MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

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Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: TRIGONOMENTRY

SUBJECT CODE: SM22A

SEMESTER: II

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TRIGONOMETRY

YEAR: I **SEMESTER: II**

Learning outcomes:

Students will acquire Knowledge

About the expansions of Trigonometric Functions, Hyperbolic Functions and sum of Trigonometric Series.

UNIT I

Expansions of powers of sin , cos - Expansions of cos$_{n}$, sin$_{n}$,cos$_{m}$ sin$_{n}$

UNIT II

Expansions of sinn , cosn , tan n - Expansions of tan($_{1}$+ $_{2}$ $_{n}$) - Expansions of sin x,
Cosx, tanx in terms of x-Sum of roots of trigonometric equations Formation of equation with trigonometric roots.

UNIT III

Hyperbolic functions-Relation between circular and hyperbolic functions - Formulas in hyperbolic functions Inverse hyperbolic functions.

UNIT IV

Inverse function of exponential functions Values of Log (u+iv) - Complex index.

UNIT V

Sums of Trigonometric series Applications of binomial, exponential, logarithmic and Gregory’s series - Difference method.

UNIT 1 : EXPANSION OF TRIGONOMETRIC FUNCTIONS

UNIT STUCTURE

1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- derive of $\cos nG$, $\sin nG$ and $\tan nG$
- derive of $\sin a$, $\cos a$ and $\tan a$ in terms of a
- derive of $\sin a^0$ and $\cos a^0$.

2 INTRODUCTION

In this unit, We will obtain the expansion of some trigonometric functions.

3 EXPANSION OF $\cos nG$ AND $\sin nG$.(N BEING A +VE INTEGER)

We know from De moivre’s Theorem, that

$(\cos G + i\sin 6)^n = \cos nG + i\sin nG$.

Expanding the left hand side by Binomial theorem, we have

$\cos n\theta + i\sin n\theta$

$= (\cos\theta)^n + {}^nC_1(\cos\theta)^{n-1}(i\sin\theta) + {}^nC_2(\cos\theta)^{n-2}(i\sin\theta)^2 + {}^nC_3(\cos\theta)^{n-3}(i\sin\theta)^3$

$+ {}^nC_4(\cos\theta)^{n-4}(i\sin\theta)^4 + \ldots + {}^nC_{n-1}(\cos\theta)\ (i\sin\theta)^{n-1} + {}^nC_n(i\sin\theta)^n$.

Now, $i^2 = -1, i^3 = i^2.i = -i, i^4 = (i^2)^2 = 1$ and so on.

and ${}^nC_{n-1} = {}^nC_1 = n, {}^nC_n = {}^nC_0 = 1$.

$\therefore$ $\cos n\theta + i\sin n\theta$

$= (\cos\theta)^n + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta$

$+ {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots + i^{n-1}n\cos\theta\sin^{n-1}\theta + i^n\sin^n\theta.$

There arise two cases, according as n is odd or even

Case I: If n is odd, then (n–1) is even $\cos n\theta + i\sin n\theta$

$= (\cos\theta)^n + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta$

$+ {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots + n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta + i(-1)^{\frac{n-1}{2}}\sin^n\theta.$

$\left[\because\ i^{n-1} = (i^2)^{\frac{n-1}{2}} = (-1)^{\frac{n-1}{2}};\ i^n = i.i^{n-1} = i.(-1)^{\frac{n-1}{2}}\right]$

Equating real and imaginary parts, we get

$\cos n\theta = (\cos\theta)^n - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots + n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta$ .... (i)

and

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots + (-1)^{\frac{n-1}{2}}\sin^n\theta.$ .... (ii)

Case II: If n is even, then(n–1) is odd, (n–2) is even $\cos n\theta + i\sin n\theta$

$+ {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots.. + i(-1)^{\frac{n-2}{2}}n\cos\theta\sin^{n-1}\theta + (-1)^{\frac{n}{2}}\sin^n\theta\cdot$

$\left[\because\ i^{n-1} = i.i^{n-2} = i.(i^2)^{\frac{n-1}{2}} = i.(-1)^{\frac{n-1}{2}};\ i^n = (i^2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}}\right]$

Equating real and imaginary parts,we get

$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots\cdot + (-1)^{\frac{n}{2}}\sin^n\theta$

....(iii)

and

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots.. + (-1)^{\frac{n-2}{2}}n\cos\theta\sin^{n-1}\theta$.

....(iv)

Example 1: a) Expand $\cos 7\theta$ in descending powers of $\cos\theta$.

b) Expand $\sin 7\theta$ in ascending powers of $\sin\theta$.

The binomial co-efficient,
${}^nC_r$ is defined by
${}^nC_r = \dfrac{n!}{r!(n-r)!}$.
where n! is known as n factorial is defined by
$n! = n(n-1)(n-2)\cdots 2.1$
for $n \geq 1$ and $0! = 1$.

Solution: We have $(\cos 7\theta + i\sin 7\theta) = (\cos\theta + i\sin\theta)^7$

Expanding the R.H.S. by Bionomial Theorem, we have

$$\cos 7\theta + i\sin 7\theta$$
$$= \cos^7\theta + {}^7C_1\cos^6\theta\,(i\sin\theta) + {}^7C_2\cos^6\theta(i\sin\theta)^2 + {}^7C_3\cos^5\theta(i\sin\theta)^3$$
$$+ {}^7C_4(\cos\theta)^3(i\sin\theta)^4 + {}^7C_5(\cos\theta)^2(i\sin\theta)^5 + {}^7C_6(\cos\theta)(i\sin\theta)^6 + {}^7C_7(i\sin\theta)^7$$

Now, $i^2 = -1$, $i^3 = -i$, $i^4 = 1, i^5 = i, i^6 = -1, i^7 = i^3.i^4 = -i$.

Also, ${}^7C_7 = 1$, ${}^7C_6 = {}^7C_1 = 7$, ${}^7C_5 = {}^7C_2 = \frac{7.6}{1.2} = 21$, ${}^7C_4 = {}^7C_3 = \frac{7.6.5}{1.2.3} = 35$.

$$\therefore \cos 7\theta + i\sin 7\theta = \cos^7\theta + 7i\cos^6\theta\sin\theta$$
$$- 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$$
$$+ 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta.$$

Equating real and imaginary parts,we get

a) $\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$
$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta)$$
$$- 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$
$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

b) $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$$
$$= 7(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin\theta - 35(1 - 2\sin^2\theta + \sin^4\theta)\sin^3\theta$$
$$+ 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta.$$
$$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

4 EXPANSION OF $\tan n\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta........}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta........}$$

Dividing the numerator and denominator by $\cos^n\theta$,we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - ....}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - ....}.$$

Last terms:

i) When n is odd: The last terms of the numerator is $(-1)^{\frac{n-1}{2}}\tan^n\theta$ and that of the denominator is $n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta$.

ii) When n is even: The last term of the denominator is $n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta$ and that of the denominator is $(-1)^{\frac{n}{2}}\tan^{n}\theta$.

Example 2: Expand $\tan 5\theta$ in powers of $\tan\theta$.

Solution: We know that

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - ....}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - ....}$$

Putting n = 5, we have

$$\tan 5\theta = \frac{{}^5C_1\tan\theta - {}^5C_3\tan^3\theta + {}^5C_5\tan^5\theta}{1 - {}^5C_2\tan^2\theta + {}^5C_4\tan^4\theta}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta - 5\tan^4\theta}$$

Example 3: Write down the value of $\tan 9\theta$ in terms of $\tan\theta$.

Solution: We know that

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \cdots\cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta + \cdots\cdots}$$

Putting n = 9, we get

$$\tan 9\theta = \frac{{}^9C_1\tan\theta - {}^9C_3\tan^3\theta + {}^9C_5\tan^5\theta - \cdots\cdots}{1 - {}^9C_2\tan^2\theta + {}^9C_4\tan^4\theta - {}^9C_6\tan^6\theta + \cdots\cdots}$$

$$= \frac{9\tan\theta - \frac{9.8.7}{3!}\tan^3\theta + \frac{9.8.7.6}{4!}\tan^5\theta - \frac{9.8}{2!}\tan^7\theta + \tan^9\theta}{1 - \frac{9.8}{4!}\tan^2\theta + \frac{9.8.7.6}{4!}\tan^4\theta - \frac{9.8.7}{3!}\tan^6\theta + 9\tan^8\theta}$$

$$= \frac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1 - 36\tan^2\theta + 126\tan^4\theta - 84\tan^6\theta + 9\tan^8\theta}$$

CHECK YOUR PROGRESS

Q.1: Expand $\cos 8\theta$ in powers of $\cos\theta$ and $\sin\theta$.

Q.2: Prove that, $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Q.3: Prove that,

$$\tan 8\theta = \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}$$

Q.4: Prove that:

$$\frac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta$$

UNIT 2 : EXPANSION OF TRIGONOMETRIC FUNCTIONS

1. EXPANSION OF $\sin\alpha$ IN TERMS OF α (RADIANS)

We know that if n is a positive integer, then

$$\sin n\theta = n\theta\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{n\theta(n\theta-\theta)(n\theta-2\theta)}{2!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 .....$$

Let $n\theta = \alpha$ and suppose that n increses without limit and θ decreases in such a way that $n\theta = \alpha$ remains constant, then

$$\sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{2!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 ....$$

Now, when $\theta \to 0, \frac{\sin\theta}{\theta} \to 1$ and $\cos\theta - 1$.

$$.\quad \sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} ....\infty .$$

2. EXPANSION $\cos\alpha$ IN TERMS OF α (RADIANS)

We know that if n is a positive integer, then

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \cdots\cdots$$

$$= \cos^n\theta - \frac{n\theta(n\theta-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{n\theta\,(n\theta-\theta)(n\theta-2\theta)(n\theta-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots\cdots$$

Let $n\theta = \alpha$ and supose that n increases without limit and θ decreases in such a way that $n\theta = \alpha$ remains constant, then

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots\cdots$$

Now when $\theta \to 0, \frac{\sin\theta}{\theta} \to 1$ and $\cos\,\theta - 1$

$$.\quad \cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\cdot\infty$$

3. EXPANSION OF $\tan\alpha$ IN TERMS OF α (RADIANS), α BEING SMALL

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots)}{(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots)}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)(1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \cdots)^{-1}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left[1 - (\frac{\alpha^2}{2} - \frac{\alpha^4}{24} + \cdots)\right]^{-1}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left\{1 + (\frac{\alpha^2}{2} - \frac{\alpha^4}{24}) + (\frac{\alpha^2}{2} - \frac{\alpha^4}{24})^2 + \cdots\right\}$$

$$= (\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots)\left(1 + \frac{\alpha^2}{2} + \frac{5\alpha^4}{24} + \cdots\right)$$

$$= \alpha + \frac{\alpha^3}{6} + \frac{2\alpha^5}{15} + \cdots$$

4. EXPANSION OF $\sin \alpha^0$

We know that $1^0 = \frac{\pi}{180}$ radians

. $\alpha^0 = \frac{\pi\alpha}{180}$ radians

Therefore $\sin \alpha^0 = \sin \frac{\pi\alpha}{180}$

$$= \frac{\pi\alpha}{180} - \frac{1}{3!}\left(\frac{\pi\alpha}{180}\right)^3 + \cdots$$

5. EXPANSION OF $\cos \alpha^0$

$$\cos \alpha^0 = \cos \frac{\pi\alpha}{180}$$

$$= 1 - \frac{1}{2!}\left(\frac{\pi\alpha}{180}\right)^2 + \frac{1}{4!}\left(\frac{\pi\alpha}{180}\right)^4 - \cdots$$

Example 1: If $\frac{\sin\theta}{\theta} = \frac{5765}{5766}$, Show that $\theta = 2^0$ approximately.

Solution: Here $\frac{5765}{5766}$ is nearly equal to 1.

$\frac{\sin\theta}{\theta} = 1$ (nearly), and so θ must be very small.

We know that, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta} = \frac{5765}{5766}$$

or $1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots = \frac{5765}{5766}$.

Neglecting powers of θ higher than θ^2, we have

$$1 - \frac{\theta^2}{6} = \frac{5765}{5766}$$

or $\frac{\theta^2}{6} = 1 - \frac{5765}{5766} = \frac{1}{5766}$.

or $\theta^2 = \frac{1}{961}$

or $\theta = \frac{1}{31}$ radians

or $\theta = \frac{1}{31} \times \frac{180}{\pi}$ degrees

or $\theta = \frac{180 \times 7}{31 \times 22}$ degrees

$= 1.84$ degrees

$\theta = 2^0$ nearly.

Example 2: Show that,

$$\sin^3\theta = \frac{3}{4}\left\{\frac{(3^2-1)}{3!}\theta^3 - \frac{(3^4-1)}{5!}\theta^5 + \cdots + \frac{(-1)^{n-1}(3^{2n}-1)}{(2n+1)!}\theta^{2n+1} + \cdots\right\}$$

Solution: We know that, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$= \frac{1}{4}\left[3\left(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5!} - \cdots\right) - \left\{(3\theta) - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \cdots\right\}\right]$$

$$= \frac{1}{4}\left[(3\theta - 3\theta) - \frac{\theta^3}{3!}(3 - 3^3) + \frac{\theta^5}{5!}(3 - 3^5) - \cdots\right]$$

$$= \frac{1}{4}\left[\frac{\theta^3}{3!}(3^3 - 3) - \frac{\theta^5}{5!}(3^5 - 3) + \frac{\theta^7}{7!}(3^7 - 3) - \cdots\right]$$

$$= \frac{3}{4}\left[\frac{\theta^3}{3!}(3^2 - 1) - \frac{\theta^5}{5!}(3^4 - 1) + \frac{\theta^7}{7!}(3^7 - 1) - \cdots\right].$$

Example 3: Prove that,

$$\sin^2\theta\cos\theta = \theta^2 - \frac{5}{6}\theta^4 + \cdots + (-1)^{n+1}\frac{3^{2n}-1}{4}\frac{\theta^{2n}}{(2n)!} + \cdots\infty$$

Solution: We have

$$\sin^2\theta\cos\theta = \frac{1}{2}\sin\theta(2\sin\theta\cos\theta) = \frac{1}{2}\sin\theta\sin2\theta$$

$$= \frac{1}{4}(2\sin\theta\sin2\theta) = \frac{1}{4}(\cos\theta - \cos3\theta)$$

$$= \frac{1}{4}\left[\left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n\frac{\theta^{2n}}{(2n)!} + \cdots\infty\right\} - \left\{1 - \frac{(3\theta)^2}{2!} + \frac{(3\theta)^4}{4!} - \cdots + (-1)^{2n}\frac{(3\theta)^{2n}}{(2n)!} + \cdots\infty\right\}\right]$$

$$= \frac{1}{4}\left[-\frac{\theta^2}{2!}(1-3^2) + \frac{\theta^4}{4!}(1-3^4) - \cdots\infty + (-1)^n\frac{\theta^{2n}}{(2n)!}(1-3^{2n}) + \cdots\infty\right]$$

$$= \frac{1}{4}\left[4\theta^2 - \frac{10}{3}\theta^4 + \cdots + \frac{(-1)^n(-1)(3^{2n}-1)\theta^{2n}}{(2n)!} + \cdots\infty\right]$$

$$= \theta^2 - \frac{5}{6}\theta^4 + \cdots + \frac{(-1)^{n+1}(3^{2n}-1)\theta^{2n}}{4(2n)!} + \cdots\cdots\infty$$

Example 4: If $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots\cdot\infty$

and $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\cdot\infty$, then show that $x^2 = y$.

