}^z$, we have $f^{(n)}(0) = \mathrm{e}^0 = 1$. So,

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example 7.8. Expand $f(z) = z^8 \mathrm{e}^{3z}$ in a Taylor series around $z = 0$.

Solution: Let $w = 3z$. So,

$$\mathrm{e}^{3z} = \mathrm{e}^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{k=0}^{\infty} \frac{3^n}{n!} z^n$$

Thus,

$$f(z) = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+8}.$$

Example 7.9. Find the Taylor series of $\sin(z)$ around $z = 0$ (Sometimes the Taylor series around 0 is called the Maclaurin series.)

Solution: We give two methods for doing this.

Method 1.

$$f^{(n)}(0) = \frac{d^n \sin(z)}{dz^n} = \begin{cases} (-1)^m & \text{for } n = 2m+1 = \text{odd, } m = 0, 1, 2, \ldots \\ 0 & \text{for } n \text{ even} \end{cases}$$

Method 2. Using

$$\sin(z) = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i},$$

we have

$$\sin(z) = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}[(1-(-1)^n)]\frac{i^n z^n}{n!}$$

(We need absolute convergence to add series like this.)

Conclusion:

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

which converges for $|z| < \infty$.

Example 7.10. Expand the rational function

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

around $z = 0$.

Solution: Note that f has a singularity at 0, so we can't expect a convergent Taylor series expansion. We'll aim for the next best thing using the following shortcut.

$$f(z) = \frac{1}{z^3} \frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left[2 - \frac{1}{1+z^2}\right].$$

Using the geometric series we have

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = 1 - z^2 + z^4 - z^6 + \ldots$$

Putting it all together

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + \ldots\right) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$$

Note. The first terms are called the singular part, i.e. those with negative powers of z. The summation is called the regular or analytic part. Since the geometric series for $1/(1+z^2)$ converges for $|z| < 1$, the entire series is valid in $0 < |z| < 1$

Example 7.11. Find the Taylor series for

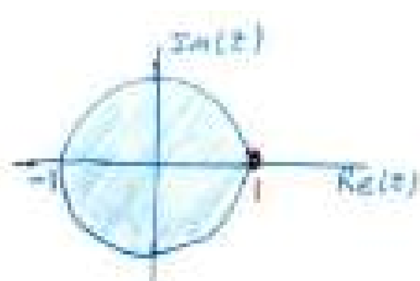
$$f(z) = \frac{e^z}{1-z}$$

around $z = 0$. Give the radius of convergence.

Solution: We start by writing the Taylor series for each of the factors and then multiply them out.

$$\begin{aligned} f(z) &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)\left(1 + z + z^2 + z^3 + \ldots\right) \\ &= 1 + (1+1)z + \left(1 + 1 + \frac{1}{2!}\right)z^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)z^3 + \ldots \end{aligned}$$

The biggest disk around $z = 0$ where f is analytic is $|z| < 1$. Therefore, by Taylor's theorem, the radius of convergence is $R = 1$.



$f(z)$ is analytic on $|z| < 1$ and has a singularity at $z = 1$.

Example 7.12. Find the Taylor series for

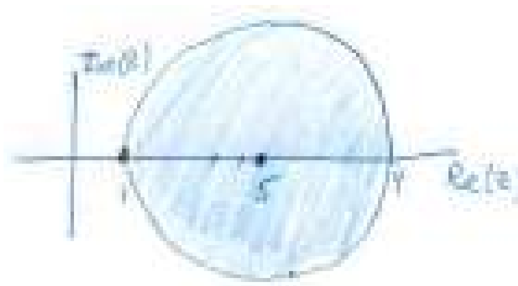
$$f(z) = \frac{1}{1-z}$$

around $z = 5$. Give the radius of convergence.

Solution: We have to manipulate this into standard geometric series form.

$$f(z) = \frac{1}{-4(1 + (z-5)/4)} = -\frac{1}{4}\left(1 - \left(\frac{z-5}{4}\right) + \left(\frac{z-5}{4}\right)^2 - \left(\frac{z-5}{4}\right)^3 + \ldots\right)$$

Since $f(z)$ has a singularity at $z = 1$ the radius of convergence is $R = 4$. We can also see this by considering the geometric series. The geometric series ratio is $(z-5)/4$. So the series converges when $|z-5|/4 < 1$, i.e. when $|z-5| < 4$, i.e. $R = 4$.



Disk of convergence stops at the singularity at $z = 1$.

Example 7.13. Find the Taylor series for

$$f(z) = \log(1 + z)$$

around $z = 0$. Give the radius of convergence.

Solution: We know that f is analytic for $|z| < 1$ and not analytic at $z = -1$. So, the radius of convergence is $R = 1$. To find the series representation we take the derivative and use the geometric series.

$$f'(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots$$

Integrating term by term (allowed by Theorem 7.1) we have

$$f(z) = a_0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots = a_0 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

Here a_0 is the constant of integration. We find it by evalating at $z = 0$.

$$f(0) = a_0 = \log(1) = 0.$$

Disk of convergence for $\log(1 + z)$ around $z = 0$.

Example 7.14. Can the series

$$\sum a_n (z-2)^n$$

converge at $z = 0$ and diverge at $z = 3$.

Solution: No! We have $z_0 = 2$. We know the series diverges everywhere outside its radius of convergence. So, if the series converges at $z = 0$, then the radius of convergence is at least 2. Since $|3 - z_0| < 2$ we would also have that $z = 3$ is inside the disk of convergence.

7.5.1 Proof of Taylor's theorem

For convenience we restate Taylor's Theorem 7.5.

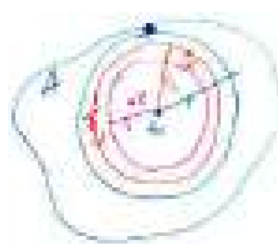
Taylor's theorem. (Taylor series) Suppose $f(z)$ is an analytic function in a region A. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in A. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{7}$$

Proof. In order to handle convergence issues we fix $0 < r_1 < r_2 < r$. We let γ be the circle $|w - z_0| = r_2$ (traversed counterclockise).



Disk of convergence extends to the boundary of A
$r_1 < r_2 < r$, but r_1 and r_2 can be arbitrarily close to r.

Take z inside the disk $|z - z_0| < r_1$. We want to express $f(z)$ as a power series around z_0. To do this we start with the Cauchy integral formula and then use the geometric series.

As preparation we note that for w on γ and $|z - z_0| < r_1$ we have

$$|z - z_0| < r_1 < r_2 = |w - z_0|,$$

so

$$\frac{|z - z_0|}{|w - z_0|} < 1.$$

Therefore,

$$\frac{1}{w - z} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

Using this and the Cauchy formula gives

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}}\,(z-z_0)^n\,dw \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n
\end{aligned}$$

The last equality follows from Cauchy's formula for derivatives. Taken together the last two equalities give Taylor's formula. QED

7.6 Singularities

Definition. A function $f(z)$ is singular at a point z_0 if it is not analytic at z_0

Definition. For a function $f(z)$, the singularity z_0 is an isolated singularity if f is analytic on the deleted disk $0 < |z - z_0| < r$ for some $r > 0$.

Example 7.15. $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has isolated singularities at $z = 0, \pm i$.

Example 7.16. $f(z) = e^{1/z}$ has an isolated singularity at $z = 0$.

Example 7.17. $f(z) = \log(z)$ has a singularity at $z = 0$, but it is not isolated because a branch cut, starting at $z = 0$, is needed to have a region where f is analytic.

Example 7.18. $f(z) = \dfrac{1}{\sin(\pi/z)}$ has singularities at $z = 0$ and $z = 1/n$ for $n = \pm 1, \pm 2, \ldots$ The singularities at $\pm 1/n$ are isolated, but the one at $z = 0$ is not isolated.

Every neighborhood of 0 contains zeros at $1/n$ for large n.

7.7 Laurent series

Theorem 7.19. (Laurent series). Suppose that $f(z)$ is analytic on the annulus

$$A: r_1 < |z - z_0| < r_2.$$

Then $f(z)$ can be expressed as a series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

The coefficients have the formulus

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$
$$b_n = \frac{1}{2\pi i}\int_\gamma f(w)(w-z_0)^{n-1}\,dw,$$

where γ is any circle $|w-z_0| = r$ inside the annulus, i.e. $r_1 < r < r_2$.

Furthermore

- The series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to an analytic function for $|z-z_0| < r_2$.
- The series $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges to an analytic function for $|z-z_0| > r_1$.
- Together, the series both converge on the annulus A where f is analytic.

The proof is given below. First we define a few terms.

Definition. The entire series is called the Laurent series for f around z_0. The series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is called the analytic or regular part of the Laurent series. The series

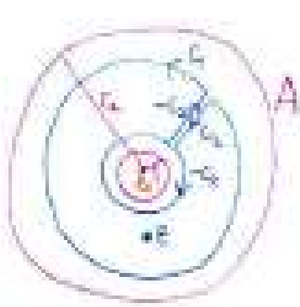
$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is called the singular or principal part of the Laurent series.

Note. Since $f(z)$ may not be analytic (or even defined) at z_0 we don't have any formulas for the coefficients using derivatives.

Proof. (Laurent series). Choose a point z in A. Now set circles C_1 and C_3 close enough to the boundary that z is inside $C_1 + C_2 - C_3 - C_2$ as shown. Since this curve and its interior are contained in A, Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i}\int_{C_1+C_2-C_3-C_2} \frac{f(w)}{w-z}\,dw$$



The contour used for proving the formulas for Laurent series.

The integrals over C_2 cancel, so we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1 - C_3} \frac{f(w)}{w-z}\, dw.$$

Next, we divide this into two pieces and use our trick of converting to a geometric series. The calculuations are just like the proof of Taylor's theorem. On C_1 we have

$$\frac{|z-z_0|}{|w-z_0|} < 1,$$

so

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z_0}\cdot \frac{1}{\left(1-\frac{z-z_0}{w-z_0}\right)}\, dw \\
&= \frac{1}{2\pi i}\int_{C_1} \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}}\,(z-z_0)^n\, dw \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}\, dw\right)(z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n.
\end{aligned}$$

Here a_n is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on z is that $|z-z_0| < r_2$. So, this series converges for all such z.

Similarly on C_3 we have

$$\frac{|w-z_0|}{|z-z_0|} < 1,$$

so

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_3} \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i}\int_{C_3} -\frac{f(w)}{z-z_0}\cdot \frac{1}{\left(1-\frac{w-z_0}{z-z_0}\right)}\, dw \\
&= -\frac{1}{2\pi i}\int_{C_3} \sum_{n=0}^{\infty} f(w)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\, dw \\
&= -\frac{1}{2\pi i}\sum_{n=0}^{\infty} \left(\int_{C_1} f(w)(w-z_0)^n\, dw\right)(z-z_0)^{-n-1} \\
&= -\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.
\end{aligned}$$

In the last equality we changed the indexing to match the indexing in the statement of the theorem. Here b_n is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on z is that $|z-z_0| > r_1$. So, this series converges for all such z.

Combining these two formulas we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1 - C_3} \frac{f(w)}{w-z}\, dw = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

The last thing to note is that the integrals defining a_n and b_n do not depend on the exact radius of the circle of integration. Any circle inside A will produce the same values. We have proved all the statements in the theorem on Laurent series. QED

7.7.1 Examples of Laurent series

In general, the integral formulas are not a practical way of computing the Laurent coefficients. Instead we use various algebraic tricks. Even better, as we shall see, is the fact that often we don't really need all the coefficients and we will develop more techniques to compute those that we do need.

Example 7.20. Find the Laurent series for

$$f(z) = \frac{z+1}{z}$$

around $z_0 = 0$. Give the region where it is valid.

Solution: The answer is simply

$$f(z) = 1 + \frac{1}{z}.$$

This is a Laurent series, valid on the infinite region $0 < |z| < \infty$.

Example 7.21. Find the Laurent series for

$$f(z) = \frac{z}{z^2+1}$$

around $z_0 = i$. Give the region where your answer is valid. Identify the singular (principal) part.

Solution: Using partial fractions we have

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{2} \cdot \frac{1}{z+i}.$$

Since $\dfrac{1}{z+i}$ is analytic at $z = i$ it has a Taylor series expansion. We find it using geometric series.

$$\frac{1}{z+i} = \frac{1}{2i} \cdot \frac{1}{1+(z-i)/(2i)} = \frac{1}{2i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

So the Laurent series is

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{4i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

The singular (principal) part is given by the first term. The region of convergence is $0 < |z-i| < 2$.

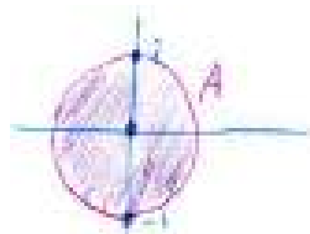
Note. We could have looked at $f(z)$ on the region $2 < |z-i| < \infty$. This would have produced a different Laurent series. We discuss this further in an upcoming example.

Example 7.22. Compute the Laurent series for

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

on the region $A: 0 < |z| < 1$ centered at $z = 0$.

Solution: This function has isolated singularities at $z = 0, \pm i$. Therefore it is analytic on the region A.



$f(z)$ has singularities at $z = 0, \pm i$.

At $z = 0$ we have

$$f(z) = \frac{1}{z^3}(1+z)(1 - z^2 + z^4 - z^6 + \ldots).$$

Multiplying this out we get

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} - 1 + z + z^2 - z^3 - \ldots.$$

The following example shows that the Laurent series depends on the region under consideration.

Example 7.23. Find the Laurent series around $z = 0$ for $f(z) = \dfrac{1}{z(z-1)}$ in each of the following regions:

(i) the region $A_1: 0 < |z| < 1$

(ii) the region $A_2: 1 < |z| < \infty$.

Solution: For (i)

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + \ldots) = -\frac{1}{z} - 1 - z - z^2 - \ldots$$

For (ii): Since the usual geometric series for $1/(1-z)$ does not converge on A_2 we need a different form,

$$f(z) = \frac{1}{z} \cdot \frac{1}{z(1 - 1/z)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Since $|1/z| < 1$ on A_2 our use of the geometric series is justified.

