MAR GREGORIOS COLLEGE
OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: INTEGRAL CALCULUS AND VECTOR ANALYSIS

SUBJECT CODE: SM22B

SEMESTER: II

PREPARED BY: PROF.S.KAVITHA

UNIVERSITY OF MADRAS
B.Sc. DEGREE COURSE IN MATHEMATICS
SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC04

CORE-IV: INTEGRAL CALCULUS AND VECTOR ANALYSIS
(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 5 **YEAR: I**
Credits : 4 **SEMESTER: II**

Learning outcomes:

Students will acquire Knowledge about

- Integration and its geometrical applications, double, triple integrals and improper integrals.
- Vector differentiation and Vector integration.

UNIT I

Reduction formulae– Types, $\int x^n e^{ax} dx$, $\int x^n cosaxdx$, $\int x^n sinaxdx$, $\int cos^n xdx$, $\int sin^n xdx$, $\int sin^m xcos^n xdx$, $\int tan^n xdx$,$\int cot^n xdx$,$\int sec^n xdx$,$\int cosec^n xdx$, $\int x^n (\log x)^m dx$- Bernoulli's formula.

Chapter 1 Section 13, 13.1 to 13.10,14,15.1.

UNIT II

Multiple Integrals- definition of the double integrals- evaluation of the double integrals- double integrals in polar coordinates – triple integrals – applications of multiple integrals – volumes of solids of revolution – areas of curved surfaces – change of variables – Jacobians.

Chapter 5 Section 1, 2.1, 2.2, 3.1, 4, 6.1, 6.2, 6.3, 7
Chapter 6 Section 1.1, 1.2, 2.1 to 2.4.

UNIT III

Beta and Gamma functions - infinite integral – definitions – recurrence formula of Γ functions - properties of β-functions - relation between β and Γ functions.
Chapter 7 Sections 1.1 to 1.4 , 2.1, 2.3, 3, 4, 5.

UNIT IV

Introduction - directional derivative- Gradient- divergence- curl- Laplacian Differential Operator.
Chapter 2 Sections 2.1 - 2.13.

UNIT V

Line, surface and volume integrals - Integral Theorems - Gauss, Greens and Stokes (Without proof) – Problems.

Chapter 3 Sections 3.1 to 3.6 and Chapter 4 Sections 4.1 to 4.5.

Content and treatment as in

1. "Calculus", Vol- II by S. Narayanan and T.K. Manicavachagampillay - S. Viswanathanpublishers– 2007 for Unit 1 , Unit 2 , Unit 3.
2. "Vector Analysis" by P.Duraipandian and KayalalPachaiyappa, S.ChandFor Unit 4, Unit 5.

Reference:-

1. Integral Calculus and differential equations : Dipak Chatterjee (TATA McGraw Hill Publishing companyLtd.).
2. Vector Algebra and Analysis by Narayanan and T.K.Manickvachagam Pillay S .Viswanathan Publishers.
3. Vector Analysis: Murray Spiegel (Schaum Publishing Company, NewYork).

e-Resources:

1. http://mathworld.wolfram.com.
2. http://www.sosmath.com.

UNIT-1
REDUCTION FORMULAE

4.1Reduction formulae for $\sin^n x$ and $\cos^n x$:

Let $I_n = \int sin^n x dx = \int sin^{n-1} x sin\, x\, dx$

Integrating by parts by taking $sin^{n-1}x$ as first function and sin x as second function.

$I_n = \sin^{n-1}x(-\cos x) - \int (n-1)\, sin^{n-2}x \,.\cos x\,(-\cos x)\,dx$

$= -\sin^{n-1} x\cos x + (n-1) \int sin^{n-2} x cos^2\; x\, dx$

$= -\; \sin^{n-1} x \cos x + (n-1) \int sin^{n-2} x\,(1 - sin^2 x)\,dx$

$= -\; \sin^{n-1} x \cos x + (n-1) \int sin^{n-2} x\, dx - (n-1) \int sin^n x\, dx$

$I_n \;= -\; \sin^{n-1} x\cos x + (n-1)\, I_{n-2} - (n-1)\, I_n$

$\Rightarrow I_n\big(1 + (n-1)\big) = -\sin^{n-1} x\, \cos x + (n-1)\, I_{n-2}$

$$\Rightarrow I_n = \frac{-Sin^{n-1}x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$

is the required reduction formula for $\int sin^n x\, dx$

Similary $\int cos^n x\, dx = \frac{cos^{n-1}x\; Sin\, x}{n} + \frac{(n-1)}{n} I_{n-2}$

Derivation of formula for $\int_0^{\pi/2} \text{Sin}^n\, x\, d\, x$

$\int sin^n\, x\, dx \;= -\frac{1}{n}\left(sin^{n-1}x cos x\right) + \left(\frac{n-1}{n}\right) I_{n-2}$ (By reduction formula for $\int sin^n x\, dx$)

$\therefore \int_0^{\pi/2} Sin^n\, x\, d\, x = -\frac{1}{n}\left[sin^{n-1}x \cos x\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} Sin^n\, x\, d\, x$

$= 0 + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} Sin^{n-2}\, x\, d\, x$

$\therefore I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ (where $I_n = \int_0^{\pi/2} Sin^n\, x\, d\, x$)

Changing n to n–2, n–4, n–6,…..in successive steps, we get

$I_{n-2} = \left(\frac{n-3}{n-2}\right) I_{n-4}$

$I_{n-4} = \left(\frac{n-5}{n-4}\right) I_{n-6}$ and so on.

$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4}\, I_{n-6}$

Case (i) If n is an even positive integer, then

$I_n = \frac{n-1}{n} \quad \frac{n-3}{n-2} \ldots\ldots \frac{5}{6}, \frac{3}{4}, \frac{1}{2} \quad \int_0^{\pi/2} 1\, d\, x$

$\int_0^{\pi/2} Sin^n\, x\; d\, x = \frac{n-1}{n} \quad \frac{n-3}{n-2} \ldots\ldots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, if n is even

Case (ii) If n is an odd positive integer, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x\; d\, x$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \left[-cosx\right]_0^{\pi/2}$$

$\therefore \int_0^{\pi/2} sin^n x\; d\, x = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$, if n is odd

Example 1 Find $I_n = \int_0^{\pi/2} cos^n x\; d\, x$

Solution: $I_n = \int_0^{\pi/2} cos^n (\frac{\pi}{2} - x)\; d\, x$ $\quad$ ($\because \int_0^a f\,(x) dx = \int_0^a f\,(a - x) dx, if$ f is continuous function on [0,a])

$$= \int_0^{\pi/2} sin^n x\; d\, x = \begin{cases} \frac{n-1}{n} \; \frac{n-3}{n-2} \; \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} \cdot 1, if\ n\ is\ odd \\ \frac{n-1}{n} \; \frac{n-3}{n-2} \ldots\ldots \frac{5}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} \; \frac{\pi}{2}, if\ n\ is\ even \end{cases}$$

Example 2 Evaluate $\int_0^{\pi/2} sin^4 x\; d\, x$

Solution: $\int_0^{\pi/2} sin^4 x\; d\, x = \frac{(4-1)\,(4-3)}{4\,(4-2)} \frac{\pi}{2}$ ($\because$ n = 4 is even) $= \frac{3\pi}{16}$

Example 3 Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^4}$

Solution: Put x = $tan\theta$ $\Rightarrow$ dx = $\sec^2 \theta\, d\theta$

When x $\to$ 0, $\theta \to 0$ and when x $\to \infty$, $\theta \to \frac{\pi}{2}$

$\therefore$ Given integral becomes

$$\int_0^{\pi/2} \frac{sec^2\theta d\,\theta}{(1 + tan^2\theta)^4} = \int_0^{\pi/2} \frac{sec^2\theta}{(sec^2\theta)^4} = \int_0^{\pi/2} \frac{Sec^2\theta}{Sec^8\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{Sec^6\theta} = \int_0^{\pi/2} cos^6\; \theta d\theta$$

$$= \frac{(6-1)\,(6-3)\,(6-5)}{6\,(6-2)(6-4)} \frac{\pi}{2} = \frac{15\pi}{32}$$

Example 4 Obtain the reduction formula for $\int sin^m\, x\, cos^n\, x\, dx$

Solution: Let $I_{m,n} = \int sin^m\, x\, cos^n\, x\, dx$

$$= \int sin^m\, x\, cos^{n-1}\, x cos\, x\, dx$$

$$= \int cos^{n-1}\, x\, (sin^m x \cos x)\, dx$$

$$= cos^{n-1} x\; \frac{sin^{m+1} x}{m+1} - \int (n-1)\, cos^{n-2} x\, (-\sin x) \cdot \frac{sin^{m+1} x}{m+1}\; dx$$

(Integrating by parts) $\left(\because \int Sin^m\, x cos\;\; x\, dx \;\; = \frac{sin^{m+1}x}{m+1}\right)$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} \;\; + \frac{n-1}{m+1}\int cos^{n-2}\, x\, sin^{m+2}\, x\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} \;\; + \frac{n-1}{m+1}\int cos^{n-2}\, x\, sin^{m}\, x\, sin^2\, x\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} \;\; + \frac{n-1}{m+1}\int cos^{n-2}\, x\, sin^{m}\, x(1 - \; cos^2\, x)\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} \;\; + \frac{n-1}{m+1}\int cos^{n-2}\, x\, sin^{m}\, x\, dx - \frac{(n-1)}{m+1}\int cos^{n}\, x\, sin^{m}\, x\, dx$$

$$\mathrm{I}_{m,n} = \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} \qquad + \frac{n-1}{m+1}\, \mathrm{I}_{m,n-2} - \frac{(n-1)}{m+1}\, \mathrm{I}_{m,n}$$

$$\left(1 + \frac{n-1}{m+1}\right)\mathrm{I}_{m,n} = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+1} \;\; + \frac{n-1}{m+n} I_{m,n-2}$$

$$\Rightarrow\; \mathrm{I}_{m,n}\,(\mathrm{m} + \;\mathrm{n}) = \sin^{m+1} x \cos^{n-1} x \; + \; (\mathrm{n}-1)\mathrm{I}_{m,n-2}$$

$$\Rightarrow \mathrm{I}_{m,n} = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+n} \;\; + \frac{n-1}{m+n}\int sin^m\, x\, cos^{n-2}\, x\, dx$$

$$\Rightarrow\; \int sin^m\, x\, cos^n\, x\, dx \; = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+n} \;\; + \frac{n-1}{m+n}\int sin^m\, x\, cos^{n-2}\, x\, dx$$

Example 5 If $\mathrm{U}_n = \int_0^{\pi/2} x^n\, sinx\, dx$ and n > 1 prove that

$$\mathrm{U}_n + \mathrm{n}(\mathrm{n}-1)\mathrm{U}_{n-2} = \mathrm{n}\left(\frac{\pi}{2}\right)^{n-1}$$

Solution: $\mathrm{U}_n \; = \int_0^{\pi/2} x^n\, sinx\, dx$

$$= x^n \int \sin x\, dx - \int_2^{\pi/2}\left\{\frac{d}{dx}\,(x^n)\,[\int \sin x\, dx]\right\} dx$$

$$= \left[x^n\,(-\cos x)\right]^{\pi/2}{}_0 - \int_0^{\pi/2} n\, x^{n-1}\,(-\cos x)\, dx$$

$$= \; -\left[\left(\frac{\pi}{2}\right)^n \cos\frac{\pi}{2} - 0\right] \; + \int_0^{\pi/2} n\, x^{n-1} \cos x\, dx$$

$$= \; \mathrm{n}\int_0^{\pi/2} \; x^{n-1} \cos x\, dx$$

$$= \mathrm{n}\left\{[x^{n-1} sinx]_0^{\pi/2} - \int_0^{\pi/2} (n-1)\, x^{n-2} \sin x\; d\; x\right\}$$

$$= \mathrm{n}\left[x^{n-1} \sin x\right]_0^{\pi/2} - n\,(n-1)\int_0^{\pi/2} \; x^{n-2} \sin x\, dx$$

$$\Rightarrow \mathrm{U}_n = \mathrm{n}\left[\left(\frac{\pi}{2}\right)^{n-1} sin\frac{\pi}{2} - 0\right] - \mathrm{n}(\mathrm{n}-1)\mathrm{U}_{n-2}$$

$$\Rightarrow \mathrm{U}_n + \mathrm{n}(\mathrm{n}-1)\mathrm{U}_{n-2} = \mathrm{n}\left(\frac{\pi}{2}\right)^{n-1}$$

Example 6 Evaluate $\int_0^{\pi/2} \; x^4 \sin x\, dx$

Solution: $U_n = \int_0^{\pi/2} \; x^4 \sin x\, dx$

Now $U_n + \mathrm{n}(\mathrm{n}-1)U_{n-2} - 2 = \mathrm{n}\left(\frac{\pi}{2}\right)^{\mathrm{n}-1}$ ....(1)

Putting n = 4 in (1), we get

$$\mathrm{U}_4 + 4(4-1)\mathrm{U}_{4-2} = 4\left(\frac{\pi}{2}\right)^{4-1}$$

$$\Rightarrow \mathrm{U}_4 + 12\mathrm{U}_2 = \frac{\pi^3}{2} \qquad \text{.........(2)}$$

Putting n = 2 in (1) , we get

$$\mathrm{U}_2 + 2(2-1)\mathrm{U}_{2-2} = 2\left(\frac{\pi}{2}\right)^{2-1}$$

$$\mathrm{U}_2 + 2\mathrm{U}_0 = \pi \qquad \text{..........(3)}$$

Now $U_0 = \int_0^{\pi/2} x^0 \sin x \, dx = \int_0^{\pi/2} \sin x \, dx$

$$= \left[-\cos x\right]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$$

Hence equation (3) becomes

$$U_2 + 2\,(1) = \pi$$

$$\Rightarrow U_2 = \pi - 2$$

$\therefore$ (2) becomes $U_4 + 12\,(\pi - 2) = \frac{\pi^3}{2}$

$$U_4 = \frac{\pi^3}{2} - 12\pi + 24$$

$$\Rightarrow \int_0^{\pi/2} x^4 \sin x \, dx = \frac{\pi^3}{2} - 12\pi + 24$$

Example 7 If $I_{m,n} = \int_0^{\pi/2} sin^m cos^n x \, dx$ then prove that

$$I_{m,n} = \frac{\mathrm{m}-1}{\mathrm{m}+\mathrm{n}} \cdot \frac{\mathrm{m}-3}{\mathrm{m}+\mathrm{n}-2} \cdot \frac{\mathrm{m}-5}{\mathrm{m}+\mathrm{n}-4} \cdots\cdots\cdots\cdots \frac{2}{3+\mathrm{n}} \cdot \frac{1}{1+\mathrm{n}}$$

where m is an odd positive integer and n is a positive integer, even or odd.

Solution: $\int sin^m x\, cos^n x \, d\,x = \int sin^{m-1} x\, (\sin x\; cos^n)\; d\,x$

$$= \frac{-cos^{n+1} x\, sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int cos^{n+1} x\, sin^{m-2} x \cos x \, dx$$

(Integrating using by parts)

$$= \frac{-cos^{n+1} x\, sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int cos^{m-2} x\, cos^n x \left(1 - sin^2 x\right) dx$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) \int sin^m x\, cos^n x \, dx = -\frac{cos^{n+1} x\, Sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int sin^{m-2} x\, cos^n x \, dx$$

$$\Rightarrow \int sin^m x\, cos^n x \, dx = -\frac{cos^{n+1} x\, Sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int sin^{m-2} x\, cos^n x \, dx$$

Now $I_{m,n} = \int_0^{\pi/2} sin^m x\, cos^n x \, dx$

$$= \left[\frac{-cos^{n+1}x\, sin^{m-1}x}{m+n}\right]_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} sin^{m-2}x\, cos^n x\, dx$$

$$\Rightarrow \int_0^{\pi/2} sin^m x\, cos^n x\, dx = \frac{m-1}{m+n} \int_0^{\pi/2} sin^{m-2}x\, cos^n x\, dx$$

Hence, $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

Replacing m by m – 2, m – 4, …….,3, 2, we obtain

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

.

.

$$I_{3,n} = \frac{2}{3+n} I_{1,n}$$

$$I_{2,n} = \frac{1}{2+n} I_{0,n}$$

From these relations, we obtain

$$I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \dots \frac{2}{3+n} I_{1,n}, & \text{if m is odd} \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \dots \frac{1}{2+n} I_{0,n}, & \text{if m is even} \end{cases}$$

Now, we have

$$I_{1,n} = \int_0^{\pi/2} \sin x\ cos^n x\, dx = -\left[\frac{cos^{n+1}x}{n+1}\right]_0^{\pi/2} = \frac{1}{n+1}$$

And $I_{0,n} = \int_0^{\pi/2} cos^n x\, dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \dots \frac{2}{3}.1, if\ n\ is\ odd \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \dots \frac{1}{2}.\frac{\pi}{2}\ if\ n\ is\ even \end{cases}$

$$\therefore I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots .. \frac{2}{3+n} . \frac{1}{1+n} \\ if\ m\ is\ odd\ \ and\ n\ may\ be\ even\ or\ odd \\ \frac{m-1}{m+n} . \frac{m-3}{m+n-2} \dots \dots \frac{1}{2+n} \frac{n-1}{n} - \frac{n-3}{n-2} \dots \dots . \frac{2}{3} \\ if\ m\ is\ even\ and\ n\ is\ odd \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \dots \frac{1}{2+n} \frac{n-1}{n} - \frac{n-3}{n-2} \dots \dots \frac{1}{2} . \frac{\pi}{2} \\ if\ m\ is\ even\ \&\ n\ is\ even \end{cases}$$

These formulae can be expressed as a single formula

$$\int_0^{\pi/2} \sin^m x\ \cos^n x\, dx = \frac{(m-1)\,(m-3)\ldots\ldots\ldots\ldots(n-1)(n-3)\ldots\ldots}{(m+n)\,(m+n-2)(m+n-4)\ldots\ldots\ldots}$$

to be multiplied by $\frac{\pi}{2}$ when m & n both are even integers.