Solution: Given, $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \cdots\cdot\infty$

$$= \frac{1+1}{1!} - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \cdots\cdot\infty$$

$$= \left(\frac{1}{1!} - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!} + \cdots\cdot\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\cdots\right)$$

$$= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\cdot\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\cdots\right)$$

$$= \cos1 + \sin1$$

and $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} - \frac{2^5}{5!} - \cdots\cdot = 1 + \sin2$

$\therefore\ x^2 = (\cos1 + \sin1)^2 = \cos^2 1 + \sin^2 1 + 2\sin1\cos1$

$= 1 + \sin2 = y$

Thus, $x^2 = y$

Example 5: Find the value of the series: $1-\frac{2}{3!}+\frac{3}{5!}-\frac{4}{7!}\cdots\cdots\infty$

Solution: The given series,

$$1-\frac{2}{3!}+\frac{3}{5!}-\frac{4}{7!}\cdots$$

$$=\frac{1}{2}\left[2-\frac{4}{3!}+\frac{6}{5!}-\frac{8}{7!}\cdots\right]$$

$$=\frac{1}{2}\left[1+1-\frac{3+1}{3!}+\frac{5+1}{5!}-\frac{7+1}{7!}\cdots\right]$$

$$=\frac{1}{2}\left[(1-\frac{3}{3!}+\frac{5}{5!}-\frac{7}{7!}\cdots)+(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots)\right]$$

$$=\frac{1}{2}\left[(1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots)+(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots)\right]$$

$$=\frac{1}{2}(\cos 1+\sin 1)$$

$$=\frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}\cos 1+\frac{1}{\sqrt{2}}\sin 1)$$

$$=\frac{1}{\sqrt{2}}(\sin\frac{\pi}{4}\cos 1+\cos\frac{\pi}{4}\sin 1)$$

$$=\frac{1}{\sqrt{2}}\sin(\frac{\pi}{4}+1).$$

CHECK YOUR PROGRESS

Q.5: If $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, Show that θ is nearly equal to 3^0.

Q.6: If $\cos\theta=\frac{1681}{1682}$,show that θ is nearly equal to $1^0.58^/$.

Q.7: Prove that,

$$\frac{1}{3}\sin^3\theta=\frac{\theta^3}{3!}-(1+3^2)\frac{\theta^5}{5!}+(1+3^2+3^4)\frac{\theta^7}{7!}-\cdots\cdots\infty.$$

Q.8: Prove that,

$$\frac{\pi^2}{2.4}-\frac{\pi^4}{2.4.6.8}+\frac{\pi^6}{2.4.6.8.10.12}-\cdots=1$$

6. LET US SUM UP

- We derive some important deductions from De-Moivre's theorem. These deductions are

a) Expansions of $\cos n\theta$ and $\sin n\theta$ in powers of $\cos\theta$ and $\sin\theta$ (θ being a positive integer).

b) Expansion of $\tan n\theta$ in powers of $\tan\theta$.

c) Expansions of $\sin\theta$ and $\cos\theta$ in series of powers of θ.

d) Expansions of $\sin\theta^0$ and $\cos\theta^0$.

e) Expansion of $\tan\theta$ in powers of θ.

7. ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: We have, $\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8$

(By DeMovire's theorem)

$$= \cos^8\theta + i\ {}^8C_1\cos^7\theta\sin\theta + i^2\ {}^8C_2\cos^6\theta\sin^2\theta + i^3\ {}^8C_3\cos^5\theta\sin^3\theta + \cdots + i^8\sin^8\theta$$

Equating the real parts, we have

$$\cos 8\theta = \cos^8\theta - {}^8C_2\cos^6\theta\sin^2\theta + {}^8C_4\cos^4\theta\sin^4\theta - {}^8C_6\cos^2\theta\sin^6\theta + \sin^8\theta$$

Ans. to Q. No. 2: We have $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$= \cos^5\theta + i5\cos^4\theta\sin\theta + i^2 10\cos^3\theta\sin^2\theta + i^3 10\cos^2\theta\sin^3\theta + i^4.5\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Equating real parts, we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

Ans. to Q. No. 3: We know that

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots\ldots..}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots\ldots..}$$

Now putting $n = 8$, we get the required result.

Ans. to Q. No. 4: First we find expansion of $\sin 8\theta$ in terms of $\sin\theta$ and $\cos\theta$ then dividing both sides by $\sin\theta$. We get the required result.

Ans. to Q. No. 5: Given $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$ which is nearly equal to 1. So that we take θ to be very small. Then in the series for $\sin\theta$, we neglect higher powers of θ.

$$\therefore \sin\theta = \theta - \frac{\theta^3}{3!}$$

$$\text{or } \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6}$$

$$\text{or } \frac{\theta^2}{6} = 1 - \frac{2165}{2166}$$

$$\text{or } \theta^2 = \frac{6}{2166} = \frac{1}{361}$$

$$\text{or } \theta = \frac{1}{19}\text{radian} = 3^0\text{nearly}$$

Ans. to Q. No. 6: Try to solve yourself.

Ans. to Q. No. 7: We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$= \frac{3}{4}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) - \frac{1}{4}\left(3\theta - \frac{3^3\theta^3}{3!} + \frac{3^5\theta^5}{5!} - \frac{3^7\theta^7}{7!} + \cdots\right)$$

$$= \frac{3^3-3}{4.3!}.\theta^3 - \frac{3^5-3}{4.5!}.\theta^5 + \frac{3^7-3}{4.7!}.\theta^7 - \cdots$$

$$= \frac{3}{4}\left[\frac{(3^2-1)}{3!}.\theta^3 - \frac{(3^4-1)}{5!}.\theta^5 + \frac{(3^6-1)}{7!}.\theta^7 - \cdots\right]$$

$$= \frac{3(3^2-1)}{4}\left[\frac{\theta^3}{3!} - (3^2+1)\frac{.\theta^5}{5!} + (3^4+3^2+1)\frac{\theta^7}{7!} - \cdots\right]$$

Hence the result.

Ans. to Q. No. 8: We know that $\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$

If $\alpha = \frac{\pi}{2}$, we have $\cos\frac{\pi}{2} = 1 - \frac{\pi^2}{2^2 2!} + \frac{\pi^4}{2^4 4!} - \frac{\pi^6}{2^6 6!} + \cdots$

$$\text{or } 0 = 1 - \frac{\pi^2}{2.4} + \frac{\pi^4}{2.4.6.8} - \frac{\pi^6}{2.4.6.8.10.12} + \cdots$$

$$\because 2^2(2!) = (2.2)(1.2) = 2.4$$

$$2^4 4! = (2.2.2.2)(4.3.2.1) = 2.4.6.8$$

or $\dfrac{\pi^2}{2.4} - \dfrac{\pi^4}{2.4.6.8} + \dfrac{\pi^6}{2.4.6.8.10.12} - \cdots = 1$

Hence the result.

8. MODEL QUESTIONS

Q.1: Prove that,

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

Q.2: Prove that,

$$\tan 8\theta = \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}.$$

Q.3: If $\dfrac{\sin\theta}{\theta} = \dfrac{863}{864}$, Show that θ is nearly $4^0 47'$.

$\dfrac{1}{15}$, Show that θ is nearly $\dfrac{42}{11}$ degrees.

UNIT 3 . HYPERBOLIC FUNCTIONS

UNIT STUCTURE

1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain about Complex hyperbolic functions
- know about Euler's formula.

2 INTRODUCTION

In this unit, we will learn about complex trigonometric and hyperbolic functions. Also, we will discuss Euler's theorem and also its some important deductions.

3 COMPLEX EXPONENTIAL FUNCTIONS

We know that if x is any real number, then

a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\infty$

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\infty$

c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\infty$

Now, for the complex number $z = x + iy$ where x and y are real we can define–

i) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\infty$

ii) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\infty$

iii) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\infty$

The other trigonometric functions for a complex quantity are defined as follows:

a) $\tan z = \frac{\sin z}{\cos z}$

b) $\cot z = \frac{\cos z}{\sin z}$

c) $\operatorname{cosec} z = \frac{1}{\sin z}$

d) $\sec z = \frac{1}{\cos z}$

4 EULER'S FORMULA

Statement: If θ be real or complex, we have $e^{i\theta} = \cos\theta + i\sin\theta$.

Proof: If z is any complex number,then by definition of e^z ,we have

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\infty$$

Replacing z by $i\theta$, we have

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

$= \cos\theta + i\sin\theta$ by definitions of $\cos\theta$ and $\sin\theta$.

Thus $e^{i\theta} = \cos\theta + i\sin\theta$.

Important Deductions from Euler's Formula:

i) $e^{-i\theta} = \cos\theta - i\sin\theta$

Proof: We have

$$e^{-i\theta} = 1 + (-i\theta) + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!} + \frac{(-i\theta)^5}{5!} + \cdots$$

$$= 1 - i\theta + \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \frac{i\theta^5}{5!} + \cdots$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots) - i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

$$= \cos\theta - i\sin\theta.$$

ii) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Proof: We have $e^{i\theta} = \cos\theta + i\sin\theta$ (1)

and $e^{-i\theta} = \cos\theta - i\sin\theta$ (2)

Adding (1) and (2), we get $2\cos\theta = e^{i\theta} + e^{-i\theta}$

So, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ (3)

iii) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Proof: Subtracting (2) from (1), we get $2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$\Rightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ (4)

Formulae (3) and (4) are known as ***Euler's Exponential Values*** for $\cos\theta$ and $\sin\theta$ respectively.

Corollary:

i) $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{e^{i\theta} - e^{-i\theta}}{2i}}{\frac{e^{i\theta} + e^{-i\theta}}{2}} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

ii) $$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{e^{i\theta} + e^{-i\theta}}{2}}{\frac{e^{i\theta} - e^{-i\theta}}{2i}} = \frac{i(e^{i\theta} + e^{-i\theta})}{(e^{i\theta} - e^{-i\theta})}$$

iii) $$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{e^{i\theta} + e^{-i\theta}}{2}} = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

iv) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\dfrac{e^{i\theta} - e^{-i\theta}}{2i}} = \dfrac{2i}{e^{i\theta} - e^{-i\theta}}$.

11.4.1 Some Standard Trigonometrical Formulae for Complex Quantities

All the trigonometrical formulae which are known to be true for real values can also be shown to be true for complex quantities also.

i) $\sin^2 z + \cos^2 z = 1$

Proof: L.H.S = $\sin^2 z + \cos^2 z$

$$= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$

$$= \left(\frac{e^{2iz} + e^{-2iz} - 2e^{2iz}.e^{-2iz}}{4i^2}\right) + \left(\frac{e^{2iz} + e^{-2iz} + 2e^{2iz}.e^{-2iz}}{4}\right)$$

$$= \left(\frac{e^{2iz} + e^{-2iz} - 2e^{2iz}.e^{-2iz}}{4(-1)}\right) + \left(\frac{e^{2iz} + e^{-2iz} + 2e^{2iz}.e^{-2iz}}{4}\right)$$

$$= -\left(\frac{e^{2iz} + e^{-2iz} - 2}{4}\right) + \left(\frac{e^{2iz} + e^{-2iz} + 2}{4}\right)$$

$$= \frac{-e^{2iz} - e^{-2iz} + 2 + e^{2iz} + e^{-2iz} + 2}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

ii) $\sin 2z = 2\sin z\cos z$

Proof: R.H.S, $2\sin z\cos z$

$$= 2.\frac{e^{iz} - e^{-iz}}{2i}.\frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{2iz} - e^{-2iz}}{2i}$$

$$= \sin 2z \text{ = L.H.S}$$

iii) $\cos 2z = 1 - 2\sin^2 z$

Proof: R.H.S = $1 - 2\sin^2 z$

$$= 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= 1 - 2\left(\frac{e^{2iz} + e^{-2iz} - 2}{-4}\right)$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z = \text{L.H.S}$$

iv) $\cos 3z = 4\cos^3 z - 3\cos z$

Proof: R.H.S $= 4\cos^3 z - 3\cos z$

$$= 4\left(\frac{e^{iz} + e^{-iz}}{2}\right)^3 - 3\left(\frac{e^{iz} + e^{-iz}}{2}\right)$$

$$= \frac{1}{2}\left(e^{iz} + e^{-iz}\right)^3 - \frac{3}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \frac{1}{2}\left[\left(e^{3iz} + e^{-3iz}\right) + 3e^{iz}.e^{-iz}\left(e^{iz} + e^{-iz}\right)\right] - \frac{3}{2}\left(e^{iz} + e^{-iz}\right)$$

(Using $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$)

$$= \frac{e^{3iz} + e^{-3iz}}{2} + 3\frac{e^{iz} + e^{-iz}}{2} - \frac{3}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \frac{e^{3iz} + e^{-3iz}}{2} = \cos 3z = \text{L.H.S}$$

v) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

Proof: R.H.S = $\sin z_1 \cos z_2 + \sin z_2 \cos z_1$

$$= \frac{e^{iz_1} - e^{-iz_1}}{2i}.\frac{e^{iz_2} + e^{-iz_2}}{2} + .\frac{e^{iz_1} + e^{-iz_1}}{2}.\frac{e^{iz_2} - e^{-iz_2}}{2i}$$

$$= \frac{1}{4i}\left[\left(e^{iz_1} - e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right) + \left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} - e^{-iz_2}\right)\right]$$

$$= \frac{1}{4i}\left[e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)} + e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}\right]$$

$$= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)}\right]$$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1 + z_2) = \text{L.H.S}$$

Similarly, one can prove the following results:

vi) $\sin 3z = 3\sin z - 4\sin^3 z$

vii) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

viii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

ix) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

5 COMPLEX HYPERBOLIC FUNCTIONS

For all values of complex variable z, the expressions

i) $\dfrac{e^z - e^{-z}}{2}$ is called Hyperbolic sine of z and is denoted as sinh z.

ii) $\dfrac{e^z + e^{-z}}{2}$ is called Hyperbolic cosine of z and is denoted as cosh z.