One lesson from this example is that the Laurent series depends on the region as well as the formula for the function.

7.8 Digression to differential equations

Here is a standard use of series for solving differential equations.

Example 7.24. Find a power series solution to the equation

$$f'(x) = f(x) + 2, \qquad f(0) = 0.$$

Solution: We look for a solution of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the initial condition we find $f(0) = 0 = a_0$. Substituting the series into the differential equation we get

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^3 + \ldots = f(x) + 2 = a_0 + 2 + a_1 x + a_2 x^2 + \ldots$$

Equating coefficients and using $a_0 = 0$ we have

$$\begin{aligned}
a_1 &= a_0 + 2 & \Rightarrow\ a1 &= 2 \\
2a_2 &= a_1 & \Rightarrow\ a_2 &= a_1/2 = 1 \\
3a_3 &= a_2 & \Rightarrow\ a_3 &= 1/3 \\
4a_4 &= a_3 & \Rightarrow\ a_4 &= 1/(3 \cdot 4)
\end{aligned}$$

In general

$$(n+1)a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)} = \frac{1}{3 \cdot 4 \cdot 5 \cdots (n+1)}.$$

You can check using the ratio test that this function is entire.

7.9 Poles

Poles refer to isolated singularities. So, we suppose $f(z)$ is analytic on $0 < |z - z_0| < r$ and has Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Definition of poles. If only a finite number of the coefficients b_n are nonzero we say z_0 is a finite pole of f. In this case, if $b_k \neq 0$ and $b_n = 0$ for all $n > k$ then we say z_0 is a pole of order k.

- If z_0 is a pole of order 1 we say it is a simple pole of f.
- If an infinite number of the b_n are nonzero we say that z_0 is an essential singularity or a pole of infinite order of f.
- If all the b_n are 0, then z_0 is called a removable singularity. That is, if we define $f(z_0) = a_0$ then f is analytic on the disk $|z - z_0| < r$.

The terminology can be a bit confusing. So, imagine that I tell you that f is defined and analytic on the punctured disk $0 < |z - z_0| < r$. Then, a priori we assume f has a singularity at z_0. But, if after computing the Laurent series we see there is no singular part we can extend the definition of f to the full disk, thereby 'removing the singularity'.

We can explain the term essential singularity as follows. If $f(z)$ has a pole of order k at z_0 then $(z-z_0)^k f(z)$ is analytic (has a removable singularity) at z_0. So, $f(z)$ itself is not much harder to work with than an analytic function. On the other hand, if z_0 is an essential singularity then no algebraic trick will change $f(z)$ into an analytic function at z_0.

7.9.1 Examples of poles

We'll go back through many of the examples from the previous sections.

Example 7.25. The rational function

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

expanded to

$$f(z) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty} (-1)^n z^{2n+1}.$$

Thus, $z = 0$ is a pole of order 3.

Example 7.26. Consider

$$f(z) = \frac{z+1}{z} = 1 + \frac{1}{z}.$$

Thus, $z = 0$ is a pole of order 1, i.e. a simple pole.

Example 7.27. Consider

$$f(z) = \frac{z}{z^2+1} = \frac{1}{2} \cdot \frac{1}{z-i} + g(z),$$

where $g(z)$ is analytic at $z = i$. So, $z = i$ is a simple pole.

Example 7.28. The function

$$f(z) = \frac{1}{z(z-1)}$$

has isolated singularities at $z = 0$ and $z = 1$. Show that both are simple poles.

Solution: In a neighborhood of $z = 0$ we can write

$$f(z) = \frac{g(z)}{z}, \quad \text{where} \quad g(z) = \frac{1}{z-1}.$$

Since $g(z)$ is analytic at 0, $z = 0$ is a finite pole. Since $g(0) \neq 0$, the pole has order 1, i.e. it is simple.

Likewise, in a neighborhood of $z = 1$,

$$f(z) = \frac{h(z)}{z-1}, \quad \text{where} \quad h(z) = \frac{1}{z}.$$

Since h is analytic at $z = 1$, f has a finite pole there. Since $h(1) \neq 0$ it is simple.

Example 7.29. Consider

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

So, $z = 0$ is an essential singularity.

Example 7.30. $\log(z)$ has a singularity at $z = 0$. Since the singularity is not isolated, it can't be classified as either a pole or an essential singularity.

7.9.2 Residues

In preparation for discussing the residue theorem in the next topic we give the definition and an example here.

Note well, residues have to do with isolated singularites.

Definition 7.31. Consider the function $f(z)$ with an isolated singularity at z_0, i.e. defined on $0 < |z - z_0| < r$ and with Laurent series

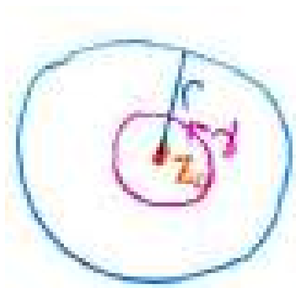
$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The residue of f at z_0 is b_1. This is denoted

$$\mathrm{Res}(f, z_0) \quad \text{or} \quad \mathop{\mathrm{Res}}_{z=z_0} f = b_1.$$

What is the importance of the residue? If γ is a small, simple closed curve that goes counterclockwise around z_0 then

$$\int_\gamma f(z) = 2\pi i b_1.$$



γ is small enough to be inside $|z - z_0| < r$, and surround z_0.

This is easy to see by integrating the Laurent series term by term. The only nonzero integral comes from the term b_1/z.

Example 7.32. The function

$$f(z) = \mathrm{e}^{1/(2z)} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \ldots$$

has an isolated singularity at 0. From the Laurent series we see that

$$\mathrm{Res}(f, 0) = \frac{1}{2}.$$

7.10 Appendix: convergence

This section needs to be completed. It will give some of the careful technical definitions and arguments regarding convergence and manipulation of series. In particular it will define the notion of uniform convergence. The short description is that all of our manipulations of power series are justified on any closed bounded region. Almost, everything we did can be restricted to a closed disk or annulus, and so was valid.

UNIT-5

RESIDUE CALCULUS

7.50 ZERO OF ANALYTIC FUNCTION

A zero of analytic function $f(x)$ is the value of z for which $f(z) = 0$.

Example 92. *Find out the zeros and discuss the nature of the singularities of*

$$f(z) = \frac{(z-2)}{z^2} \sin\left(\frac{1}{z-1}\right)$$ (*R.G.P.V. Bhopal, III Semester, Dec. 2004*)

Solution. Poles of $f(z)$ are given by equating to zero the denominator of $f(z)$ i.e. $z = 0$ is a pole of order two.

zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ *i.e.*, $(z-2)\sin\left(\frac{1}{z-1}\right) = 0$

$\Rightarrow$ Either $z - 2 = 0$ or $\sin\left(\frac{1}{z-1}\right) = 0$

$\Rightarrow$ $z = 2$ and $\frac{1}{z-1} = n\pi$

$\Rightarrow$ $z = 2,$ $z = \frac{1}{n\pi} + 1,\ n = \pm 1, \pm 2, \ldots\ldots\ldots$

Thus, $z = 2$ is a simple zero. The limit point of the zeros are given by

$$z = \frac{1}{n\pi} + 1 \quad (n = \pm 1, \pm 2, \ldots\ldots\ldots) \text{ is } z = 1.$$

Hence $z = 1$ is an isolated essential singularity. **Ans.**

7.51 PRINCIPAL PART

$$\text{If } f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

then the term $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ is called the principal part of the function $f(z)$ at $z = z_0$

7.52 SINGULAR POINT

A point at which a function $f(z)$ is not analytic is known as a singular point or **singularity** of the function.

For example, the function $\frac{1}{z-2}$ has a singular point at $z - 2 = 0$ or $z = 2$.

Isolated singular point. If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

For example, the function $\frac{1}{(z-1)(z-3)}$ has two isolated singlar points, namely $z = 1$ and $z = 3$. [Put $(z-1)(z-3) = 0 \Rightarrow z = 1, 3$].

Example of non-isolated singularity. Function $\frac{1}{\sin\frac{\pi}{z}}$ is not analytic at the points where

$\sin\frac{\pi}{z}$ = 0, *i.e.*, at the points $\frac{\pi}{z} = n\pi$ *i.e.*, the points $z = \frac{1}{n}$ $(n = 1, 2, 3, ...)$. Thus $z = 1, \frac{1}{2}, \frac{1}{3}, ..., z = 0$ are the points of singularity. $z = 0$ is the **non-isolated singularity** of the function $\frac{1}{\sin\frac{\pi}{z}}$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$, where n is very large.

Pole of order *m*. Let a function $f(z)$ have an isolated singular point $z = a$, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + ... + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \frac{b_{m+2}}{(z-a)^{m+2}} + ... \quad ... (1)$$

In some cases it may happen that the coefficients $b_{m+1} = b_{m+2} = b_{m+3}$ = 0, then (1) reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + ... + \frac{b_m}{(z-a)^m}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + b_3(z-a)^{m-3} + ... + b_m\}$$

then $z = a$ is said to be a **pole of order *m*** of the function $f(z)$, when $m = 1$, the pole is said to be **simple pole**. In this case

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{b_1}{z-a}$$

If the number of the terms of negative powers in expansion (1) is infinite, then $z = a$ is called an essential singular point of $f(z)$.

7.53 REMOVABLE SINGULARITY

If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$

$\Rightarrow$ $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_n(z-a)^n + ...$

Here the coefficients of negative powers are zero *i.e.* Laurent series does not contain negative power of $(z - a)$ then $z = a$ is called a removable signularity *i.e.*, $f(z)$ can be made analytic by redefining $f(a)$ suitably *i.e.* if $\lim_{x\to 0} f(z)$ exists.

Remark. This type of singularity can be made to disappear by defining the function suitably *e.g.*, $f(z) = \frac{\sin(z-a)}{z-a}$ has removable singularity at $z = a$ because

$$\frac{\sin(z-a)}{z-a} = \frac{1}{z-a}\left\{(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} ......\infty\right\} = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} .....\infty$$

has no term containing negative powers of $(z - a)$. However this singularity can be rermoved and the function can be made analytic by defining $f(z) = \frac{\sin(z-a)}{z-a} = 1$ at $z = a$

7.54 WORKING RULE TO FIND SINGULARITY

Step 1. If $\underset{z\to a}{\text{Lt}} f(z)$ exists and is finite then $z = a$ is a **removable singular point.**

Step 2. If $\underset{z \to a}{\text{Lt}} f(z)$ does not exist then $z = a$ is an **essential singular point.**

Step 3. If $\underset{z \to a}{\text{Lt}} f(z)$ is infinite then $f(z)$ has a **pole at $z = a$.** The order of the pole is same as the number of negative power terms in the series expansion of $f(z)$.

Example 93. *Define the singularity of a function. Find the singularity (ties) of the functions*

(*i*) $f(z) = \sin \dfrac{1}{z}$ (*ii*) $g(z) = \dfrac{e^{\frac{1}{z}}}{z^2}$ (*U.P. III Semester, 2009-2010*)

Solution. See Art. 8.2 on page 254 for definition.

(*i*) We know that

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} + \dots\dots\dots + (-1)^n \frac{1}{(2n+1)!\, z^{2n+1}}$$

Obviously, there is a number of singularity.

$$\sin \frac{1}{z} \text{ is not analytic at } z = 0. \qquad \left(\frac{1}{z} = \infty \text{ at } z = 0\right)$$

Hence, $\sin \dfrac{1}{z}$ has a singularity at $z = 0$.

(ii) Here, we have $g(z) = \dfrac{e^{\frac{1}{z}}}{z^2}$

We know that, $\left(\dfrac{1}{z^2}\right)\left(e^{\frac{1}{z}}\right) = \dfrac{1}{z^2}\left(1 + \dfrac{1}{z} + \dfrac{1}{2!\, z^2} + \dfrac{1}{3!\, z^3} + \dots + \dfrac{1}{n!\, z^n} + \dots\right)$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!\, z^4} + \frac{1}{3!\, z^5} + \dots\dots\dots \frac{1}{n!\, z^{n+2}} + \dots\dots$$

Here, $f(z)$ has infinite number of terms in negative powers of z.

Hence, $f(z)$ has essential singularity at $z = 0$. **Ans.**

Example 94. *Find the pole of the function* $\dfrac{e^{z-a}}{(z-a)^2}$

Solution. $\dfrac{e^{z-a}}{(z-a)^2} = \dfrac{1}{(z-a)^2}\left[1 + (z-a) + \dfrac{(z-a)^2}{2!} + \dots\right]$

The given function has negative power 2 of $(z - a)$.

So, the given function has a pole at $z = a$ of order 2. **Ans.**

Example 95. *Find the poles of* $f(z) = \sin\left(\dfrac{1}{z-a}\right)$

Solution. $\sin\left(\dfrac{1}{z-a}\right) = \dfrac{1}{z-a} - \dfrac{1}{3!}\dfrac{1}{(z-a)^3} + \dfrac{1}{5!}\dfrac{1}{(z-a)^5} - \dots\dots$

The given function $f(z)$ has infinite number of terms in the negative powers of $z - a$.

So, $f(z)$ has essential singularity at $z = a$. **Ans.**

Example 96. *Discuss singularity of* $\dfrac{1}{1-e^z}$ *at* $z = 2\pi i$.

Solution. We have, $f(z) = \dfrac{1}{1-e^z}$

The poles are determined by putting the denominator equal to zero.

i.e., $1 - e^z = 0$

$\Rightarrow$ $e^z = 1 = (\cos 2n\pi + i \sin 2n\pi) = e^{2n\pi i}$

$\Rightarrow$ $z = 2n\pi i$ $(n = 0, \pm 1, \pm 2, ...)$

Clearly $z = 2\pi i$ is a simple pole. **Ans.**

Example 97. *Discuss singularity of* $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z = a$ *and* $z = \infty$.
(*R.G.P.V., Bhopal, III Semester, Dec. 2002*)

Solution. Let $$f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^2}$$

The poles are given by putting the denominator equal to zero.

i.e., $\sin \pi z\,(z-a)^2 = 0 \quad \Rightarrow \quad (z-a)^2 = 0 \quad$ or $\quad \sin \pi z = 0 = \sin n\pi$

$\Rightarrow$ $z = a, \ \pi z = n\pi,$ $(n \in I)$

$\Rightarrow$ $z = a, n$

$f(z)$ has essential singularity at $z = \infty$.