Example 8 Find $\int_0^{\pi/2} \sin^6 x\ \cos^5 x\, dx$

Solution: Here m = 6 and n = 5

$$\int_0^{\pi/2} \sin^6 x\ \cos^5 x\, dx = \frac{(6-1)\,(6-3)\,(6-5)\,(5-1)\,(5-3)}{(6+5)\,(6+5-2)\,(6+5-4)\,(6+5-6)(6+5-8)\,(6+5-10)} = \frac{8}{693}$$

Example 9 Evaluate $\int_0^{\pi} x\, \sin^7 x \cos^4 x\, dx$

Solution: Let I = $\int_0^{\pi} x\, \sin^7 x \cos^4 x\, dx$

$$= \int_0^{\pi} (\pi - x)\, \sin^7 (\pi - x)\, \cos^4 (\pi - x)\, dx \quad (\because \int_0^a f(x)dx = \int_0^a f(a-x)dx)$$

$$= \int_0^{\pi} (\pi - x) \sin^7 x \cos^4 x\, dx$$

$$= \pi \int_0^{\pi} \sin^7 x \cos^4 dx - \int_0^{\pi} x \sin^7 x \cos^4 x\, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sin^7 x \cos^4 dx$$

$$= 2\int_0^{\pi/2} \sin^7 x \cos^4 x dx \quad \because \int_0^{2a} f\,(x) dx = \begin{cases} 2\int_0^a f\,(x)dx, & if\ f\,(2a-x) = f\,(x) \\ 0, & if\ f\,(2a-x) = -f\,(x) \end{cases}$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \sin^7 x \cos^4 dx$$

$$= \frac{\pi\,(7-1)\,(7-3)\,(7-5)\,(4-1)(4-3)}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)} = \frac{16\,\pi}{385}$$

$$\left(using \int_0^{\pi/2} \sin^m x \cos^n dx = \frac{(m-1)\,(m-3)\ldots\ldots(n-1)\,(n-3)\ldots\ldots}{(m+n)\,(m+n-2)\,(m+n-4)\ldots\ldots\ldots\ldots} \right)$$

Example 10 Evaluate $\int_0^4 x^3 \sqrt{4x - x^2}\ dx$

Solution: Let I = $\int_0^4 x^3 \sqrt{4x - x^2}\ dx = \int_0^4 x^3 \sqrt{x\,(4-x)}\ dx$

$$= \int_0^4 x^3 \sqrt{x} \sqrt{(4-x)}\ dx$$

$$= \int_0^4 x^{7/2}\,(4-x)^{1/2}\ dx$$

Putting x = 4 $\sin^2\theta$ $\Rightarrow$ dx = 8sin $\theta \cos\theta$ d θ

Hence I = $\int_0^{\pi/2} 4^{7/2} \sin^7 \theta (4 - 4\sin^2\theta)^{1/2}\, 8 \sin\theta \cos\theta\, d\theta$

$$= \int_0^{\pi/2} 4^{7/2}\, 4^{1/2}\, 8 \sin^7 \theta\, (1 - \sin^2\theta)^{1/2} \sin\theta \cos\theta\, d\theta$$

$$= 8.\ 4^4 \int_0^{\pi/2} \sin^8 x \cos^2 dx$$

$$= 8.4^4 \frac{(8-1)\,(8-3)\,(8-5)\,(8-7)}{(8+2)\,(8+2-2)(8+2-4)(8+2-6)(8+2-8)} \frac{\pi}{2} = \frac{8.4^4.7.5.3}{10.8.6.4.2} \frac{\pi}{2} = 28\,\pi$$

Example 11 Evaluate $\int_0^{\infty} \frac{x^6 - x^3}{(1+x^3)^5} x^2 \; dx$

Solution: Let I = $\int_0^{\infty} \frac{(x^6 - x^3)}{(1+x^3)^5} x^2 \; dx$

Put $x^3 = \tan^2\theta \Rightarrow 3x^2\, dx = 2 \tan\theta \sec^2\theta d\theta$

Then I = $\int_0^{\pi/2} \frac{(tan^4\theta - tan^2\theta)}{(1+tan^2\theta)^5} \frac{2}{3} \tan\theta \sec^2\theta d\theta$

$= \frac{2}{3}\int_0^{\pi/2} \frac{tan^5\theta}{(sec^2\theta)^5} \; Sec^2\theta\, d\theta - \frac{2}{3}\int_0^{\pi/2} \frac{tan^3\theta}{(sec^2\theta)^5} sec^2\theta \, d\theta$

$= \frac{2}{3}\int_0^{\pi/2} \frac{tan^5\theta}{sec^8\theta} d\theta \quad - \frac{2}{3}\int_0^{\pi/2} \frac{tan^3\theta}{sec^8\theta} d\theta$

$= \frac{2}{3}\int_0^{\pi/2} sin^5\theta\, cos^3\theta d\theta - \frac{2}{3}\int_0^{\pi/2} sin^3\theta\, cos^5\theta d\theta$

$= \frac{2}{3}\int_0^{\pi/2} sin^5\theta\, cos^3\theta d\theta - \frac{2}{3}\int_0^{\pi/2} sin^3\left(\frac{\pi}{2} - \theta\right) cos^5\left(\frac{\pi}{2} - \theta\right) d\theta$

$= \frac{2}{3}\int_0^{\frac{\pi}{2}} sin^5\theta\, cos^3\theta d\theta - \frac{2}{3}\int_0^{\frac{\pi}{2}} cos^3\theta\, sin^5\theta d$

$\left[\because \int_0^a f(x) d x = \int_0^a f(a-x) d x\right]$

= 0

Example 12 Evaluate $\int_0^{\pi/2} sin^5 x\, d x$

Solution: We know $\int sin^n x\, d x = -\frac{sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int sin^{n-2} x\, dx$

$\therefore \int_0^{\pi/4} sin^5 x\, d x = \left[\frac{-sin^{5-1}x \cos x}{5}\right]_0^{\pi/4} + \frac{5-1}{5}\int_0^{\pi/4} sin^{5-2} x\, d x$

$= \frac{-1}{5}\left[sin^4 x \cos x\right]_0^{\pi/4} + \frac{4}{5}\int_0^{\pi/4} sin^3 x\, d x$

$= \frac{-1}{5}\left[\left(\frac{1}{\sqrt{2}}\right)^4 \left(\frac{1}{\sqrt{2}}\right)\right] + \frac{4}{5}\int_0^{\pi/4} sin^3 x\, d x$ ……..(1)

Now $\int_0^{\pi/4} sin^3 x\, d x = \left[-\frac{Sin^{3-1}x \cos x}{3}\right]_0^{\pi/4} + \frac{3-1}{3}\int_0^{\pi/4} sin^{3-2} x\, d x$

$= -\frac{1}{3}\left[sin^2 x \cos x\right]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} sin^3 x\, d x$

$= -\frac{1}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{2}}\right] + \frac{2}{4}\left(-\cos x\right)_0^{\pi/4}$

$= \frac{-1}{3.2\sqrt{2}} \quad \frac{-2}{4}\left(\frac{1}{\sqrt{2}} - 1\right)$

Putting this value is (1) ,we get

$\int_0^{\pi/4} sin^5 x\, d x = -\frac{1}{5}\left[\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{\sqrt{2}}\right] + \frac{4}{5}\left[\frac{-1}{6\sqrt{2}} - \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right)\right]$

$= \frac{-1}{5.4\sqrt{2}} - \frac{4}{5}\left[\frac{1}{6\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{2}\right]$

Example 13 Evaluate $\int_0^1 \frac{x^5}{2\sqrt{1-x^2}} dx$

Solution: Put x= Sin θ $\Rightarrow$ dx = cosθ d θ

Then the given integral becomes

$$\frac{1}{2}\int_0^{\pi/2} \frac{sin^5\theta}{\sqrt{1-sin^2\theta}} \cos\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} sin^5\,\theta \, d\theta$$

$$= \frac{1}{2} \frac{(5-1)\ (5-3)}{5\,(5-2)\,(5-4)} = \frac{4}{15}$$

Example 14 Evaluate $\int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1+sin\theta)^2 d\theta$

Solution: $\int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1-sin\theta)^2 d\theta = \int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1+sin^2\theta+2sin\theta) d\theta$

$$= \int_{-\pi/2}^{\pi/2} cos^3\,\theta\, d\theta + \int_{-\pi/2}^{\pi/2} cos^3\,\theta sin^2\theta + 2\int_{-\pi/2}^{\pi/2} cos^3\,\theta \sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} cos^3\,\theta\, d\theta + 2\int_0^{\pi/2} cos^3\, sin^2\,\theta + 0$$

$$= \frac{2(3-1)}{3\,(3-2)} + 2\frac{(2-1)\,(3-1)}{(3+2)\,(3+2-2)\,(3+2-4)}$$

$$= \frac{4}{3} + \frac{4}{15} = \frac{8}{5}$$

Exercise7A

1. Evaluate $\int_0^{2a} x^3\,(2ax-x^2)^{3/2}\,dx$ $\left(\text{Ans. } \frac{9\pi a^7}{16}\right)$
2. Evaluate $\int_0^{\infty} \frac{x^3}{(1+x^2)^{9/2}} dx$ $\left(\text{Ans } \frac{2}{35}\right)$
3. Evaluate $\int_0^{\pi/2} (cos2\theta)^{3/2} \cos\theta \quad d\theta$ $\left(\text{Ans } \frac{3\pi}{16\sqrt{2}}\right)$
4. Evaluate $\int_0^{\pi/2} sin^4 x\, cos3x\, dx$ $\left(\text{Ans } \frac{-13}{35}\right)$
5. Evaluate $\int_0^{a} x^2\,\sqrt{ax-x^2}\,dx$ $\left(\text{Ans } \frac{5\pi a^4}{128}\right)$
6. Evaluate $\int_0^{\pi/2} \frac{cos^2\theta}{cos^2\theta+4\,sin^2\theta}$ dθ $\left(\text{Ans } \frac{\pi}{6}\right)$
7. Evaluate $\int_0^{\pi} \frac{sin^4\theta\sqrt{1-\cos\theta}}{(1+\cos\theta)^2}$ dθ $\left(\text{Ans } \frac{64\sqrt{2}}{15}\right)$
8. Evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}$ dx $\left(\text{Ans } \frac{3\pi}{128}\right)$

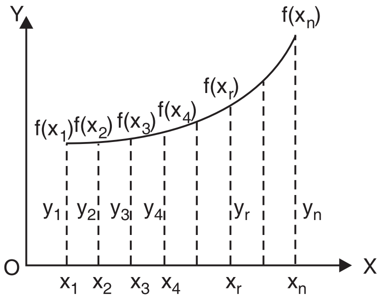
UNIT-2

Multiple Integrals

2.1 DOUBLE INTEGRATION

We know that

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x\to 0}} [f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots$$
$$+ f(x_n)\;\delta\; x_n]$$



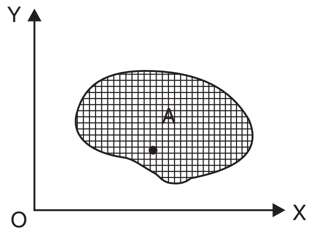
Let us consider a function $f(x, y)$ of two variable x and y defined in the finite region A of xy-plane. Divide the region A into elementary areas.

$$\delta A_1,\ \delta A_2,\ \delta A_3,\ \ldots\ldots\ \delta A_n$$

Then $$\iint_A f(x, y)\,dA$$

$$= \lim_{\substack{n\to\infty \\ \delta A\to 0}} \left[f(x_1, x_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots\ldots + f(x_n, y_n)\,\delta A_n\right]$$

2.2 EVALUATION OF DOUBLE INTEGRAL



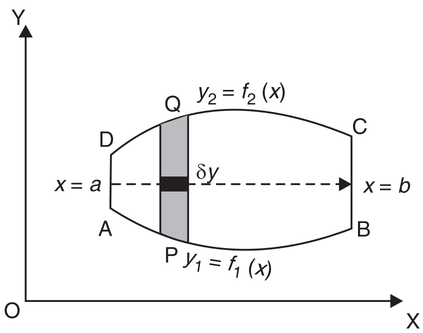
Double integral over region A may be evaluated by two successive integrations.

If A is described as $f_1(x) \le y \le f_2(x)$ $[y_1 \le y \le y_2]$
and $a \le x \le b$,

Then $$\iint_A f(x, y)\,dA = \int_a^b \int_{y_1}^{y_2} f(x, y)\,dx\,dy$$

(1) First Method

$$\iint_A f(x, y)\,dA = \int_a^b \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx$$



$f(x, y)$ is first integratred with respect to y treating x as constant between the limits a and b.

In the region we take an elementary area $\delta x \delta y$. Then integration w.r.t y (x keeping constant). converts small rectangle $\delta x\delta y$ into a strip PQ ($y\,\delta x$). While the integration of the result w.r.t. x corresponding to the sliding to the strip PQ, from AD to BC covering the while region $ABCD$.

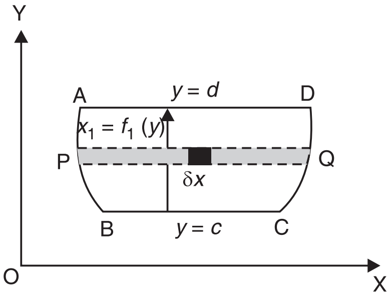
Second method

$$\iint_A f(x, y)\,dxdy = \int_c^d \boxed{\int_{x_1}^{x_2} f(x, y)\,dx}\,dy$$

Here $f(x,y)$ is first integrated w.r.t x keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated with respect to y between the limits c and d

Take a small area $\delta x \delta y$. The integration w.r.t. x between the limits x_1, x_2 keeping y fixed indicates that integration is done, along PQ. Then the integration of result w.r.t y corresponds to sliding the strips PQ from BC to AD covering the whole region $ABCD$.

Note. For constant limits, it does not matter whether we first integrate w.r.t x and then w.r.t y or vice versa.



Example 1. *Evaluate* $\int_0^1 \int_0^x (x^2 + y^2)\, dA$, *where dA indicates small area in xy-plane.*

(*Gujarat, I Semester, Jan. 2009*)

Solution. Let
$$I = \int_0^1 \int_0^x (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^x dx$$
$$= \int_0^1 \left[x^2 (x - 0) + \frac{1}{3}(x^3 - 0) \right] dx = \int_0^1 \left[x^3 + \frac{x^3}{3} \right] dx$$
$$= \int_0^1 \frac{4}{3} x^3\, dx = \frac{4}{3}\left[\frac{x^4}{4} \right]_0^1 = \frac{1}{3}[1 - 0] = \frac{1}{3} \text{ sq. units.}$$
Ans.

Example 2. *Evaluate* $\int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1 - x - y)^{1/2}\, dy\, dx$. (*M.U., II Semester 2002*)

Solution. Here, we have
$$I = \int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1 - x - y)^{1/2}\, dy\, dx \qquad ...(1)$$
Putting $(1 - x) = c$ in (1), we get
$$I = \int_{-1}^{1} x^{1/3}\, dx \int_0^c y^{-1/2}\, (c - y)^{1/2}\, dy \qquad ...(2)$$
Again putting $y = ct \Rightarrow dy = c\, dt$ in (2), we get
$$I = \int_{-1}^{1} x^{\frac{1}{3}}\, dx \int_0^1 c^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, (c - ct)^{\frac{1}{2}}\, c\, dt$$
$$= \int_{-1}^{1} x^{1/3}\, dx \int_0^1 c^{-1/2}\, t^{-1/2}\, c^{1/2}\, (1 - t)^{1/2}\, c\, dt$$
$$= \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^c t^{-1/2} (1 - t)^{1/2}\, dt = \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^1 t^{1/2 - 1}\, (1 - t)^{3/2 - 1}\, dt$$
$$= \int_{-1}^{1} c\, x^{\frac{1}{3}}\, dx\, \beta\left(\frac{1}{2}, \frac{3}{2}\right) \qquad \left[\int_0^1 x^{l-1} (1 - x)^{m-1}\, dx = \beta(l, m) \right]$$
$$= \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{3}{2}\right)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\sqrt{\pi}\, \frac{1}{2} \sqrt{\pi}}{1}$$
$$= \int_{-1}^{1} c\, x^{1/3} \frac{\pi}{2}\, dx = \frac{\pi}{2} \int_{-1}^{1} x^{1/3} \cdot c\, dx$$

Putting the value of c, we get

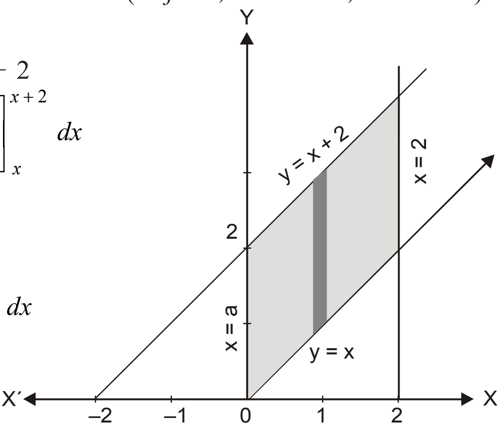
$$I = \frac{\pi}{2}\int_{-1}^{1} x^{1/3}(1-x)\,dx = \frac{\pi}{2}\int_{-1}^{1}(x^{1/3}-x^{4/3})\,dx = \frac{\pi}{2}\left[\frac{x^{4/3}}{\frac{4}{3}} - \frac{x^{7/3}}{\frac{7}{3}}\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\frac{3}{4}(1) - \frac{3}{7}(1) - \frac{3}{4}(-1) + \frac{3}{7}(-1)\right] = \frac{\pi}{2}\left[\frac{9}{14}\right] = \frac{9\pi}{28}$$ **Ans.**

Example 3. *Evaluate* $\iint_R (x+y)\,dy\,dx$, *R is the region bounded by* $x = 0$, $x = 2$, $y = x$, $y = x + 2$. (*Gujarat, I Semester, Jan. 2009*)

Solution. Let $I = \iint_R (x+y)\,dy\,dx$

The limits are $x = 0$, $x = 2$, $y = x$ and $y = x + 2$



$$I = \int_0^2 dx \int_x^{x+2}(x+y)\,dy = \int_0^2 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx$$

$$= \int_0^2 \left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 \left[x^2 + 2x + \frac{1}{2}(x^2+4x+4) - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 [2x + 2x + 2]\,dx$$

$$= 2\int_0^2 (2x+1)\,dx = 2[x^2 + x]_0^2 = 2[4+2] = 12$$ **Ans.**

Example 4. *Evaluate* $\iint_R xy\,dx\,dy$

where R is the quadrant of the circle $x^2 + y^2 = a^2$ *where* $x \ge 0$ *and* $y \ge 0$.