Thus, $\sinh z = \dfrac{e^z - e^{-z}}{2}$ and $\cosh z = \dfrac{e^z + e^{-z}}{2}$

The other hyperbolic functions are defined in terms of hyperbolic sine and cosine as follows:

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{\frac{e^z - e^{-z}}{2}}{\frac{e^z + e^{-z}}{2}} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{\frac{e^z + e^{-z}}{2}}{\frac{e^z - e^{-z}}{2}} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{1}{\frac{e^z + e^{-z}}{2}} = \frac{2}{e^z + e^{-z}}$$

$$\operatorname{cosech} z = \frac{1}{\sinh z} = \frac{1}{\frac{e^z - e^{-z}}{2}} = \frac{2}{e^z - e^{-z}}$$

Note : (a) $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$ (b) $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$

.5.1 Expansions of sinhz and coshz in Powers of z

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\infty\right) - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots\infty\right)\right]$$

$$= \frac{1}{2}\left[2z + 2.\frac{z^3}{3!} + 2.\frac{z^5}{5!} + \cdots\right]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Similarly, $\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) + \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[2 + 2.\frac{z^2}{2!} + 2.\frac{z^4}{4!} + \cdots\right]$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

5.2 Relations between Hyperbolic and Circular Functions

The hyperbolic functions can be expressed in terms of circular functions as follows:

We have, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Putting $\theta = ix$ in these relations,we get

$$= \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{-(e^{x} - e^{-x})}{2i}$$

$$= \frac{i^2\left(e^{x} - e^{-x}\right)}{2i}$$

$$= i\left(\frac{e^{x} - e^{-x}}{2}\right) = i\sinh x$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2} = \cosh x$$

Again, $\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x$

$$\cot(ix) = \frac{\cos(ix)}{\sin(ix)} = \frac{\cosh x}{i\sinh x} = \frac{i\cosh x}{i^2 \sinh x} = -i\coth x$$

$$\sec(ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \sec hx$$

$$\cos ec(ix) = \frac{1}{\sin(ix)} = \frac{1}{i\sinh x} = \frac{i}{i^2 \sinh x} = -i\cos echx$$

Note: We have $\sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

Putting $\theta = ix$ in these relations,we get

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i\frac{e^{ix} - e^{-ix}}{2i} = i\sin x$$

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

Again, $\tanh(ix) = \frac{\sinh(ix)}{\cosh(ix)} = \frac{i\sin x}{\cos x} = i\tan x$

.5.3 Some Important Formulas for Hyperbolic Functions

i) $\cosh^2 x - \sinh^2 x = 1$

Proof: L.H.S, $\cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{4}{4}$$

= 1

= R.H.S

ii) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

Proof: We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\cosh^2 x$,we get

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\Rightarrow \operatorname{sech}^2 x = 1 - \tanh^2 x$

iii) $\operatorname{cosech}^2 x = 1 - \coth^2 x$

Proof: We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\sinh^2 x$,we get

$\coth^2 x - 1 = \operatorname{cosech}^2 x$

So, $\operatorname{cosech}^2 x = 1 - \coth^2 x$

iv) $\sinh 2x = 2\sinh x \cosh x$

Proof: R.H.S $= 2\sinh x \cosh x$

$$= 2.\frac{e^x - e^{-x}}{2}.\frac{e^x + e^{-x}}{2}$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x =$ L.H.S.

v) $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \cosh^2 x + \sinh^2 x$

Proof: We have, $\cosh^2 x + \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$= \cosh 2x$

Again, $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= \cosh^2 x + (\cosh^2 x - 1)$ $\qquad \because \cosh^2 x - \sinh^2 x = 1$

$= 2\cosh^2 x - 1$

Also, $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= (1 + \sinh^2 x) + \sinh^2 x$

$= 1 + 2\sinh^2 x$ $\qquad \because \cosh^2 x - \sinh^2 x = 1$

Thus,we have $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

$= \cosh^2 x + \sinh^2 x$

vi) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Proof: L.H.S = $\sinh(x+y)$

$$= \frac{1}{i}\sinh i(x+y) \qquad \Theta \sinh\theta = \frac{1}{i}\sin(i\theta)$$

$$= \frac{1}{i}\sin(ix+iy)$$

$$= \frac{1}{i}[\sin ix \cos iy + \cos ix \sin iy]$$

$$= \frac{1}{i}[i \sinh x \cosh y + \cosh x i \sinh y]$$

$$= \sinh x \cosh y + \cosh x \sinh y = \text{R.H.S.}$$

vii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

Proof: Try yourself

viii) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

Proof: L.H.S, $\cosh(x-y)$

$$= i\cos(x-y) \qquad \Theta \cosh\theta = \cos(i\theta)$$

$$= \cos i(x-y)$$

$$= \cos(ix-iy)$$

$$= \cos ix \cos iy + \sin ix \sin iy$$

$$= \cosh x \cosh y + i \sinh x i \sinh y$$

$$= \cosh x \cosh y - \sinh x \sinh y = \text{R.H.S}$$

ix) $\cosh(x+y) = \cosh x \cosh y - \sinh x \sinh y$

Proof: Try yourself

x) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1+\tanh x \tanh y}$

Proof: L.H.S, $\tanh(x+y)$

$$= \frac{\sinh(x+y)}{\cosh(x+y)}$$

$$= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$$

Dividing the numerator and denominator by $\cosh x \cosh y$, we get

$$= \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$$

xi) $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1-\tanh x \tanh y}$

Proof: Try yourself

Note: Some more formulas for hyperbolic functions

a) $\sinh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

b) $\cosh 2x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

c) $\tanh 2x = \dfrac{2\tanh x}{1 - \tanh^2 x}$

d) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

e) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

f) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

5.4 Perodicity of Hyperbolic Functions

a) We know that $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\therefore \sinh(x + 2n\pi i) = \dfrac{e^{x+2n\pi i} - e^{-(x+2n\pi i)}}{2}$, where n is any integer

$= \dfrac{1}{2}\left[e^x . e^{2n\pi i} - e^{-x} . e^{-2n\pi i}\right]$

$= \dfrac{1}{2}\left[e^x .1 - e^{-x} .1\right] \quad \because e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi$

$= \dfrac{e^x - e^{-x}}{2}$

$= 1 + i(0)$

$= \sinh x \quad = 1$

$\because \ e^{-2n\pi i} = \cos 2n\pi - i \sin 2n\pi$

$= 1 - i.0 \quad = 1$

Thus, $\sinh x$ remains unchanged, when x is increased by an integral multiple of $2\pi i$.

Hence, $\sinh x$ is a periodic function and its period is $2\pi i$.

b) We know that $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\therefore \cosh(x + 2n\pi i) = \dfrac{e^{x+2n\pi i} + e^{-(x+2n\pi i)}}{2}$

$$= \frac{e^x.e^{2n\pi i} + e^{-x}.e^{-2n\pi i}}{2}$$

$$= \frac{e^x.1 + e^{-x}.1}{2}$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Thus, $\cosh x$ remains unchanged,when x is increased by any integral multiple of $2\pi i$.

Hence, $\cosh x$ is a periodic function and its period of $2\pi i$.

c) We know that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\therefore \tanh(x + n\pi i) = \frac{e^{x+n\pi i} - e^{-(x+n\pi i)}}{e^{x+n\pi i} + e^{-(x+n\pi i)}}$$

$$= \frac{e^x.e^{n\pi i} - e^{-x}.e^{-n\pi i}}{e^x.e^{n\pi i} + e^{-x}.e^{-n\pi i}},$$

Multiplying the numerator and denominator by $e^{n\pi i}$, we get

$$= \frac{e^x.e^{2n\pi i} - e^{-x}.1}{e^x.e^{2n\pi i} + e^x.1}$$

$$= \frac{e^x - e^{-x}}{e^x + e^x}$$

$$= \tanh x$$

Thus, $\tanh x$ remains unchanged,when x is increased by an integral multiple of πi.

Hence, $\tanh x$ is a periodic function and its period is πi.

Note1: $\operatorname{cosech} x, \operatorname{sech} x$ and $\coth x$ being reciprocals of $\sinh x, \cosh x$ and $\tanh x$ respectively,are also periodic functions and their periods are respectively $2\pi i$, $2\pi i$ and πi.

Note 2: The periods of hyperbolic functions are imaginary.

Example 1: Separate the following into real and imaginary parts

a) $\sinh(x + iy)$ b) $\cosh(x + iy)$ c) $\tanh(x + iy)$

Solution: (a) We know that $i \sinh\theta = \sin i\theta$

$$\sinh(x + iy) = \frac{1}{i} \sin i(x + iy)$$

$$= \frac{i}{i^2} \sin(ix - y)$$

$$= -i(\sin ix \cos y - \cos ix \sin y)$$

$$= -i(i \sinh x \cos y - \cosh x \sin y)$$

$$= \sinh x \cos y + i \cosh x \sin y$$

where real part $= \sinh x \cos y$ and Imaginary part $= \cosh x \sin y$.

b) We know that $\cosh\theta = \cos i\theta$

$$\cosh(x + iy) = \cos i(x - iy) = \cos(ix - y)$$

$$= \cos ix \cos y + \sin ix \sin y$$

$$= \cosh x \cos y + i \sinh x \sin y$$

where real part $= \cosh x \cos y$ and Imaginary part $= \sinh x \sin y$

c) We have $\tanh(x + iy) = \frac{1}{i} \tan i(x + iy)$ since $i \tanh\theta = \tan i\theta$

$$= -i \frac{\sin(ix - y)}{\cos(ix - y)}$$

$$= -i \frac{\sin 2ix - \sin 2y}{\cos 2ix + \cos 2y}$$

$$= -i \frac{i \sinh 2x - \sin 2y}{\cosh 2x + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i . \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

Example 2: Show that $\frac{1 + \tanh x}{1 - \tanh x} = \cosh 2x + \sinh x$

Solution: L.H.S $= \frac{1 + \tanh x}{1 - \tanh x} = \frac{1 + \frac{\sinh x}{\cosh x}}{1 - \frac{\sinh x}{\cosh x}}$

$$= \frac{\cosh x + \sinh x}{\cosh x - \sinh x}$$

$$= \frac{\cosh x + \sinh x}{\cosh x - \sinh x} \times \frac{\cosh x + \sinh x}{\cosh x + \sinh x}$$

$$= \frac{\cosh^2 x + \sinh^2 x + 2 \sinh x \cosh x}{\cosh^2 x - \sinh^2 x}$$

$$= \frac{\cosh 2x + \sinh 2x}{1}$$

$$= \cosh 2x + \sinh 2x = \text{R.H.S}$$

Example 3: Show that,

$$(1+\cosh x+\sinh x)^n = 2^n \cosh^n\left(\frac{x}{2}\right)\left\{\cosh\left(\frac{nx}{2}\right)+\sinh\left(\frac{nx}{2}\right)\right\}$$

Solution: $\text{L.H.S} = (1+\cosh x+\sinh x)^n$

$$= \left[1+\cos(ix)+\frac{1}{i}\sin(ix)\right]^n$$

$$\Theta \cosh x = \cos(ix) \text{ and } \sinh x = \frac{1}{i}\sin(ix)$$

$$= \left[1+\left\{2\cos^2\left(\frac{ix}{2}\right)-1\right\}+\left(\frac{1}{i}\right)2\sin\left(\frac{ix}{2}\right)\cos\left(\frac{ix}{2}\right)\right]^n$$

$\Theta \cos x = 2\cos^2 x - 1$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= \left[1+\left\{2\cosh^2\left(\frac{x}{2}\right)-1\right\}+\left(\frac{1}{i}\right)2\left(i\sinh\frac{x}{2}\right)\cosh\frac{x}{2}\right]^n$$

$$= \left[2\cosh^2\left(\frac{x}{2}\right)+2\sinh\frac{x}{2}\cosh\frac{x}{2}\right]^n$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cosh\left(\frac{x}{2}\right)+\sinh\left(\frac{x}{2}\right)\right]^n$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cos\left(\frac{ix}{2}\right)+\frac{1}{i}\sin\left(\frac{ix}{2}\right)\right]^n$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cos\left(\frac{ix}{2}\right)+\frac{i}{i^2}\sin\left(\frac{ix}{2}\right)\right]^n$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cos\left(\frac{ix}{2}\right)-i\sin\left(\frac{ix}{2}\right)\right]^n$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cos\left(\frac{inx}{2}\right)-i\sin\left(\frac{inx}{2}\right)\right] \text{ (By DeMovire's theorem)}$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cosh\left(\frac{nx}{2}\right)-i\left\{i\sinh\left(\frac{nx}{2}\right)\right\}\right]$$

$$= 2^n \cosh^n\left(\frac{x}{2}\right)\left[\cosh\left(\frac{nx}{2}\right)+\sinh\left(\frac{nx}{2}\right)\right] = \text{R.H.S}$$

Example 4: If $(x+iy)=\tan(A+iB)$, Prove that

i) $x^2+y^2+2x\cot 2A=1$

ii) $x^2+y^2-2y\coth 2B+1=0$

iii) $x\cot 2A+y\coth 2B=1$

Solution: We have $x+iy=\tan(A+iB)$ (a)

So, $x-iy=\tan(A-iB)$ (b)

i) Now, $\tan 2A=\tan[(A+iB)+(A-iB)]$

$$=\frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$=\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$

$$=\frac{2x}{1-x^2-y^2}$$

So, $\frac{1}{\cot 2A}=\frac{2x}{1-x^2-y^2}$

Or, $1-x^2-y^2=2x\cot 2A$

$\therefore\ 2x\cot 2A=1-x^2-y^2$ (1)

ii) $\tan 2iB=\tan[(A+iB)-(A-iB)]$

$$=\frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$=\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$=\frac{2iy}{1+x^2+y^2}$$

So, $i\tanh 2B=\frac{2iy}{1+x^2+y^2}$

$\therefore \tanh 2B=\frac{2y}{1+x^2+y^2}$

So, $\frac{1}{\coth 2B}=\frac{2y}{1+x^2+y^2}$

$\therefore\ 1+x^2+y^2=2y\coth 2B$ (2)

iii) From (1), $2x\cot 2A=1-x^2-y^2$

From (2), $2y\coth 2B=1+x^2+y^2$

Adding,we get $2x\cot 2A+2y\coth 2B=2$

Or, $x\cot 2A+y\coth 2B=1$

5 Inverse Hyperbolic Functions

Let $\sinh x = u$. Then x is said to be the inverse hyperbolic sine of u. It is denoted by $\sinh^{-1} u$. Thus, if $\sinh x = u$, then $x = \sinh^{-1} u$. In this way we define $\cosh^{-1} u$, $\tanh^{-1} u$, and so on.

To prove that, $\sinh^{-1} u = \log\left(u + \sqrt{1+u^2}\right)$

Let $\sinh x = u$

$\therefore\ x = \sinh^{-1} u$

Also $u = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$\therefore 2u = e^x - \dfrac{1}{e^x}$

Or, $e^{2x} - 2ue^x - 1 = 0$

$\therefore e^x = \dfrac{2u \pm \sqrt{4u^2+4}}{2} = u \pm \sqrt{u^2+1}$

$\therefore x = 2n\pi i + \log\left(u \pm \sqrt{u^2+1}\right)$

Now, $\log\left(u - \sqrt{u^2+1}\right) = \log \dfrac{\left(u - \sqrt{u^2+1}\right)\left(u + \sqrt{u^2+1}\right)}{\left(u + \sqrt{u^2+1}\right)}$

$$= \log \frac{-1}{\left(u + \sqrt{u^2+1}\right)}$$

$$= \log(-1) - \log\left(u + \sqrt{u^2+1}\right)$$

$$\therefore x \begin{cases} = 2n\pi i + \log\left(u + \sqrt{u^2+1}\right) \\ \text{or } 2n\pi i + (2m-1)\pi i - \log\left(u + \sqrt{u^2+1}\right) \end{cases}$$

$\therefore$ the general value of $\sinh^{-1} u$ is $r\pi + (-1)^r \log\left(u + \sqrt{u^2+1}\right)$ where r is any integer; and the principal value is $\log\left(u + \sqrt{u^2+1}\right)$.

To prove that, $\cosh^{-1} u = \log\left(u + \sqrt{u^2-1}\right)$

et $\cosh x = u$

$\therefore\ x = \cosh^{-1} u$

Also $u = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$\therefore 2u = e^x + \dfrac{1}{e^x}$

Or, $e^{2x} - 2ue^x + 1 = 0$

$$\therefore e^x = \frac{2u \pm \sqrt{4u^2 - 4}}{2} = u \pm \sqrt{u^2 - 1}$$

$$\therefore x = 2n\pi i + \log\left(u \pm \sqrt{u^2 - 1}\right)$$

Now, $\log\left(u - \sqrt{u^2 - 1}\right) = \log \dfrac{\left(u - \sqrt{u^2 - 1}\right)\left(u + \sqrt{u^2 - 1}\right)}{\left(u + \sqrt{u^2 - 1}\right)}$

$$= \log \frac{1}{\left(u + \sqrt{u^2 - 1}\right)}$$

$$= -\log\left(u + \sqrt{u^2 + 1}\right)$$

$$\therefore x = 2n\pi i + \log\left(u + \sqrt{u^2 - 1}\right)$$

. the general value of $\cosh^{-1} u$ is $2n\pi i \pm \log\left(u + \sqrt{u^2 + 1}\right)$, where n is any integer; and the principal value is $\log\left(u + \sqrt{u^2 + 1}\right)$.