Also, $z = a$ being repeated twice gives the double pole. **Ans.**

Example 98. *Determine the poles of the function*

$$f(z) = \frac{1}{z^4+1}$$ (*R.G.P.V., Bhopal, III Semester, June 2003*)

Solution. $$f(z) = \frac{1}{z^4+1}$$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

i.e., $z^4 + 1 = 0 \quad \Rightarrow \quad z^4 = -1$

$$z = (-1)^{\frac{1}{4}} = (\cos \pi + i \sin \pi)^{\frac{1}{4}}$$

$$= [\cos(2n+1)\pi + i \sin(2n+1)\pi]^{\frac{1}{4}} \quad \text{[By De Moiver's theorem]}$$

$$= \left[\cos \frac{(2n+1)\pi}{4} + i \sin \frac{(2n+1)\pi}{4}\right]$$

If $n = 0$, Pole at $z = \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right] = \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)$

If $n = 1$, Pole at $z = \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)$

If $n = 2$, Pole at $z = \left[\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right)$

If $n = 3$, Pole at $z = \left[\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right] = \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right)$ **Ans.**

Example 99. *Show that the function* e^z *has an isolated essential singularity at* $z = \infty$.
(*R.G.P.V., Bhopal, III Semester, Dec. 2003*)

Solution. Let $f(z) = e^z$

Putting $z = \dfrac{1}{t}$, we get $f\left(\dfrac{1}{t}\right) = e^{\frac{1}{t}} = 1 + \dfrac{1}{t} + \dfrac{1}{2!\,t^2} + \dfrac{1}{3!\,t^3} + ...$

Here, the principal part of $f\left(\dfrac{1}{t}\right)$;

$$\frac{1}{t} + \frac{1}{2\,!\,t^2} + \frac{1}{3\,!\,t^3} + \ldots$$

Contains infinite number of terms.

Hence $t = 0$ is an isolated essential singularity of $e^{\frac{1}{t}}$ and $z = \infty$ is an isolated essential singularity of e^z. **Ans.**

EXERCISE 7.11

Find the poles or singularity of the following functions:

1. $\dfrac{1}{(z-2)(z-3)}$ **Ans.** 2 simple poles at $z = 2$ and $z = 3$.
2. $\dfrac{e^z}{(z-2)^3}$ **Ans.** Pole at $z = 2$ of order 3.
3. $\dfrac{1}{\sin z - \cos z}$ **Ans.** Simple pole at $z = \dfrac{\pi}{4}$
4. $\cot \dfrac{1}{z}$ **Ans.** Essential singularity at $z = 0$
5. $z \operatorname{cosec} z$ **Ans.** Non-isolated essential singularity
6. $\sin \dfrac{1}{z}$ **Ans.** Essential singularity

Choose the correct alternative:

7. Let $f(z) = \dfrac{1}{(z-2)^4 (z+3)^6}$, then $z = 2$ and $z = -3$ are the poles of order :

(*a*) 6 and 4 (*b*) 2 and 3 (*c*) 3 and 4 (*d*) 4 and 6 **Ans.** (*d*)

(*R.G.P.V., Bhopal III Semester, June 2007*)

7.55 THEOREM

If $f(z)$ has a pole at $z = a$, then $|f(z)| \to \infty$ as $z \to a$.

Proof. Let $z = a$ be a pole of order m of $f(z)$. Then by Laurent's theorem

$$f(z) = \sum_0^\infty a_n (z-a)^n + \sum_1^m b_n (z-a)^{-n}$$

$$= \sum_0^\infty a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

$$= \sum_0^\infty a_n (z-a)^n + \frac{1}{(z-a)^m}[b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \ldots + b_{m-1}(z-a) + b_m]$$

$$= \sum_0^\infty a_n (z-a)^n + \frac{\varphi(z)}{(z-a)^m}$$

Now $\varphi(z) \to b_m$ as $z \to a$.

Hence $|f(z)| \to \infty$ as $z \to a$. **Proved.**

Example 100. *If an analytic function $f(z)$ has a pole of order m at $z = a$, then $\dfrac{1}{f(z)}$ has a zero of order m at $z = a$.*

Solution. If $f(z)$ has a pole of order m at $z = a$, then

$$f(z) = \frac{\varphi(z)}{(z-a)^m} \quad \text{where } \varphi(z) \text{ is analytic and non-zero at } z = a.$$

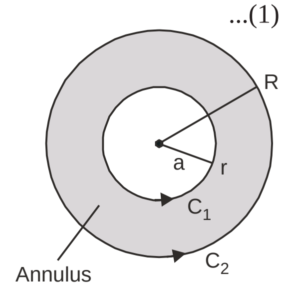
$$\therefore \quad \frac{1}{f(z)} = \frac{(z-a)^m}{\varphi(z)}$$

Clearly, $\dfrac{1}{f(z)}$ has a zero of order m at $z = a$, since $\phi(a) \neq 0$. **Proved.**

7.56 DEFINITION OF THE RESIDUE AT A POLE

Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z - a| < R$ can be expanded within the annulus. Laurent's series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad ...(1)$$



where $$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad ...(2)$$

and $$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{-n+1}}\,dz \quad ...(3)$$

$|z - a| = r$ being the circle C_1.

Particularly, $$b_1 = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz$$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. It is denoted by symbol Res. $(z = a) = b_1$.

7.57 RESIDUE AT INFINITY

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_C f(z)\,dz$ where the integration is taken round C in anti-clockwise direction.

where C is a large circle containing all finite singularities of $f(z)$.

7.58 METHOD OF FINDING RESIDUES

(*a*) **Residue at simple pole**

(*i*) If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \lim_{z\to a}(z-a)\,f(z)$$

Proof. $$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + \frac{b_1}{z-a}$$

$$\Rightarrow \quad (z-a)\,f(z) = a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + ... + b_1$$

$$\Rightarrow \quad b_1 = (z-a)\,f(z) - [a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + ...]$$

Taking limit as $z \to a$, we have $b_1 = \lim_{z\to a}(z-a)\,f(z)$

$$\textbf{Res } (\textbf{\textit{at z = a}}) = \lim_{z\to a} (z-a)\,f(z)$$ **Proved.**

(*ii*) If $f(z)$ is of the form $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(a) = 0$, but $\phi(a) \neq 0$

$$\textbf{Res } (\textbf{\textit{at z = a}}) = \frac{\phi(a)}{\psi'(a)}$$

Proof . $$f(z) = \frac{\phi(z)}{\psi(z)}$$

$$\text{Res (at } z = a) = \lim_{z\to a}(z-a)\,f(z) = \lim_{z\to a}(z-a)\frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z\to a} \frac{(z-a)[\phi(a) + (z-a)\,\phi'(a) + ...]}{\psi(a) + (z-a)\psi'(a) + \frac{(z-a)^2}{2!}\psi''(a) + ...} \quad \text{(By Taylor's Theorem)}$$

$$= \lim_{z\to a} \frac{(z-a)\,[\phi(a) + (z-a)\,\phi'(a) + ...]}{(z-a)\psi'(a) + \frac{(z-a)^2}{2!}\psi''(a) + ...} \quad [\text{since } \Psi(a) = 0]$$

$$= \lim_{z\to a} \frac{\phi(a)+(z-a)\,\phi'(a)+\ldots}{\psi'(a)+\frac{z-a}{2!}\psi''(a)+\ldots}$$

$$\textbf{Res (at } z = a\textbf{)} = \frac{\phi(a)}{\psi'(a)}$$ **Proved.**

(*b*) **Residue at a pole of order *n*.** If $f(z)$ has a pole of order n at $z = a$, then

$$\textbf{Res (at } z = a\textbf{)} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Proof. If $z = a$ is a pole of order n of function $f(z)$, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^n}$$

Multiplying by $(z - a)^n$, we get

$$(z-a)^n f(z) = a_0(z-a)^n + a_1(z-a)^{n+1} + a_2(z-a)^{n+2} + \ldots$$

$$+ \quad b_1(z-a)^{n-1} + b_2(z-a)^{n-2} + b_3(z-a)^{n-3} + \ldots + b_n$$

Differentiating both sides w.r.t. 'z' $n - 1$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\,b_1$$

$$\Rightarrow \qquad b_1 = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

$$\textbf{Residue } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$ **Proved.**

(*c*) **Residue at a pole *z* = *a* of any order (simple or of order *m*)**

$$\textbf{Res } f(a) = \textbf{coefficient of } \frac{1}{t}$$

Proof. If $f(z)$ has a pole of order m, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 \ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

If we put $z - a = t$ or $z = a + t$, then

$$f(a+t) = a_0 + a_1 t + a_2 t^2 + \ldots + \frac{b_1}{t} + \frac{b_2}{t^2} + \ldots + \frac{b_m}{t^m}$$

Res $f(a) = b_1$, **Res *f* (*a*) = coefficient of** $\frac{1}{t}$ **Proved.**

Rule. Put $z = a + t$ in the function $f(z)$, expand it in powers of t. Coefficient of $\frac{1}{t}$ is the residue of $f(z)$ at $z = a$.

(*d*) **Residue of *f* (*z*) (at *z* = ∞)** $= \lim_{z\to\infty} \{-z\, f(z)\}$

or The residue of $f(z)$ at infinity $= -\frac{1}{2\pi i}\int_c f(z)\, dz$

7.59 RESIDUE BY DEFINITION

Example 101. *Find the residue at* $z = 0$ *of* $z \cos\frac{1}{z}$.

Solution. Expanding the function in powers of $\frac{1}{z}$, we have

$$z \cos \frac{1}{z} = z\left[1 - \frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} - .....\right] = z - \frac{1}{2z} + \frac{1}{24z^3} - ......$$

This is the Laurent's expansion about $z = 0$.

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$. So the residue of $z \cos \frac{1}{z}$ at $z = 0$ is $-\frac{1}{2}$. **Ans.**

Example 102. *Find the residue of* $f(z) = \frac{z^3}{z^2 - 1}$ *at* $z = \infty$.

Solution. We have, $f(z) = \frac{z^3}{z^2 - 1}$

$$f(z) = \frac{z^3}{z^2\left(1 - \frac{1}{z^2}\right)} = z\left(1 - \frac{1}{z^2}\right)^{-1} = z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + ...\right) = z + \frac{1}{z} + \frac{1}{z^3} + ...$$

$$\text{Residue at infinity} = -\left(\text{coeff. of } \frac{1}{z}\right) = -1.$$ **Ans.**

7.60 FORMULA: RESIDUE $= \lim\limits_{z \to a} (z - a)\, f(z)$

Example 103. *Determine the pole and residue at the pole of the function* $f(z) = \frac{z}{z-1}$

Solution. The poles of $f(z)$ are given by putting the denominator equal to zero.

$\therefore \quad z - 1 = 0 \quad \Rightarrow \quad z = 1$

The function $f(z)$ has a simple pole at $z = 1$.

Residue is calculated by the formula

$$\text{Residue} = \lim_{z \to a} (z - a)\, f(z)$$

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z \to 1} (z - 1)\left(\frac{z}{z - 1}\right) = \lim_{z \to 1} (z) = 1$$

Hence, $f(z)$ has a simple pole at $z = 1$ and residue at the pole is 1. **Ans.**

Example 104. *Evaluate the residues of* $\frac{z^2}{(z-1)(z-2)(z-3)}$ *at z = 1, 2, 3 and infinity and show that their sum is zero.* (*R.G.P.V., Bhopal, III Semester Dec. 2002*)

Solution. Let $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

$\therefore (z - 1)(z - 2)(z - 3) = 0 \Rightarrow z = 1, 2, 3$

$$\text{Residue of } f(z) \text{ at } (z = 1) = \lim_{z \to 1} (z - 1)\, f(z) = \lim_{z \to 1} (z - 1) . \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z \to 1} \frac{z^2}{(z-2)(z-3)} = \frac{1}{2}$$

$$\text{Residue of } f(z) \text{ at } (z = 2) = \lim_{z \to 2} (z - 2)\, f(z) = \lim_{z \to 2} (z - 2) \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to 2} \frac{z^2}{(z-1)\,(z-3)} = \frac{4}{(1)\,(-1)} = -4$$

Residue of $f(z)$ at $(z = 3)$ $= \lim_{z\to 3} (z-3)\; f(z)$

$$= \lim_{z\to 3} (z-3) \frac{z^2}{(z-1)\,(z-2)\,(z-3)} = \lim_{z\to 3} \frac{z^2}{(z-1)\,(z-2)} = \frac{9}{2}$$

Residue of $f(z)$ at $(z = \infty)$ $= \lim_{z\to\infty} -z\; f(z) = \frac{-z\,(z^2)}{(z-1)\,(z-2)\,(z-3)}$

$$= \lim_{z\to\infty} \frac{-1}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)} = -1$$

Sum of the residues at all the poles of $f(z) = \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$

Hence, the sum of the residues is zero. **Proved.**

7.61 FORMULA: RESIDUE OF $f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n\; f(z)]\right\}_{z=a}$

Example 105. *Find the residue of a function*

$$f(z) = \frac{z^2}{(z+1)^2\,(z-2)} \text{ at its double pole.}$$

Solution. We have, $f(z) = \frac{z^2}{(z+1)^2\,(z-2)}$

Poles are determined by putting denominator equal to zero.

i.e.; $(z+1)^2\,(z-2) = 0$

$\Rightarrow$ $z = -1, -1$ and $z = 2$

The function has a double pole at $z = -1$

$$\text{Residue at } (z=-1) = \lim_{z\to -1} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z^2}{(z+1)^2\,(z-2)}\right\}\right]$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z-2}\right)\right]_{z=-1} = \left(\frac{(z-2)\,2z - z^2\cdot 1}{(z-2)^2}\right)_{z=-1} = \left[\frac{z^2-4z}{(z-2)^2}\right]_{z=-1} = \frac{(-1)^2-4(-1)}{(-1-2)^2}$$

Residue at $(z = -1) = \frac{1+4}{9} = \frac{5}{9}$ **Ans.**

Example 106. *Find the residue of* $\frac{1}{(z^2+1)^3}$ *at* $z = i$.