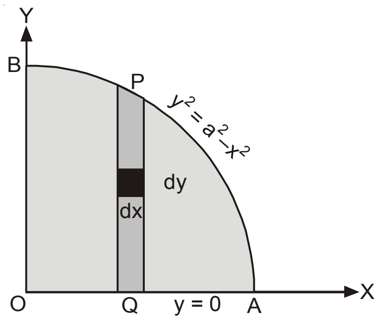
(*A.M.I.E.T.E, Summer 2004, 1999*)

Solution. Let the region of integration be the first quadrant of the circle OAB.

$$\iint_R xy\,dx\,dy \quad (x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2})$$

First we integrate w.r.t. y and then w.r.t. x.

The limits for y are 0 and $\sqrt{a^2 - x^2}$ and for x, 0 to a.



$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy = \int_0^a x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\,dx = \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{8}$$ **Ans.**

Example 5. *Evaluate* $\iint_s \sqrt{xy - y^2}\,dy\,dx$,

where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).

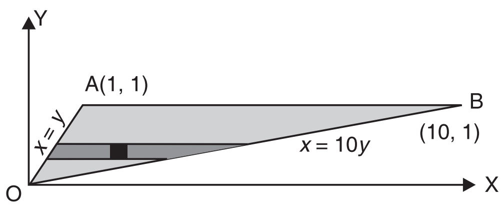
Solution. Let the verties of a triangle OBA be (0, 0) (10, 1) and (1, 1).

Equation of OA is $x = y$.

Equation of OB is $x = 10y$.

The region of ΔOBA, given by the limits

$y \le x \le 10y$ and $0 \le y \le 1$.



$$\iint_s \sqrt{xy - y^2}\,dy\,dx = \int_0^1 dy \int_y^{10y} (xy - y^2)^{1/2}\,dx$$

$$= \int_0^1 dy \left[\frac{2}{3}\,\frac{1}{y}\,(xy - y^2)^{3/2}\right]_y^{10y} = \int_0^1 \frac{2}{3}\,\frac{1}{y}\,(9y^2)^{3/2}\ dy = 18\int_0^1 y^2\ dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 = \frac{18}{3} = 6 \qquad \textbf{Ans.}$$

Example 6. *Evaluate* $\iint_A x^2\ dx\ dy$, *where A is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8.* (*A.M.I.E., Summer 2001*)

Solution. The line OP, $y = x$ and the curve PS, $xy = 16$ intersect at (4, 4).

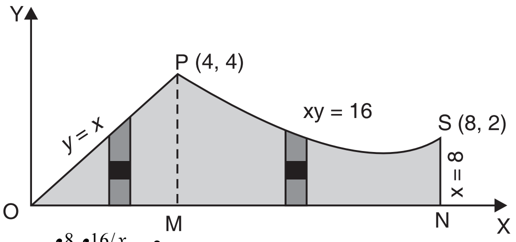
The line SN, $x = 8$ intersects the hyperbola at S (8, 2). $y = 0$ is x-axis.

The area A is shown shaded.

Divide the area in to two part by PM perpendicular to OX.

For the area OMP, y varies from 0 to x, and then x varies from 0 to 4.

For the area PMNS, y -series from 0 to $16/x$ and then x varies from 4 to 8.



$$\therefore \qquad \iint_A x^2\ dx\ dy = \int_0^4\int_0^x x^2\ dx\ dy + \int_4^8\int_0^{16/x} x^2\ dx\ dxy$$

$$= \int_0^4 x^2\ dx\int_0^x dy + \int_4^8 x^2\ dx\int_0^{16/x} dy = \int_0^4 x^2\,[y]_0^x\ dx + \int_4^8 x^2\,[y]_0^{16/x}\ dx$$

$$= \int_0^4 x^3\ dx + \int_4^8 16\,x\ dx = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64 + 8\,(8^2 - 4^2) = 64 + 384 = 448. \ \textbf{Ans.}$$

Example 7. *Evaluate* $\iint (x+y)^2\ dx\ dy$ *over the area bounded by the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

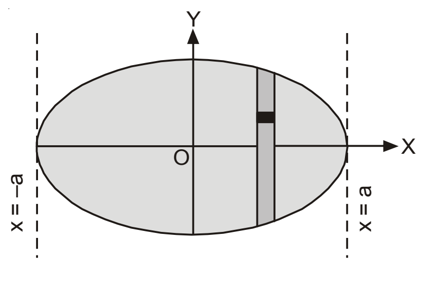
(*U.P. Ist Semester Compartment 2004*)

Solution. For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \Rightarrow \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$\therefore$ The region of integration can be expressed as

$$-a \le x \le a \text{ and } -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$



$$\therefore \quad \iint (x+y)^2\ dx\ dy = \iint (x^2 + y^2 + 2xy)\ dx\ dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy)\ dx\ dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2)\ dx\ dy \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\ dy\ dx$$

$$= \int_{-a}^{a}\int_{0}^{b/a\sqrt{a^2-x^2}} 2\,(x^2 + y^2)\ dy\ dx + 0$$

[Since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a}\left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_0^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2 \times \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

[On putting $x = a \sin \theta$ and $dx = a \cos \theta \, d\theta$]

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{b}{a}\,.\,a^2\sin^2\theta\,.\,a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right)\times a\cos\theta\, d\theta$$

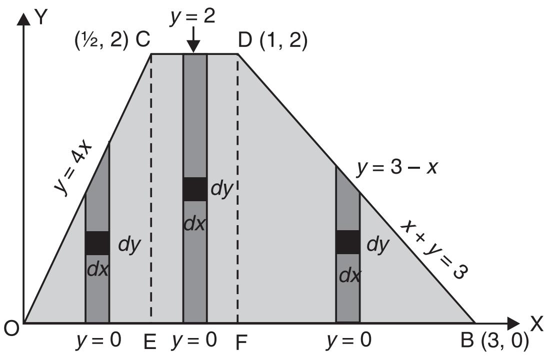
$$= 4\int_0^{\frac{\pi}{2}}\left(a^3 b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right)d\theta = 4\left[a^3b\,.\,\frac{1}{4}\,.\,\frac{1}{2}\,.\,\frac{\pi}{2} + \frac{ab^3}{3}\,.\,\frac{3}{4}\,.\,\frac{1}{2}\,.\,\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}(a^3b + ab^3) = \frac{\pi}{4}ab\,(a^2+b^2)$$ **Ans.**

Example 8. *Evaluate* $\iint (x^2 + y^2)\, dx\, dy$ *throughout the area enclosed by n the curves* $y = 4x$, $x + y = 3$, $y = 0$ *and* $y = 2$.

Solution. Let OC represent $y = 4x$; BD, $x + y = 3$; OB, $y = 0$, and CD, $y = 2$. The given integral is to be evaluated over the area A of the trapezium $OCDB$.

Area $OCDB$ consists of area OCE, area $ECDF$ and area FDB.



The co-ordinates of C, D and B are $\left(\frac{1}{2}, 2\right)$ (1, 2) and (3, 0) respectively.

$$\therefore \quad \iint_A (x^2+y^2)\,dy\,dx$$

$$= \iint_{OCE}(x^2+y^2)\,dy\,dx + \iint_{ECDE}(x^2+y^2)\,dy\,dx + \iint_{FDB}(x^2+y^2)\,dy\,dx$$

$$= \underbrace{\int_0^{½}dx\int_0^{4x}(x^2+y^2)\,dy}_{I_1} + \underbrace{\int_{½}^{1}dx\int_0^{2}(x^2+y^2)\,dy}_{I_2} + \underbrace{\int_1^{3}dx\int_0^{3-x}(x^2+y^2)\,dy}_{I_3}$$

Now, $$I_1 = \int_0^{½}dx\int_0^{4x}(x^2+y^2)\,dy = \int_0^{½}\left[x^2y + \frac{y^3}{3}\right]_0^{4x}dx = \int_0^{½}\frac{76}{3}x^3\,dx$$

$$= \frac{76}{3}\int_0^{½}x^3\,dx = \frac{76}{3}\left[\frac{x^4}{4}\right]_0^{½} = \frac{76}{3}\left[\frac{1}{4}\,.\,\frac{1}{16}\right] = \frac{19}{48}$$

$$I_2 = \int_{½}^{1}dx\int_{½}^{1}(x^2+y^2)\,dy = \int_{½}^{1}\left[x^2y + \frac{y^3}{3}\right]_0^{2}dx = \int_{½}^{1}\left(2x^2 + \frac{8}{3}\right)dx$$

$$= \left[\frac{2x^3}{3} + \frac{8}{3}x\right]_{½}^{1} = \left[\left(\frac{2}{3}+\frac{8}{3}\right) - \left(\frac{2}{3}\,.\,\frac{1}{8} + \frac{8}{3}\,.\,\frac{1}{2}\right)\right] = \frac{23}{12}$$

$$I_3 = \int_1^{3}dx\int_0^{3-x}(x^2+y^2)\,dy = \int_1^{3}\left[x^2y + \frac{y^3}{3}\right]_0^{3-x}dx = \int_0^{3}\left[x^2\,(3-x) + \frac{(3-x)^3}{3}\right]dx$$

$$= \int_1^{3}\left[3x^2 - x^3 + \frac{(3-x)^3}{3}\right]dx = \left[x^3 - \frac{x^4}{4} - \frac{(3-x)^4}{3}\right]_1^{3}$$

$$= \left[27 - \frac{81}{4} - 0 - 1 + \frac{1}{4} + \frac{16}{12}\right] = \frac{22}{3}$$

$$\therefore \quad \int_A \int (x^2 + y^2)\, dy\, dx = I_1 + I_2 + I_3 = \frac{19}{48} + \frac{23}{12} + \frac{22}{3} = \frac{463}{48} = 9\frac{31}{48}. \qquad \textbf{Ans.}$$

EXERCISE 2.1

Evaluate

1. $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$ **Ans.** $e^2 - 1$

2. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$ **Ans.** $\frac{a^4}{6}$

3. $\int_0^a \int_0^{\sqrt{a^2 - y^2}} dx\, dy$ **Ans.** $\frac{\pi a^2}{4}$

4. $\int_0^1 \int_{y^2}^{y} (1 + xy^2)\, dx\, dy$ **Ans.** $\frac{41}{210}$

5. $\int_0^{2a} \int_0^{\sqrt{2ax - x}} xy\, dy\, dx$ **Ans.** $\frac{2a^4}{3}$

6. $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\, dy\, dx$ **Ans.** $\frac{5\pi a^4}{8}$

7. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$ **Ans.** $\frac{\pi a^3}{4}$

8. $\int_0^1 \int_0^{\sqrt{\frac{1}{2}(1 - y^2)}} \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}$ **Ans.** $\frac{\pi}{4}$

9. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dx\, dy}{(1 + e^y)\sqrt{a^2 - x^2 - y^2}}$ **Ans.** $\frac{\pi}{2} \log \frac{2e^a}{1 + e^a}$

10. $\int_0^a \int_y^a \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$ **Ans.** $\frac{a^2}{2} \log(\sqrt{2} + 1)$

11 $\int_{x=0}^{1} \int_{y=0}^{2} (x^2 + 3xy^2)\, dx\, dy$ (*A.M.I.E.T.E., June 2009*) **Ans.** $\frac{14}{3}$

12. $\iint_A (5 - 2x - y)\, dx\, dy$, where A is given by $y = 0$, $x + 2y = 3$, $x = y^2$. **Ans.** $\frac{217}{60}$

13. $\iint_A xy\, dx\, dy$, where A is given by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$, $y = x$. **Ans.** $\frac{7}{12}$

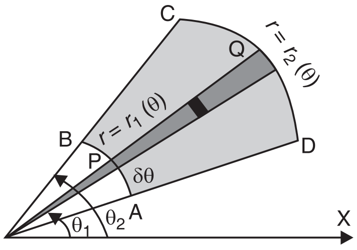
14. $\iint_A \sqrt{4x^2 - y^2}\, dx\, dy$, where A is the triangle given by $y = 0$, $y = x$ and $x = 1$. **Ans.** $\frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$

15. $\iint_R x^2\, dx\, dy$, where R is the two-dimensional region bounded by the curves $y = x$ and $y = x^2$. **Ans.** $\frac{1}{20}$

16. $\iint_A \sqrt{xy(1 + x - y)}\, dx\, dy$ where A is the area bounded by $x = 0$, $y = 0$ and $x + y = 1$. **Ans.** $\frac{2\pi}{105}$

2.3 EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

We have to evalaute $\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, dr\, d\theta$ over the region bounded by the staight lines $\theta = \theta_1$ and $\theta = \theta_2$ and the curves $r = r_1$ (q) and $r = r_2$ (θ). We first integrate with respect to r between the limits $r = r_1(\theta)$ and $r = r_2(\theta)$ and taking θ as constant. Then the resulting expression is integrated with respect to θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.

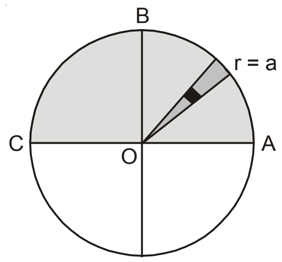


The area of integration is *ABCD*. On integrating first with respect to r, the strip extends from P to Q and the integration with respect to θ means the rotation ot this strip PQ from AD to BC.

Example 9. *Transform the integral to cartesian form and hence evaluate*

$\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta.$ (*M.U., II Semester 2000*)

Solution. Here, we have



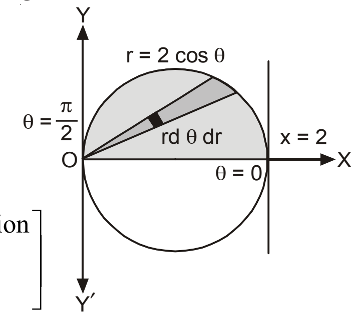
$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\, d\theta \quad ...(1)$$

Here the region *i.e.*, semicircle *ABC* of integration is bounded by $r = 0$, *i.e.*, x-axis.

$r = a$ *i.e.*, circle, $\theta = 0$ and $\theta = \pi$ *i.e.*, x-axis in the second quadrant.

$$\int\int (r\sin\theta)(r\cos\theta)(r\, d\theta\, dr)$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, d\theta\, dr$ in (1), we get



$$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_{-a}^{a} x\, dx \int_0^{\sqrt{a^2-x^2}} y\, dy$$

$$= \int_{-a}^{a} x\, dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} = \int_{-a}^{a} x\, dx \, \frac{(a^2-x^2)}{2}$$

$$= \frac{1}{2}\int_{-a}^{a} (a^2 x - x^3)\, dx = 0 \textbf{ Ans.} \quad \left[\begin{array}{l}\text{Since } f(x) \text{ is odd function} \\ \int_{-a}^{a} f(x)\, dx = 0\end{array}\right]$$

Example 10. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Solution. $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

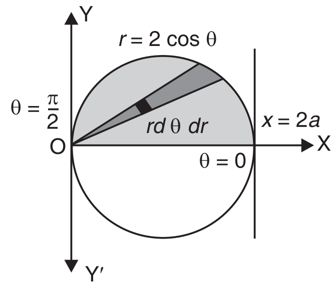
Limits of $\quad y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0 \quad ...(1)$

(1) represents a circle whose centre is (1, 0) and radius = 1.

Lower limit of y is 0 *i.e., x*-axis.

Region of integration is upper half circle.

Let us convert (1) into polar co-ordinate by putting



$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 - 2r\cos\theta = 0 \Rightarrow r = 2\cos\theta$$

Limits of r are 0 to $2\cos\theta$

Limits of θ are 0 to $\frac{\pi}{2}$

$$\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2 (r\, d\theta\, dr) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3\, dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 4\times\frac{3\times1\times\pi}{4\times2\times2} = \frac{3\pi}{4} \qquad \textbf{Ans.}$$

Example 11. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ *by changing to polar coordinates.*

Solution. In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

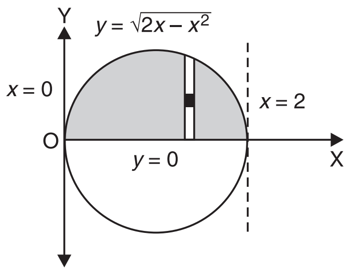
$$y = \sqrt{2x-x^2}$$

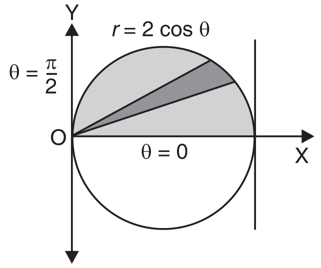
$$\Rightarrow \qquad y^2 = 2x - x^2$$
$$\Rightarrow \qquad x^2 + y^2 = 2x$$

In polar co-ordinates, we have $r^2 = 2r \cos \theta \quad \Rightarrow \quad r = 2 \cos \theta$.