CHECK YOUR PROGRESS

Q.1: Separate the following into real and imaginary parts–

a) $\coth(x + iy)$ b) $\operatorname{sech}(x + iy)$

c) $\operatorname{cosech}(x + iy)$

Q.2: If $\tan\theta = \tanh x \cot y$ and $\tan\varphi = \tanh x \tan y$,

prove that $\dfrac{\sin 2\theta}{\sin 2\varphi} = \dfrac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$

Q.3: If $u + iv = \cot(x + iy)$, show that,

$$v = -\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

6 LET US SUM UP

- For the complex number $z = x + iy$ where x and y are real we can define–

a) $e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\cdots\infty$

b) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\cdots\infty$

c) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\infty$ etc.

- The complex hyperbolic sine and hyperbolic cosine functions are defined by:

  $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$.

  The complex hyperbolic tangent,cotangent,secant,and cosecant are defined in terms of $\sinh z$ and $\cosh z$:

  $\tanh z = \frac{\sinh z}{\cosh z}$, $\coth z = \frac{\cosh z}{\sinh z}$ $\operatorname{sech} z = \frac{1}{\cosh z}$ and $\operatorname{cosech} z = \frac{1}{\sinh z}$.

- The relation between Hyperbolic and Trigonometric functions are as follows : $\sin(ix) = i\sinh x$, $\cos(ix) = \cosh x$, $\tan(ix) = i\tanh x$ etc.

- We discuss some important formula for hyperbolic functions:

  a) $\cosh^2 x - \sinh^2 x = 1$ b) $\sec h^2 x = 1 - \tanh^2 x$

  c) $\operatorname{cos\,ech}^2 x = \coth^2 x - 1$ d) $\sinh 2x = 2\sinh x \cosh x$

  e) $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \cosh^2 x + \sinh^2 x$ etc.

- We discuss periodicity of hyperbolic function.i.e., $\sinh x, \cosh x$ and $\tanh x$ respectively,are periodic functions and their periods are respectively $2\pi i$, $2\pi i$ and πi. Their reciprocals $\operatorname{cosech} x, \operatorname{sech} x$ and $\coth x$ respectively, are also periodic functions and their periods are respectively $2\pi i$, $2\pi i$ and πi.

- If θ be real or complex, by Euler's theorem we have $e^{i\theta} = \cos\theta + i\sin\theta$.

- We derived important deductions from Euler's theorem.

7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) we have $\coth(x+iy) = \frac{\cosh(x+iy)}{\sinh(x+iy)}$

$$= \frac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)} = i.\frac{\cos(ix-y)}{\sin(ix-y)}$$

$$= i.\frac{2\sin(ix+y)\cos(ix-y)}{2\sin(ix+y)\sin(ix-y)} = i.\frac{\sin 2ix + \sin 2y}{\cos 2y - \cos 2ix} = i.\frac{i\sinh 2x + \sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{-\sinh 2x}{\cos 2y - \cosh 2x} + i.\frac{\sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{\sinh 2x}{\cosh 2x - \cos 2y} - i.\frac{\sin 2y}{\cosh 2x - \cos 2y}$$

b) we have $\operatorname{sech}(x + iy) = \frac{1}{\cosh(x + iy)} = \frac{1}{\cos i(x + iy)} = \frac{1}{\cos(ix - y)}$

$$= \frac{2\cos(ix + y)}{2\cos(ix + y)\cos(ix - y)} = \frac{2(\cos ix \cos y - \sin ix \sin y)}{\cos 2ix + \cos 2y} = \frac{2(\cosh x \cos y - i \sinh x \sin y)}{\cosh 2x + \cos 2y}$$

$$= \frac{2\cosh x \cos y}{\cosh 2x + \cos 2y} - i.\frac{2\sinh x \sin y}{\cosh 2x + \cos 2y}$$

c) We have

$$\operatorname{cosech}(x + iy) = \frac{1}{\sinh(x + iy)} = \frac{1}{\frac{1}{i}\sin i(x + iy)} = \frac{i}{\sin(ix - y)}$$

$$= i.\frac{2\sin(ix + y)}{2\sin(ix + y)\sin(ix - y)} = i.\frac{2(\sin ix \cos y + \cos ix \sin y)}{\cos 2y - \cos 2ix} = i.\frac{2(i \sinh x \cos y + \cosh x \sin y)}{\cos 2y - \cosh 2x}$$

$$= -\frac{2\sinh x \cos y}{\cos 2y - \cosh 2x} - i.\frac{2\sinh x \sin y}{\cos 2y - \cosh 2x}$$

$$= \frac{2\sinh x \cos y}{\cosh 2x - \cos 2y} - i\frac{2\sinh x \sin y}{\cosh 2x - \cos 2y}$$

Ans. to Q. No. 2: We have $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

$$= \frac{2\tanh x \cot y}{1 + \tanh^2 x \cot^2 y}$$

$$= \frac{2\frac{\sinh x}{\cosh x}\frac{\cos y}{\sin y}}{1 + \frac{\sinh^2 x}{\cosh^2 x}\frac{\cos^2 y}{\sin^2 y}} = \frac{2\sinh x \cosh x \sin y \cos y}{\cosh^2 x \sin^2 y + \sinh^2 x \cos^2 y}$$

$$= \frac{\frac{1}{2}.(2\sinh x \cosh x)(2\sin y \cos y)}{\frac{\cosh 2x + 1}{2}.\frac{1 - \cos 2y}{2} + \frac{\cosh 2x - 1}{2}.\frac{1 + \cos 2y}{2}}$$

$$= \frac{2\sinh 2x \sin 2y}{\cosh 2x - 2\cos 2y} = \frac{\sinh 2x \sin 2y}{\cosh 2x - \cos 2y} \qquad (1)$$

Similarly, we can prove that $\sin 2\varphi = \frac{\sinh 2x \sin 2y}{\cosh 2x + \cos 2y}$ (2)

Dividing (1) by (2),we get $\dfrac{\sin 2\theta}{\sin 2\varphi} = \dfrac{\cosh 2x - \cos 2y}{\cosh 2x - \cos 2y}$. Hence proved.

Ans. to Q. No. 3: We have $u + iv = \cot(x + iy)$

$$= \frac{\cos(x + iy)}{\sin(x+iy)} = \frac{2\cos(x - iy)\sin(x - iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin(2x) - \sin(2iy)}{\cos(2iy) - \cos(2x)} = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x} - i\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

Equating imaginary parts on both sides, we get

$v = -\dfrac{\sinh 2y}{\cosh 2y - \cos 2x}$. Hence the result.

9 MODEL QUESTIONS

Q.1: Separate into real and imaginary parts

a) $\sec(x + iy)$ b) $\operatorname{cosec}(x + iy)$

c) $e^{\sin(x+iy)}$ d) $\sin^2(x+iy)$

Q.2: Prove that, $\tan\left(\dfrac{u+iv}{2}\right) = \dfrac{\sin u + i\sinh v}{\cos u + i\cosh v}$

Q.3: If $\cosh(\alpha + i\beta) = x + iy$, show that

a) $\dfrac{x^2}{\cosh^2\alpha} + \dfrac{y^2}{\sinh^2\alpha} = 1$ b) $\dfrac{x^2}{\cos^2\beta} - \dfrac{y^2}{\sin^2\beta} = 1$

Q.4: $\sin(\theta + i\varphi)\sin(\alpha + i\beta) = 1$, Prove that

$\cosh^2\varphi\tanh^2\beta = \cos^2\theta$ and $\tanh^2\varphi\cosh^2\beta = \cos^2\alpha$.

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UNIT 4 : LOGARITHM OF COMPLEX NUMBER

UNIT STUCTURE

1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know logarithm of complex quantities
- find $\log_e(x+iy)$ and $\text{Log}_e(x+iy)$.

2 INTRODUCTION

In this unit, we will discuss the definition of logarithm of a complex number using the fundamental concepts of logarithm of a real function.

3 LOGARITHM OF COMPLEX QUANTITIES

We know that if a and x are real numbers such that $e^x = a$ (1)

then x is called the logarithm of a to the base e.

we can write (1) as $x = \log_e a$

Now, we will discuss logarithm of complex numbers.

If z and w are two complex numbers such that $w = e^z$ (2)

then z is called the logarithm of w to the base e.

we can write (2) as $z = \log_e w$.

Therefore $z = \log_e w$ if and only if $w = e^z$.

Note: Logarithm of a complex number is a complex number.

Theorem: Logarithm of a complex number is a many-valued function.

Proof: Let $\log_e w = z$. Then $e^z = w$.

We know that $e^{2n\pi i} = \cos 2n\pi + i\sin 2n\pi = 1 + i0 = 1$,

where n is any integer.

$\therefore \quad e^z = e^z.1 = e^z.e^{2n\pi i} = e^{z+2n\pi i}$

Now, $w = e^z = e^{z+2n\pi i}$

By definition, we have $\log_e w = z + 2n\pi i$

Hence, the logarithm of a complex number has infinite values.

4 PRINCIPAL AND GENERAL VALUES OF LOGARITHM

We have $\log_e w = z + 2n\pi i$, the value $z + 2n\pi i$ is called the general value of $\log_e w$ and is denoted by $\mathrm{Log}_e w$.

Thus, $\mathrm{Log}_e w = z + 2n\pi i = 2n\pi i + \log_e w$

If $w = x + iy$, then $\mathrm{Log}_e(x+iy) = 2n\pi i + \log_e(x+iy)$ (3)

If we put n = 0 in (3), we get the principal value of z, i.e., $\log_e w$.

Note: We denote the general value of $\log_e w$ by $\mathrm{Log}_e w$, using the first letter L as the capital and the principal value by $\log_e w$, using the first letter l as the small letter.

5 TO FIND $\log_e(x+iy)$ AND $\mathrm{Log}_e(x+iy)$

Let $z = x + iy$ be a non-zero complex number.

Suppose $\log_e z = \alpha + i\beta = \log_e(x+iy) = \alpha + i\beta$ (4)

$\therefore \; x + iy = e^{\alpha + i\beta}$

$$= e^{\alpha}.e^{i\beta}$$

$$= e^{\alpha}(\cos\beta + i\sin\beta)$$

$\Rightarrow x + iy = e^{\alpha}\cos\beta + ie^{\alpha}\sin\beta$

Equating real and imaginary parts from both sides,we get

$$x = e^{\alpha}\cos\beta \quad (5)$$

$$y = e^{\alpha}\sin\beta \quad (6)$$

Squaring and adding (5) & (6), we get

$$x^2+y^2=e^{2\alpha}\cos^2\beta+e^{2\alpha}\sin^2\beta \;=e^{2\alpha}\left(\cos^2\beta+\sin^2\beta\right)$$

$$\Rightarrow x^2+y^2=e^{2\alpha}$$

$$\Rightarrow 2\alpha=\log_e\left(x^2+y^2\right)$$

$$\Rightarrow \alpha=\frac{1}{2}\log_e\left(x^2+y^2\right)$$

Dividing (6) by (5), we get $\dfrac{y}{x}=\dfrac{e^{\alpha}\sin\beta}{e^{\alpha}\cos\beta} \;\Rightarrow \dfrac{y}{x}=\dfrac{\sin\beta}{\cos\beta}=\tan\beta$

$$\therefore \beta=\tan^{-1}\frac{y}{x}$$

From (4), we get $\log_e(x+iy)=\dfrac{1}{2}\log_e\left(x^2+y^2\right)+i\tan^{-1}\dfrac{y}{x}$ (7)

Now we find the value of $\mathrm{Log}_e(x-iy)$

We have $\mathrm{Log}_e(x+iy)=2n\pi i+\log_e(x-iy)$ from (3)

$$=2n\pi i+\frac{1}{2}\log_e\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}$$

$$\therefore \mathrm{Log}_e(x+iy)=\frac{1}{2}\log_e\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}+2n\pi i$$

where n is any integer.

6 PROPERTIES OF LOGARITHM

We know some properties of logarithm of real numbers.Now we state some properties for complex numbers.The properties are as follows:

i) $\mathrm{Log}_e(zw)=\mathrm{Log}_e z-\mathrm{Log}_e w$

ii) $\mathrm{Log}_e\left(\dfrac{z}{w}\right)=\mathrm{Log}_e z-\mathrm{Log}_e w$

iii) $\mathrm{Log}_e z^{w}=w\log_e z+2n\pi i$

iv) $\mathrm{Log}_w z=\dfrac{\mathrm{Log}_e z}{\mathrm{Log}_e w}$

Logarithm of a Negative Number:

We know that $e^{(2n+1)\pi i}=\cos(2n+1)\pi+i\sin(2n+1)\pi=-1$

. $\mathrm{Log}(-1)=(2n+1)\pi i$ and $\log(-1)=\pi i$.

Again if $x=e^{y}$ then $\log x=y$

Now, $-x=(-1).x=e^{(2n+1)\pi i}.e^{y}=e^{(2n+1)\pi i+y}$

. $\mathrm{Log}(-x)=(2n+1)\pi i+\log x$ and $\log(-x)=\pi i+\log x$.

Example 1: Find the value of $\text{Log}(1+i)$

Solution: Let $1+i = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = 1$ and $r\sin\theta = 1$

$\therefore \quad r^2\cos^2\theta + r^2\sin^2\theta = 1^2 + 1^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$= \; r = \sqrt{2}$

Also $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{1} \Rightarrow \tan\theta = 1 \;\Rightarrow \theta = \dfrac{\pi}{4}$

$$\therefore 1+i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right]$$

$$\therefore \text{Log}(1+i) = \log\left[\sqrt{2}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}\right]$$

$$= \log\left[\sqrt{2}.e^{\left(2n\pi + \frac{\pi}{4}\right)i}\right]$$

$$= \log\sqrt{2} + \log e^{\left(2n\pi + \frac{\pi}{4}\right)i}$$

$$= \log\sqrt{2} + \left(2n\pi + \frac{\pi}{4}\right)i.$$

Example 2: Prove that $\log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i\theta$

Solution: L.H.S $= \log(1+\cos 2\theta + i\sin 2\theta)$

$= \log(2\cos^2\theta + i2\sin\theta\cos\theta)$

$= \log\{2\cos\theta(\cos\theta + i\sin\theta)\}$

$= \log(2\cos\theta) + \log(\cos\theta + i\sin\theta)$

$= \log(2\cos\theta) + \log(e^{i\theta})$

$= \log(2\cos\theta) + i\theta$

= R.H.S Proved.