Solution. Let $f(z) = \frac{1}{(z^2+1)^3}$

The poles of $f(z)$ are determined by putting denominator equal to zero.

i.e.; $(z^2+1)^3 = 0$

$\Rightarrow$ $(z+i)^3\,(z-i)^3 = 0$

$\Rightarrow$ $z = \pm\, i$

Here, $z = i$ is a pole of order 3 of $f(z)$.

Residue at $z = i$:

$$= \lim_{z\to i} \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3 \frac{1}{(z^2+1)^3}\right]\right\} = \lim_{z\to i} \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)\right\}$$

$$= \lim_{z \to i} \frac{1}{2}\left(\frac{3 \times 4}{(z+i)^5}\right) = \frac{1}{2} \times \frac{12}{(i+i)^5} = \frac{6}{32i} = \frac{3}{16i} = -\frac{3i}{16}$$

Hence, the residue of the given function at $z = i$ is $-\frac{3i}{16}$. **Ans.**

7.62 FORMULA: RES. (AT $z = a$) = $\frac{\phi(a)}{\psi'(a)}$

Example 107. *Determine the poles and residue at each pole of the function $f(z) = \cot z$.*

Solution. $f(z) = \cot z = \frac{\cos z}{\sin z}$

The poles of the function $f(z)$ are given by

$$\sin z = 0, \quad z = n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Residue of $f(z)$ at $z = n\pi$ is $= \frac{\cos z}{\frac{d}{dz}(\sin z)} = \frac{\cos z}{\cos z} = 1$ $\left[\text{Res. at } (z = a) = \frac{\phi(a)}{\psi'(a)}\right]$ **Ans.**

Example 108. *Determine the poles of the function and residue at the poles.*

$$f(z) = \frac{z}{\sin z}$$

Solution. $f(z) = \frac{z}{\sin z}$

Poles are determined by putting $\sin z = 0 = \sin n\pi \quad \Rightarrow \quad z = n\pi$

$$\text{Residue} = \left(\frac{z}{\cos z}\right)_{z=n\pi} \qquad \left[\text{Residue} = \frac{\phi(a)}{\psi'(a)}\right]$$

$$= \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$$

Hence, the residue of the given function at pole $z = n\pi$ is $\frac{n\pi}{(-1)^n}$. **Ans.**

7.63 FORMULA: RESIDUE = COEFFICIENT OF $\frac{1}{t}$

where $z = \frac{1}{t}$

Example 109. *Find the residue of* $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ *at a pole of order 4.*

Solution. The poles of $f(z)$ are determined by putting the denominator equal to zero.

$\therefore$ $(z-1)^4(z-2)(z-3) = 0 \quad \Rightarrow \quad z = 1, 2, 3$

Here $z = 1$ is a pole of order 4.

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} \qquad \ldots(1)$$

Putting $z - 1 = t$ or $z = 1 + t$ in (1), we get

$$f(1+t) = \frac{(1+t)^3}{t^4(t-1)(t-2)} = \frac{1}{t^4}(t^3 + 3t^2 + 3t + 1)(1-t)^{-1}\frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{t} + \frac{3}{t^2} + \frac{3}{t^3} + \frac{1}{t^4}\right)(1 + t + t^2 + t^3 + \ldots) \times \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} \ldots\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}+\frac{1}{t^4}\right)\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\frac{15}{8}t^3+\ldots\right)=\frac{1}{2}\left(\frac{1}{t}+\frac{9}{2t}+\frac{21}{4}\frac{1}{t}+\frac{15}{8}\frac{1}{t}\right)+\ldots$$

$$= \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right)\frac{1}{t} \qquad \left[\text{Res } f(a) = \text{coeffi. of } \frac{1}{t}\right]$$

Coefficient of $\frac{1}{t} = \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right)=\frac{101}{16}$,

Hence, the residue of the given function at a pole of order 4 is $\frac{101}{16}$. **Ans.**

Example 110. *Find the residue of* $f(z) = \frac{z\,e^z}{(z-a)^3}$ *at its pole.*

Solution. The pole of $f(z)$ is given by $(z-a)^3 = 0$ *i.e.,* $z = a$
Here $z = a$ is a pole of order 3.
Putting $z - a = t$ where t is small.

$$f(z) = \frac{ze^z}{(z-a)^3} \Rightarrow f(z) = \frac{(a+t)\,e^{a+t}}{t^3} = \left(\frac{a}{t^3}+\frac{1}{t^2}\right)e^{a+t} = e^a\left(\frac{a}{t^3}+\frac{1}{t^2}\right)e^t \qquad (z = a + t)$$

$$= e^a\left(\frac{a}{t^3}+\frac{1}{t^2}\right)\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\ldots\right) = e^a\left[\frac{a}{t^3}+\frac{a}{t^2}+\frac{a}{2t}+\frac{1}{t^2}+\frac{1}{t}+\frac{1}{2}+\ldots\right]$$

$$= e^a\left[\frac{1}{2}+\left(\frac{a}{2}+1\right)\frac{1}{t}+(a+1)\frac{1}{t^2}+(a)\,\frac{1}{t^3}+\ldots\right]$$

Coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2}+1\right)$

Hence the residue at $z = a$ is $e^a\left(\frac{a}{2}+1\right)$. **Ans.**

EXERCISE 7.12

1. Determine the poles of the following functions. Find the order of each pole.

(*i*) $\frac{z^2}{(z-a)(z-b)(z-c)}$ **Ans.** Simple poles at $z = a$, $z = b$, $z = c$

(*ii*) $\frac{z-3}{(z-2)^2(z+1)}$ **Ans.** Pole at $z = 2$ of second order and $z = -1$ of first order.

(*iii*) $\frac{z\,e^{iz}}{z^2+a^2}$ **Ans.** Poles at $z = \pm\, ia$, order 1.

(*iv*) $\frac{1}{(z-1)(z-2)}$ **Ans.** $z = 2$, $z = 1$

Find the residue of

2. $\frac{z^3}{(z-2)(z-3)}$ at its poles. **Ans.** 19

3. $\frac{z^2}{z^2+a^2}$ at $z = ia$. **Ans.** $\frac{1}{2}ia$

4. $\frac{1}{(z^2+a^2)^2}$ at $z = ia$ **Ans.** $-\frac{i}{4a^3}$

5. tan z at its pole. **Ans.** $f\left(n+\dfrac{\pi}{2}\right) = -1$ at its pole

6. $z^2 e^{1/z}$ at the point $z = 0$. **Ans.** $\dfrac{1}{6}$

7. $z^2 \sin\left(\dfrac{1}{z}\right)$ at $z = 0$ **Ans.** $-\dfrac{1}{6}$

8. $\dfrac{1}{z^2(z-i)}$ at $z = i$ **Ans.** -1

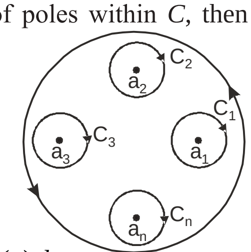
9. $\dfrac{e^{2z}}{1+e^z}$ at its pole **Ans.** -1

10. $\dfrac{1+e^z}{\sin z + z\cos z}$ at $z = 0$ **Ans.** 1

11. $\dfrac{1}{z(e^z-1)}$ at its poles **Ans.** $-\dfrac{1}{2}$

7.64 CAUCHY'S RESIDUE THEOREM *(MDU, DEC. 2008)*

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then $\int_c f(z)\,dz = 2\pi i$ **(sum of residues at the poles within C).**



Proof. Let $C_1, C_2, C_3, \ldots, C_n$ be the non-intersecting circles with centres at $a_1, a_2, a_3, \ldots, a_n$ respectively, and radii so small that they lie entirely within the closed curve C. Then $f(z)$ is analytic in the multiple connected region lying between the curves C and $C_1, C_2, \ldots, C_n$.

Applying Cauchy's theorem

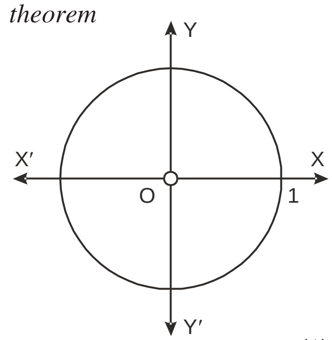
$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \ldots + \int_{c_n} f(z)\,dz.$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \text{Res } f(a_3) + \ldots + \text{Res } f(a_n)]$$ **Proved.**

Example 111. *Evaluate the following integral using residue theorem*

$$\int_c \frac{1+z}{z\,(2-z)}\,dz$$

where c is the circle $|z| = 1$.



Solution. The poles of the integrand are given by putting the denominator equal to zero.

$$z(2-z) = 0 \text{ or } z = 0,\ 2$$

The integrand is analytic on $|z| = 1$ and all points inside except $z = 0$, as a pole at $z = 0$ is inside the circle $|z| = 1$.

Hence by residue theorem

$$\int_c \frac{1+z}{z(2-z)}\,dz = 2\pi i \quad [\text{Res } f(0)] \qquad \ldots (1)$$

$$\text{Residue } f(0) = \lim_{z\to 0} z.\frac{1+z}{z\,(2-z)} = \lim_{z\to 0}\frac{1+z}{2-z} = \frac{1}{2}$$

Putting the value of Residue $f(0)$ in (1), we get

$$\int_c \frac{1+z}{z(2-z)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$$ **Ans.**

Example 112. *Determine the poles of the following function and residue at each pole:*

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$ *and hence evaluate*

$$\int_c \frac{z^2\,dz}{(z-1)^2(z+2)}$$ *where c*: $|z| = 3$. *(R.G.P.V. Bhopal, III Sem. Dec. 2007)*

Solution. $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

Poles of $f(z)$ are given by $(z-1)^2(z+2) = 0$ *i.e.* $z = 1, 1, -2$
The pole at $z = 1$ is of second order and the pole at $z = -2$ is simple.

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d}{dz}\frac{(z-1)^2 z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z\to 1}\frac{d}{dz}\frac{z^2}{z+2} = \lim_{z\to 1}\frac{(z+2)\,2z - 1.z^2}{(z+2)^2}$$

$$= \lim_{z\to 1}\frac{z^2+4z}{(z+2)^2} = \frac{1+4}{(1+2)^2} = \frac{5}{9}$$

$$\text{Residue of } f(z) \text{ (at } z = -2) = \lim_{z\to -2}\frac{(z+2)\,z^2}{(z-1)^2(z+2)} = \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{(-2-1)^2} = \frac{4}{9}$$ **Ans.**

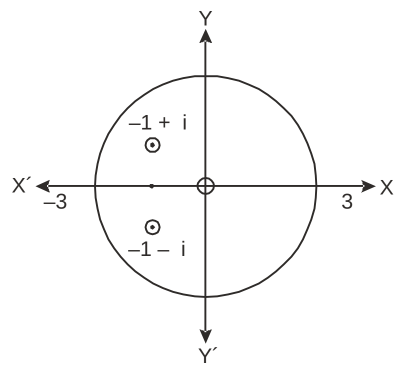
$$\int_c \frac{z^2 dz}{(z-1)^2(z+2)} = 2\pi i\left(\frac{5}{9}+\frac{4}{9}\right) = 2\pi i$$ **Ans.**

Example 113. *Using Residue theorem, evaluate* $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{zt}dz}{(z^2+2z+2)}$

where C is the cirlce $|\,z\,| = 3$. (*U.P., III Semester, Dec. 2009*)

Solution. Here, we have

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}dz}{z^2\,(z^2+2z+2)}$$



Poles are given by

$z = 0$ (double pole)

$z = -1 \pm i$ (simple poles)

All the four poles are inside the given circle.

$$\frac{1}{2\pi i}\int \frac{e^{zt}\,dz}{z^2\,(z^2+2z+2)}$$

$$\text{Residue at z = 0 is } \lim_{z\to 0}\frac{d}{dz}z^2\,\frac{e^{zt}}{z^2\,(z^2+2z+2)}$$

$$= \lim_{z\to 0}\frac{d}{dz}\frac{e^{zt}}{z^2+2z+2}$$

$$= \lim_{z\to 0}\frac{(z^2+2z+2)\,t\,e^{zt} - (2z+2)e^{zt}}{(z^2+2z+2)^2}$$

$$= \frac{2\,t\,e^0 - 2\,e^0}{4} = \frac{(t-1)}{2}$$

$$z^2+2z+2 = 0$$
$$\Rightarrow\ z^2+2z+1 = -1$$
$$\Rightarrow\ (z+1)^2 = -1$$
$$\Rightarrow\ z+1 = \pm i$$
$$\Rightarrow\ z = -1 \pm i$$

Residue at $z = -1 + i$

$$= \lim_{z\to -1+i}\frac{(z+1-i)\,e^{zt}}{z^2\,(z+1-i)\,(z+1+i)} = \lim_{z\to -1+i}\frac{e^{zt}}{z^2\,(z+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2\,(-1+i+1+i)} = \frac{e^{(-1+i)\,t}}{(1-2i-1)\,(2i)} = \frac{e^{(-1+i)\,t}}{4}$$

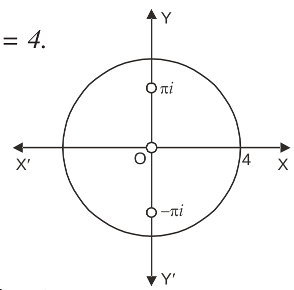
$$\int \frac{e^{2zt}}{z^2\,(z^2+2z+2)}\,dz = 2\pi i \quad \text{(Sum of the Residues)}$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int \frac{e^{2zt}}{z^2\,(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\,(e^{it} + e^{-it}) = \frac{t-1}{2} + \frac{e^{-t}}{4}\,(2\cos t)$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cos t$$ **Ans.**

Example 114. *Evaluate* $\oint_C \frac{1}{\sinh z}\,dz$, *where C is the circle* $|z| = 4$.