$\therefore$ For the region of integarion, r varies from 0 to $2 \cos \theta$ and θ varies from 0 to $\dfrac{\pi}{2}$.

In the given integral, replacing x by $r \cos \theta$, y by $r \sin \theta$, $dy\, dx$ by $r\, dr\, d\theta$, we have





$$I = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r \cos \theta . r\, dr\, d\theta}{r} = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cos \theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos \theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2 \cos^3 \theta\, d\theta = 2 . \frac{2}{3} = \frac{4}{3}. \qquad \textbf{Ans.}$$

EXERCISE 2.2

Evaluate the folloing:

1. $\int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin \theta\, d\theta\, dr$ **Ans.** $\dfrac{8}{3} \pi a^3$

2. $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos \theta\, dr\, d\theta$ **Ans.** $\dfrac{5}{8} \pi\, a^3$

3. $\iint_A \dfrac{r\, dr\, d\theta}{\sqrt{r^2 + a^2}}$ where A is a loop of $r^2 = a^2 \cos 2\theta$ **Ans.** $2a - \dfrac{\pi a}{2}$

4. $\iint_A r^2 \sin \theta\, d\theta\, dr$ where A is $r = 2a \cos \theta$ above initial line. (*A.M.I.E. Winter 2001*) **Ans.** $\dfrac{2a^3}{3}$

5. Calculate the integral $\iint \dfrac{(x-y)^2}{x^2 + y^2}\, dx\, dy$ over the circle $x^2 + y^2 \le 1$. **Ans.** $\pi - 2$

6. $\iint (x^2 + y^2)\, x\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ by changing to polar coordinates.

Ans. $\dfrac{a^2}{5}$

7. $\iint_R \sqrt{x^2 + y^2}\, dx\, dy$ by changing to polar coordinates, R is the region in the xy-plane bounded by the circles $x^2 + y^2 = 4$ (*AMIETE, Dec. 2009*) **Ans.** $\dfrac{38\pi}{3}$

8. Convert into polar coordinates

$$\int_0^{2a} \int_0^{2ax - x^2} dx\, dy$$

Ans. $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$

9. $\iint r^3\, dr\, d\theta$, over the area bounded between the circles $r = 2b \cos \theta$ and $r = 2b \cos \theta$. **Ans.** $\dfrac{3\pi}{2} (a^4 - b^4)$

10. $\iint r \sin \theta\, dr\, d\theta$ over the area of the cardiod $r = a\,(1 + \cos \theta)$ above the initial line. **Ans.** $\dfrac{5}{8} \pi\, a^3$

11. $\int\int_A x^2\, dr\, d\theta$, where A is the area between the circles $r = a\cos\theta$ and $r = 2a\cos\theta$. **Ans.** $\dfrac{28a^3}{9}$

12. Transform the integral $\int_0^1\int_0^x f(x, y)\, dy\, dx$ to the integral in polar co-ordinates.

Ans. $\int_0^{\pi/4}\int_0^{\sec\theta} f(r, \theta)\, r\, d\theta\, dr$

2.4 CHANGE OF ORDER OF INTEGRATION

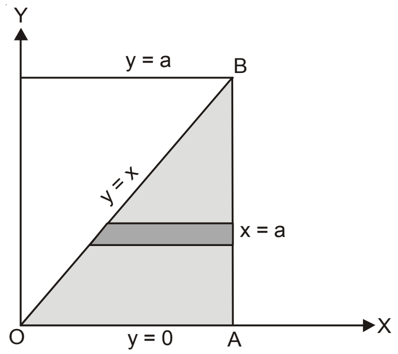
On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.

Some of the problems connected with double integrals, which seem to be complicated, can be made easy to handle by a change in the order of integration.

Example 12. *Evaluate* $\int_0^a\int_y^a \dfrac{x}{x^2+y^2}\, dx\, dy$ *by changing the order of integration.*

(*AMIETE, June 2010, Nagpur University, Summer 2008*)

Solution. Here we have



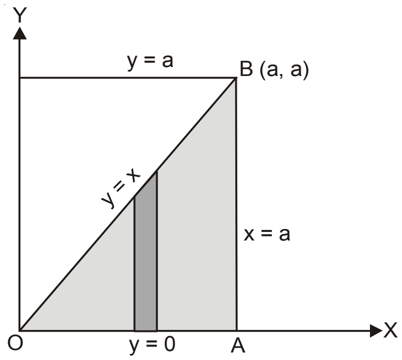
$$I = \int_0^a\int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

Here $x = a, x = y, y = 0$ and $y = a$

The area of integration is OAB.

On changing the order of integration Lower limit of $y = 0$ and upper limit is $y = x$.

Lower limit of $x = 0$ and upper limit is $x = a$.



$$I = \int_0^a x\,dx \int_0^{y=x} \frac{1}{x^2+y^2}\, dy$$

$$= \int_0^a x\,dx\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{y=x}$$

$$= \int_0^a \frac{x}{x}\,dx\left(\tan^{-1}\frac{x}{x} - \tan^{-1}0\right)$$

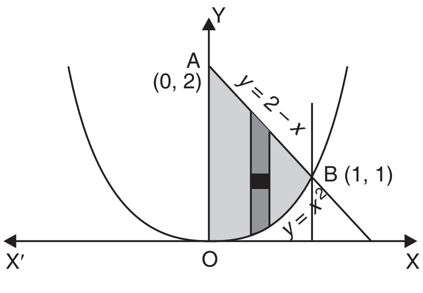
$$= \int_0^a dx\left(\frac{\pi}{4}\right) = \frac{\pi}{4}[x]_0^a = \frac{a\pi}{4} \quad \textbf{Ans.}$$

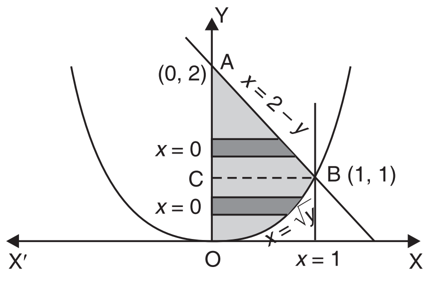
Example 13. *Change the order of integration in*

$I = \int_0^1\int_{x^2}^{2-x} xy\, dx\, dy$ *and hence evaluate the same.*

(*A.M.I.E.T.E., June 2010, 2009, U.P. I Sem., Dec., 2004*)

Solution. $I = \int_0^1\int_{x^2}^{2-x} xy\, dx\, dy$





The region of integration is shown by shaded portion in the figure bounded by parabola $y = x^2$ and the line $y = 2 - x$.
The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is B (1, 1).
In the figure below (left) we have taken a strip parallel to y-axis and the order of integration is

$$\int_0^1 x\, dx \int_{x^2}^{2-x} y\, dy$$

In the second figure above we have taken a strip parallel to x-axis in the area OBC and second strip in the area ABC. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area ABC are 0 and $2 - y$.

$$= \int_0^1 y\, dy \int_0^{\sqrt{y}} x\, dx + \int_1^2 y\, dx \int_0^{2-y} x\, dx = \int_0^1 y\, dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_0^{\sqrt{y}} y\, dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2\, dy + \frac{1}{2}\int_1^2 y\,(2-y)^2\, dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy$$

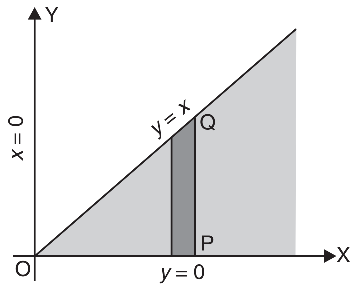
$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96 - 128 + 48 - 24 + 16 - 3}{12}\right] = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$ **Ans.**

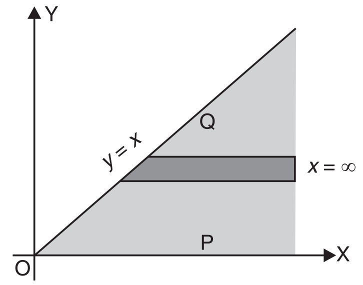
Example 14. *Evaluate the integral* $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy$ *by changing the order of integration* (*U.P. I Semester Dec., 2005*)

Solution. Limits are given

$$y = 0 \text{ and } y = x$$
$$x = 0 \text{ and } x = \infty$$



Here, the elementary strip PQ extends from $y = 0$ to $y = x$ and this vertical strip slides from $x = 0$ to $x = \infty$.
The region of integration is shown by shaded portion in the figure bounded by $y = 0$, $y = x$, $x = 0$ and $x = \infty$.
On changing the order of integration, we first integrate with respect to x along a horizontal strip RS which extends from $x = y$ to $x = \infty$ and this horizontal strip slides from $y = 0$ to $y = \infty$ to cover the given region of integration.



New limits :

$$x = y \quad \text{and} \quad x = \infty$$
$$y = 0 \quad \text{and} \quad y = \infty$$

We first integrate with respect to x.
Thus,

$$\int_0^\infty dy \int_y^\infty x e^{-\frac{x^2}{y}}\, dx = \int_0^\infty dy \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-\frac{x^2}{y}}\right) dx$$

$$= \int_0^\infty dy \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty = \int_0^\infty dy \left[0 + \frac{y}{2} e^{-\frac{y^2}{2}}\right] = \int_0^\infty \frac{y}{2} e^{-y}\, dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \left(\frac{1}{2}\right)(e^{-y})\right]_0^{\infty}$$ (Integrating by parts)

$$= \left[(0-0) - \left(0 - \frac{1}{2}\right)\right] = \frac{1}{2}$$ **Ans.**

Example 15. *Change the order of the integration*

$$\int_0^{\infty}\int_0^{x} e^{-xy}\ y\ dy\ dx$$ (*B.P.U.T.; I Semester 2008*)

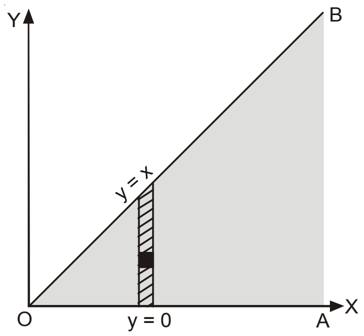
Solution. Here, we have

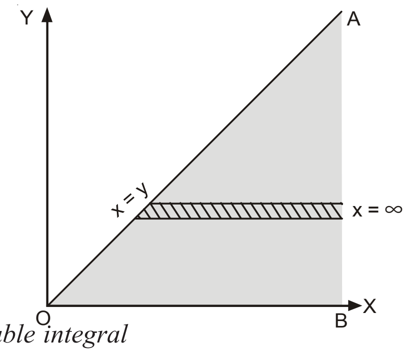
$$\int_0^{\infty}\int_0^{x} e^{-xy}\ y\ dy\ dx$$

Here the region OAB of integration is bounded by $y = 0$ (x-axis), $y = x$ (a straight line), $x = 0$, *i.e.*, y axis. A strip is drawn parallel to y-axis, y varies 0 to x and x varies 0 to ∞.

On changing the order of integration, first we integrate w.r.t. x and then w.r.t. y.

A strip is drawn parallel to x-axis. On this strip x varies from y to ∞ and y varies from 0 to ∞.





Hence $\displaystyle\int_0^{\infty}\int_0^{x} e^{-xy}\ y\ dy\ dx = \int_0^{\infty} y\ dy \int_y^{\infty} e^{-xy} dx$

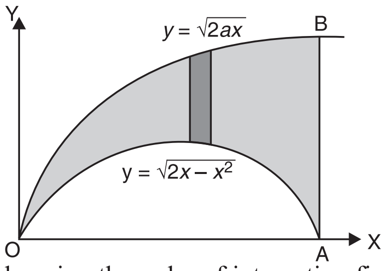
$$= \int_0^{\infty} y\ dy \left(\frac{e^{-xy}}{-y}\right)_y^{\infty}$$

$$= \int_0^{\infty} \frac{y\ dy}{-y}[0 - e^{y^2}]$$

$$= \int_0^{\infty} e^{-y^2}\ dy = \frac{1}{2}\sqrt{\pi}$$ **Ans.**

Example 16. *Change the order of integration in the double integral*

$$\int_0^{2a}\int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} V\ dx\ dy$$



Solution. Limits are given as

$$x = 0,\ x = 2a$$

$$y = \sqrt{2\ ax}$$

and $\quad y = \sqrt{2\ ax - x^2} \quad \Rightarrow \quad y^2 = 2\ ax$

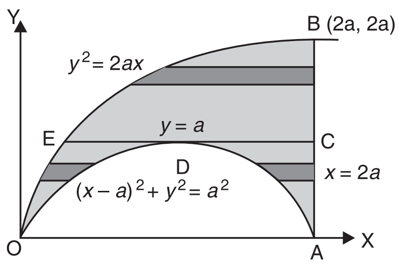
and $\quad (x - a)^2 + y^2 = a^2$

The area of integration is the shaded portion OAB. On changing the order of integration first we have to integrate w.r.t. x, The area of integration has three portions BCE, ODE and ACD.

$$\int_0^{2a} dx \int_{\sqrt{2\ ax - x^2}}^{\sqrt{2ax}} V\ dy$$

$$= \int_0^{2a} dy \int_{y^2/2a}^{2a} V\ dx + \int_0^{a} dy \int_{y^2/2a}^{a+\sqrt{a^2+y^2}} V\ dx$$

$$+ \int_0^{a} dy \int_{a+\sqrt{a^2-y^2}}^{2a} V\ dx$$ **Ans.**



EXERCISE 2.3

Change the order of integration and hence evaluate the following:

1. $\int_0^a \int_0^x \frac{\cos y\, dy}{\sqrt{(a-x)(a-y)}}\, dx$ **Ans.** $(a) \int_0^a dy \int_y^a \frac{\cos y\, dx}{\sqrt{(a-x)(a-y)}}$ (b) $2 \sin a$.

2. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-x} (x^2+y^2)\, dy\, dx$ **Ans.** $(a) \int_0^a dy \int_0^{2\sqrt{ay}} (x^2+y^2)\, dx + \int_a^{3a} dy \int_0^{3a-y} (x^2+y^2)\, dx$ (b) $\frac{314\, a^4}{35}$.

3. $\int_0^1 \int_{x^2}^{x} (x^2+y^2)^{-1/2}\, dy\, dx$ **Ans.** $\int_0^1 dy \int_y^{\sqrt{y}} (x^2+y^2)^{-1/2}\, dx.$

4. $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x, y)\, dx\, dy$ **Ans.** $\int_0^a dx \int_{\sqrt{a^2-x^2}}^{a} f(x, y)\, dy + \int_a^{2a} dx \int_{x-a}^{a} f(x, y)\, dy$

5. $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy$ **Ans.** $\int_0^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy$

6. $\int_0^1 \int_x^{2-x} \frac{x}{y} dy\, dx$ **Ans.** $\int_0^a \frac{dy}{y} \int_0^y x dx + \int_1^2 \frac{dy}{y} \int_0^{2-y} x dx; \log \frac{4}{e}$

7. $\int_0^b \int_y^a \frac{x\, dy\, dx}{x^2+y^2}$ (*M.P. 2003*)

8. $\int_0^a \int_0^{bx/a} x\, dy\, dx$ **Ans.** $(a) \int_0^b dy \int_{ay/b}^{a} x\, dx$ (b) $\frac{1}{3} a^2 b$

9. $\int_0^5 \int_{2-x}^{2+x} f(x, y)\, dx\, dy$ **Ans.** $\int_0^2 dy \int_{2-y}^{5} f(x, y)\, dx + \int_2^7 dy \int_{y-2}^{5} f(x, y)\, dx$

10. $\int_0^\infty \int_{-y}^{y} (y^2-x^2)\, e^{-y}\, dx\, dy$ **Ans.** $\int_{-\infty}^{\infty} dx \int_{-x}^{x} (y^2-x^2)\, e^{-y}\, dy$ (*A.M.I.E., Summer 2000*)

11. $\int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} \times yx\, dy$ (*A.M.I.E.T.E., June 2009*)

12. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dx\, dy$ (*U.P. I Semester, Dec., 2007*) **Ans.** $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy + \int_0^{2a-y} xy\, dx\, dy, \frac{3a^2}{8}$

13. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$ **Ans.** $\int_0^{2a} x\, dx \int_0^{\sqrt{a^2-(x-a)^2}} y\, dy, \frac{2}{3} a^4$

[**Hint:** Put $x = a \sin^2 \theta \Rightarrow dx = 2\, a \sin \theta \cos \theta\, d\, \theta$]

14. $\int_0^1 \int_{-1}^{1-y} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\, dx\, dy$ **Ans.** $\int_{-1}^{1} x^{\frac{1}{3}}\, dx \int_0^{1-x} y^{-\frac{1}{2}}\, (1-x-y)^{\frac{1}{2}}\, dy, -\frac{3\pi}{7}$

15. $\int_0^{2a} dx \int_0^{\frac{x^2}{4a}} (x+y)^3\, dy$ **Ans.** $\int_0^a dy \int_{\sqrt{4a\, y}}^{2a} (x+y)^3\, dx$

16. $\int_0^1 \int_0^y (x^2+y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\, dx\, dy$ **Ans.** $\int_0^1 dx \int_x^{2-x} (x^2+y^2)\, dy, \frac{5}{3}$

17. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dx\, dy$ by changing into polar coordinates. **Ans.** $\frac{\pi\, a^5}{20}$

(*U.P., I Semester, Dec. 2007, A.M.I.E., Summer 2001*)

18. $\int_0^1 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy + \int_0^2 \int_y^2 \frac{1}{x^2+y^2}\, dx\, dy = \int_R \int \frac{1}{x^2+y^2}\, dy\, dx$

Recognise the region R of integration on the R.H.S. and then evaluate the integral on the right in the order indicated. (*AMIETE, Dec. 2004*)

Ans. Region R is $x = 0$, $x = y$, $y = 1$ and $y = 2, \frac{\pi}{4} \log 2.$

19. Express as single integral and evaluate :

$\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\, dx\, dy + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} x\, dx\, dy$ **Ans.** $\int_0^{\frac{a}{\sqrt{2}}} dy \int_y^{\sqrt{a^2-y^2}} x\, dx, \frac{5a^3}{6\sqrt{2}}$

20. Express as single integral and evaluate :

$$\int_0^1 \int_0^y (x^2 + y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2 + y^2)\, dx\, dy \qquad \textbf{Ans. } \int_0^1 dx \int_x^{2-x} (x^2 + y^2)\, dy, \frac{5}{3}$$

21. If $f(x, y)\, dx\, dy$, where R is the circle $x^2 + y^2 = a^2$, is R equivalent to the repeated integral.

(*AMIE winter 2001*) [**Ans.** $\int_0^{2\pi}\int_0^1 (r, \theta)\, r\, dr\, d\theta.$]

2.5 CHANGE OF VARIABLES

Sometimes the problems of double integration can be solved easily by change of independent variables. Let the double integral as be $\iint_R f(x, y)\, dx\, dy.$ It is to be changed by the new variables u, v.