Example 3: Prove that $\sin(\log i^i) = -1$

Solution: We have $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{i\frac{\pi}{2}}$

$\therefore \log i = i\dfrac{\pi}{2}$

Now $i^i = e^{i\log i} = e^{i\left(i\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}}$

$\therefore \log i^i = -\dfrac{\pi}{2}$

$\therefore \sin\left(\log i^{i}\right) = \sin\left(-\frac{\pi}{2}\right) = -1.$

Example 4: Prove that $\operatorname{Log}\left(\operatorname{Log} e^{i\theta}\right) = \log(2n\pi + \theta) + i\left(2k + \frac{1}{2}\right)\pi$

Solution: We have $\operatorname{Log} e^{i\theta} = 2n\pi i + i\theta = (2n\pi + \theta)i = (2n\pi + \theta).e^{i\frac{\pi}{2}}$ where n is any integer

$\therefore \operatorname{Log}\left(\operatorname{Log} e^{i\theta}\right) = \log(2n\pi + \theta) + i\frac{\pi}{2} + 2k\pi i$, where k is any integer

$$= \log(2n\pi + \theta) + i\left(2k + \frac{1}{2}\right)\pi.$$

Example 5: Find the general value of $\log(-3)$.

Solution: Let $-3 = -3 + i0 = r(\cos\theta + i\sin\theta)$

. $-3 = r\cos\theta$ and $0 = r\sin\theta$

. $r^2 = 9 = r = 3$

Putting $r = 3$, we get $\cos\theta = -1$ and $\sin\theta = 0$

. $\theta = \pi$

$\therefore -3 = 3(\cos\pi + i\sin\pi) = 3.e^{i\pi}$

Hence, $\operatorname{Log}(-3) = \log\{3.e^{i\pi}.e^{2n\pi i}\}$

$$= \log 3 + \log e^{(2n\pi+\pi)i}$$

$$= \log 3 + (2n + 1)\pi i$$

Putting $n = 0$, we get the principal value of $\operatorname{Log}(-3)$ i.e., $\log(-3)$.

. $\log(-3) = \log 3 + i\pi$

Example 6: Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right)$

Solution: Let $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$ i.e., $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Now, $\log\frac{a+ib}{a-ib} = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log\frac{e^{i\theta}}{e^{-i\theta}}$

$$= \log e^{2i\theta} = 2i\theta$$

$$= 2i\tan^{-1}\left(\frac{b}{a}\right).$$

Example 7: Find the value of $\log\frac{(1+i)(1+i\sqrt{3})}{\sqrt{3}+i}$

Solution: $\log\frac{(1+i)(1+i\sqrt{3})}{\sqrt{3}+i}=\log\left\{\frac{1-\sqrt{3}+i(1+\sqrt{3})}{\sqrt{3}+i}\right\}$

$$=\log\frac{\{1-\sqrt{3}+i(1+\sqrt{3})\}(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}$$

$$=\log\frac{\{(1-\sqrt{3})\sqrt{3}+(1+\sqrt{3})\}+i\{(1+\sqrt{3})\sqrt{3}-(1-\sqrt{3})\}}{3-i^2}$$

$$=\log\frac{(2\sqrt{3}-2)+i(2\sqrt{3}+2)}{4}$$

$$=\log\left\{\frac{(\sqrt{3}-1)}{2}+i\frac{(\sqrt{3}+1)}{2}\right\}$$

$$=\log\left\{\frac{1}{2}(\sqrt{3}-1)+i\frac{1}{2}(\sqrt{3}+1)\right\}$$

$$=\frac{1}{2}\log\left\{\frac{1}{4}(\sqrt{3}-1)^2+\frac{1}{4}(\sqrt{3}+1)^2\right\}+i\tan^{-1}\frac{\frac{1}{2}(\sqrt{3}+1)}{\frac{1}{2}(\sqrt{3}-1)}$$

$$=\frac{1}{2}\log 2+i\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$=\frac{1}{2}\log 2+i\tan^{-1}\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$$=\frac{1}{2}\log 2+i\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$$

$$=\frac{1}{2}\log 2+i\frac{5\pi}{12}.$$

CHECK YOUR PROGRESS

Q.1: Define logarithm of a complex number.

Q.2: Prove that, $\log(\cos\theta+i\sin\theta)=i\theta$,$-\pi<\theta<\pi$

Q.3: Obtain the general value of $\text{Log}_i i$

Q.4: Prove that $\tan\left(i\log\frac{a+ib}{a-ib}\right)=\frac{2ab}{a^2-b^2}$

Q.5: Find the value of $\text{Log}(4+3i)$

7 LET US SUM UP

- If z and w are two complex numbers such that $w = e^z$, then z is called the logarithm of q to the base e. We can write it as $z = \log_e w$.
- logw is a many valued function. We have $\log_e w = z - 2n\pi i$, the value $z + 2n\pi i$ is called the general value of $\log_e w$ and is denoted by $\text{Log}_e w$.

  Thus, $\text{Log}_e w = z + 2n\pi i = 2n\pi i + \log_e w$

8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: If z and w are two complex numbers such that $w = e^z$, then z is called the logarithm of w to the base e.

Ans to Q No 2: We know that $\log_e(x+iy) = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

$$\therefore \log(\cos\theta + i\sin\theta) = \frac{1}{2}\log(\cos^2\theta + \sin^2\theta) + i\tan^{-1}\frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{2}\log 1 + i\tan^{-1}\tan\theta = i\theta$$

Ans. to Q. No. 3: We know that, $\text{Log}_i i = \frac{\text{Log}_e i}{\text{Log}_e i}$

Now, $\text{Log}_e i = \log_e i + 2n\pi i$

$$= i\frac{\pi}{2} + 2n\pi i$$

$$= i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$= i\left(\frac{\pi + 4n\pi}{2}\right)$$

$$= i\pi\left(\frac{1+4n}{2}\right)$$

Similarly, $\text{Log}_e i = i\pi\left(\frac{1+4m}{2}\right)$

$$\therefore \mathrm{Log}_i i = \frac{i\pi\left(\frac{1+4n}{2}\right)}{i\pi\left(\frac{1+4m}{2}\right)} = \frac{4n+1}{4m+1}$$, m and n are any integers.

Ans. to Q. No. 4: Try yourself (hint : put $a = r\cos\theta$ and $b = r\sin\theta$)

Ans. to Q. No. 5: Let $4 + 3i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get $r\cos\theta = 4; r\sin\theta = 3$

Squaring and adding, $r^2 = 16 + 9 = 25 \Rightarrow r = 5$

Dividing, $\tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}$

$$\therefore \mathrm{Log}(4+3i) = \mathrm{Log}[r(\cos\theta + i\sin\theta)] = \mathrm{Log}(re^{i\theta}) = 2n\pi i + \log(re^{i\theta})$$

$$= 2n\pi i + \log r + \log e^{i\theta}$$

$$= 2n\pi i + \log 5 + i\theta$$

$$= 2n\pi i + \log 5 + i\tan^{-1}\frac{3}{4}.$$

12.10 MODEL QUESTIONS

Q.1: Prove that:

i) $\log(-1) = i\pi$

ii) $\log(i) = i\frac{\pi}{2}$

iii) $\log(1+i) = \frac{1}{2}\log_e 2 + i\frac{\pi}{4}$

iv) $\log(1-i) = \frac{1}{2}\log_e 2 - i\frac{\pi}{4}$

v) $\log(\sqrt{3}+i) = \log_e 2 + i\frac{\pi}{6}$

vi) $\log(1-\sqrt{3}i) = \log_e 2 - i\frac{\pi}{3}$

vii) $\mathrm{Log}(-2) = \log_e 2 + (2n+1)\pi i$

viii) $\mathrm{Log}(-5) = \log_e 5 + (2n+1)\pi i$

ix) $\log(1-\cos\theta+i\sin\theta)=\log(2\sin\theta)+i\left(\frac{\pi}{2}-\theta\right)$

x) $\mathrm{Log}(-e)=1+i(2n+1)\pi$, where n is any integer.

Q.2: Prove that:

i) $\mathrm{Log}\left(\sqrt{i}\right)=(8n+1)i\frac{\pi}{4}$

ii) $\log(1+i\tan\alpha)=\log\sec\alpha+i\alpha; 0<\alpha<\frac{\pi}{2}$

Q.3: Prove that:

$$\log\cos(x+iy)=\frac{1}{2}\log\frac{1}{2}(\cosh y+\cos 2x)-i\tan^{-1}(\tan x\tanh y)$$

Q.4: Separate i^{i+1} into real and imaginary parts.

Q.5: Prove that: $\log\frac{1}{1-e^{i\theta}}=\log\left(\frac{1}{2}\mathrm{cosec}\,\theta\right)+i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

Q.6: Prove that: $\log\frac{\cos(x-iy)}{\cos(x+iy)}=2i\tan^{-1}(\tan x\tanh y)$

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UNIT 5 : TRIGONOMETRIC SERIES

UNIT STUCTURE

1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about Gregory's series
- describe the summation of trigonometric series.

2 INTRODUCTION

In previous unit, we discussed DeMovire's Theorem and its some important deductions. We will introduce Gregory's series. Finally, we will discuss summation of trigonometric series.

3 GREGORY'S SERIES

Statement: If θ lies within the closed interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$,

i.e., if $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$,

then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \cdots\infty$

Proof: We have $(1+i\tan\theta) = (1+i\frac{\sin\theta}{\cos\theta}) = \frac{1}{\cos\theta}(\cos\theta + i\sin\theta)$

$$= \sec\theta . e^{i\theta}$$

Now, taking logarithm of both sides, we have

$\log(1+i\tan\theta) = \log(\sec\theta . e^{i\theta})$ [log(AB) = logA+logB]

$$= \log\sec\theta + \log e^{i\theta}$$

$$= \log\sec\theta + i\theta \quad (1)$$

Now, since θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, $\tan\theta$ lies between –1 and 1,

i.e., $\tan\theta$ is numerically not greater than 1.

We have from (1)

$\log\sec\theta + i\theta = \log(1+i\tan\theta)$ (By Logarithmic series (4.7.1)

$$= i\tan\theta - \frac{1}{2}(i\tan\theta)^2 + \frac{1}{3}(i\tan\theta)^3 - \frac{1}{4}(i\tan\theta)^4 + \cdots\text{to }\infty$$

$$= i\tan\theta + \frac{\tan^2\theta}{2} - \frac{i\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + \cdots\text{to }\infty$$

$$= i(\tan\theta - \frac{\tan^3\theta}{3} + \cdots\infty) + (\frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \cdots\infty) \quad (2)$$

Equating imaginary part on both sides, we get

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} \cdots\cdots\cdots\infty \quad (3)$$

(3) is known as Gregory's series.

Some Important Deduction:

i) Now we put $\tan\theta = x$

So that $\theta = \tan^{-1}x$

Then we have from (3)

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots\cdots\infty$ where $-1 \le x \le 1$

ii) We equate the real parts on both sides of (2), we get

$$\log\sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \cdots\cdots\infty$$

3.1 General Theorem on Gregory's Series

Statement: If θ lies between $n\pi - \frac{1}{4}\pi$ and $n\pi + \frac{1}{4}\pi$

i.e., $n\pi - \frac{1}{4}\pi \le \theta \le n\pi + \frac{1}{4}\pi$,

then, $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\cdots\infty$

Proof: We put $\theta = n\pi + \alpha$, then $\alpha = \theta - n\pi$

The given condition reduces to $-\frac{\pi}{4} \le \alpha \le \frac{\pi}{4}$.

Hence, $1 + i\tan\theta = 1 + i\tan(n\pi + \alpha)$

$$= 1 + i\tan\alpha = 1 + i\frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{\cos\alpha + i\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha . e^{i\alpha}$$

Now, taking logarithm $\log(1 + i\tan\theta) = \log(\sec\alpha . e^{i\alpha})$

$\Theta\ n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}$

$\therefore \tan(n\pi - \frac{\pi}{4}) \le \tan\theta \le \tan(n\pi + \frac{\pi}{4})$

$\therefore -1 \le \tan\theta \le 1$

$|i\tan\theta| \le 1$

$\log(1 + i\tan\theta)$ can be expanded in powers of $\tan\theta$.

$$\log(1 + i\tan\theta) = i\tan\theta - \frac{1}{2}(i\tan\theta)^2 + \frac{1}{3}(i\tan\theta)^3 - \frac{1}{4}(i\tan\theta)^4 + \cdots\cdots\infty$$

$$= i\tan\theta + \frac{\tan^2\theta}{2} - \frac{i\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + \cdots \text{to } \infty$$

$$= i(\tan\theta - \frac{\tan^3\theta}{3} + \cdots\infty) + (\frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \cdots\infty) \ldots\ldots.$$

$$= \log(\sec\alpha) + i\alpha$$

Equating the imaginary parts on both sides, we have

$$\alpha = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\cdots\infty$$

Or, $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\cdots\infty$

Example 1: Prove that: $2\sqrt{3}.\left[1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right] = \pi$

Solution: L.H.S $= 2\sqrt{3}.\left[1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right]$

$$= 2\sqrt{3}.\left[1 - \frac{1}{3\left(\sqrt{3}\right)^2} + \frac{1}{5.\left(\sqrt{3}\right)^4} - \frac{1}{7.\left(\sqrt{3}\right)^6} + \cdots\right]$$

$$= 2.\sqrt{3}.\sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3\left(\sqrt{3}\right)^3} + \frac{1}{5\left(\sqrt{3}\right)^5} - \frac{1}{7\left(\sqrt{3}\right)^7} + \cdots\right]$$

$$= 6\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 6.\frac{\pi}{6}$$

$= \pi$ = R.H.S

Example 2: Sum the series

i) $1 - \frac{1}{3.4^2.} + \frac{1}{5.4^4} - \cdots\cdots\infty$

ii) $1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^2} + \cdots\cdots\infty$

Solution: i) The given series–

$$1 - \frac{1}{3.4^2.} + \frac{1}{5.4^4} - \cdots\cdots\infty$$

$$= 4\left[\frac{1}{4} - \frac{1}{3.4^3.} + \frac{1}{5.4^5} - \cdots\cdots\infty\right]$$

$= 4\tan^{-1}\frac{1}{4}$ (By Gregory's series)

ii) The given series

$$1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\cdots\infty$$

$$= 1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\cdots\infty$$

$$= \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3.(\sqrt{3})^3} + \frac{1}{5.(\sqrt{3})^5} - \frac{1}{7.(\sqrt{3})^7} + \cdots\right] \text{ (Gregory's series)}$$

$$= \sqrt{3}\left\{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right\} = \sqrt{3}.\frac{\pi}{6} = \frac{(\pi\sqrt{3})}{6}$$

Example 3: Prove that:

$$\frac{\pi}{4} = \left[\frac{2}{3} + \frac{1}{7}\right] - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right] - \cdots$$

Solution: R.H.S, $\left[\frac{2}{3} + \frac{1}{7}\right] - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right] - \cdots$

$$= 2\left[\frac{1}{3} - \frac{1}{3}.\frac{1}{3^3} + \frac{1}{5}.\frac{1}{3^5} - \cdots\right] + \left[\frac{1}{7} - \frac{1}{3}.\frac{1}{7^3} + \frac{1}{5}.\frac{1}{7^5} - \cdots\right]$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} \qquad \text{(Gregory's series)}$$

$$= \tan^{-1}\frac{\left(\frac{2}{3}\right)}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left[\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}.\frac{1}{7}}\right]$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{R.H.S}$$

Example 4: Prove that

$$\frac{\pi}{4} = \frac{17}{12} - \frac{713}{81 \times 343} + \cdots + \frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}.9^{1-n} + 7^{1-2n}\right\} + \cdots$$

Solution: The nth term of the series is given by

$$T_n = \frac{(-1)^{n+1}}{2n-1}\left[\frac{2}{3}.9^{1-n} + 7^{1-2n}\right]$$

$$= \frac{(-1)^{n+1}}{2n-1}\left[2.\frac{1}{3^{2n-1}} + \frac{1}{7^{2n-1}}\right]$$

$$\therefore T_1 = \left(\frac{2}{3}+\frac{1}{7}\right),\ T_2 = -\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right),\ T_3 = \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)$$

$$T_4 = -\frac{1}{7}\left(\frac{2}{3^7}+\frac{1}{7^7}\right), \text{ and so on.}$$

Hence, $S = T_1 + T_2 + T_3 + \cdots + T_n + \cdots\infty$

$$= 2\left[\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\cdots\infty\right]+\left[\frac{1}{7}-\frac{1}{3.7^3}+\frac{1}{7.7^7}+\cdots\infty\right]$$

$$= 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2.\frac{1}{3}}{1-\frac{1}{3^2}}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}$$

$$= \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{L.H.S}$$

CHECK YOUR PROGRESS

Q.1: Prove that: $\dfrac{\pi}{8} = \dfrac{1}{1.3}+\dfrac{1}{5.7}+\dfrac{1}{9.11}+\cdots\cdots\infty$

Q.2: Prove that:

$$\frac{\pi}{4} = \left(\frac{2}{3}+\frac{1}{7}\right)-\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)+\cdots\infty$$

Q.3: Prove that: $1-\dfrac{1}{3.4^2}+\dfrac{1}{5.4^4}-\cdots\cdot\infty = 4\tan^{-1}\dfrac{1}{4}$

Q.4: If θ lies between 0 and $\dfrac{\pi}{2}$, prove that

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^2\frac{\theta}{2}-\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}-\cdots\infty$$

4 SUMMATION OF TRIGONOMETRICAL SERIES

Here, we shall discuss important methods for summing up trigonometric series which may be finite or infinite. There are two important methods for summation. These are (a) $C+iS$ method, (b) the difference method.