Solution. Here, $f(z) = \frac{1}{\sinh z}$.

Poles are given by

$$\sinh z = 0$$

$$\Rightarrow \quad \sin iz = 0$$

$$\Rightarrow \quad z = n\pi i \text{ where } n \text{ is an integer.}$$

Out of these, the poles $z = -\pi i$, 0 and πi lie inside the circle $|z| = 4$.

The given function $\frac{1}{\sinh z}$ is of the form $\frac{\phi(z)}{\psi(z)}$

Its pole at $z = a$ is $\frac{\phi(a)}{\psi'(a)}$.

Residue (at $z = -\pi i$)

$$= \frac{1}{\cosh(-\pi i)} = \frac{1}{\cos i(-\pi i)} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$$

$$\text{Residue (at } z = 0) = \frac{1}{\cosh 0} = \frac{1}{1} = 1$$

$$\text{Residue (at } z = \pi i) = \frac{1}{\cosh(\pi i)} = \frac{1}{\cos i\,(\pi i)} = \frac{1}{\cos(-\pi)}$$

$$= \frac{1}{\cos \pi} = \frac{1}{-1} = -1$$

Residue at $-\pi i$, 0, πi are respectively –1, 1 and –1.

Hence, the required integral = $2\pi i\,(-1 + 1 - 1) = -2\pi i$. **Ans.**

Example 115. *Evaluate* $\int_c \frac{dz}{z \sin z}$: c *is the unit circle about origin.*

Solution. $$\frac{1}{z\,\sin z} = \frac{1}{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right]} = \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right]}$$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120}\ldots\right)\right]^{-1} = \frac{1}{z^2}\left[1 + \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \left(\frac{z^2}{6} - \frac{z^4}{120}\ldots\right)^2 \ldots\right]$$

$$= \frac{1}{z^2}\left[1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \ldots\right] = \frac{1}{z^2} + \frac{1}{6} - \frac{z^2}{120} + \frac{z^2}{36}\ldots = \frac{1}{z^2} + \frac{1}{6} + \frac{7}{360}z^2 \ldots$$

This shows that $z = 0$ is a pole of order 2 for the function $\dfrac{1}{z \sin z}$ and the residue at the pole is zero, (coefficient of $\dfrac{1}{z}$).

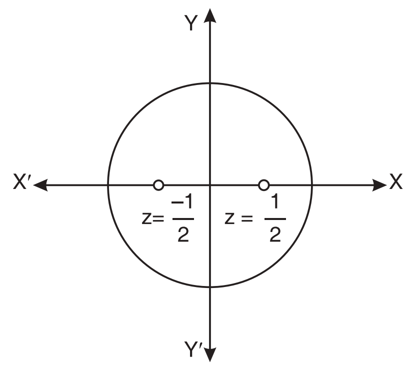
Now the pole at $z = 0$ lies within C.

$\therefore \quad \displaystyle\int \frac{1}{z \sin z}\, dz = 2\pi i$ (Sum of Residues) = 0 **Ans.**

Example 116. *Evaluate* $\displaystyle\int_C \frac{e^z}{\cos \pi z}$, *where C is the unit circle* $|z| = 1$. (*M.D.U. 2005, 2007,2008*)

Solution. Here $f(z) = \dfrac{e^z}{\cos \pi z}$

$$= \frac{e^z}{\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \ldots\right)}$$



It has simple poles at $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\ldots,$ of which only $z = \pm\frac{1}{2}$ lie inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = \lim_{z \to \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) e^z}{\cos \pi z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) e^z + e^z}{-\pi \sin \pi z} \qquad \text{[By L' Hopital's Rule]}$$

$$= \frac{e^{1/2}}{-\pi}.$$

Similarly, residue of $f(z)$ at $z = -\frac{1}{2}$ is $\dfrac{e^{-1/2}}{\pi}$.

$\therefore$ By residue theorem $\displaystyle\oint_C \frac{e^z}{\cos \pi z}\, dz = 2\pi i$ (sum of residues)

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = -4i \sinh \frac{1}{2}. \quad \textbf{Ans.}$$

EXERCISE 7.13

Evaluate the following complex integrals:

1. $\displaystyle\int_c \frac{1 - 2z}{z(z-1)(z-2)}\, dz$, where c is the circle $|z| = 1.5$ (*MDU Dec. 2006*) **Ans.** $3\pi i$

2. $\displaystyle\int_c \frac{z^2 e^{zt}}{z^2 + 1}\, dz$, where c is the circle $|z| = 2$ **Ans.** $-2\pi i \sin t$

3. $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$, where c is the circle $|z-i|=2$. **Ans.** $-\frac{2\pi i}{9}$

4. $\int_c \frac{2z^2+z}{z^2-1}dz$, where c is the circle $|z-1|=1$. **Ans.** $3\pi i$

5. $\int_c \frac{e^{2z}+z^2}{(z-1)^5}dz$, where c is the circle $|z|=2$ **Ans.** $\frac{4\pi e^2 i}{3}$

6. $\int_c \frac{dz}{(z^2+1)(z^2-4)}$, where c is the circle $|z|=1.5$ **Ans.** 0

7. $\int_c \frac{4z^2-4z+1}{(z-2)(z^2+4)}dz$, where c is the circle $|z|=1$ **Ans.** 0

8. $\int_c \frac{\sin z}{z^6}dz$, where c is the circle $|z|=2$ **Ans.** $\frac{\pi i}{60}$

9. Let $\left[\frac{P(z)}{Q(z)}\right]$, where both $P(z)$ and $Q(z)$ are complex polynomial of degree two. If $f(0)=f(-1)=0$ and only singularity of $f(z)$ is of order 2 at $z=1$ with residue -1, then find $f(z)$.

Ans. $f(z) = -\frac{1}{3}\frac{z(z+1)}{(z-1)^2}$

10. $\oint_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}dz$, where C is the circle $|z|=3$ (*MDU. Dec. 2008*) **Ans.** $4\pi i(\pi+1)$

11. $\int_C \frac{1-\cos 2(z-3)}{(z-3)^3}dz$, where C: $|z-3|=1$. (*MDU. Dec. 2004*) **Ans.** $4\pi i$

7.65 EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

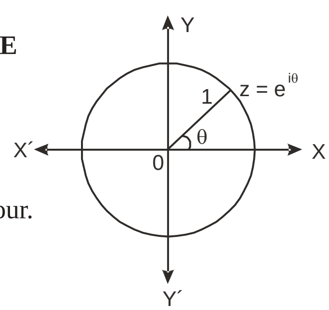
A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of the residues of } f(z) \text{ at the poles within } C)$$

We call the curve, a contour and the process of integration along a contour is called contour integration.

7.66 INTEGRATION ROUND UNIT CIRCLE OF THE TYPE

$$\int_0^{2\pi} f(\cos\theta,\ \sin\theta)d\theta$$



where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Convert $\sin\theta$, $\cos\theta$ into z.

Consider a circle of unit radius with centre at origin, as contour.

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left[z-\frac{1}{z}\right], \qquad z = re^{i\theta} = 1.\,e^{i\theta} = e^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left[z+\frac{1}{z}\right]$$

As we know

$$z = e^{i\theta},\ dz = e^{i\theta}\,i\,d\theta = z\,i\,d\theta \text{ or } d\theta = \frac{dz}{iz}$$

The integrand is converted into a function of z.

Then apply Cauchy's residue theorem to evaluate the integral.

Some examples of these are illustrated below.

Example 117. *Evaluate the integral:*

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$$ (*R.G.P.V., Bhopal, III Semester, June 2007*)

Solution. $$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-3\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

$$= \int_0^{2\pi} \frac{2\,d\theta}{10-3e^{i\theta}-3e^{-i\theta}}$$

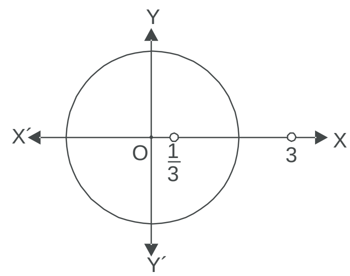
$$\left[\begin{array}{l} e^{i\theta} = z \;\Rightarrow\; i.e.^{i\theta}\,d\theta = dz \\ d\theta = \frac{dz}{iz} \end{array}\right]$$

$$= \int_C \frac{1}{10-3z-\frac{3}{z}}\,\frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{10z-3z^2-3}$$

[C is the unit circle $|z| = 1$]

$$= -\frac{1}{i}\int_C \frac{dz}{3z^2-10z+3}$$

$$= -\frac{1}{i}\int_C \frac{dz}{(3z-1)(z-3)} = i\int_C \frac{dz}{(3z-1)(z-3)}$$



Let $$I = \int_C \frac{dz}{(3z-1)(z-3)}$$

Poles of the integrand are given by

$$(3z-1)(z-3) = 0 \quad\Rightarrow\quad z = \frac{1}{3}, 3$$

There is only one pole at $z = \frac{1}{3}$ inside the unit circle C.

Residue at $\left(z = \frac{1}{3}\right)$ $$= \lim_{z\to\frac{1}{3}} \left(z-\frac{1}{3}\right) f(z) = \lim_{z\to\frac{1}{3}} \frac{\left(z-\frac{1}{3}\right)}{(3z-1)(z-3)} = \lim_{z\to\frac{1}{3}} \frac{1}{3(z-3)}$$

$$= \frac{1}{3\left(\frac{1}{3}-3\right)} = -\frac{1}{8}$$

Hence, by Cauchy's Residue Theorem

$$I = 2\pi i \text{ (Sum of the residues within Contour)} = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{4}$$

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = i\left(\frac{-\pi i}{4}\right) = \frac{\pi}{4}$$ **Ans.**

Example 118. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ if $a > |b|$ (*U.P. III Semester 2009-2010*)

Solution. Let $$I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$$

$$= \int_0^{2\pi} \frac{1}{a+b\,\frac{e^{i\theta}-e^{-i\theta}}{2i}}\,d\theta$$ $$\left[\text{Writing } e^{i\theta} = z,\ d\theta = \frac{dz}{iz}\right]$$

$$= \int_C \frac{1}{a + \frac{b}{2i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz} \qquad \text{(where } C \text{ is the unit circle } |z| = 1)$$

$$= \int_c \frac{1}{2iaz + bz^2 - b}\, dz = \frac{1}{b}\int \frac{2}{z^2 + \frac{2aiz}{b} - 1}\, dz$$

$$= \int_c \frac{2}{bz^2 + 2aiz - b}\, dz$$

$$= \frac{1}{b}\int_c \frac{2}{(z-\alpha)(z-\beta)}\, dz \qquad \left[bz^2 + 2aiz - b = b\left\{z^2 + \frac{2aiz}{b} - 1\right\}\right]$$

Where $\alpha + \beta = -\frac{2ai}{b}$

$\alpha\beta = -1$

$|\alpha| < 1$ then $|\beta| > 1$

i.e.; Pole lies at $z = \alpha$ in the unit circle.

$$\left[\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta \\ &= -\frac{4a^2}{b^2} + 4 \\ \alpha - \beta &= 2\frac{\sqrt{b^2-a^2}}{b}\end{aligned}\right]$$

Residue (at $z = a$) $= \lim_{z\to\alpha} (z-\alpha) \dfrac{2}{(z-\alpha)(z-\beta)}$

$$= \frac{2}{\alpha-\beta} = \frac{b}{\sqrt{b^2-a^2}} = \frac{b}{i\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{1}{a + b\sin\theta}\, d\theta = \frac{1}{b}\int_c \frac{2}{z^2 + 2\frac{aiz}{b} - 1}\, dz = 2\pi i \frac{b}{bi\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad \textbf{Ans.}$$

Example 119. *Use the complex variable technique to find the value of the integral :*

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$$ (*R.G.P.V., Bhopal, III Semester, Dec. 2003*)

Solution. Let $I = \int_0^{2\pi} \dfrac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \dfrac{d\theta}{2+\dfrac{e^{i\theta}+e^{-i\theta}}{2}} = \int_0^{2\pi} \dfrac{2d\theta}{4+e^{i\theta}+e^{-i\theta}}$

Put $e^{i\theta} = z$ so that $e^{i\theta}(i\,d\theta) = dz \Rightarrow iz\,d\theta = dz \Rightarrow d\theta = \dfrac{dz}{iz}$

$$I = \int_C \frac{2\frac{dz}{iz}}{4+z+\frac{1}{z}}$$ where c denotes the unit circle $|z| = 1$.

$$= \frac{1}{i}\int_c \frac{2\,dz}{z^2+4z+1}$$

The poles are given by putting the denominator equal to zero.

$$z^2+4z+1 = 0 \text{ or } z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$$

The pole within the unit circle C is a simple pole at $z = -2+\sqrt{3}$. Now we calculate the residue at this pole.