The relation of x, y with u, v are given as $x = f(u, v)$, $y = \Psi(u, v)$. Then the double integration is converted into.

$\iint_{R'} f\,\{\phi(u, v), \Psi(u, v)\}\, |J|\, du\, dv,$ where

$$dx\, dy = |J|\, du\, dv = \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} du\, dv$$

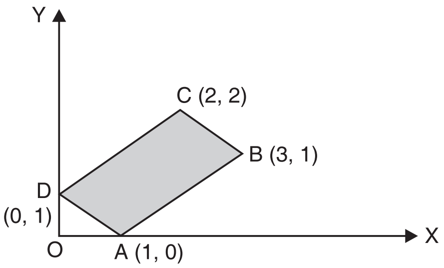
Example 17. *Evaluate* $\iint_R (x + y)^2\, dx\, dy,$ *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation $u = x + y$ and $v = x - 2y$.*

(*U.P., I Semester, 2003*)

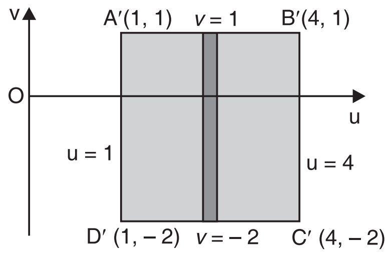
Solution. The region of integration is a parallelogram $ABCD$, where A (1, 0), B (3, 1), C (2, 2) and D (0, 1) in xy-plane.

The new region of integration is a rectangle $A'B'C'D'$ in uv-plane

xy-plane	$A \equiv (x, y)$ $A \equiv (1, 0)$	$B \equiv (x, y)$ $B \equiv (3, 1)$	$C \equiv (x, y)$ $C \equiv (2, 2)$	$D \equiv (x, y)$ $D \equiv (0, 1)$
uv-plane	$A' \equiv (u, v)$ $A' \equiv (x + y, x - 2y)$ $A' \equiv (1 + 0, 1 - 2 \times 0)$ $A' \equiv (1, 1)$	$B' \equiv (u, v)$ $B' \equiv (x + y, x - 2y)$ $B' \equiv (3 + 1, 3 - 2 \times 1)$ $B' \equiv (4, 1)$	$C' \equiv (u, v)$ $C' \equiv (u, v)$ $C' \equiv (2 + 2, 2 - 2 \times 2)$ $C' \equiv (4, -2)$	$D' \equiv (u, v)$ $D' \equiv (0 + 1, 0 - 2 \times 1)$ $D' \equiv (1, -2)$



(xy-plane)



(uv-plane)

and $\left\{\begin{array}{l} u = x + y \\ v = x - 2y \end{array}\right\} \Rightarrow \quad x = \frac{1}{3}(2u + v)$ and $y = \frac{1}{3}(u - v)$

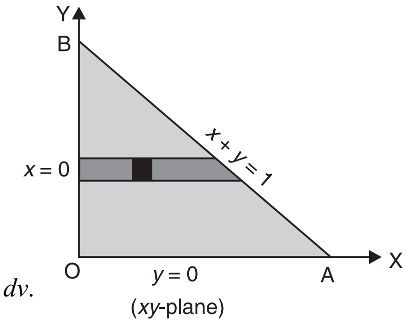
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ \dfrac{1}{3} & -\dfrac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$dx\,dy = |J|\,du\,dv = \frac{1}{3}\,du\,dv$$

$$\iint_R (x+y)^2\,dx\,dy = \int_{-2}^{1}\int_{1}^{4} u^2 . \frac{1}{3}\,du\,dv = \int_{-2}^{1}\frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\,dv = 7\,[v]_{-2}^{1} = 7\times 3 = 21 \textbf{ Ans.}$$

Example 18. *Using the transformation $x + y = u$, $y = uv$, show that*

$\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy = \frac{2\pi}{105}$, *integration being taken over the area of the tringle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.*



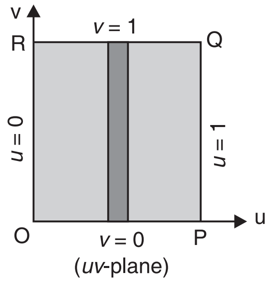
Solution. $\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy$

$$x + y = u \text{ or } x = u - y = u - uv,$$

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$dx\,dy = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} du\,dv = u\,du\,dv.$$

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad u\,(1-v) = 0 \\ &\Rightarrow \quad u = 0,\ v = 1 \\ y = 0 \quad &\Rightarrow \quad uv = 0 \\ &\Rightarrow \quad u = 0,\ v = 0 \\ x + y = 1 \quad &\Rightarrow \quad u = 1 \end{aligned}$$



Hence, the limits of u are from 0 to 1 and the limits of v are from 0 to 1.

The area of integration is a square $OPQR$ in uv-plane.

On putting $x = u - uv$, $y = uv$, $dx\,dy = u\,du\,dv$ in (1), we get

$$\iint (u-uv)^{1/2}\,(uv)^{1/2}\,(1-v)^{1/2}\,u\,du\,dv$$

$$= \int_0^1 u^2\,(1-u)^{1/2}\,du \int_0^1 v^{1/2}\,(1-v)^{1/2}\,dv = \frac{\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)} \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(5)}$$

$$= \frac{2.\,\Gamma\left(\frac{3}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)} \times \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2} = \frac{1}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} \times \frac{\frac{1}{2}\sqrt{\pi}.\,\frac{1}{2}\sqrt{\pi}}{1} = \frac{2\pi}{105} \qquad \textbf{Ans.}$$

EXERCISE 2.4

1. Evaluate $\int_0^\infty\int_0^\infty e^{-(x+y)} \sin\left(\frac{\pi y}{x+y}\right) dx\,dy$ by means of the transformation $u = x + y$, $v = y$ from (x, y) to (u, v) **Ans.** $\frac{1}{\pi}$

2. Using the transformation $x + y = u$, $y = uv$, show that $\int_0^1\int_0^{1-x} \frac{y}{e^{x+y}}\,dy\,dx = \frac{1}{2}(e-1)$

(*A.M.I.E. Winter 2001*)

3. Using the transformation $u = x - y$, $v = x + y$, prove that $\iint_R \cos\frac{x-y}{x+y}\,dx\,dy = \frac{1}{2}\sin 1$ where R is bounded by $x = 0$, $y = 0$, $x + y = 1$

$\left[\textbf{Hint :}\ x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(v-u) \text{ so that } |J| = \frac{1}{2}\right]$

2.6 AREA IN CARTESIAN CO-ORDINATES

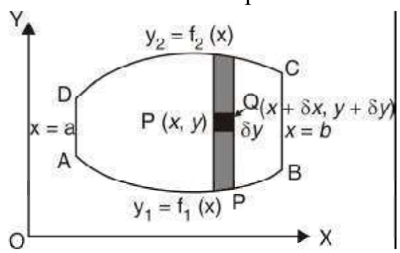
Let the curves AB and CD be $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

Let the ordinates AD and BC be $x = a$ and $x = b$.

So the area enclosed by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and $x = a$ and $x = b$ is $ABCD$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points, thent the area of the small rectangle $PQ = \delta x . \delta y$.

Area of the vertical strip $= \lim\limits_{\partial y \to 0} \sum\limits_{y_1}^{y_2} \delta x\, \delta y = \delta x \int_{y_1}^{y_2} dy$ δx the width of the strip is constant throughout.

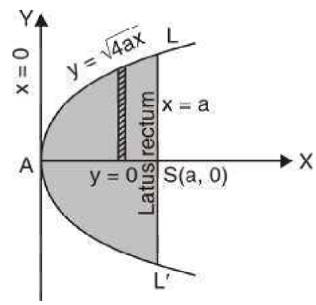


If we add all the strips from $x = a$ to $x = b$, we get

The area $ABCD = \lim\limits_{\partial x \to 0} \sum\limits_{a}^{b} \delta x \int_{y_1}^{y_2} dy = \int_a^b dx \int_{y_1}^{y_2} dy$

$$\boxed{\textbf{Area} = \int_a^b \int_{y_1}^{y_2} dx\, dy}$$

Example 19. *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*



Solution. Required area $= 2$ (area (ASL)

$$= 2 \int_0^a \int_0^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_0^a 2\sqrt{ax}\, dx$$

$$= 4\sqrt{a} \left(\frac{x^{3/2}}{3/2} \right)_0^a = \frac{8a^2}{3}$$

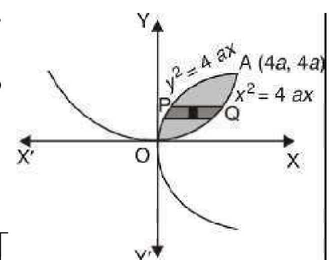
Example 20. *Find the area between the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.*

Solution. $y^2 = 4ax$ ...(1)

$x^2 = 4ay$ ...(2)

On solving the equations (1) and (2) we get the point of intersection $(4a, 4a)$.

Divide the area into horizontal strips of width δy, x varies from $P, \dfrac{y^2}{4a}$ to $Q, \sqrt{4\,ay}$ and then y varies from $O(y = 0)$ to A $(y = 4a)$.



$\therefore$ The required area $= \int_0^{4a} dy \int_{y^2/4a}^{\sqrt{4ay}} dx$

$$= \int_0^{4a} dy\, [x]_{y^2/4a}^{\sqrt{4ay}} = \int_0^{4a} dy \left[\sqrt{4ay} - \frac{y^2}{4a} \right] = \left[\sqrt{4a}\, \frac{y^{3/2}}{\frac{3}{2}} - \frac{y^3}{12\,a} \right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{(4a)^3}{12a} \right] = \left[\frac{32}{3} a^2 - \frac{16}{3} a^2 \right] = \frac{16}{3} a^2$$ **Ans.**

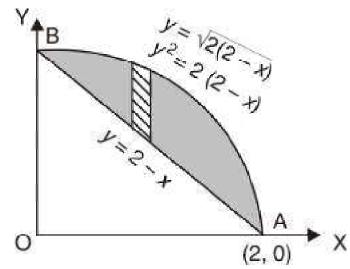
Example 21. *Find by double integration the area enclosed by the pair of curves*

$y = 2 - x$ and $y^2 = 2(2 - x)$

Solution. $y = 2 - x$

$y^2 = 2\,(2 - x)$

On solving the equations (1) and (2), we get the points of intersection (2, 0) and (0, 2).



$$A = \int \int dx\, dy$$

$$\text{The required area} = \int_0^2 dx \int_{2-x}^{\sqrt{2(2-x)}} dy$$

$$= \int_0^2 dx\, [y]_{2-x}^{\sqrt{2(2-x)}} = \int_0^2 dx\, [\sqrt{4-2x} - 2 + x]$$

$$= \left[\frac{2}{3 \times -2}(4-2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= \left[-\frac{1}{3}(4-2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2 = \left(-4 + \frac{4}{2}\right) + \frac{8}{3} = \frac{2}{3}$$ **Ans.**

EXERCISE 2.5

Use double integration in the following questions:

1. Find the area bounded by $y = x - 2$ and $y^2 = 2x + 4$. **Ans.** 18.

2. Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant. **Ans.** $(\pi - 2)a^2/4$

3. Find the area of a plate in the form of quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **Ans.** $\frac{\pi ab}{4}$

4. Find the area included between the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$. **Ans.** $\frac{8\sqrt{ab}}{3}$

(*A.M.I.E.T.E., Summer 2001*)

5. Find the area bounded by (*a*) $y^2 = 4 - x$ and $y^2 = x$. **Ans.** $\frac{16\sqrt{2}}{3}$

(*b*) $x - 2y + 4 = 0$, $x + y - 5 = 0$, $y = 0$ (*A.M.I.E., Winter 2001*) **Ans.** $\frac{27}{2}$

6. Find the area enclosed by the leminscate $r^2 = a^2 \cos 2\theta$. **Ans.** a^2

7. Find the area common to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 2ax$. **Ans.** $\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]a^2$

8. Find the area included between the curves $y = x^2 - 6x + 3$ and $y = 2x + 9$.

(*A.M.I.E., Summer 2001*) **Ans.** $\frac{88\sqrt{22}}{3}$

9. Determine the area of region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$. **Ans.** $\frac{28}{3} - 4\log 2$

(*U.P. I Semester 2003*)

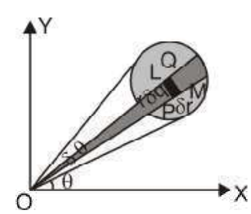
2.7 AREA IN POLAR CO-ORDINATES

$$\text{Area} = \iint r\, d\theta\, dr$$

Let us consider the area enclosed by the curve $r = f(\theta)$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points.

Draw ares PL and QM, radii r and $r + \delta r$.



$$PL = r\delta\theta,\ PM = \delta r$$

Area of rectangle $\quad PLQM = PL \times PM$

$$= (r\delta\theta)\,(d\delta r) = r\,\delta\theta\,\delta r.$$

The whole area A is composed of such small rectangles.

Hence,

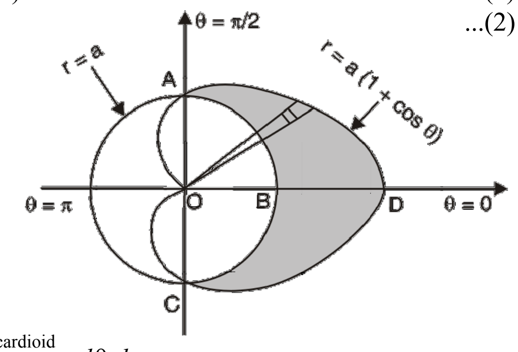
$$A = \lim_{\substack{\partial r \to 0 \\ \delta\theta \to 0}} \sum \sum r\, \delta\theta.\, \delta r = \iint r\, d\theta\, dr$$

Example 22. *Find by double integration, the area lying inside the cardioid $r = a\,(1 + \cos\theta)$ and outside the circle $r = a$.* (*Nagpur University, Winter 2000*)

Solution.

$$r = a\,(1 + \cos\theta) \qquad ...(1)$$

$$r = a \qquad ...(2)$$



Solving (1) and (2), by eliminating r, we get

$$a(1 + \cos\theta) = a \quad \Rightarrow \quad 1 + \cos\theta = 1$$

$$\cos\theta = 0 \quad \Rightarrow \quad \theta = -\frac{\pi}{2} \text{ or } \frac{\pi}{2}$$

limits of r are a and $a(1 + \cos\theta)$

limits of θ are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Required area = Area ABCDA

$$= \int_{-\pi/2}^{\pi/2} \int_{r \text{ for circle}}^{\text{for cardioid}} r\, d\theta\, dr$$

$$= \int_{-\pi/2}^{\pi/2} \int_{a}^{a(1+\cos\theta)} r\, d\theta\, dr \qquad = \int_{-\pi/2}^{\pi/2} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} [(1 + \cos\theta)^2 - 1]\, d\theta \qquad = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} (\cos^2\theta + 2\cos\theta)\, d\theta$$

$$= a^2 \int_0^{\pi/2} (\cos^2\theta + 2\cos\theta)\, d\theta \qquad = a^2 \left[\int_0^{\pi/2} \cos^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta\right]$$

$$= a^2 \left[\frac{\pi}{4} + 2\,(\sin\theta)_0^{\pi/2}\right] = a^2 \left[\frac{\pi}{4} + 2\right] \quad = \frac{a^2}{4}(\pi + 8) \qquad \textbf{Ans.}$$

Example 23. *Find by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a\,(1 - \cos\theta)$.*

Solution. We have,

$$r = a\sin\theta \qquad ...(1)$$

$$r = a\,(1 - \cos\theta) \qquad ...(2)$$

Solving (1) and (2) by eliminating r, we have

$$\sin\theta = 1 - \cos\theta \Rightarrow \sin\theta + \cos\theta = 1$$

Squaring above, we get

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$$

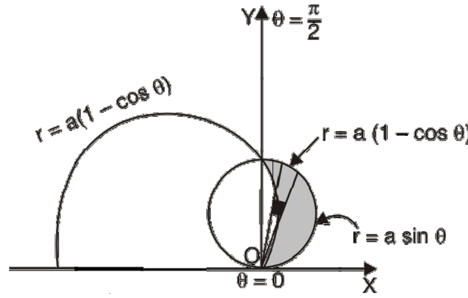
$$\Rightarrow \qquad 1 + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } \pi = \theta = 0 \text{ or } \frac{\pi}{2}$$

The required area is shaded portion in the fig.