4.1 $C+iS$ Method

Consider the series:

$$C = a_0 \cos\alpha + a_1 \cos(\alpha+\beta) + a_2 \cos(\alpha+2\beta) + \cdots \qquad (1)$$

and

$$S = a_0 \sin\alpha + a_1 \sin(\alpha+\beta) + a_2 \sin(\alpha+2\beta) + \cdots \qquad (2)$$

The above series may be finite or infinite.The coefficients a_0, a_1, a_2 ... and α, β,... may be any numbers real or complex.

In the series (1), we have terms which contain cosines of numbers.It is called cosine series and its sum is denoted by C. The series (2) contains sines of numbers.It is called sine series and its sum is denoted by S.

Now, using Euler,s Theorem

$$C+iS = a_0(\cos\alpha + i\sin\alpha) + a_1[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + a_2[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)] + \cdots\infty$$

$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \cdots\infty \qquad (3)$$

$$C-iS = a_0(\cos\alpha - i\sin\alpha) + a_1[\cos(\alpha+\beta) - i\sin(\alpha+\beta)] + a_2[\cos(\alpha+2\beta) - i\sin(\alpha+2\beta)] + \cdots\infty$$

$$= a_0 e^{-i\alpha} + a_1 e^{-i(\alpha+\beta)} + a_2 e^{-i(\alpha+2\beta)} + \cdots\infty \qquad (4)$$

From the series (3) and (4),we use

$$C = \frac{1}{2}[(C+iS) + (C-iS)]$$

and $$S = \frac{1}{2i}[(C+iS) - (C-iS)]$$

to find the values of C and S respectively.

4.2 Series Based on Geometric or Arithmetico-Geometric Series

Sum of n terms in G.P

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

$$= a\left(\frac{1-r^n}{1-r}\right) \text{ or } a\left(\frac{r^n-1}{r-1}\right) \text{ according as } r < 1 \text{ or } r > 1.$$

Sum of the infinite Geometric series:

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n + \cdots\infty$$

$$= \frac{a}{1-r}, \quad \text{if } |r| < 1 \text{ i.e., } -1 < r < 1.$$

4.3 Sum of a Series of Sines(or Cosines) of Angles in Arithmetical Progression

Let

$$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \sin(\alpha+3\beta) + \cdots + \sin\{\alpha+(n-1)\beta\}$$

We assume that,

$$C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cos(\alpha+3\beta) + \cdots + \cos\{\alpha+(n-1)\beta\}$$

So,

$$C + iS = (\cos\alpha + i\sin\alpha) + \{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\} + \{\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\} + \cdots$$

$$+ [\cos\{\alpha+(n-1)\beta\} + i\sin\{\alpha+(n-1)\beta\}]$$

$$= e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)} + \cdots + e^{i\{\alpha+(n-1)\beta\}}$$

$$= e^{i\alpha}\{1 + e^{i\beta} + e^{2i\beta} + \cdots + e^{(n-1)i\beta}\}$$

$$= e^{i\alpha}\left(\frac{1-e^{ni\beta}}{1-e^{i\beta}}\right)$$

$$= e^{i\alpha}\left(\frac{1-e^{ni\beta}}{1-e^{i\beta}}\right)\frac{1-e^{i\beta}}{1-e^{i\beta}}$$

$$= \frac{e^{i\alpha} - e^{i(\alpha-\beta)} - e^{i(\alpha+n\beta)} + e^{i\{\alpha+(n-1)\beta\}}}{1-\left(e^{i\beta}+e^{-i\beta}\right)+1}$$

$$= \frac{e^{i\alpha} - e^{i(\alpha-\beta)} - e^{i(\alpha+n\beta)} + e^{i\{\alpha+(n-1)\beta\}}}{2-2\cos\beta}$$

$$= \frac{(\cos\alpha + i\sin\alpha) - \{\cos(\alpha-\beta) + i\sin(\alpha-\beta)\} - \{\cos(\alpha+n\beta) + i\sin(\alpha+n\beta)\} + [\cos\{\alpha+(n-1)\beta\} + i\sin\{\alpha+(n-1)\beta\}]}{2(1-\cos\beta)}$$

$$=\frac{[\cos\alpha-\cos(\alpha-\beta)-\cos(\alpha+n\beta)-\cos\{\alpha+(n-1)\beta\}]}{2(1-\cos\beta}+i\frac{[\sin\alpha-\sin(\alpha-\beta)-\sin(\alpha-n\beta)+\sin\{\alpha+(n-1)\beta\}]}{2(1-\cos\beta)}$$

Equating real and imaginary parts, we get

$$C=\frac{\cos\alpha-\cos(\alpha-\beta)-\cos(\alpha-n\beta)-\cos\{\alpha+(n-1)\beta\}}{2(1-\cos\beta)}$$

$$=\frac{[\cos\alpha-\cos\{\alpha-(n-1)\beta\}]-\{\cos(\alpha-\beta)-\cos(\alpha+n\beta)\}}{2(1-\cos\beta)}$$

$$=\frac{2\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\cos\left\{\left(\frac{n-1}{2}\right)\beta\right\}-2\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\cos\left\{\left(\frac{n+1}{2}\right)\beta\right\}}{4\sin^2\frac{\beta}{2}}$$

$$=\frac{\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\left[\cos\frac{n-1}{2}\beta-\cos\frac{n+1}{2}\beta\right]}{2\sin^2\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right).2\sin\frac{n\beta}{2}\sin\frac{\beta}{2}}{2\sin^2\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

and $S=\dfrac{\sin\alpha-\sin(\alpha-\beta)-\sin(\alpha-n\beta)+\sin\{\alpha+(n-1)\beta\}}{2(1-\cos\beta)}$

$$=\frac{[\sin\alpha+\sin\{\alpha+(n-1)\beta\}]-\{\sin(\alpha-\beta)+\sin(\alpha+n\beta)\}}{2(1-\cos\beta)}$$

$$=\frac{2\sin\left\{\alpha+\left(\frac{n+1}{2}\right)\beta\right\}\cos\left\{\left(\frac{n-1}{2}\right)\beta\right\}-2\sin\left\{\alpha-\left(\frac{n-1}{2}\right)\beta\right\}\cos\left\{\left(\frac{n+1}{2}\right)\beta\right\}}{4\sin^2\frac{\beta}{2}}$$

$$=\frac{\sin\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\left[\cos\frac{n-1}{2}\beta-\cos\frac{n+1}{2}\beta\right]}{2\sin^2\frac{\beta}{2}}$$

$$=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)2\sin\frac{n\beta}{2}\sin\frac{\beta}{2}}{2\sin^2\frac{\beta}{2}}$$

$$=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Hence,

$$\sin\alpha+\sin(\alpha+\beta)+\cdots+\sin\{\alpha+(n-1)\beta\}=\frac{\sin\left\{\alpha+\frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \qquad (1)$$

$$\cos\alpha+\cos(\alpha+\beta)+\cdots+\cos\{\alpha+(n-1)\beta\}=\frac{\cos\left\{\alpha+\frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \qquad (2)$$

Particular case (i): Putting $\beta=\alpha$ in (1) and (2), we get

$$\sin\alpha+\sin 2\alpha+\cdots+\sin n\alpha=\frac{\sin\left(\frac{n+1}{2}\alpha\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

and $\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha=\dfrac{\cos\left(\frac{n+1}{2}\alpha\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$

ii) If $\beta=\frac{2\pi}{n}$, then $\sin\frac{n\beta}{2}=\sin\pi=0$ in (1) and (2),

then, $\sin\alpha+\sin\left(\alpha+\frac{2\pi}{n}\right)+\sin\left(\alpha+\frac{4\pi}{n}\right)+\cdots$ to n terms $=0$

and

$$\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cos\left(\alpha+\frac{4\pi}{n}\right)+\cdots \text{ to n terms } =0$$

Example 1: Sum to n terms of the series

$$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha-2\beta)-\sin(\alpha+3\beta)-\cdots\cdots$$

Solution: Let

$S=\sin\alpha-\sin(\alpha-\beta)+\sin(\alpha+2\beta)-\sin(\alpha+3\beta)+\cdots\cdots$ to n terms

$=\sin\alpha+\sin(\pi+\alpha+\beta)+\sin(2\pi+\alpha-2\beta)+\sin(3\pi+\alpha+3\beta)+\cdots$ to n terms

$$=\frac{\sin\left\{\alpha+\frac{n-1}{2}(\pi+\beta)\right\}\sin\frac{n}{2}(\pi+\beta)}{\sin\frac{\pi+\beta}{2}}$$

$$=\frac{\sin\left\{\alpha+\frac{(n-1)(\pi+\beta)}{2}\right\}\sin\frac{n(\pi+\beta)}{2}}{\cos\frac{\beta}{2}}$$

Example 2: Sum to n terms of the series:

$$\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\cdots\cdots$$

Solution: $S=\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\cdots\cdots \text{to n terms}$

$$=\frac{1-\cos2\alpha}{2}+\frac{1-\cos2(\alpha+\beta)}{2}+\frac{1-\cos2(\alpha+2\beta)}{2}+\cdots\cdots \text{to n terms}$$

$$=\frac{1}{2}(1+1+1+\cdots\cdots \text{to n terms})-\frac{1}{2}\left[\cos2\alpha+\cos2(\alpha+\beta)+\cos2(\alpha+2\beta)+\cdots\cdots \text{to n terms}\right]$$

$$=\frac{n}{2}-\frac{1}{2}\cdot\frac{\cos\left\{2\alpha+\frac{n-1}{2}.2\beta\right\}.\sin\left(\frac{n}{2}.2\beta\right)}{\sin\left(\frac{2\beta}{2}\right)}$$

$$=\frac{n}{2}-\frac{1}{2}\cdot\frac{\cos\{2\alpha+(n-1).\beta\}.\sin(n\beta)}{\sin\beta}$$

Example 3: Sum the series:

$$1+\frac{\cos\theta}{\cos\theta}+\frac{\cos2\theta}{\cos^2\theta}+\frac{\cos3\theta}{\cos^3\theta}+\cdots\cdots \text{to n terms}$$

Solution: $C=1+\frac{\cos\theta}{\cos\theta}+\frac{\cos2\theta}{\cos^2\theta}+\frac{\cos3\theta}{\cos^3\theta}+$ .... to n terms.

$$S=\frac{\sin\theta}{\cos\theta}+\frac{\sin2\theta}{\cos^2\theta}+\frac{\sin3\theta}{\cos^3\theta}+\cdots\cdots \text{to n terms}$$

Now,

$$C+iS=1+\frac{(\cos\theta+i\sin\theta)}{\cos\theta}+\frac{(\cos2\theta+i\sin2\theta)}{\cos^2\theta}+\frac{(\cos3\theta+i\sin3\theta)}{\cos^3\theta}+\cdots\cdots \text{to n terms}$$

$$=1+e^{i\theta}\sec\theta+e^{2i\theta}\sec^2\theta+e^{3i\theta}\sec^3\theta+\cdots\cdots \text{to n terms}$$

$$=\frac{(e^{i\theta}\sec\theta)^n-1}{e^{i\theta}\sec\theta-1}$$

$$=\frac{(e^{ni\theta}\sec^n\theta-1)(e^{-i\theta}\sec\theta-1)}{(e^{i\theta}\sec\theta-1)(e^{-i\theta}\sec\theta-1)}$$

$$=\frac{\sec^{n+1}\theta\, e^{(n-1)i\theta}-\sec^{n}\theta\, e^{ni\theta}-\sec\theta\, e^{-i\theta}+1}{\sec^{2}\theta-\sec\theta(e^{i\theta}+e^{-i\theta})+1}$$

$$=\frac{\sec^{n+1}\theta\, e^{(n-1)i\theta}-\sec^{n}\theta\, e^{ni\theta}-\sec\theta\, e^{-i\theta}+1}{\sec^{2}\theta-\sec\theta 2\cos\theta+1}$$

$$=\frac{\sec^{n+1}\theta\, e^{(n-1)i\theta}-\sec^{n}\theta\, e^{ni\theta}-\sec\theta\, e^{-i\theta}+1}{\sec^{2}\theta-1}$$

$$=\frac{\sec^{n+1}\theta\, e^{(n-1)i\theta}-\sec^{n}\theta\, e^{ni\theta}-\sec\theta\, e^{-i\theta}+1}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta[\cos(n-1)\theta+i\sin(n-1)\theta]-\sec^{n}\theta(\cos n\theta+i\sin n\theta)-\sec\theta(\cos\theta+i\sin\theta)+1}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta\cos(n-1)\theta-\sec^{n}\theta\cos n\theta-\sec\theta\cos\theta+1}{\tan^{2}\theta}+i\frac{\sec^{n+1}\theta\sin(n-1)\theta-\sec^{n}\theta\sin n\theta-\sec\theta\cos\theta}{\tan^{2}\theta}$$

Equating real parts on both sides,we get

$$C=\frac{\sec^{n+1}\theta\cos(n-1)\theta-\sec^{n}\theta\cos n\theta-\sec\theta\cos\theta+1}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta\cos(n-1)\theta-\sec^{n}\theta\cos n\theta-1+1}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta\cos(n-1)\theta-\sec^{n}\theta\cos n\theta}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta[\cos(n-1)\theta-\cos\theta\cos n\theta]}{\tan^{2}\theta}$$

$$=\frac{\sec^{n+1}\theta\sin n\theta\sin\theta}{\tan^{2}\theta}$$

$$=\frac{\sec^{n}\theta\sin\theta}{\tan\theta}$$

CHECK YOUR PROGRESS

Q.5: Sum the series:

a) $\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\cdots$ to n terms

b) $\cos^2\alpha+\cos^2(\alpha+\beta)+\cos^2(\alpha+2\beta)+\cdots$ to n terms

c) $\cos^2\alpha+\cos^2 2\alpha+\cos^2 3\alpha+\cdots$ to n terms

d) $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\cdots$ to n terms

e) $\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cdots$ to n terms

4.4 Summation of Series using Binomial Series

We should remember the following formulae :

i) When n is a positive integer and x,a are any complex number, we have

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \cdots + a^n$$

$$(x-a)^n = x^n - nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 - \cdots + (-1)^n a^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + (-1)^n x^n$$

ii) When n is any rational index and x is a complex number such that $|x| < 1$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \infty$$