Residue at $(z = -2+\sqrt{3}) = \lim_{z\to(-2+\sqrt{3})} \dfrac{1}{i} \dfrac{(z+2-\sqrt{3})2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$$= \lim_{z\to(-2+\sqrt{3})} \frac{2}{i(z+2+\sqrt{3})} = \frac{2}{i(-2+\sqrt{3}+2+\sqrt{3})} = \frac{1}{\sqrt{3}i}$$

Hence by Cauchy's Residue Theorem, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 2\pi i \text{ (sum of the residues within the contour)}$$

$$= 2\pi i \, \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$ **Ans.**

Example 120. *Using complex variable techniques evaluate the real integral*

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta$$

Solution. If we write $z = e^{i\theta}$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\quad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right),\quad d\theta = \frac{dz}{iz}$$

and so $I = \int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{5-4\cos\theta}\, d\theta$

$$\text{I = Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta - i\sin 2\theta}{5-4\cos\theta}\, d\theta$$ $\left[\begin{array}{l}\text{where } c \text{ is a circle of unit}\\ \text{radius with centre } z = 0\end{array}\right]$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1-e^{2i\theta}}{5-4\cos\theta}\, d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_c \frac{1-z^2}{5-2(z+\frac{1}{z})}\left(\frac{dz}{iz}\right) = \text{Real part of } \frac{1}{2i}\int_c \frac{1-z^2}{5z-2z^2-2}\, dz$$

$$= \text{Real part of } \frac{1}{2i}\int_c \frac{z^2-1}{2z^2-5z+2}\, dz$$

Poles are determined by $2z^2-5z+2=0$ or $(2z-1)(z-2) = 0$ or $z = \frac{1}{2},\ 2$

So inside the contour c there is a simple pole at $z = \frac{1}{2}$

Residue at the simple pole $\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^2-1}{(2z-1)(z-2)}$

$$= \lim_{z\to\frac{1}{2}} \frac{z^2-1}{2(z-2)} = \frac{\frac{1}{4}-1}{2\left(\frac{1}{2}-2\right)} = \frac{1}{4}$$

I = Real part of $\frac{1}{2i}\int_c \frac{(z^2-1)}{2z^2-5z+2}\, dz = \frac{1}{2i}2\pi i$ (sum of the residues)

$$\Rightarrow \qquad \int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = \pi\left(\frac{1}{4}\right) = \frac{\pi}{4}$$ **Ans.**

Example 121. *Using contour integration, evaluate the real integral*

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$$ (*R.G.P.V., Bhopal, III Semester, Dec. 2004*)

Solution. Let $I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta} d\theta$

$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+2(e^{i\theta}+e^{-i\theta})} d\theta$

writing $e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$ where C is the unit circle $|z| = 1$.

$= \text{Real part of } \frac{1}{2}\int_C \frac{1+2z}{5+2\left(z+\frac{1}{z}\right)} \frac{dz}{iz} = \text{Real part of } \frac{1}{2}\int_C \frac{-i(1+2z)}{2z^2+5z+2} dz$

$= \text{Real part of } \frac{1}{2}\int_C \frac{-i(2z+1)}{(2z+1)(z+2)} dz = \text{Real part of } -\frac{i}{2}\int_C \frac{1}{z+2} dz$

Pole is given by $z + 2 = 0$ *i.e.* $z = -2$.
Thus there is no pole of $f(z)$ inside the unit circle C. Hence $f(z)$ is analytic in C.

By Cauchy's Theorem $\int_C f(z)\, dz = 0$ if $f(z)$ is analytic in C.

$\therefore$ I = Real part of zero = 0

Hence, the given integral = 0 **Ans.**

Example 122. *Using complex variables, evaluate the real integral*

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}, \quad \textit{where } p^2 < 1.$$ (*Kerala 2005; MDU Dec. 2008*)

Solution. $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_0^{2\pi} \frac{d\theta}{1-2p\frac{(e^{i\theta}-e^{-i\theta})}{2i}+p^2}$

Let $I = \int_0^{2\pi} \frac{d\theta}{1+ip(e^{i\theta}-e^{-i\theta})+p^2}$

Writing $z = e^{i\theta},\ dz = ie^{i\theta}d\theta = i\,z\,d\,\theta,\ d\,\theta = \frac{dz}{z\,i}$

$I = \int_C \frac{1}{1+i\,p\left(z-\frac{1}{z}\right)+p^2} \frac{dz}{z\,i}$ where c is the unit circle $|z| = 1$.

$= \int_C \frac{dz}{zi - pz^2 + p + p^2 z\,i} = \int_C \frac{dz}{-pz^2+ip^2z+zi+p} = \int_C \frac{dz}{(iz+p)(izp+1)}$

Poles are given by $(iz+p)(ipz+1) = 0$

$\Rightarrow$ $z = -\frac{p}{i} = ip$ and $z = -\frac{1}{pi} = \frac{i}{p}$ $\qquad |ip| < 1$ and $\left|\frac{i}{p}\right| > 1$ as $p^2 < 1$

pi is the only pole inside the unit circle.

Residue $(z = p\,i) = \lim_{z\to pi} \frac{(z-pi)}{(iz+p)(i\,z\,p+1)} = \lim_{z\to pi}\left[\frac{1}{i\,(i\,z\,p+1)}\right] = \frac{1}{i}\frac{1}{(-p^2+1)}$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = 2\pi i\left(\frac{1}{i}\frac{1}{1-p^2}\right) = \frac{2\pi}{1-p^2}$$ **Ans.**

Example 123. *Apply calculus of residue to prove that*:

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}, \qquad (a^2 < 1)$$

(*MDU. May 2007, 2003, R.G.P.V., Bhopal, III Semester, June 2003*)

Solution. Let $I = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2} = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-a\,(e^{i\theta}+e^{-i\theta})+a^2}$

$= \text{Real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{(1-ae^{i\theta})\,(1-ae^{-i\theta})}\, d\theta$

$= \text{Real part of } \oint_C \frac{z^2}{(1-az)\left(1-\frac{a}{z}\right)} \frac{dz}{iz}$ $\left[\text{Put } e^{i\theta} \text{ so that } d\theta = \frac{dz}{iz}\right]$

$= \text{Real part of } \oint_C \frac{-iz^2}{(1-az)\,(z-a)}\, dz$ [C is the unit circle $|z| = 1$]

Poles of $\frac{-iz^2}{(1-az)\,(z-a)}$ are given by

$$(1-az)\,(z-a) = 0$$

Thus, $z = \frac{1}{a}$ and $z = a$ are the simple poles. Only $z = a$ lies within the unit circle C as $a < 1$.

The residue of $f(z)$ at $(z = a)$ $= \lim_{z\to a} (z-a) \frac{-iz^2}{(1-az)\,(z-a)} = \lim_{z\to a} \frac{-iz^2}{(1-az)} = -\frac{ia^2}{1-a^2}$

Hence, by Cauchy's Residue Theorem, we have

$\oint_C f(z)\, dz = 2\pi i$ [Sum of residues within the contour]

$= 2\pi i\left(-\frac{ia^2}{1-a^2}\right) = \frac{2\pi a^2}{1-a^2}$ which is purely real.

Thus, $I = \text{Real part of } \oint_C f(z)\, dz = \frac{2\pi a^2}{1-a^2}$

Hence, $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}.$ **Proved.**

Example 124. *Evaluate:* $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ *by using contour integration.*

(*R.G.P.V., Bhopal, III Semester, June 2007*)

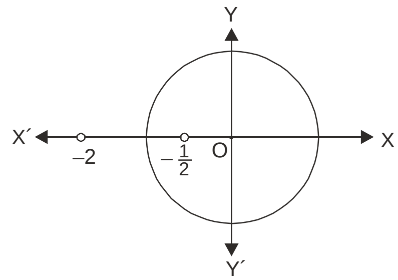
Solution.

Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

$$= \text{Real part of } \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta}\, d\theta$$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{5+2\,(e^{i\theta}+e^{-i\theta})}\, d\theta \qquad \left[\begin{array}{r} e^{i\theta} = z \\ \Rightarrow \quad i.e.^{i\theta}\, d\theta = dz \\ \Rightarrow d\theta = \dfrac{dz}{ie^{i\theta}} = \dfrac{dz}{iz} \end{array}\right]$$

$$= \text{Real part of } \oint_C \frac{z^2}{5+2\left(z+\frac{1}{z}\right)} \frac{dz}{iz} \qquad [C \text{ is the unit circle } |\, z\,| = 1]$$



$$= \text{Real part of } \oint_C \frac{z^2}{5z+2z^2+2} \frac{dz}{i}$$

$$= \text{Real part of } \oint_C \frac{-iz^2}{2z^2+5z+2}\, dz$$

$$= \text{Real part of } \oint_C \frac{-iz^2}{(2z+1)\,(z+2)}\, dz$$

Poles are determined by putting denominator equal to zero.

$$(2z+1)\,(z+2) = 0 \qquad \Rightarrow \quad z = -\frac{1}{2}, -2$$

The only simple pole at $z = -\frac{1}{2}$ is inside the contour.

Residue at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{-iz^2}{(2z+1)\,(z+2)}$

$$= \lim_{z\to -\frac{1}{2}} \frac{-iz^2}{2(z+2)} = \frac{-i\left(-\frac{1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)} = \frac{-i}{12}$$

By Cauchy's Integral Theorem

$$\int_C f(z)\, dz = 2\pi i \quad (\text{Sum of the residues within } C)$$

$$= 2\pi\, i\left(\frac{-i}{12}\right) = \frac{\pi}{6}, \text{ which is real}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$$ **Ans.**

Example 125. *Evaluate contour integration of the real integral*

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta.$$ (*U.P., III Sem., 2009, R.G.P.V., Bhopal, III Semester, Dec. 2007*)
(*MDU, Dec. 2010*)

Solution. $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$ = Real part of $\int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta} d\theta$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{3i\theta}}{5-2(e^{i\theta}+e^{-i\theta})} d\theta$$ On writing $z = e^{i\theta}$ and $d\theta = \frac{dz}{iz}$

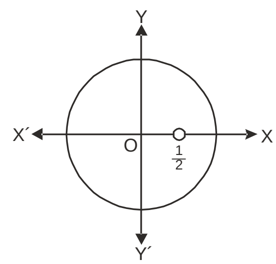
$$= \text{Real part of } \int_c \frac{z^3}{5-2\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$ c is the unit circle.

$$= \text{Real part of } \frac{1}{i}\int_c \frac{z^3}{5z-2z^2-2}dz \quad = \text{Real part of } -\frac{1}{i}\int \frac{z^3}{2z^2-5z+2}dz$$

$$= \text{Real part of } i\int \frac{z^3}{(2z-1)(z-2)}dz$$

Poles are given by $(2z-1)(z-2)=0$ *i.e.* $z=\frac{1}{2},\ z=2$

$z=\frac{1}{2}$ is the only pole inside the unit circle.



$$\text{Residue }\left(\text{at } z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{i\left(z-\frac{1}{2}\right)z^3}{(2z-1)(z-2)}$$

$$= \lim_{z\to\frac{1}{2}} \frac{iz^3}{2\,(z-2)} = \frac{i\,\frac{1}{8}}{2\left(\frac{1}{2}-2\right)} = -\frac{i}{24}$$

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = \text{Real part of } 2\pi i\left(-\frac{i}{24}\right) = \frac{\pi}{12}$$ **Ans.**

Question. *Evaluate* : $\int_0^{\infty}\frac{\cos 3\theta}{5+4\cos\theta}\,d\theta$ (*U.P. III Semester, Dec. 2008, 2006*)

Example 126. *Use the residue theorem to show that*

$$\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$ *where* $a>0,\ b>0,\ a>b$.

(*R.G.P.V., Bhopal, III Semester, June 2004*)

Solution. $$\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2} = \int_0^{2\pi}\frac{d\theta}{\left(a+b\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2}$$

Put $e^{i\theta}=z$, so that $e^{i\theta}(i\,d\theta)=dz \quad \Rightarrow \quad iz\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

$$= \int_c \frac{1}{\left\{a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right\}^2}\frac{dz}{iz}$$ where c is the unit circle $|z|=1$.

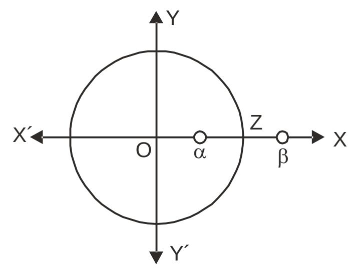
$$\int_c \frac{1}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{iz} = \int_c \frac{-4iz}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{(2z)^2}$$

$$= \int_c \frac{-4iz\,dz}{(bz^2+2az+b)^2} = \frac{-4i}{b^2}\int_c \frac{z\,dz}{\left(z^2+\frac{2az}{b}+1\right)^2}$$

The poles are given by putting the denominator equal to zero.

i.e., $$\left(z^2 + \frac{2a}{b} z + 1\right)^2 = 0$$

$\Rightarrow$ $$(z-\alpha)^2 (z-\beta)^2 = 0$$



where $$\alpha = \frac{-\frac{2a}{b} + \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$\beta = \frac{-\frac{2a}{b} - \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

There are two poles, at $z = \alpha$ and at $z = \beta$, each of order 2.

Since $|\alpha\beta| = 1$ or $|\alpha|\,|\beta| = 1$ if $|\alpha| < 1$ then $|\beta| > 1$.

There is only one pole [$|\alpha| < 1$] of order 2 within the unit circle c.