Limits of r are $a(1 - \cos\theta)$ and $a\sin\theta$, limits of θ are 0 and $\frac{\pi}{2}$.

$$\text{Required area} = \int_0^{\frac{\pi}{2}} \int_{a\,(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} a^2\, [\sin^2\theta - (1 - \cos\theta)^2]\, d\theta$$



$$= \frac{a^2}{2}\int_0^{\pi/2}(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta]\,d\theta$$

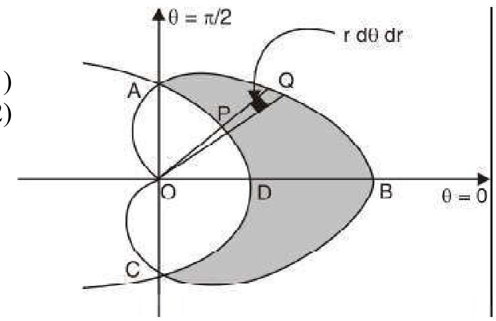
$$= \frac{a^2}{2}\int_0^{\pi/2}(-2\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2} -2\cos^2\theta\,d\theta + \int_0^{\pi/2} 2\cos\theta\,d\theta\right]$$

$$= \frac{a^2}{2}\left[\left(-2.\frac{\pi}{4}\right) + 2\,(\sin\theta)_0^{\pi/2}\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\left(\sin\frac{\pi}{2} - \sin 0\right)\right] = \frac{a^2}{2}\left[-\frac{\pi}{2}+2\right] = a^2\left(1-\frac{\pi}{4}\right)$$ **Ans.**

Example 24. *Find by double integration, the area lying inside a cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.*



Solutio. We have,

$$r = 1 + \cos\theta \quad ...(1)$$

$$r(1 + \cos\theta) = 1 \quad ...(2)$$

Solving (1) and (2), we get

$$(1 + \cos\theta)(1 + \cos\theta) = 1$$

$$(1 + \cos\theta)^2 = 1$$

$$1 + \cos\theta = 1$$

$$\cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$$

limits of r are $1 + \cos\theta$ and $\frac{1}{1+\cos\theta}$ limits of θ are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Required area = Area *ADCBA* (Shaded portion)

$$= \int_{-\pi/2}^{\pi/2}\int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r\,d\theta\,dr = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos^2\theta + 2\cos\theta) - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right]d\theta$$

$$= 2\times\frac{1}{2}\int_0^{\pi/2}\left[(1+\cos^2\theta+2\cos\theta) - \frac{1}{4}\sec^4\frac{\pi}{2}\right]d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos^2\theta+2\cos\theta) - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right]d\theta$$

$$= \int_0^{\pi/2}\left[\left(1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right) - \frac{1}{4}\left(1+\tan^2\frac{\pi}{2}\right)\sec^2\frac{\pi}{2}\right]d\theta$$

$$= \int_0^{\pi/2}\left[1+\frac{1}{2}+\frac{\cos 2\theta}{2}+2\cos\theta - \frac{1}{4}\left(\sec^2\frac{\theta}{2}+\tan^2\frac{\theta}{2}\times\sec^2\frac{\theta}{2}\right)\right]d\theta$$

$$= \left[\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta-\frac{1}{4}\left(2\tan\frac{\theta}{2}+\frac{2}{3}\tan^3\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2}+\frac{\pi}{4}+0+2\sin\frac{\pi}{2}-\frac{1}{2}\tan\frac{\pi}{4}-\frac{1}{6}\tan^3\frac{\pi}{4}\right] = \left[\frac{3\pi}{4}+2-\frac{1}{2}-\frac{1}{6}\right] = \left[\frac{3\pi}{4}+\frac{4}{3}\right]$$ **Ans.**

EXERCISE 2.6

1. Find the area of cardioid $r = a(1 + \cos\theta)$. **Ans.** $\frac{3\pi a^2}{2}$
2. Find the area of the curve $r^2 = a^2 \cos 2\theta$. **Ans.** a^2
3. Find the area enclosed by the curve $r = 2\, a \cos\theta$ **Ans.** πa^2
4. Find the area enclosed by the curve $r = 3 + 2\cos\theta$. **Ans.** $11\,\pi$
5. Find the area enclosed by the curve

   $r^3 = a^2 \cos^2\theta + b^2 \sin^2\theta$. **Ans.** $\frac{\pi}{2}(a^2 + b^2)$
6. Show that the area of the region included between the cardioides $r = a(1 + \cos\theta)$ and $r = a\,(1 - \cos\theta)$ is $\frac{a^2}{2}(3\pi - 8)$.
7. Find the area outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$. **Ans.** $(\pi + 8)$
8. Find the area inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$. **Ans.** $2a^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$
9. Find the area inside the circle $r = 4\sin\theta$ and outside the lemniscate $r^2 = 8\cos 2\theta$.

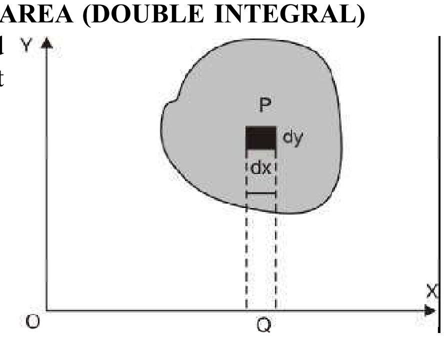
   Ans. $\left(\frac{8}{3}\pi + 4\sqrt{3} - 4\right)$

2.8 VOLUME OF SOLID BY ROTATION OF AN AREA (DOUBLE INTEGRAL)

When the area enclosed by a curve $y = f(x)$ is revolved about an axis, a solid is generated, we have to find out the volume of solid generated.

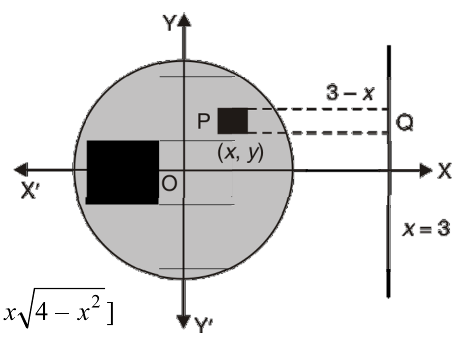
Volume of the solid generated about x-axis

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} 2\pi\, PQ\, dx\, dy$$



Example 25. *Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.*

Solution. $x^2 + y^2 = 4$



$$V = \int\int (2\pi\, PQ)\, dx\, dy = 2\pi \int\int (3 - x)\, dx\, dy$$

$$= 2\pi \int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (3 - x)\, dy$$

$$= 2\pi \int_{-2}^{+2} dx\, (3y - x\,y)_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}$$

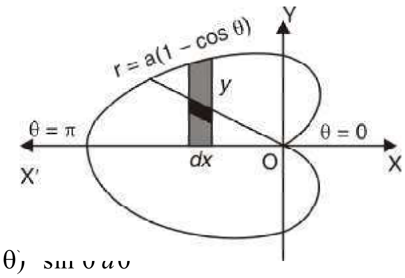
$$= 2\pi \int_{-2}^{+2} dx\, [3\sqrt{4-x^2} - x\sqrt{4-x^2} + 3\sqrt{4-x^2} - x\sqrt{4-x^2}]$$

$$= 4\pi [3\sqrt{4-x^2} - x\sqrt{4-x^2}]\, dx = 4\pi \left[3\frac{x}{2}\sqrt{4-x^2} + 3\times\frac{4}{2}\sin^{-1}\frac{x}{2} + \frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^{2}$$

$$= 4\pi\left[6\times\frac{\pi}{2} + 6\times\frac{\pi}{2}\right] = 24\,\pi^2$$ **Ans.**

Example 26. *Calculate by double integration the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.* (*AMIETE, June 2010*)

Solution. $r = a\,(1 - \cos\theta)$



$$V = 2\pi \int \int y\, dx\, dy \Rightarrow V = 2\pi \int \int (r\, d\theta\, dr)\, y$$

$$= 2\pi \int d\theta \int r\, dr\, (r \sin\theta)$$

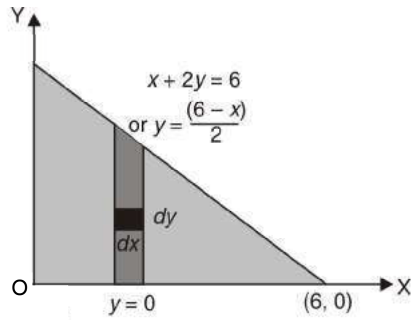
$$= 2\pi \int_0^{\pi} \sin\theta\, d\theta \int_0^{a(1-\cos\theta)} r^2\, dr$$

$$= 2\pi \int_0^{\pi} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^{a\,(1-\cos\theta)} = \frac{2\pi}{3}\int_0^{\pi} a^3\,(1-\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi\, a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi} = \frac{2\pi\, a^3}{12}[16] = \frac{8}{3}\pi\, a^3$$ **Ans.**

Example 27. *A pyramid is bounded by the three co-ordinate planes and the plane $x + 2y + 3z = 6$. Compute this volume by double integration.*

Solution. $x + 2y + 3z = 6$ ...(1)



$x = 0$, $y = 0$, $z = 0$ are co-ordinate planes.

The line of intersection of plane (1) and xy plane ($z = 0$) is

$$x + 2y = 6 \quad ...(2)$$

The base of the pyramid may be taken to be the triangle bounded by x-axis, y-axis and the line (2).

An elementary area on the base is $dx\, dy$.

Consider the elementary rod standing on this area and having height z, where

$$3z = 6 - x - 2y \text{ or } z = \frac{6 - x - 2y}{3}$$

Volume of the rod = $dx\, dy$, z, Limits for z are 0 and $\frac{6-x-2y}{3}$.

Limits of y are 0 and $\frac{6-x}{2}$ and limits of x are 0 and 6.

$$\text{Required volume} = \int_0^6 \int_0^{\frac{6-x}{2}} z\, dx\, dy = \int_0^6 dx \int_0^{\frac{6-x}{2}} \frac{6-x-2y}{3}\, dy$$

$$= \frac{1}{3}\int_0^6 dx \left(6x - xy - y^2\right)_0^{\frac{6-x}{2}} = \frac{1}{3}\int_0^6 \left(\frac{6(6-x)}{2} - \frac{x(6-x)}{2} - \left(\frac{6-x}{2}\right)^2\right) dx$$

$$= \frac{1}{3}\int_0^6 \left(\frac{36-6x}{2} - \frac{6x-x^2}{2} - \frac{36+x^2-12x}{4}\right) dx$$

$$= \frac{1}{12}\int_0^6 (72 - 12x - 12x + 2x^2 - 36 - x^2 + 12x)\, dx$$

$$= \frac{1}{12}\int_0^6 (x^2 - 12x + 36)\, dx = \frac{1}{12}\left[\frac{x^3}{3} - \frac{12x^2}{2} + 36x\right]_0^6 = \frac{1}{12}[72 - 216 + 216] = 6$$ **Ans.**

EXERCISE 2.7

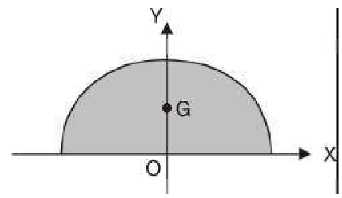
1. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by revolving area of the circle $x^2 + y^2 = a^2$. **Ans.** $\frac{4}{3}\pi a^3$

2.9 CENTRE OF GRAVITY

$$\overline{x} = \frac{\int\int \rho\, x\, dx\, dy}{\int\int \rho\, dx\, dy}, \quad \overline{y} = \frac{\int\int \rho\, y\, dx\, dy}{\int\int \rho\, dx\, dy}$$

Example 28. *Find the position of the C.G. of a semi-circular lamina of radius a if its density varies as the square of the distance from the diameter.* (*AMIETE, Dec. 2010*)

Solution. Let the bounding diameter be as the x-axis and a line perpendicular to the diameter and passing through the centre is y-axis. Equation of the circle is $x^2 + y^2 = a^2$. By symmetry $\bar{x} = 0$.



$$\bar{y} = \frac{\int\int y\,\rho\,dx\,dy}{\int\int \rho\,dx\,dy} = \frac{\int\int (\lambda\,y^2)\,y\,dx\,dy}{\int\int(\lambda\,y^2)\,dx\,dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\,dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\,dy}$$

$$= \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left(\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}}} = \frac{3\int_{-a}^{a}(a^2-x^2)^2\,dx}{4\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx}$$

Put $x = a \sin\theta$

$$= \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a^2 - a^2\sin^2\theta)^2\,a\cos\theta\,d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a^2 - a^2\sin^2\theta)^{3/2}\,a\cos\theta\,d\theta} = \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^5\cos^5\theta\,d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} a^4\cos^4\theta\,d\theta}$$

$$= \frac{3a}{4}\,\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}} = \left(\frac{3a}{4}\right)\left(\frac{8}{15}\right)\left(\frac{16}{3\pi}\right) = \frac{32a}{15\pi}$$

Hence C.G. is $\left(0, \frac{32\,a}{15\,\pi}\right)$ **Ans.**

Example 29. *Find C.G. of the area in the positive quadrant of the curve*

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Solution. For C.G. of area; $\bar{x} = \dfrac{\int\int x\,dx\,dy}{\int\int dx\,dy}$, $\bar{y} = \dfrac{\int\int y\,dx\,dy}{\int\int dx\,dy}$

$$\bar{x} = \frac{\int_0^a x\,dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy}{\int_0^a dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy} = \frac{\int_0^a x\,dx\,[y]_0^{(a^{2/3}-x^{2/3})^{3/2}}}{\int_0^a dx\,[y]_0^{(a^{2.3}-x^{2/3})^{3/2}}}$$

[Put $x = a\cos^3\theta$]

$$= \frac{\int_0^a x\,dx\,(a^{2/3}-x^{2/3})^{3/2}}{\int_0^a dx\,(a^{2/3}-x^{2/3})^{3/2}} = \frac{\int_{\frac{\pi}{2}}^{0} a\cos^3\theta\,(a^{2/3} - a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\theta)}{\int_{\frac{\pi}{2}}^{0}(a^{2/3} - a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\theta)}$$

$$= \frac{\int_0^{\frac{\pi}{2}} 3a^3\cos^3\theta\sin^3\theta\cos^2\theta\sin\theta\,d\theta}{\int_0^{\frac{\pi}{2}} 3a^2\sin^3\theta\cos^2\theta\sin\theta\,d\theta} = \frac{a\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^5\theta\,d\theta}{\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta} = \frac{\dfrac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{6}{2}\right)}{2\,\Gamma\left(\frac{4}{2}\right)}\,a}{\dfrac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(4)}}$$

$$= \frac{\Gamma 3\,\Gamma 4\, a}{\Gamma \frac{3}{2}\,\Gamma \frac{11}{2}} = \frac{(2)\,(6)\,a}{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\pi} = \frac{256\,a}{315\,\pi}, \quad \text{Similarly,} \quad \bar{y} = \frac{256\,a}{315\,\pi}$$

Hence, C.G. of the area is $\left(\dfrac{256\,a}{315\,\pi}, \dfrac{256\,a}{315\,\pi}\right)$.

Example 30. *Find by double integration, the centre of gravity of the area of the cardioid $r = a\,(1 + \cos\,\theta)$.*

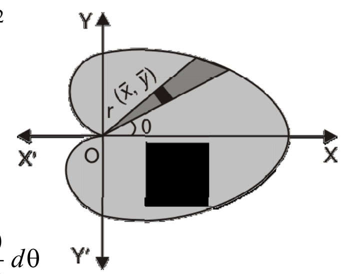
Solution. Let $(\bar{x}, \bar{y})$ be the C.G. the cardioid

By Symmetry, $\bar{y} = 0$.

$$\bar{x} = \frac{\int_A\int x\,dx\,dy}{\int_A\int dx\,dy} = \frac{\int_A\int x\,dx\,dy}{\int_A\int dx\,dy}$$

$$= \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} (r\cos\theta)\,(r\,d\theta\,dr)}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} r\,d\theta\,dr} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\int_0^{a\,(1+\cos\theta)} r^2\,dr}{\int_{-\pi}^{\pi}d\theta\int_0^{a(1+\cos\theta)} r\,dr}$$

$$= \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\left[\frac{r^3}{3}\right]_0^{a\,(a+\cos\theta)}}{\int_{-\pi}^{\pi}d\theta\left(\frac{r^2}{2}\right)_0^{a\,(1+\cos\theta)}} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta.\,\frac{a^3}{3}\,(1+\cos\theta)^3}{\int_{-\pi}^{\pi}d\theta\,\frac{a^2}{2}\,(1+\cos\theta)^2}$$



$$= \frac{\frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(1+2\cos^2\frac{\theta}{2}-1\right)^3 d\theta}{\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\cos^2\frac{\theta}{2}-1\right)d\theta}$$

$$= \frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(8\cos^6\frac{\theta}{2}\right)d\theta \div \frac{a^2}{2}\int_{-\pi}^{\pi}4\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{8\,a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 2a^2\int_{-\pi}^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{2\times 8a^3}{3}\int_0^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2}(2\cos^8 t-\cos^6 t)\,(2\,dt) \div 4a^2\int_0^{\pi/2}\cos^4 t\,(2\,dt)$$

$$= \frac{32\,a^3}{3}\left[\frac{2\times 7\times 5\times 3\times 1}{8\times 6\times 4\times 2}\,\frac{\pi}{2} - \frac{5\times 3\times 1}{6\times 4\times 2}\,\frac{\pi}{2}\right] \div 8a^2\left(\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}\right)$$

$$= \frac{32a^3}{3}\left(\frac{35\,\pi}{128}-\frac{5\,\pi}{32}\right) \div 8a^2\left(\frac{3\pi}{16}\right) = \frac{8a^3}{3}\times\frac{15\,\pi}{128}\times\frac{16}{8a^2\times 3\pi} = \frac{5a}{24}$$ **Ans.**

2.10 CENTRE OF GRAVITY OF AN ARC

Example 31. *Find the C.G. of the arc of the curve*

$x = a\,(\theta + \sin\,\theta)$, $y = a(1 - \cos\,\theta)$ in the positive quadrant.