Also, $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 - \cdots \infty$

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots \infty$$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1.2}{3.6}x^2 + \frac{1.2.5}{3.6.9}x^3 - \cdots \infty$$

and $(1-x)^{-\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{1.2}{3.6}x^2 + \frac{1.2.5}{3.6.9}x^3 + \cdots \infty$

Example 1: Sum the Series: $\sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1.3}{2.4}\sin 5\alpha + \cdots \text{ to } \infty$

Solution: Let $S = \sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1.3}{2.4}\sin 5\alpha + \cdots \text{ to } \infty$

Now, $C = \cos\alpha + \frac{1}{2}\cos 3\alpha + \frac{1.3}{2.4}\cos 5\alpha + \cdots \text{ to } \infty$

Then, using Binomial Theorem

$$C+iS=(\cos\alpha+i\sin\alpha)+\frac{1}{2}(\cos 3\alpha+i\sin 3\alpha)+\frac{1.3}{2.4}(\cos 5\alpha+i\sin 5\alpha)+\cdots\text{ to }\infty$$

$$=e^{i\alpha}+\frac{1}{2}e^{3i\alpha}+\frac{1.3}{2.4}e^{5i\alpha}+\cdots\text{ to }\infty$$

$$=e^{i\varepsilon}\left(1+\frac{1}{2}e^{2i\alpha}+\frac{1.3}{2.4}e^{4i\alpha}+\cdots\text{ to }\infty\right)$$

$$=e^{i\alpha}\left[1-e^{2i\alpha}\right]^{-\frac{1}{2}}$$

$$=(\cos\alpha+i\sin\alpha)\left[2\sin^2\alpha-i2\sin\alpha\cos\alpha\right]^{-\frac{1}{2}}$$

$$=(\cos\alpha+i\sin\alpha)\left[\left(2\sin^2\alpha\right)^{-\frac{1}{2}}(\sin\alpha-i\cos\alpha)^{-\frac{1}{2}}\right]$$

$$=\left(2\sin^2\alpha\right)^{-\frac{1}{2}}(\cos\alpha+i\sin\alpha)\left[\cos\left\{-\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right)\right\}-i\sin\left\{-\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right)\right\}\right]$$

$$=\left(2\sin^2\alpha\right)^{-\frac{1}{2}}(\cos\alpha+i\sin\alpha)\left[\cos\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)+i\sin\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)\right]$$

$$=\left(2\sin^2\alpha\right)^{-\frac{1}{2}}\left[\cos\left(\alpha+\frac{\pi}{4}-\frac{1}{2}\alpha\right)+i\sin\left(\alpha+\frac{\pi}{4}-\frac{1}{2}\alpha\right)\right]$$

(Since $(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi)$)

$$=\left(2\sin^2\alpha\right)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)+i\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)\right]$$

Equating the imaginary parts on both sides,we get

$$S=\left(2\sin^2\alpha\right)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)$$

Example 2: Sum the series

$$1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots\infty$$

Solution: Let $C=1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots$

$$S=-\frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta-\frac{1.3.5}{2.4.6}\sin 3\theta+\cdots$$

Then,

$$C+iS=1-\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1.3}{2.4}(\cos 2\theta+isn2\theta)-\frac{1.3.5}{2.4.6}(\cos 3\theta+i\sin 3\theta)+\cdots$$

$$=1-\frac{1}{2}e^{i\theta}+\frac{1.3}{2.4}e^{2i\theta}-\frac{1.3.5}{2.4.6}e^{3i\theta}+\cdots$$

$$= 1 - \frac{1}{2}x + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \cdots \text{ where } \quad x = e^{i\theta}$$

$$= (1+x)^{-\frac{1}{2}}$$

$$= \left(1+e^{i\theta}\right)^{-\frac{1}{2}}$$

$$= (1+\cos\theta + i\sin\theta)^{-\frac{1}{2}}$$

$$= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right) \qquad \text{[By De Moivre's Theorem]}$$

Equating real parts, we get $C = \left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\cos\frac{\theta}{4}$

$$= \frac{\cos\frac{\theta}{4}}{\left(2\cos\frac{\theta}{2}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$$

4.5 Summation of Series using of Exponential Series

The following series are frequently used.

i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

ii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$

iii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty$

iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty$

v) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots\infty$

vi) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots\infty$

Example 1: Sum the Series

$$1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha\cos 2\beta}{2!} - \frac{\cos^3\alpha\cos 3\beta}{3!} + \cdots\infty$$

Solution: Let

$$C = 1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha\cos 2\beta}{2!} - \frac{\cos^3\alpha\cos 3\beta}{3!} + \cdots\infty$$

and $S = -\cos\alpha\sin\beta + \cos^2\alpha\dfrac{\sin 2\beta}{2!} - \cos^3\alpha\dfrac{\sin 3\beta}{3!} + \cdots\infty$

Then,

$$C + iS = 1 - \cos\alpha(\cos\beta + i\sin\beta) + \frac{\cos^2\alpha(\cos 2\beta + i\sin 2\beta)}{2!} - \frac{\cos^3\alpha(\cos 3\beta + i\sin 3\beta)}{3!} + \cdots\infty$$

$$= 1 - \cos\alpha e^{i\beta} + \frac{\cos^2\alpha}{2!}e^{2i\beta} - \frac{\cos^3\alpha}{3!}e^{3i\beta} + \cdots\infty$$

$$= e^{-\cos\alpha e^{i\beta}}$$

$$= e^{-\cos\alpha(\cos\beta + i\sin\beta)}$$

$$= e^{-\cos\alpha\cos\beta}.e^{-i\cos\alpha\sin\beta}$$

$$= e^{-\cos\alpha\cos\beta}\{\cos(\cos\alpha\sin\beta) - i\sin(c\cos\alpha\sin\beta)\}$$

Equating real parts on both sides, we get

$$C = e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\sin\beta)$$

Example 2: Find the sum of the series

$$1 + \frac{c^2}{2!}\cos 2\theta + \frac{c^4}{4!}\cos 4\theta + \cdots\infty$$

Solution: Let $C = 1 + \dfrac{c^2}{2!}\cos 2\theta + \dfrac{c^4}{4!}\cos 4\theta + \cdots\infty$

and $S = 1 + \dfrac{c^2}{2!}\sin 2\theta + \dfrac{c^4}{4!}\sin 4\theta + \cdots\infty$

Then $C + iS = 1 + \dfrac{c^2}{2!}(\cos 2\theta + i\sin 2\theta) + \dfrac{c^4}{4!}(\cos 4\theta + i\sin 4\theta) + \cdots\infty$

$$= 1 + \frac{c^2}{2!}e^{2i\theta} + \frac{c^4}{4!}e^{4i\theta} + \cdots\infty$$

$$= \cosh(ce^{i\theta})$$

$$= \cosh\{c(\cos\theta + i\sin\theta)\}$$

$$= \cos\{ic(\cos\theta + i\sin\theta)\}$$

$$= \cos(ic\cos\theta - c\sin\theta)$$

$$= \cosh(c\cos\theta)\cos(c\sin\theta) + i\sinh(c\cos\theta)\sin(c\sin\theta)$$

Equating real parts, we get $C = \cosh(c\cos\theta)\cos(c\sin\theta)$

4.6 Summation of series using Logarithmic series and Gregory's series

i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\infty$

ii) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\infty$

iii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\infty$ where $-1 \le x \le 1$

Example: Sum the Series:

i) $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \cdots\infty$

ii) $\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \cdots\infty$

Solution: Let the series (i) and (ii) be denoted by C and S respectively.

i.e $C = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \cdots\infty$

and $S = \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \cdots\infty$

Then

$$C + iS = (\cos\theta + i\sin\theta) - \frac{1}{2}(\cos 2\theta + i\sin 2\theta) + \frac{1}{3}(\cos 3\theta + i\sin 3\theta) - \cdots\infty$$

$$= e^{i\theta} - \frac{1}{2}e^{2i\theta} + \frac{1}{3}e^{3i\theta} - \cdots\infty$$

$= \log(1+e^{i\theta})$ by Logarithmic series

$$= \log(1 + \cos\theta + i\sin\theta)$$

$$= \frac{1}{2}\log\{(1+\cos\theta)^2 + \sin^2\theta\} + i\tan^{-1}\frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{1}{2}\log\left(1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right) + i\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

$$= \frac{1}{2}\log(2+2\cos\theta) + i\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$

$$= \frac{1}{2}\log\{2(1+\cos\theta)\} + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\log\left(2.2\cos^2\frac{\theta}{2}\right) + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\log\left(2.\cos\frac{\theta}{2}\right)^2 + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \log\left(2.\cos\frac{\theta}{2}\right) + i\frac{\theta}{2}$$

Equating real and imaginary parts, we get $C = \log\left(2.\cos\frac{\theta}{2}\right)$

i.e i) $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \cdots\infty = \log\left(2\cos\frac{\theta}{2}\right)$ and $S = \frac{\theta}{2}$

i.e ii) $\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \cdots\infty = \frac{\theta}{2}$

CHECK YOUR PROGRESS

Q.6: Sum the series:

i) $\cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \cdots\infty$

ii) $e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\alpha + \cdots\infty$

iii) $1 + x\cos\theta + \frac{x^2}{2!}\cos 2\theta + \frac{x^3}{3!}\cos 3\theta + \cdots\infty$

iv) $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \cdots\infty$

4.7 Difference Method

In order to sum a series,sometimes it is convenient to split up each term as the difference of two expressions such that one expression of each difference occurs in succeeding with an opposite sign. The splitting is done in such a way that when all the terms of the series are added together, the component terms cancel in pairs. Finally we are left with two terms, one from the first term and one from last term.

Suppose, we have to find the sum of $u_1 + u_2 + u_3 + \cdots + u_n$

First we write $u_n = f(n+1) - f(n)$ (1)

From (1) putting $n = 1, 2, 3, \ldots, n$, we get

$$u_1 = f(2) - f(1)$$
$$u_2 = f(3) - f(2)$$
$$u_3 = f(4) - f(3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_n = f(n+1) - f(n)$$

Adding vertically, we get $u_1 + u_2 + u_3 + \cdots + u_n = f(n+1) - f(n)$

(since all the intermediate terms cancel in pairs)

Example1: Sum the series to n terms

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \cdots + \tan^{-1}\frac{1}{1+n(n+1)}$$

Solution: Here

$$T_n = \tan^{-1}\frac{1}{1+n(n+1)} = \tan^{-1}\frac{n+1-n}{1+n(n+1)}$$

$$= \tan^{-1}(n+1) - \tan^{-1}(n) \qquad \left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right)$$

Now, $T_1 = \tan^{-1}2 - \tan^{-1}1$

$$T_2 = \tan^{-1}3 - \tan^{-1}2$$
$$T_3 = \tan^{-1}4 - \tan^{-1}3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$$

Adding, we get $S_n = T_1 + T_2 + T_3 + \cdots + T_n = \tan^{-1}(n+1) - \tan^{-1}1$

Example 2: Sum the series:

$$\sin\theta\sec 3\theta + \sin 3\theta\sec 3^2\theta + \sin 3^2\theta\sec 3^3\theta + \cdots \text{to n terms}$$

Solution: We have $T_1 = \sin\theta\sec 3\theta = \dfrac{\sin\theta}{\cos 3\theta} = \dfrac{1}{2}.\dfrac{2\sin\theta\cos\theta}{\cos\theta\cos 3\theta}$

$$= \frac{1}{2}\frac{\sin 2\theta}{\cos 3\theta\cos\theta} = \frac{1}{2}\frac{\sin(3\theta - \theta)}{\cos 3\theta\cos\theta}$$

$$= \frac{1}{2}\left[\frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\cos 3\theta\cos\theta}\right]$$

$$= \frac{1}{2}[\tan 3\theta - \tan\theta]$$

Similarly, $T_2 = \dfrac{1}{2}[\tan 3^2\theta - \tan 3\theta]$

$$T_3 = \frac{1}{2}[\tan 3^3\theta - \tan 3^2\theta]$$

$$T_4 = \frac{1}{2}[\tan 3^4\theta - \tan 3^3\theta]$$

....

....

$$T_n = \frac{1}{2}[\tan 3^n\theta - \tan 3^{n-1}\theta]$$

Adding these,we have the required sum

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

$= \dfrac{1}{2}[\tan 3^n\theta - \tan\theta]$, other terms cancelling each other.

CHECK YOUR PROGRESS

Q.7: Sum the series to n terms by Difference Method.

a) $\tan^{-1}\dfrac{2}{4} + \tan^{-1}\dfrac{2}{9} + \tan^{-1}\dfrac{2}{16} + \cdots + \tan^{-1}\dfrac{2}{(n+1)^2}$

b) $\sin\alpha\sin 3\alpha + \sin\dfrac{\alpha}{2}\sin\dfrac{3\alpha}{2} + \sin\dfrac{\alpha}{2^2}\sin\dfrac{3\alpha}{2^2} + \cdots$

c) $\tan\theta\tan^2\dfrac{1}{2}\theta + 2\tan\dfrac{1}{2}\theta\tan^2\dfrac{1}{2^2}\theta + 2^2\tan\dfrac{1}{2^2}\theta\tan^2\dfrac{1}{2^3}\theta + \cdots$

5 LET US SUM UP

- If θ lies within the closed interval $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, i.e., if $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, then the series $\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\frac{1}{7}\tan^7\theta+\cdots\infty$ is called Gregory's series.We also derive important results from Gergory's series.
- We discuss important methods for summing up trigonometric series which may be finite or infinite.There are two important methods for summation.These are (a) $C+iS$ method (b) The difference method.

6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: We have $\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\cdots\infty$

$$=\frac{1}{2}\left[\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\cdots\infty\right]$$

$$=\frac{1}{2}\left[\frac{3-1}{1.3}+\frac{7-5}{5.7}+\frac{11-7}{9.11}+\cdots\infty\right]$$

$$=\frac{1}{2}\left[\left\{\frac{3}{1.3}-\frac{1}{1.3}\right\}+\left\{\frac{7}{5.7}-\frac{5}{5.7}\right\}+\left\{\frac{11}{9.11}-\frac{9}{9.11}\right\}+\cdots\infty\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots\infty\right]$$

$$=\frac{1}{2}\tan^{-1}1$$

$$=\frac{1}{2}.\frac{1}{4}\pi=\frac{1}{8}\pi$$

Ans. to Q. No. 2: From R.H.S,

$$=\left[\frac{2}{3}+\frac{1}{7}\right]-\frac{1}{3}\left[\frac{2}{3^3}+\frac{1}{7^3}\right]+\frac{1}{5}\left[\frac{2}{3^5}+\frac{1}{7^5}\right]-\cdots\infty$$

$$=2\left[\frac{1}{3}-\frac{1}{3}.\frac{1}{3^3}+\frac{1}{5}.\frac{1}{3^5}-\cdots\right]+\frac{1}{7}\left[\frac{1}{7}-\frac{1}{3}.\frac{1}{7^3}+\frac{1}{5}.\frac{1}{7^5}-\cdots\right]$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} \qquad \text{(By Gregory's series)}$$

$$= \tan^{-1}\frac{\left(2/3\right)}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right] = \tan^{-1}1 = \frac{\pi}{4}$$

= The L.H.S

Ans. to Q. No. 3: $\text{L.H.S} = 1\text{-}\frac{1}{3.4^2} + \frac{1}{5.4^4} - \cdots \text{ad.inf}$

$$= 4\left[\frac{1}{4} - \frac{1}{3.4^3} + \frac{1}{5.4^5} - \cdots \text{ad.inf}\right]$$

$$= 4\tan^{-1}\frac{1}{4} \quad \text{By Gregory,s series because } \frac{1}{4} < 1.$$

$$= \text{R.H.S}$$

Ans to Q No 4 : We have

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan^2\frac{\theta}{2}\right) \qquad (1)$$

Since θ lies between 0 and $\frac{\pi}{2}$

$\therefore \frac{1}{2}\theta$ lies between θ and $\frac{\pi}{4}$, so that $\tan^2\frac{\theta}{2} < 1$.