Residue (at the double pole $z = \alpha) = \lim_{z\to\alpha} \frac{d}{dz} (z-\alpha)^2 \frac{(-4iz)}{b^2 (z-\alpha)^2 (z-\beta)^2}$

$$= \lim_{z\to\alpha} \frac{d}{dz} \frac{-4iz}{b^2 (z-\beta)^2}$$

$$= -\frac{4i}{b^2} \lim_{z\to\alpha} \frac{(z-\beta)^2 .1 - 2(z-\beta)z}{(z-\beta)^4} = \frac{-4i}{b^2} \lim_{z\to\alpha} \frac{z-\beta-2z}{(z-\beta)^3} = \frac{-4i}{b^2} \lim_{z\to\alpha} \frac{-(z+\beta)}{(z-\beta)^3}$$

$$= \frac{4i}{b^2} \frac{(\alpha+\beta)}{(\alpha-\beta)^3} = \frac{4i}{b^2} \frac{\alpha+\beta}{[(\alpha+\beta)^2 - 4\alpha\beta]^{\frac{3}{2}}} = \frac{4i}{b^2} \frac{\frac{-2a}{b}}{\left[\left(-\frac{2a}{b}\right)^2 - 4(1)\right]^{\frac{3}{2}}}$$

$$= \frac{-8ai}{(4a^2 - 4b^2)^{\frac{3}{2}}} = -\frac{ai}{(a^2 - b^2)^{\frac{3}{2}}}$$

Hence, $\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = 2\pi i \times \frac{-ai}{(a^2 - b^2)^{3/2}} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$ **Proved.**

Example 127. *Evaluate by Contour integration:*

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta .$$

Solution. Let
$$I = \int_0^{2\pi} e^{\cos\theta} [\cos(\sin\theta - n\theta) + i \sin(\sin\theta - n\theta)]\, d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)} d\theta = \int_0^{2\pi} e^{\cos\theta + i\sin\theta} \cdot e^{-ni\theta} d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}} \cdot e^{-in\theta} d\theta \qquad ...(1)$$

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ then,

$$I = \int_C e^z \cdot \frac{1}{z^n} \cdot \frac{dz}{iz} = -i \int_C \frac{e^z}{z^{n+1}} dz$$

Pole is at $z = 0$ of order $(n + 1)$.
It lies inside the unit circle.
Residue of $f(z)$ at $z = 0$ is

$$= \frac{1}{(n+1-1)!}\left[\frac{d^n}{dz^n}\left\{z^{n+1}\cdot\frac{-ie^z}{z^{n+1}}\right\}\right]_{z=0} = \frac{-i}{n!}\left[\frac{d^n}{dz^n}(e^z)\right]_{z=0} = \frac{-i}{n!}\left(e^z\right)_{z=0} = \frac{-i}{n!}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i\left(\frac{-i}{n!}\right) = \frac{2\pi}{n!}$$

Comparing real part of $\int_0^{2\pi} e^{\cos\theta}\,[\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)]\,d\theta = \frac{2\pi}{n!}$,

we have

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$$ **Ans.**

EXERCISE 7.14

Evaluate the following integrals:

1. $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$ (*R.G.P.V., Bhopal, III Semester, June 2008*) **Ans.** $\frac{2\pi}{b^2}\{a-\sqrt{a^2-b^2}\},\ a>b>0$

2. $\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}\,d\theta$ **Ans.** $\frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n,\ n>0$

3. $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ **Ans.** $\frac{2\pi}{\sqrt{3}}$

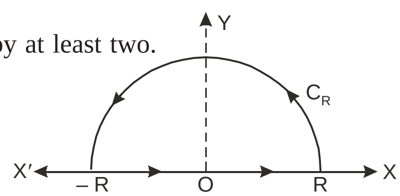
4. $\int_0^{2\pi}\frac{4}{5+4\sin\theta}\,d\theta$ **Ans.** $\frac{8\pi}{5}$

5. $\int_0^{\pi}\frac{d\theta}{17-8\cos\theta}$ **Ans.** $\frac{\pi}{15}$

6. $\int_0^{\pi}\frac{d\theta}{a+b\cos\theta}$, *where* $a>|b|$. *Hence or otherwise evaluate* $\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}$. **Ans.** $\frac{\pi}{\sqrt{a^2-b^2}}$; π

7.67 EVALUATION OF $\int_{-\infty}^{\infty}\frac{f_1(x)}{f_2(x)}\,dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x.

Such integrals can be reduced to contour integrals, if
(*i*) $f_2(x)$ has no real roots.
(*ii*) the degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.



Procedure: Let $f(x) = \frac{f_1(x)}{f_2(x)}$

Consider $\int_C f(z)\,dz$

where C is a curve, consisting of the upper half C_R of the circle $|z| = R$, and part of the real axis from $-R$ to R.

If there are no poles of $f(z)$ on the real axis, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its circumference C_R in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's theorem of residues, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{(sum of the residues of } f(z) \text{ at the poles within } C)$$

i.e. $$\int_{-R}^{R} f(x)\,dx + \int_{CR} f(z)\,dz = 2\pi i\,\text{(sum of residues within } C)$$

$$\Rightarrow \quad \int_{-R}^{R} f(x)\,dx = -\int_{CR} f(z)\,dz + 2\pi i\ \text{(sum of residues within } C)$$

$\therefore \qquad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = -\lim_{R\to\infty}\int_{CR} f(z)\,dz + 2\pi i$ (sum of residues within C) ... (1)

Now, $\qquad \lim_{R\to\infty}\int_{CR} f(z)\,dz = \int_0^{\pi} f(Re^{i\theta})\,Ri\,e^{i\theta}\,d\theta$

$= 0$ when $R \to \infty$

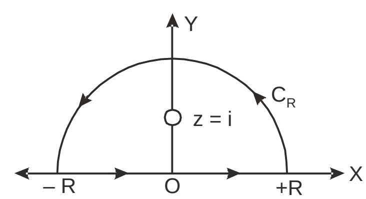
(1) reduces $\quad \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ (sum of residues within C)

Example 128. *Evaluate* $\int_0^{\infty}\frac{\cos mx}{(x^2+1)}dx$. (*R.G.P.V., Bhopal, III Semester, Dec. 2006*)

Solution. $\int_0^{\infty}\frac{\cos mx}{x^2+1}dx$

Consider the integral $\int_C f(z)\,dz$, where

$f(z) = \frac{e^{imz}}{z^2+1}$, taken round the closed contour c consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R.



Poles of $f(z)$ are given by

$$z^2 + 1 = 0 \text{ i.e. } z^2 = -1 \text{ i.e. } z = \pm i$$

The only pole which lies within the contour is at $z = i$.
The residue of $f(z)$ at $z = i$

$$= \lim_{z\to i}\frac{(z-i)e^{imz}}{(z^2+1)} = \lim_{z\to i}\frac{e^{imz}}{z+i} = \frac{e^{-m}}{2i}$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{ sum of the residues}$$

$$\Rightarrow \quad \int_C \frac{e^{imz}}{z^2+1}dz = 2\pi i \times \frac{e^{-m}}{2i} \qquad \Rightarrow \quad \int_{-R}^{R}\frac{e^{imx}}{x^2+1}dx = \pi e^{-m}$$

Equating real parts, we have

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+1}dx = \pi e^{-m} \qquad \Rightarrow \quad \int_0^{\infty}\frac{\cos mx}{x^2+1}dx = \frac{\pi e^{-m}}{2}$$ **Ans.**

Example 129. *Evaluate* $\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}\,dx$ (*U.P. III Semester 2009-2010*)

Solution. Here, we have $\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}\,dx$

Let us consider $\int_c \frac{z\sin\pi z}{z^2+2z+5}\,dz$

The pole can be determined by putting the denominator equal to zero.

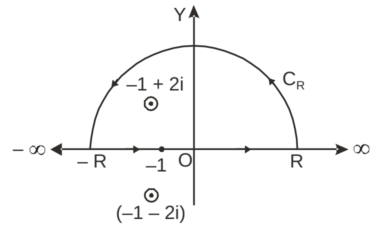
$$z^2+2z+5=0 \quad \Rightarrow \quad z = \frac{-2\pm\sqrt{4-20}}{2} \quad \Rightarrow \quad z = -1\pm 2i$$

Out of two poles, only $z = -1 + 2i$ is inside the contour.
Residue at $z = -1 + 2i$

$$= \lim_{z\to -1+2i}(z+1-2i)\frac{z\sin\pi z}{z^2+2z+5} = \lim_{z\to -1+2i}(z+1-2i)\frac{z\sin\pi z}{(z+1-2i)(z+1+2i)}$$

$$= \lim_{z \to -1+2i} \frac{z \sin \pi z}{(z+1+2i)} = \frac{(-1+2i) \sin \pi (-1+2i)}{(-1+2i+1+2i)}$$

$$= \frac{(-1+2i) \sin \pi (-1+2i)}{4i}$$



$$\int_{-R}^{R} \frac{z \sin \pi z}{z^2 + 2z + 5} dz = 2\pi i \text{ (Residue)}$$

$$= 2\pi i \frac{(-1+2i) \sin \pi (-1+2i)}{4i} = \frac{\pi}{2} (2i - 1) \sin (-\pi + 2\pi i)$$

$$= \frac{\pi}{2} (2i - 1) (-\sin 2\pi i) \qquad \left[\begin{aligned} \sin(-\pi+\theta) &= -\sin(\pi - \theta) \\ &= -\sin\theta \end{aligned} \right.$$

$$= \frac{\pi}{2} (1-2i) \sin 2\pi i = \frac{\pi}{2} (1-2i)\, i \sinh 2\pi$$

$$= \frac{\pi}{2} (i + 2) \sinh 2\pi \qquad \text{(Taking real parts)}$$

Hence $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} = \pi \sinh 2\pi$ **Ans.**

Example 130. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$. (*MDU, Dec. 2006*)

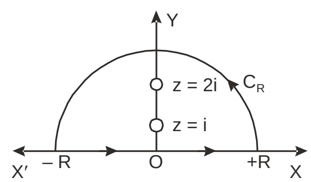
Solution. We consider $\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C f(z)\, dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to $+R$.

The integral has simple poles at

$$z = \pm i, \; z = \pm 2i$$

of which $z = i$, $2i$ only lie inside C.



The residue (at $z = i$) $= \lim_{z \to i} (z - i) \frac{z^2}{(z+i)(z-i)(z^2+4)}$

$$= \lim_{z \to i} \frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{2i(-1+4)} = \frac{-1}{6i}$$

The residue (at $z = 2i$) $= \lim_{z \to 2i} (z - 2i). \frac{z^2}{(z^2+1)(z+2i)(z-2i)}$

$$= \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{(2i)^2}{(-4+1)(2i+2i)} = \frac{1}{3i}$$

By theorem of residue;

$$\int_C f(z)\, dz = 2\pi i [\text{Res } f(i) + \text{Res } f(2i)] = 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i} \right) = \frac{\pi}{3}$$

i.e. $\int_{-R}^{R} f(x) dx + \int_{C_R} f(z)\, dz = \frac{\pi}{3}$ ... (1)

Hence by making $R \to \infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} f(x) dx + \lim_{z \to \infty} \int_{C_R} f(z)\, dz = \frac{\pi}{3}$$

Now $R \to \infty$, $\int_{C_R} f(z)\, dz$ vanishes.

For any point on C_R as $|z| \to \infty,\ f(z) = 0$

$$\lim_{|z|\to\infty} \int_{C_R} f(z)\, dz = 0, \quad \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{3}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$ **Ans.**

Example 131. *Using the complex variable techniques, evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$ (*AMIETE, June 2010, U.P. III Semester, Dec. 2006*)

Solution. $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$

Consider $\int_C f(z)\, dz$, where $f(z) = \frac{1}{z^4+1}$

taken around the closed contour consisting of real axis and upper half C_R, *i.e.* $|z| = R$. Poles of $f(z)$ are given by

$$z^4 + 1 = 0 \text{ i.e. } z^4 = -1 = (\cos\pi + i\sin\pi)$$

$$\Rightarrow \qquad z^4 = [\cos(2n+1)\pi + i\sin(2n+1)\pi]$$

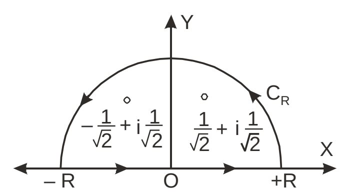
$$z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}} = \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right]$$

If $n = 0$, $\quad z_1 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{\pi}{4}}$

$n = 1$, $\quad z_2 = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{3\pi}{4}}$

$n = 2$, $\quad z_3 = \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$n = 3$, $\quad z_4 = \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$



There are four poles, but only two poles at z_1 and z_2 lie within the contour.

$$\text{Residue}\left(\text{at } z = e^{\frac{i\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{i\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=e^{\frac{i\pi}{4}}} = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}}$$

$$= \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$\text{Residue}\left(\text{at } z = e^{\frac{3i\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{3i\pi}{4}}} = \frac{1}{[4z^3]_{z=\frac{3i\pi}{4}}} = \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$= \frac{1}{4} e^{-i\frac{9\pi}{4}} = \frac{1}{4}\left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles within c)}$$

$$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \text{ (sum of the residues)}$$

$$\int_{-R}^{R} \frac{1}{x^4+1}\,dx + \int_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i \text{ (sum of the residues)}$$

Now, $\left|\int_{C_R} \frac{1}{z^4+1}\,dz\right| \le \int_{C_R} \frac{1}{|z^4+1|}\,|dz|$

$$\le \int_{C_R} \frac{1}{(|z^4|-1)}\,|dz| \qquad [\text{Since } z = Re^{i\theta}, |dz| = |Re^{i\theta}\, i\, d\theta| = Rd\theta]$$

$$\le \int_0^{\pi} \frac{1}{R^4-1} R\,d\theta \le \frac{R}{R^4-1}\int_0^{\pi} d\theta$$

$$\le \frac{R\pi}{R^4-1} = \frac{\pi/R^3}{1-1/R^4} \qquad \text{which} \to 0$$

as $R \to \infty$.

Hence, $\int_{-R}^{R} \frac{1}{x^4+1}\,dx = 2\pi i$ (Sum of the residues within contour)

As $R \to \infty$

Hence, $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i$ (Sum of the residues within contour) ... (1)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{2} i\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{\pi i}{2}\left(-i\frac{2}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

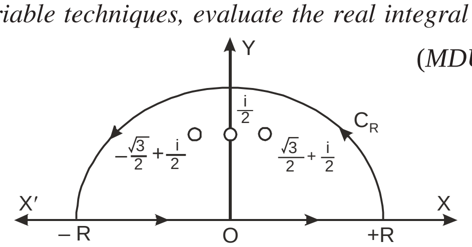
Hence, the given integral = $\frac{\pi}{\sqrt{2}}$ **Ans.**

Example 132. *Using complex variable techniques, evaluate the real integral*

$$\int_0^{\infty} \frac{dx}{1+x^6}$$ (*MDU May, 2006*)

Solution. Let $f(z) = \frac{1}{1+z^6}$

We consider $\int_C \frac{1}{1+z^6}\,dz$



where C is the contour consisting of the semi-circle C_R of radius R together with the part of real axis from $-R$ to R.