Solution. We know that, $\bar{x} = \dfrac{\int x\,ds}{\int ds}, \bar{y} = \dfrac{\int y\,ds}{\int ds}$

Now, $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$= \sqrt{\{a^2 (1+\cos\theta)^2 + a^2 \sin^2\theta\}}\, d\theta = a\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$= a\sqrt{1 + 2\cos\theta + 1}\, d\theta = a\sqrt{2(1+\cos\theta)}\, d\theta = a\sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 2a\cos\frac{\theta}{2}\, d\theta$

$$\bar{x} = \frac{\int x\, dx}{\int ds} = \frac{\int_0^\pi a(\theta + \sin\theta)\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\, d\theta} = \frac{a\int_0^\pi\left(\theta + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta\right)}{\left[2\sin\frac{\pi}{2}\right]_0^\pi}$$

$$= \frac{a}{2}\int_0^\pi\left[\theta\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right] d\theta = \frac{a}{2}\int_0^{\frac{\pi}{2}}(2t\cos t + 2\sin t\cos^2 t)\, 2\, dt$$

$$= 2a\left[t\sin t + \cos t - \frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} = 2a\left[\frac{\pi}{2} - 1 + \frac{1}{3}\right] = a\left[\pi - \frac{4}{3}\right]$$

$$\bar{y} = \frac{\int y\, ds}{\int ds} = \frac{\int_0^\pi a(1-\cos\theta)\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\, d\theta} = \frac{a\int_0^\pi 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi \cos\frac{\theta}{2}\, d\theta}$$

$$= \frac{a\, r\left[\sin^3\frac{\theta}{2}\right]_0^\pi}{3\left[2\sin\frac{\theta}{2}\right]_0^\pi} = \frac{4a}{3\times 2} = \frac{2a}{3}$$ Hence, C.G. of the arc is $\left[a\left(\pi - \frac{4}{3}\right), \frac{2a}{3}\right]$ **Ans.**

EXERCISE 2.8

1. Find the centre of gravity of the area bounded by the parabola $y^2 = x$ and the line $x + y = 2$. **Ans.** $\left(\frac{8}{5}, -\frac{1}{2}\right)$

2. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$. **Ans.** $\left(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$

3. Find the centroid of the area enclosed by the parabola $y^2 = 4\, ax$, the axis of x and latus rectum. **Ans.** $\left(\frac{3a}{20}, \frac{3a}{16}\right)$

4. Find the centroid of the loop of curve $r^2 = a^2 \cos 2\theta$. **Ans.** $\left(\frac{\pi a\sqrt{2}}{8}, 0\right)$

5. Find the centroid of solid formed by revolving about the x-axis that part of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which lies in the first quadrant. **Ans.** $\left(\frac{3a}{8}, 0\right)$

6. Find the average density of the sphere of radius a whose density at a distance r from the centre of the sphere is $\rho = \rho_0\left[1 + k\frac{r^3}{a^3}\right]$. $\rho_0\left(1 + \frac{k}{2}\right)$

7. The density at a point on a circular lamina varies as the distance from a point O on the circumference. Show that the C.G. divides the diameter through O in the ratio 3 : 2.

1.11 TRIPLE INTEGRATION

Let a function $f(x, y, z)$ be a continuous at every point of a finite region S of three dimensional space. Consider n sub-spaces $\delta S_1, \delta S_2, \delta S_3, \ldots \delta S_n$ of the space S.

If (x_r, y_r, z_r) be a point in the rth subspace.

The limit of the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta S_r$, as $n \to \infty$, $\delta S_r \to 0$ is known as the triple integral of $f(x, y, z)$ over the space S.

Symbolically, it is denoted by

$$\iiint_S f(x, y, z)\, dS$$

It can be calculated as $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\; dx\, dy\, dz$. First we integrate with respect to z treating x, y as constant between the limits z_1 and z_2. The resulting expression (function of x, y) is integrated with respect to y keeping x as constant between the limits y_1 and y_2. At the end we integrate the resulting expression (function of x only) within the limits x_1 and x_2.

$$\boxed{\int_{x_1=a}^{x_2=b} \Psi(x)\, dx \;\boxed{\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} \phi(x, y)\, dy \;\boxed{\int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x, y, z)\, dz}}}$$

First we integrate from inner most integral w.r.t. z, then we integrate with respect to y and finally the outer most with respect to x.

But the above order of integration is immaterial provided the limits change accordingly.

Example 32. *Evaluate* $\iiint_R (x + y + z)\; dx\, dy\, dz$, *where* $R : 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

Solution. $\int_0^1 dx\int_1^2 dy\int_2^3 (x + y + z)\, dz \;=\; \int_0^1 dx\int_1^2 dy\left[\frac{(x+y+z)^2}{2}\right]_2^3$

$$= \frac{1}{2}\int_0^1 dx\int_1^2 dy\,[(x+y+3)^2 - (x+y+2)^2] = \frac{1}{2}\int_0^1 dx\int_1^2 (2x+2y+5)\,.\,1.dy$$

$$= \frac{1}{2}\int_0^1 dx\left[\frac{(2x+2y+5)^2}{4}\right]_1^2 = \frac{1}{8}\int_0^1 dx\,[(2x+4+5)^2 - (2x+2+5)^2]$$

$$= \frac{1}{8}\int_0^1 (4x+16)\,.\,2\,dx = \int_0^1 (x+4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^1 = \frac{1}{2} + 4 = \frac{9}{2}$$ **Ans.**

Example 33. *Evaluate the integral :* $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$.

Solution. $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\int_0^{x+\log y} e^z dz = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy (e^z)_0^{x+\log y}$$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\,(e^{x+\log y} - 1) = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy\,(e^{\log y}\,.\,e^x - 1)$$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y\,(y e^x - 1)\, dy \;=\; \int_0^{\log 2} e^x dx\left[(ye^x - 1)\, e^y - \int e^x\,.\,e^y\, dy\right]_0^x$$

$$= \int_0^{\log 2} e^x dx\left[(ye^x - 1)\, e^y - e^{x+y}\right]_0^x \;=\; \int_0^{\log 2} e^x dx\,[(xe^x - 1)\, e^x - e^{2x} + 1 + e^x]$$

$$= \int_0^{\log 2} e^x dx\,[xe^{2x} - e^x - e^{2x} + 1 + e^x] = \int_0^{\log 2} (xe^{3x} - e^{3x} + e^x)\, dx$$

$$= \left[x\,\frac{e^{3x}}{3} - \int 1 \, . \, \frac{e^{3x}}{3}\,dx - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2} = \left[\frac{x}{3}\,e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \frac{\log 2}{3}\,e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{\log 2}{3}\,e^{\log 2^3} - \frac{e^{\log 2^3}}{9} - \frac{e^{\log 2^3}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 = \frac{8}{3}\log 2 - \frac{19}{9}$$ **Ans.**

Example 34. *Evaluate* $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz.$

(*M.U. II Semester, 2005, 2003, 2002*)

Solution. $I = \int_0^{\log 2} \int_0^x e^{x+y}\left[e^z\right]_0^{x+y} dx\,dy$

$$= \int_0^{\log 2} \int_0^x e^{x+y}(e^{x+y} - 1)\,dx\,dy = \int_0^{\log 2} \int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right] dx\,dy$$

$$= \int_0^{\log 2} \left[e^{2x} \, . \, \frac{e^{2y}}{2} - e^x \, . \, e^y \right]_0^x dx = \int_0^{\log 2} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x \right]_0^{\log 2} = \left[\frac{e^{4\log 2}}{8} - \frac{e^{2\log 2}}{2} - \frac{e^{2\log 2}}{4} + e^{\log 2} \right] - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right)$$

$$= \left(\frac{e^{\log 16}}{8} - \frac{e^{\log 4}}{2} - \frac{e^{\log 4}}{4} + e^{\log 2} \right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right)$$

$$= \left(\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2 \right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right) = \frac{5}{8}$$ **Ans.**

Example 35. *Evaluate* $\iiint_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$

where R denotes the region bounded by $x = 0$, $y = 0$, $z = 0$ *and* $x + y + z = a$, $(a > 0)$

Solution. $\iiint_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$

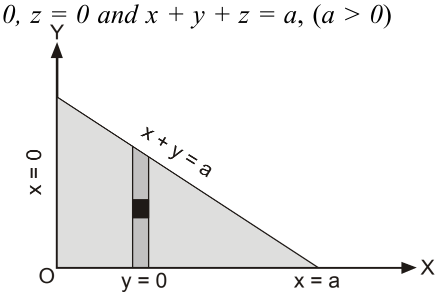
$x + y + z = a$ or $z = a - x - y$

Upper limit of $z = a - x - y$

On x-y plane, $x + y + z = a$ becomes $x + y = a$ as shown in the figure.

Upper limit of $y = a - x$

Upper limit of $x = a$



$$= \int_{x=0}^{a} dx \int_{y=0}^{a-x} dy \int_{z=0}^{a-x-y} (x^2 + y^2 + z^2)\,dz = \int_0^a dx \int_0^{a-x} dy \left(x^2 z + y^2 z + \frac{z^3}{3} \right)_0^{a-x-y}$$

$$= \int_0^a dx \int_0^{a-x} dy \left[x^2(a - x - y) + y^2(a - x - y) + \frac{(a - x - y)^3}{3} \right]$$

$$= \int_0^a dx \int_0^{a-x} \left[x^2(a - x) - x^2 y + (a - x)\,y^2 - y^3 + \frac{(a - x - y)^3}{3} \right] dy$$

$$= \int_0^a dx \left[x^2(a - x)\,y - \frac{x^2 y^2}{2} + (a - x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(a - x - y)^4}{12} \right]_0^{a-x}$$

$$= \int_0^a dx\left[x^2(a-x)^2 - \frac{x^2}{2}(a-x)^2 + (a-x)\frac{(a-x)^3}{3} - \frac{(a-x)^4}{4} + \frac{(a-x)^4}{12}\right]$$

$$= \int_0^a \left[\frac{x^2}{2}(a-x)^2 + \frac{(a-x)^4}{6}\right]dx = \int_0^a \left[\frac{1}{2}(a^2x^2 - 2ax^3 + x^4) + \frac{(a-x)^4}{6}\right]dx$$

$$= \left[\frac{1}{2}a^2\frac{x^3}{3} - \frac{ax^4}{4} + \frac{x^5}{10} - \frac{(a-x)^5}{30}\right]_0^a = \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} + \frac{a^5}{30} = \frac{a^5}{20}$$ **Ans.**

Example 36. *Compute* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *if the region of integration is bounded by the coordinate planes and the plane* $x + y + z = 1$. (*M.U., II Semester 2007, 2006*)

Solution. Let the given region be R, then R is expressed as

$0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1.$

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{-2(x+y+z+1)^2}\right]_0^{1-x-y}$$

$$= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2}\right]$$

$$= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right]dy = -\frac{1}{2}\int_0^1 dx \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1 dx\left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1}\right] = -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

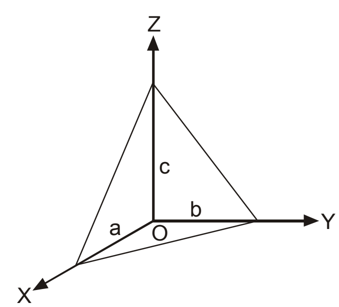
$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1 = -\frac{1}{2}\left[\frac{1}{2} - \log 2 + \frac{1}{8}\right] = -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$ **Ans.**

Example 37. *Evaluate* $\iiint x^2yz\ dx\,dy\,dz$ *throughout the volume bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (*M.U. II Semester* 2003, 2002, 2001)

Solution. Here, we have

$$I = \iiint x^2yz\ dx\,dy\,dz \quad ...(1)$$



Putting $x = au, \quad y = bv, \quad z = cw$

$dx = a\,du, \quad dy = b\,dv, \quad dz = c\,dw$ in (1), we get

$$I = \iiint a^2bc\,u^2vw\,a\,bc\,du\,dv\,dw$$

Limits are for $u = 0, 1$ for $v = 0, 1-u$ and for $w = 0, 1-u-v$

$u + v + w = 1$

$$I = \int_{u=o}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} a^3b^2c^2\,u^2vw\,du\,dv\,dw = \int_0^1\int_0^{1-u} a^3b^2c^2\,u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v(1-u-v)^2\,du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v\left[(1-u)^2 - 2(1-u)v + v^2\right]du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2[(1-u)^2 v - 2(1-u)v^2 + v^3]\,du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}\right] du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4}{12}\,du = \frac{a^3b^2c^2}{24}\int_0^1 u^{3-1}(1-u)^{5-1}\,du$$

$$= \frac{a^3b^2c^2}{24}\beta(3,5) = \frac{a^3b^2c^2}{24}\cdot\frac{\overline{|3}\,\overline{|5}}{\overline{|8}} = \frac{a^3b^2c^2}{24}\cdot\left(\frac{2!\,4!}{7!}\right) = \frac{a^3b^2c^2}{2520}.$$ **Ans.**

2.12 INTEGRATION BY CHANGE OF CARTESIAN COORDINATES INTO SPHERICAL COORDINATES

Sometime it becomes easy to integrate by changing the cartesian coordinates into spherical coordinates.

The relations between the cartesian and spherical polar co-ordinates of a point are given by the relations

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ dx\,dy\,dz &= |J|\,dr\,d\theta\,d\phi \\ &= r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

Note. 1. Spherical coordinates are very useful if the expression $x^2 + y^2 + z^2$ is involved in the problem.

2. In a sphere $x^2 + y^2 + z^2 = a^2$ the limits of r are 0 and a and limits of θ are 0, π and that of ϕ are 0 and 2π.

Example 38. *Evaluate the integral* $\iiint (x^2 + y^2 + z^2)\ dx\ dy\ dz$ *taken over the volume enclosed by the sphere* $x^2 + y^2 + z^2 = 1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$$\iiint (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^1 r^4 dr = \int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\left(\frac{r^5}{5}\right)_0^1 = \frac{1}{5}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi} = \frac{2}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}(\phi)_0^{2\pi} = \frac{4\pi}{5}$$ **Ans.**

Example 39. *Evaluate* $\iiint (x^2 + y^2 + z^2)\ dx\ dy\ dz$ *over the first octant of the sphere* $x^2 + y^2 + z^2 = a^2$. (*M.U. II Semester 2007*)

Solution. Here, we have

$$I = \iiint (x^2+y^2+z^2)\,dx\,dy\,dz \qquad ...(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$ in (1), we get

Limits of r are 0, a for θ are 0, $\frac{\pi}{2}$ for ϕ are 0, $\frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 . r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^a r^4 \, dr$$

$$\left(\begin{aligned} x^2 + y^2 + z^2 &= r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2\theta \\ &= r^2 \sin^2\theta + r^2 \cos^2\theta = r^2 \end{aligned} \right)$$

$$= [\phi]_0^{\pi/2} [-\cos\theta]_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a = \frac{\pi}{2} . (1) . \frac{a^5}{5} = \pi . \frac{a^5}{10}.$$ **Ans.**

Example 40. *Evaluate* $\iiint \frac{dx\, dy\, dz}{x^2 + y^2 + z^2}$ *throughout the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*M.U. II Semester 2002, 2001*)

Solution. Here, we have

$$I = \iiint \frac{dx\, dy\, dz}{x^2 + y^2 + z^2} \qquad ...(1)$$

Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ and $dx\, dy\, dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$ in (1), we get

The limits of r are 0 and a, for θ are 0 and $\frac{\pi}{2}$ for ϕ are 0 and $\frac{\pi}{2}$ in first octant.

$$I = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{r^2} \qquad [\text{Sphere } x^2 + y^2 + z^2 \text{ lies in 8 quadrants}]$$

$$I = 8 \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^a dr = 8 [\phi]_0^{\pi/2} [-\cos\theta]_0^{\pi/2} [r]_0^a = 8\left(\frac{\pi}{2} - 0\right)(0+1)(a+0)$$

$$= 8 \frac{\pi}{2} . 1 . a = 4\pi a$$ **Ans.**

EXERCISE 2.9

Evaluate the following :

1. $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx\, dy\, dz$ (*M.U., II Semester 2002*) **Ans.** 48

2. $\int_0^4 \int_0^x \int_0^{x+y} z\, dz\, dy\, dx$ (*R.G.P.V. Bhopal I Sem. 2003*) **Ans.** 70

3. $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)\, dx\, dy\, dz$ **Ans.** 6

4. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$ (*AMIETE, June 2006*) **Ans.** 1

5. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x - y + z)\, dx\, dy\, dz$ (*AMIETE, Summer 2004*) **Ans.** 0

6. $\iiint_R (x - y - z)\ dx\ dy\ dz$, where $R : 1 \le x \le 2;\ 2 \le y \le 3;\ 1 \le z \le 3$ **Ans.** 2

7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dx\, dy\, dz$ (*AMIETE, Dec. 2007*) **Ans.** 26 **8.** $\int_0^1 dx \int_0^2 dy \int_1^2 x^2\, yz\, dz$ **Ans.** 1

9. $\iiint x^2 yz\ dx\ dy\ dz$ throughout the volume bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

(*M.U. II Semester, 2003*) **Ans.** $\frac{1}{2520}$

10. $\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\, dy\, dx$ **Ans.** $\frac{1}{3}$ **11.** $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$ **Ans.** $\frac{1}{2}(e^2 - 8e + 13)$

12. $\iiint_T y\,dx\,dy\,dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$. (*AMIETE Dec. 2008*)