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan^2\frac{\theta}{2}\right)$$

Since θ lies between 0 and $\frac{\pi}{2}$

$\therefore \frac{1}{2}\theta$ lies between θ and $\frac{\pi}{4}$, so that $\tan^2\frac{\theta}{2} < 1$.

Therefore, $\tan^{-1}\left(\tan^2\frac{\theta}{2}\right)$ can be expanded by Gregory's series.

$$\tan^{-1}\left(\tan^2\frac{\theta}{2}\right)=\tan^2\frac{\theta}{2}-\frac{1}{3}\left(\tan^2\frac{\theta}{2}\right)^3+\frac{1}{5}\left(\tan^2\frac{\theta}{2}\right)^5-\cdots$$

$$=\ \tan^2\frac{\theta}{2}-\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}-\cdots \qquad (2)$$

Hence, from (1) and (2), we have

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)=\ \tan^2\frac{\theta}{2}-\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}-\cdots$$ Hence proved.

Ans. to Q. No. 5: a) Let $S_n=\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\cdots+\sin^2 n\alpha$

$$=\frac{1-\cos 2\alpha}{2}+\frac{1-\cos 4\alpha}{2}+\frac{1-\cos 6\alpha}{2}+\cdots+\frac{1-\cos 2n\alpha}{2}$$

$$=\frac{n}{2}-\frac{1}{2}(\cos 2\alpha+\cos 4\alpha+\cos 6\alpha+\cdots+\cos 2n\alpha)$$

$$=\frac{n}{2}-\frac{1}{2}\frac{\cos\frac{2\alpha+2n\alpha}{2}\sin\frac{n(2\alpha)}{2}}{\sin\frac{2\alpha}{2}}$$

$$=\frac{n}{2}-\frac{1}{2}\cos(n+1)\alpha\sin n\alpha\operatorname{cosec}\alpha$$

b) Let

$$S_n=\cos^2\alpha+\cos^2(\alpha+\beta)+\cos^2(\alpha+2\beta)+\cdots+\cos^2(\alpha+(n-1)\beta)$$

$$=\frac{1}{2}(1+\cos 2\alpha)+\frac{1}{2}\{(1+\cos(2\alpha+2\beta)\}+\frac{1}{2}\{1+\cos(2\alpha+4\beta)\}$$

$$+\cdots+\frac{1}{2}\{1+\cos(2\alpha+2(n-1)\alpha\}$$

$$=\frac{n}{2}+\frac{1}{2}[\cos 2\alpha+\cos(2\alpha+2\beta)+\cos(2\alpha+4\beta)+\cdots+\cos(2\alpha+2(n-1)\alpha]$$

$$=\frac{n}{2}+\frac{1}{2}\frac{\cos\frac{2\alpha+2\alpha+2(n-1)\beta}{2}\sin\frac{n(2\beta)}{2}}{\sin\frac{2\beta}{2}}$$

$$=\frac{n}{2}+\frac{1}{2}\cos(2\alpha+(n-1)\beta)\sin n\beta\operatorname{cosec}\beta$$

c) Let $S_n=\cos^2\alpha+\cos^2 2\alpha+\cos^2 3\alpha+\cdots$

$$=\frac{1}{2}(1+\cos 2\alpha)+\frac{1}{2}(1+\cos 4\alpha)+\frac{1}{2}(1+\cos 6\alpha)+\cdots+\frac{1}{2}(1+\cos 2n\alpha)$$

$$=\frac{n}{2}+\frac{1}{2}\left[\cos 2\alpha+\cos 4\alpha+\cos 6\alpha+\cdots+\cos 2n\alpha\right]$$

$$=\frac{n}{2}+\frac{1}{2}\frac{\cos\frac{2\alpha+2n\alpha}{2}\sin\frac{n(2\alpha)}{2}}{\sin\frac{2\alpha}{2}}$$

$$=\frac{n}{2}+\frac{1}{2}\frac{\cos\alpha(n+1)\sin n\alpha}{\sin\alpha}$$

d) Try yourself

Let $S_n=\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\cdots$

$$=\frac{1}{4}\left[\frac{3\sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}-\frac{\sin\frac{3(n+1)\alpha}{2}\sin\frac{3n\alpha}{2}}{\sin\frac{3\alpha}{2}}\right]$$

e) Try yourself

Let $S_n=\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cdots$

$$=\frac{1}{2}\frac{\sin 2n\alpha}{\sin\alpha}$$

Ans. to Q. No. 6: i) Let $C=\cos\alpha+x\cos(\alpha+\beta)+\frac{x^2}{2!}\cos(\alpha+2\beta)+\cdots\infty$

and $S=\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{2!}\sin(\alpha+2\beta)+\cdots\infty$

then

$$C+iS=(\cos\alpha+i\sin\alpha)+x[\cos(\alpha+\beta)+i\sin(\alpha+\beta)]+\frac{x^2}{2!}[\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)]+\cdots\infty$$

$$=e^{i\alpha}+xe^{i(\alpha+\beta)}+\frac{x^2}{2!}e^{i(\alpha+2\beta)}+\cdots\infty$$

$$=e^{i\alpha}\left[1+xe^{i\beta}+\frac{x^2}{2!}e^{2i\beta}+\cdots\infty\right]$$

$=e^{i\alpha}\left(e^{x.e^{i\beta}}\right)$ this is an exponential series

$$=e^{i\alpha}.e^{x(\cos\beta+i\sin\beta)}$$

$$=e^{x\cos\beta}.e^{i(\alpha+x\sin\beta)}$$

$$=e^{x\cos\beta}\left[\cos(\alpha+x\sin\beta)+i\sin(\alpha+x\sin\beta)\right]$$

Equating real parts on both sides,

we get $C=e^{x\cos\beta}.\cos(\alpha+x\sin\beta)$

ii) Let $C = e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\alpha + \cdots\infty$

$$S = e^{\alpha}\sin\beta - \frac{e^{3\alpha}}{3}\sin 3\beta + \frac{e^{5\alpha}}{5}\sin 5\alpha + \cdots\infty$$

$$C + iS = e^{\alpha}(\cos\beta + i\sin\beta) - \frac{e^{3\alpha}}{3}(\cos 3\beta + i\sin 3\beta) + \frac{e^{5\alpha}}{5}(\cos 5\beta + i\sin 5\beta) + \cdots\infty$$

$$= e^{\alpha}e^{i\beta} - \frac{e^{3\alpha}}{3}e^{i3\beta} + \frac{e^{5\alpha}}{5}e^{i5\alpha} + \cdots\infty$$

$$= e^{\alpha+i\beta} - \frac{e^{3(\alpha+i\beta)}}{3} + \frac{e^{5(\alpha+i\beta)}}{5} + \cdots\infty$$

iii) Let $S = \sin\alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \cdots\infty$

$$C = \cos\alpha - \frac{\cos(\alpha + 2\beta)}{2!} + \frac{\cos(\alpha + 4\beta)}{4!} - \cdots\infty$$

$$C + iS = (\cos\alpha + i\sin\alpha) - \frac{[\cos(\alpha + 2\beta) + i\sin(\alpha + 2\beta)]}{2!} + \frac{[\cos(\alpha + 4\beta) + i\sin(\alpha + 4\beta)]}{4!} - \cdots\infty$$

$$= e^{i\alpha} - \frac{e^{i(\alpha+2\beta)}}{2!} + \frac{e^{i(\alpha+4\beta)}}{4!} - \cdots\infty$$

$$= e^{i\alpha}\left[1 - \frac{e^{i2\beta}}{2!} + \frac{e^{i4\beta}}{4!} - \cdots\infty\right]$$

$= e^{i\alpha}\cos(e^{i\beta})$ By cosine series

$$= (\cos\alpha + i\sin\alpha)\cos(\cos\beta + i\sin\beta)$$

$$= (\cos\alpha + i\sin\alpha)[\cos(\cos\beta)\cos(i\sin\beta) - \sin(\cos\beta)\sin(i\sin\beta)]$$

$$= (\cos\alpha + i\sin\alpha)[\cos(\cos\beta)\cosh(\sin\beta) - \sin(\cos\beta)i\sinh(\sin\beta)]$$

$$= (\cos\alpha + i\sin\alpha)[\cos(\cos\beta)\cosh(\sin\beta) - i\sin(\cos\beta)\sinh(\sin\beta)]$$

$$= \cos\alpha\cos(\cos\beta)\cosh(\sin\beta) - i\cos\alpha\sin(\cos\beta)\sinh(\sin\beta) +$$

$$= \cos\alpha\cos(\cos\beta)\cosh(\sin\beta) - i\cos\alpha\sin(\cos\beta)\sinh(\sin\beta) +$$
$$i\sin\alpha\cos(\cos\beta)\cosh(\sin\beta) + \sin\alpha\sin(\cos\beta)\sinh(\sin\beta)$$
$$[\cos\alpha\cos(\cos\beta)\cosh(\sin\beta) + \sin\alpha\sin(\cos\beta)\sinh(\sin\beta)] +$$

Equating imaginary parts, we get

$$S = \sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)$$

iii) Let $C = 1 + x\cos\theta + \frac{x^2}{2!}\cos 2\theta + \frac{x^3}{3!}\cos 3\theta + \cdots\infty$

and $S = x\sin\theta + \frac{x^2}{2!}\sin 2\theta + \frac{x^3}{3!}\sin 3\theta + \cdots\infty$

Then $C+iS = 1+x(\cos\theta+i\sin\theta)+\dfrac{x^2}{2!}(\cos 2\theta+i\sin 2\theta)+\dfrac{x^3}{3!}(\cos 3\theta+i\sin 3\theta)+\cdots\infty$

$$= 1+xe^{i\theta}+\frac{x^2}{2!}e^{i2\theta}+\frac{x^3}{3!}e^{3i\theta}+\cdots\infty$$

$= e^{\left(xe^{i\theta}\right)}$ By expoential series

$$= e^{x(\cos\theta+i\sin\theta)}$$

$$= e^{x\cos\theta}.e^{ix\sin\theta}$$

$$= e^{x\cos\theta}[\cos(x\sin\theta)+i\sin(x\sin\theta)]$$

$$= e^{x\cos\theta}\cos(x\sin\theta)+ie^{x\cos\theta}\sin(x\sin\theta)$$

Equating the real parts on both sides,we get

$$C = e^{x\cos\theta}\cos(x\sin\theta)$$

iv) Let $S = x\sin\theta-\dfrac{1}{2}x^2\sin 2\theta+\dfrac{1}{3}x^3\sin 3\theta-\cdots\infty$

and $C = x\cos\theta-\dfrac{1}{2}x^2\cos 2\theta+\dfrac{1}{3}x^3\cos 3\theta-\cdots\infty$

Then $C+iS = x(\cos\theta+i\sin\theta)-\dfrac{1}{2}x^2(\cos 2\theta+i\sin 2\theta)+\dfrac{1}{3}x^3(\cos 3\theta+i\sin 3\theta)-\cdots\infty$

$$= xe^{i\theta}-\frac{1}{2}x^2e^{2i\theta}+\frac{1}{3}x^3e^{3i\theta}-\cdots\infty$$

$= \log\left(1+xe^{i\theta}\right)$ By Logarithmic series

$$= \log(1+x\cos\theta+ix\sin\theta)$$

$$= \log\sqrt{(1+x\cos\theta)^2+(x\sin\theta)^2}+i\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}$$

(Separating real and imaginary parts)

Equating imaginary parts, we get

$S = \tan^{-1}\dfrac{x\sin\theta}{1+x\cos\theta}$ (except when $x\cos\theta = -1$)

Ans. to Q. No. 7: a) Here $T_n = \tan^{-1}\dfrac{2}{(n+1)^2} = \tan^{-1}\dfrac{2}{n^2+n+1}$

$$= \tan^{-1}\frac{2}{1+n(n+2)} = \tan^{-1}\frac{(n+2)-n}{1+n(n+2)}$$

$$= \tan^{-1}(n+2)-\tan^{-1}n$$

Putting $n = 1, 2, 3, ..., n$, we have

$$T_1 = \tan^{-1}3-\tan^{-1}1$$

$$T_2 = \tan^{-1}4-\tan^{-1}2$$

$$T_3 = \tan^{-1}5 - \tan^{-1}3$$

....

....

$$T_n = \tan^{-1}(n+2) - \tan^{-1}n$$

Adding, we get the required sum to n terms

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

$$= \tan^{-1}(n+2) + \tan^{-1}(n+1) - \tan^{-1}2 - \tan^{-1}1$$

b) Here, $T_1 = \sin\alpha\sin3\alpha = \frac{1}{2}.2\sin\alpha\sin3\alpha = \frac{1}{2}[\cos2\alpha - \cos4\alpha]$

$$T_2 = \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} = \frac{1}{2}.2\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} = \frac{1}{2}[\cos\alpha - \cos2\alpha]$$

$$T_3 = \frac{1}{2}\left[\cos\frac{1}{2}\alpha - \cos\alpha\right]$$

....

....

$$T_n = \frac{1}{2}\left[\cos\left(\frac{\alpha}{2^{n-1}}\right) - \cos\left(\frac{\alpha}{2^{n-3}}\right)\right]$$

Adding up the above n relations, we the required sum

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2^{n-1}}\right) - \cos4\alpha\right]$$

d) Try yourself .

$$\text{Required sum} = \tan\theta - 2^n\tan\left(\frac{\theta}{2^n}\right)$$

8 MODEL QUESTIONS

Q.1: Find the sum of the following series:

a) $\sin\alpha + x\sin(\alpha+\beta) + \dfrac{x^2}{2!}\sin(\alpha+2\beta) + \cdots\infty$

b) $\sin\alpha.\cos\alpha - \dfrac{1}{2}\sin^2\alpha\cos 2\alpha + \dfrac{1}{3}\sin^3\alpha\cos 3\alpha + \cdots\infty$

c) $\sin\alpha + \dfrac{1}{3}\sin 3\alpha + \dfrac{1}{3^2}\sin 5\alpha + \cdots\infty$

d) $1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots\propto$

e) $1 + \dfrac{1}{3}\cos x + \dfrac{1}{9}\cos 2x + \dfrac{2}{27}\cos 3x + \cdots\infty$

f) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{2}{9} + \tan^{-1}\dfrac{4}{33} + \cdots$ to n terms

g) $\tan^{-1}\dfrac{4}{7} + \tan^{-1}\dfrac{4}{19} + \tan^{-1}\dfrac{4}{39} + \cdots + \tan^{-1}\dfrac{4}{4n^2+3}$

*** ***** ***