Poles are given by $1 + z^6 = 0$

$$z^6 = -1 = \cos\pi + i\sin\pi = \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$$

$$= e^{(2n+1)\pi i}$$

$$z = e^{\frac{2n+1}{6}\pi i} = \left[\cos\frac{2n\pi+\pi}{6} + i\sin\frac{2n\pi+\pi}{6}\right] \text{ where } n = 0, 1, 2, 3, 4, 5$$

If $\quad n = 0, \quad z = e^{\frac{\pi i}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

If $\quad n = 1, \quad z = e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

If $\quad n = 2, \quad z = e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{i}{2}$

If $\quad n = 3, \quad z = e^{i\frac{7\pi}{6}} = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = \frac{-\sqrt{3}}{2} - \frac{i}{2}$

If $\quad n = 4, \quad z = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

If $\quad n = 5, \quad z = e^{i\frac{11\pi}{6}} = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$

Only first three poles *i.e.*, $e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}$ are inside the contour.

$$\text{Residue}\left(\text{at } z = e^{\frac{i\pi}{6}}\right) = \lim_{z \to e^{\frac{i\pi}{6}}} \frac{1}{\frac{d}{dz}(1+z^6)} = \lim_{z \to e^{\frac{i\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{\frac{-i5\pi}{6}}$$

$$\text{Residue}\left(\text{at } z = e^{\frac{i\pi}{2}}\right) = \lim_{z \to e^{\frac{i\pi}{2}}} \frac{1}{\frac{d}{dz}(1+z^6)} = \lim_{z \to e^{\frac{i\pi}{2}}} \frac{1}{6z^5} = \frac{1}{6} e^{\frac{-i5\pi}{2}}$$

$$\text{Residue}\left(\text{at } z = e^{\frac{i5\pi}{6}}\right) = \lim_{z \to e^{\frac{i5\pi}{6}}} \frac{1}{\frac{d}{dz}(1+z^6)} = \lim_{z \to e^{\frac{i5\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{\frac{-i25\pi}{6}}$$

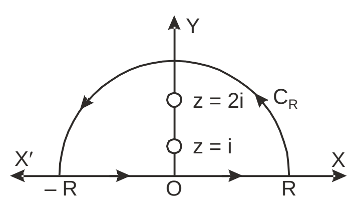
$$\text{Sum of the residues} = \frac{1}{6}\left[e^{\frac{-5i\pi}{6}} + e^{-\frac{i5\pi}{2}} + e^{-\frac{i25\pi}{6}}\right] = \frac{1}{6}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} + 0 - i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{1}{6}(-2i) = -\frac{i}{3}$$

$$\Rightarrow \quad \int_C \frac{dz}{1+z^6} = 2\pi i \text{ (Residue)} = 2\pi i\left(-\frac{i}{3}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3} \quad \Rightarrow \quad \int_0^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3} \quad \textbf{Ans.}$$

Example 133. *Using complex variables, evaluate the real integral*

$$\int_0^{\infty} \frac{\cos 3x \, dx}{(x^2+1)(x^2+4)}$$



Solution. Let $\quad f(z) = \frac{e^{3iz}}{(z^2+1)(z^2+4)}$

Poles are given by

$$(z^2+1)(z^2+4) = 0$$

i.e., $\quad z^2 + 1 = 0 \text{ or } z = \pm i$

$$z^2 + 4 = 0 \text{ or } z = \pm 2i$$

Let C be a closed contour consisting of the upper half C_R of a large circle $|z| = R$ and the real axis from $-R$ to $+R$. The poles at $z = i$ and $z = 2i$ lie within the contour.

Residue (at $z = i$) = $\lim_{z\to i} \frac{(z-i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to i} \frac{e^{3iz}}{(z+i)(z^2+4)} = \frac{e^{-3}}{6i}$

Residue (at $z = 2i$) = $\lim_{z\to 2i} \frac{(z-2i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to 2i} \frac{e^{3iz}}{(z^2+1)(z+2i)} = \frac{e^{-6}}{-12i}$

By theorem of Residue $\int_C f(z)dz = 2\pi i$ [Sum of Residues]

$$\int_{-R}^{R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} + \int_{C_R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} = 2\pi i\left[\frac{e^{-3}}{6i} + \frac{e^{-6}}{-12i}\right]$$

$$\left[\int_{C_R} \frac{e^{3iz}\,dz}{(z^2+1)(z^2+4)} = 0 \text{ as } z = Re^{i\theta} \text{ and } R \to \infty\right]$$

$$\int_{-R}^{R} \frac{e^{3ix}}{(x^2+1)(x^2+4)}dx = \pi\left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$$

$$\int_0^{\infty} \frac{\cos 3x\,dx}{(x^2+1)(x^2+4)} = \text{Real part of } \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{3ix}dx}{(x^2+1)(x^2+4)}$$

$$= \text{Real part of } \frac{\pi}{2}\left(\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right)$$

Hence, given integral = $\frac{\pi}{2}\left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$ **Ans.**

Example 134. *Evaluate:* $\int_0^{\infty} \frac{dx}{(a^2+x^2)^2}$ *(MDU. Dec. 2009)*

Sol. Consider the integral $\int_C f(z)\,dz$ where $f(z) = \frac{1}{(a^2+z^2)^2}$

Poles of $f(z)$ are given by putting denominator equal to zero.

$(a^2 + z^2)^2 = 0 \quad \Rightarrow \quad a^2 + z^2 = 0 \quad \Rightarrow \quad z = \pm\, ai$ each repeated twice

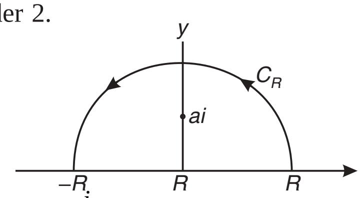
Sine there is no pole on the real axis, therefore we may take the contour C consisting of the semicirle C_R which is the upper half of a large circle $|z| = R$, and the real axis from $-R$ to R.

Here by Cauchy's residues theorem we have

$$\oint_C f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

or $$\int_{-R}^{R} \frac{1}{(a^2+x^2)^2}dx + \int_{C_R} \frac{dz}{(a^2+z^2)^2} = 2\pi i \text{ (sum of residues)} \quad \ldots (1)$$

The only pole within the contour C is $z = ai$, and is of order 2.



Here $f(z) = \frac{1}{(z-ai)^2(z+ai)^2} = \frac{\phi(z)}{(z-ai)^2}$

where $\phi(z) = \frac{1}{(z+ai)^2} \quad \Rightarrow \quad \phi'(z) = -\frac{2}{(z+ai)^3}$

$\therefore$ Residue at the double pole ($z = ai$) = $\frac{\phi'(ai)}{1!} = -\frac{2}{(2ai)^3} = -\frac{i}{4a^3}$

and $$\left|\int_{C_R} \frac{1}{(a^2+z^2)^2}\,dz\right| \le \int_{C_R} \frac{|dz|}{|a^2+z^2|^2} \le \int_{C_R} \frac{|dz|}{(|z|^2 - a^2)^2} = \int_0^{\pi} \frac{Rd\theta}{(R^2-a^2)^2}$$

$$= \frac{\pi R}{(R^2 - a^2)^2} \to 0 \text{ and } R \to \infty \text{ since } z = Re^{i\theta}$$

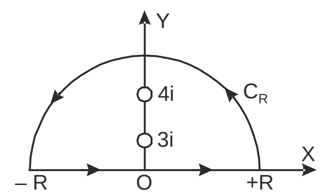
Hance when $R \to \infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{1}{(a^2 + x^2)^2}\, dx = 2\pi i \text{ (residue)} = 2\pi i \left(\frac{-i}{4a^3}\right) = \frac{\pi}{2a^3}$$

$$\Rightarrow \qquad \int_0^{\infty} \frac{1}{(a^2 + x^2)^2}\, dx = \frac{\pi}{4a^3}$$ **Ans.**

Example 135. *Using complex variable techniques, evaluate the real integral*

$$\int_0^{\infty} \frac{\cos 2x}{(x^2 + 9)^2 (x^2 + 16)}\, dx$$



Solution. Consider the integral $\int_C f(z)\, dz$,

where $$f(z) = \frac{e^{2iz}}{(z^2 + 9)^2 (z^2 + 16)},$$

taken around the closed contour C consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R.

Poles of $f(z)$ are given by

$$(z^2 + 9)^2 (z^2 + 16) = 0$$

i.e. $$(z + 3i)^2 (z - 3i)^2 (z + 4i)(z - 4i) = 0$$

i.e. $$z = 3i, -3i, 4i, -4i$$

The poles which lie within the contour are $z = 3i$ of the second order and $z = 4\,i$ simple pole.

Residue of $f(z)$ at $z = 3i$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left\{(z - 3i)^2 \frac{e^{2iz}}{(z - 3i)^2 (z + 3i)^2 (z^2 + 16)}\right\}\right]_{z=3i} = \left[\frac{d}{dz}\left\{\frac{e^{2iz}}{(z + 3i)^2 (z^2 + 16)}\right\}\right]_{z=3i}$$

$$= \left[\frac{(z + 3i)^2 (z^2 + 16) 2ie^{2iz} - e^{2iz}[2(z + 3i)(z^2 + 16) + 2z(z + 3i)^2]}{(z + 3i)^4 (z^2 + 16)^2}\right]_{z=3i}$$

$$= \left[\frac{(z + 3i)(z^2 + 16) 2i\, e^{2iz} - e^{2iz}[2(z^2 + 16) + 2z(z + 3i)]}{(z + 3i)^3 (z^2 + 16)^2}\right]_{z=3i}$$

$$= \frac{6i \times 7 \times 2i\, e^{-6} - e^{-6}(2 \times 7 + 6i \times 6i)}{(6i)^3 (7)^2} = \frac{e^{-6}[-84 + 22]i}{216 \times 49} = \frac{e^{-6}(-62)i}{216 \times 49} = -\frac{i31e^{-6}}{108 \times 49}$$

Residue of $f(z)$ at $(z = 4i)$ = $$\lim_{z \to 4i} (z - 4i) \frac{e^{2iz}}{(z^2 + 9)^2 (z - 4i)(z + 4i)}$$

$$= \frac{e^{-8}}{(-16 + 9)^2 (4i + 4i)} = \frac{e^{-8}}{49 \times 8i} = \frac{-ie^{-8}}{392}$$

Sum of the residues = $$-\frac{i31e^{-6}}{108 \times 49} - \frac{ie^{-8}}{392}$$

Hence by Cauchy's Residue Theorem, we have

$$\int_C f(z)\, dz = 2\pi i \times \text{Sum of the residues within } C$$

i.e. $$\int_{-R}^{R} f(x) dx + \int_{C_R} f(z)\, dz = 2\pi i \times \text{sum of residues}$$

or $$\int_{-R}^{R} \frac{e^{2ix}}{(x^2+9)^2(x^2+16)}\,dx + \int_{C_R} \frac{e^{2iz}}{(z^2+9)^2(z^2+16)}\,dz = 2\pi i \times \text{ Sum of residues} \quad \ldots (1)$$

Now let $R \to \infty$, so as to show that the second integral in above relation vanishes. For any point on C_R, *as* $|z| \to \infty$

Let $$F(z) = \frac{1}{z^6} \frac{e^{2iz}}{\left(1+\frac{9}{z^2}\right)^2\left(1+\frac{16}{z^2}\right)}$$

$$\lim_{|z|\to\infty} F(z) = 0 \qquad \text{or} \qquad \int_{C_R} \frac{e^{2iz}}{(z^2+9)^2(z^2+16)}\,dz = 0 \text{ as } z \to \infty$$

Hence by making $R \to \infty$, relation (1) becomes

$$\therefore \quad \int_{-\infty}^{\infty} \frac{e^{2ix}}{(x^2+9)^2(x^2+16)}\,dx = 2\pi i\left[\frac{-i31e^{-6}}{108\times 49} - i\,\frac{e^{-8}}{392}\right] = \frac{2\pi}{196}\left[\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right]$$

Equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)} = \frac{\pi}{98}\left(\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right)$$

$$\int_{0}^{\infty} \frac{\cos 2x}{(x^2+9)^2(x^2+16)}\,dx = \frac{\pi}{196}\left(\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right) \quad \left[\because \int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx \text{ If } f(x) \text{ is even function.}\right]$$ **Ans.**

EXERCISE 7.15

Evaluate the following :

1. $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ **Ans.** $\frac{\pi}{2}$ **2.** $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$ **Ans.** $\frac{\pi}{2}$

3. $\int_0^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)}\,dx$ **Ans.** $\frac{\pi}{2(a^2-b^2)}[a^2e^{-a} - b^2e^{-b}]$

4. $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx,\quad a > b > 0$ **Ans.** $\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$

5. Show that $\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a}$

6. Show that $\int_0^{\infty} \frac{x^3 \sin x}{(x^2+a^2)}\,dx = -\frac{\pi}{4}(a-2)\,a^{-a},\ a > 0$

Evaluate the following :

7. $\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx,\quad m > 0,\ a > 0$ **Ans.** $\frac{\pi}{a^2}(2-e^{-ma})$

8. $\int_0^{\infty} \frac{x^2}{x^6+1}\,dx$ (*MDU, 2008, 2005*) **Ans.** $\frac{\pi}{2}$ **9.** $\int_0^{\infty} \frac{x \sin ax}{x^4+a^4}\,dx$ **Ans.** $\frac{\pi}{2a^2} e^{-\frac{a^2}{\sqrt{2}}} \sin\frac{a^2}{\sqrt{2}}$

10. $\int_0^{\infty} \frac{x^6}{(a^4+x^4)^2}\,dx$ **Ans.** $\frac{3\pi\sqrt{2}}{16a},\ a > 0$ **11.** $\int_0^{\infty} \frac{\cos x^2 + \sin x^2 - 1}{x^2}\,dx$

12. $\int_0^{\infty} \frac{\cos mx}{x^4+x^2+1}\,dx$ **Ans.** $\frac{\pi}{\sqrt{3}} \sin\frac{1}{2}\left(m+\frac{\pi}{3}\right)e^{-\frac{1}{2}m\sqrt{3}}$ **13.** $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx$ **Ans.** $\pi \log 2$