13. $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\,dz\,dy\,dx$ **Ans.** $\frac{1}{3}\left[\frac{e^{12}}{6} - \frac{e^6}{3} - \frac{1}{6} + \frac{1}{3}\right] - \frac{1}{2}[e^4 - 1] + [e^2 - 1]$ (*M.U. II Sem., 2003*)

14. $\iiint (x + y + z)\ dx\ dy\ dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. **Ans.** $\frac{1}{8}$

15. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\,dx\,dy\,dz$ **Ans.** $\frac{a^5}{60}$

16. $\int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$ **Ans.** $8\sqrt{2}\pi$

17. $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)\,dz\,dx\,dy$ (*M.U. II Semester, 2000, 02*) **Ans.** 0

18. $\int_0^2 \int_0^y \int_{x-y}^{x+y} (x + y + z)\,dx\,dy\,dz$ (*M.U. II Semester 2004*) **Ans.** 16

19. $\iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\ dx\ dy\ dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans. $\frac{\pi^2}{4} abc$

20. $\iiint \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\ dx\ dy\ dz$ over the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **Ans.** $\frac{4\pi}{3} abc$

21. $\iiint x^{l-1} y^{m-1} z^{n-1}\ dx\ dy\ dz$ throughout the volume of the tetrahedron

$x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$. **Ans.** $\frac{1}{(l+m+n)} \cdot \frac{\lceil l \lceil m \lceil n}{\lceil l+m+n}$

22. $\iiint \frac{dx\,dy\,dz}{\sqrt{1 - x^2 - y^2 - z^2}}$ taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$, lying in the first octant. **Ans.** $\frac{\pi^2}{8}$

23. $\int_0^{\pi} 2d\theta \int_0^{a(1+\cos\theta)} r\,dr \int_0^h \left[1 - \frac{r}{a(1+\cos\theta)}\right] dz$ **Ans.** $\frac{\pi a^2}{2} h$

24. $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\,d\theta\,dr\,dz$ **Ans.** $\frac{5a^3}{64}$

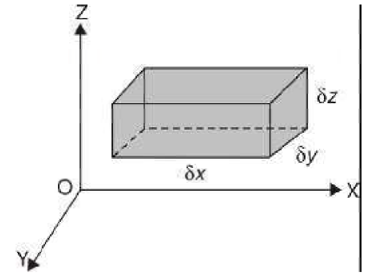
25. $\iiint z^2\,dx\,dy\,dz$ over the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 + z^2 = ax$. **Ans.** $\frac{2a^5\pi}{15}$

26. $\iiint_V \frac{dx\,dy\,dz}{(1 + x^2 + y^2 + z^2)^2}$ where V is the volume in the first octant. **Ans.** $\frac{\pi^2}{8}$

27. $\iiint \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}}$ over the volume bounded by the spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 25$. (*M.U. II Semester, 2001, 03*) **Ans.** $4\pi \log (5/4)$

28. $\iiint_T z^2\,dx\,dy\,dz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the paraboloid $x^2 + y^2 = z$ and the plane $z = 0$. **Ans.** $\frac{\pi a^8}{12}$

2.13 VOLUME = $\iiint dx\, dy\, dz$.



The elementary volume δv is $\delta x\ .\ \delta y\ .\ \delta z$ and therefore the volume of the whole solid is obtained by evaluating the triple integral.

$$\delta V = \delta x\ \delta y\ \delta z$$

$$\boxed{V = \iiint dx\, dy\, dz.}$$

Note : (*i*) Mass = volume × density = $\iiint \rho\ \mathrm{dx\ dy}\ dz$ if ρ is the density.

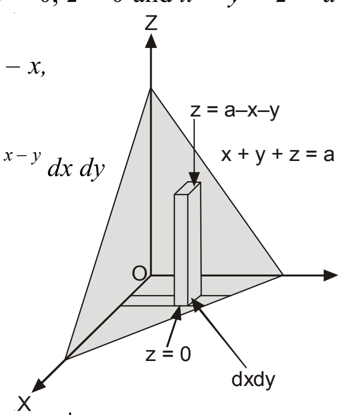
(*ii*) In cylindrical co-ordinates, we have $\boxed{V = \iiint_V r\ dr\ d\varphi\ dz}$

(*iii*) In spherical polar co-ordinates, we have $\boxed{V = \iiint_V r^2 \sin\theta \mathrm{dr}\ \mathrm{d}\theta\ \mathrm{d}\varphi}$

Example 41. *Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = a.* (*M.U. II Semester, 2005, 2000*)

Solution. Here, we have a solid which is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$ planes.

The limits of z are 0 and $a - x - y$, the limits of y are 0 and $1 - x$, the limits of x are 0 and a.



$$V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dx\, dy\, dz = \int_{x=0}^{a} \int_{y=0}^{a-x} [z]_0^{a-x-y}\, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a - x - y)\, dx\, dy$$

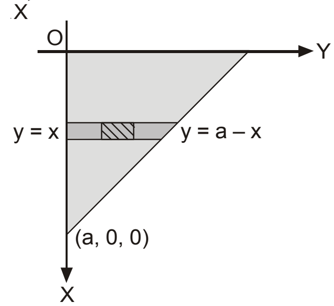
$$= \int_{x=0}^{a} \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

$$= \int_0^a \left[a^2 - ax - ax + x^2 - \frac{a^2}{2} + ax - \frac{x^2}{2}\right] dx$$

$$= \int_0^a \left(\frac{a^2}{2} - ax + \frac{x^2}{2}\right) dx$$

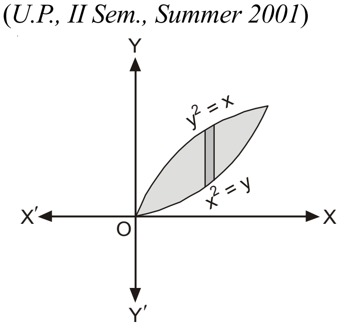
$$= \left[\frac{a^2}{2} . x - \frac{ax^2}{2} + \frac{x^3}{6}\right]_0^a = a^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{a^3}{6}.\quad \textbf{Ans.}$$



Example 42. *Find the volume of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$.* (*U.P., II Sem., Summer 2001*)

Solution. We have,

$$y^2 = x$$
$$x^2 = y$$
$$z = 12 + y - x^2$$



$$V = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{12+y-x^2} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\, dy$$

$$= \int_0^1 dx \left(12y + \frac{y^2}{2} - x^2 y\right)_{x^2}^{\sqrt{x}}$$

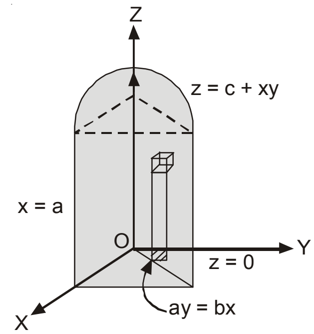
$$= \int_0^1 \left(12\sqrt{x} + \frac{x}{2} - x^{5/2} - 12x^2 - \frac{x^4}{2} + x^4\right) dx$$

$$= \left[\frac{2}{3} \times 12x^{3/2} + \frac{x^2}{4} - \frac{2}{7}x^{7/2} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5}\right]_0^1$$

$$= 8 + \frac{1}{4} - \frac{2}{7} - 4 - \frac{1}{10} + \frac{1}{5} = 4 + \frac{1}{4} - \frac{2}{7} - \frac{1}{10} + \frac{1}{5} = \frac{560 + 35 - 40 - 14 + 28}{140} = \frac{569}{140}$$ **Ans.**

Example 43. *A triangular prism is formed by planes whose equations are ay = bx, y = 0 and x = a. Find the volume of the prism between the planes z = 0 and surface z = c + xy.*

(*M.U. II Semester 2000; U.P., Ist Semester, 2009 (C.O) 2003*)



Solution. Required volume $= \int_0^a \int_0^{\frac{bx}{a}} \int_0^{c+xy} dz\, dy\, dx$

$$= \int_0^a \int_0^{\frac{bx}{a}} (c + xy)\, dy\, dx$$

$$= \int_0^a \left(cy + \frac{xy^2}{2}\right)_0^{\frac{bx}{a}} dx$$

$$= \int_0^a \left(\frac{cbx}{a} + \frac{b^2}{2a^2}x^3\right) dx = \frac{bc}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{abc}{2} + \frac{b^2a^2}{8} = \frac{ab}{8}(4c + ab)$$ **Ans.**

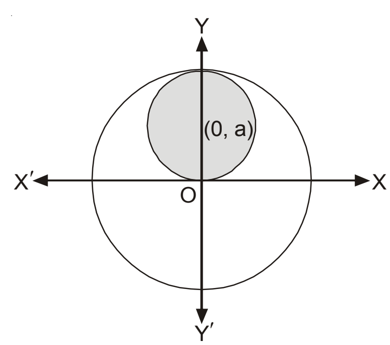
2.14 VOLUME OF SOLID BOUNDED BY SPHERE OR BY CYLINDER

We use spherical coordinates (r, θ, ϕ) and the cylindrical coordinates are (ρ, ϕ, z) and the relations are $x = \rho \cos \phi$, $y = \rho \sin \phi$.

Example 44. *Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.*

Solution.
$$x^2 + y^2 + z^2 = 4a^2 \quad \text{...(1)}$$
$$x^2 + y^2 - 2ay = 0 \quad \text{...(2)}$$

Considering the section in the positive quadrant of the *xy*-plane and taking *z* to be positive (that is volume above the *xy*-plane) and changing to polar co-ordinates, (1) becomes



$$r^2 + z^2 = 4a^2 \quad \Rightarrow \quad z^2 = 4a^2 - r^2$$

$$\therefore \qquad z = \sqrt{4a^2 - r^2}$$

(2) becomes $r^2 - 2ar \sin\theta = 0 \Rightarrow r = 2a \sin\theta$

$$\text{Volume} = \iiint dx\, dy\, dz$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\, dr \int_0^{\sqrt{4a^2 - r^2}} dz \qquad \text{(Cylindrical coordinates)}$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,[z]_0^{\sqrt{4a^2-r^2}} = 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,.\sqrt{4a^2-r^2}$$

$$= 4\int_0^{\pi/2} d\theta \left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^{2a\sin\theta} = \frac{4}{3}\int_0^{\pi/2}\left[-(4a^2-4a^2\sin^2\theta)^{3/2}+8a^3\right]d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}(-8a^3\cos^3\theta+8a^3)\,d\theta = \frac{8\times 4a^3}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\left(1-\frac{1}{4}\cos 3\theta-\frac{3}{4}\cos\theta\right)d\theta$$

$$= \frac{32a^3}{3}\left[\theta-\frac{1}{12}\sin 3\theta-\frac{3}{4}\sin\theta\right]_0^{\pi/2} = \frac{32a^3}{3}\left(\frac{\pi}{2}+\frac{1}{12}-\frac{3}{4}\right) = \frac{32a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right] \textbf{ Ans.}$$

Example 45. *Find the volume enclosed by the solid*

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3} = 1$$

Solution. The equation of the solid is

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3} = 1$$

Putting
$$\left(\frac{x}{a}\right)^{1/3} = u \quad\Rightarrow\quad x = a\,u^3 \quad\Rightarrow\quad dx = 3\,au^2\,du$$
$$\left(\frac{y}{b}\right)^{1/3} = v \quad\Rightarrow\quad y = b\,v^3 \quad\Rightarrow\quad dy = 3\,bv^2\,dv$$
$$\left(\frac{z}{c}\right)^{1/3} = w \quad\Rightarrow\quad z = c\,w^3 \quad\Rightarrow\quad dz = 3\,c\,w^2\,dw$$

The equation of the solid becomes

$$u^2+v^2+w^2 = 1 \qquad \text{...(1)}$$

$$V = \iiint dx\,dy\,dz \qquad \text{...(2)}$$

On putting the values of *dx, dy* and *dz* in (2), we get

$$V = \iiint 27abc\,u^2v^2w^2du\,dv\,dw \qquad \text{...(3)}$$

(1) represents a sphere.
Let us use spherical coordinates.

$$u = r\sin\theta\cos\phi, \qquad v = r\sin\theta\sin\phi,$$
$$w = r\cos\theta, \qquad du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$$

On substituting spherical coordinates in (3), we have

$$V = 27abc\,.\,8\int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta\cos^2\phi\,.\,r^2\sin^2\theta\sin^2\phi$$
$$.\,r^2\cos^2\theta\,.\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 216\,abc\int_{r=0}^{1} r^8\,dr\int_{\phi=0}^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\int_{\theta=0}^{\pi/2}\sin^5\theta\cos^2\theta\,d\theta$$

$$= 216\,abc\left[\frac{r^9}{9}\right]_0^1.\left(\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 3}\right)\left(\frac{\Gamma 3\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{9}{2}}\right) = 24\,abc\,.\,\frac{1}{2}\,.\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}\,.\,\frac{1}{2}\,.\,\frac{\Gamma 3\,\Gamma\frac{3}{2}}{\Gamma\frac{9}{2}}$$

$$= 6abc\,.\frac{\left[\left(\frac{1}{2}\right)\Gamma\frac{1}{2}\right]^2}{2\,!}\,.\frac{2!\,\Gamma\frac{3}{2}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\frac{3}{2}\,\Gamma\frac{3}{2}} \;=\; 6abc\,.\frac{1}{4}\,.\,\pi\,\frac{1}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4}{35}\,abc\,\pi$$

Ans.

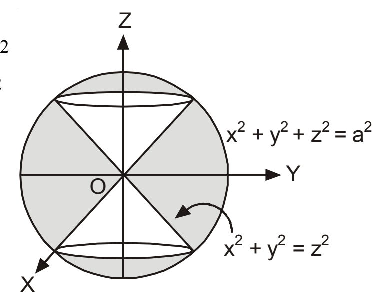
Example 46. *Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.* (*U.P. II Semester 2002*)

Solution. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ...(1)

and that of the cone is $x^2 + y^2 = z^2$ ...(2)

In polar coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

The equation (1) in polar co-ordinates is



$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 + (r\cos\theta)^2 = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta + r^2 \cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 (\sin^2\theta + \cos^2\theta) = a^2$$

$$\Rightarrow \quad r^2 = a^2 \quad \Rightarrow \quad r = a$$

The equation (2) in polar co-ordinates is

$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = (r\cos\theta)^2$$

$$\Rightarrow \quad r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta \;\Rightarrow\; r^2\sin^2\theta = r^2\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta = 1 \;\Rightarrow\; \tan\theta = 1 \Rightarrow \quad \theta = \pm\frac{\pi}{4}$$

Thus equations (1) and (2) in polar coordinates are respectively,

$$r = a \qquad \text{and} \qquad \theta = \pm\frac{\pi}{4}$$

The volume in the first octant is one fourth only.

Limits in the first octant : r varies 0 to a, θ from 0 to $\frac{\pi}{4}$ and ϕ from 0 to $\frac{\pi}{2}$.

The required volume lies between $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$.

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^a$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta\,.\frac{a^3}{3} = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\phi\,[-\cos\theta]_0^{\frac{\pi}{4}} = \frac{4a^3}{3}(\phi)_0^{\frac{\pi}{2}}\left[-\frac{1}{\sqrt{2}}+1\right]$$

$$= \frac{2}{3}\pi a^3\left(1-\frac{1}{\sqrt{2}}\right)$$

Ans.

2.15 VOLUME OF SOLID BOUNDED BY CYLINDER OR CONE

We use cylindrical coordinates (r, θ, z).

Example 47. *Find the volume of the solid bounded by the parabolic $y^2 + z^2 = 4x$ and the plane $x = 5$.*

Solution. $y^2 + z^2 = 4x$, $x = 5$

$$V = \int_0^5 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{4x-y^2}}^{\sqrt{4x-y^2}} dz = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy \int_0^{\sqrt{4x-y^2}} dz$$

$$= 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\,[z]_0^{\sqrt{4x-y^2}} = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\,\sqrt{4x-y^2}$$

$$= 4\int_0^5 dx\left[\frac{y}{2}\sqrt{4x-y^2}+\frac{4x}{2}\sin^{-1}\frac{y}{2\sqrt{x}}\right]_0^{2\sqrt{x}} = 4\int_0^5\left[0+2x\left(\frac{\pi}{2}\right)\right]dx = 4\pi\int_0^5 x\,dx$$

$$= 4\pi\left[\frac{x^2}{2}\right]_0^5 = 50\pi$$ **Ans.**

Example 48. *Calculate the volume of the solid bounded by the following surfaces :*

$z = 0,\quad x^2 + y^2 = 1,\quad x + y + z = 3$

Solution. $x^2 + y^2 = 1$ ...(1)

$x + y + z = 3$ ...(2)

$z = 0$ ...(3)

$$\text{Required Volume} = \iiint dx\,dy\,dz = \iint dx\,dy\,[z]_0^{3-x-y} = \iint (3-x-y)\,dx\,dy$$

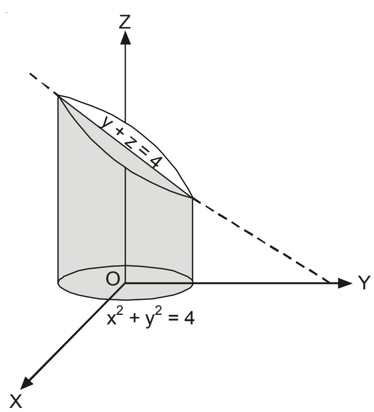
On putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$, we get

$$= \iint (3 - r\cos\theta - r\sin\theta)\,r\,d\theta\,dr = \int_0^{2\pi} d\theta \int_0^1 (3r - r^2\cos\theta - r^2\sin\theta)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{3r^2}{2}-\frac{r^3}{3}\cos\theta-\frac{r^3}{3}\sin\theta\right)_0^1 = \int_0^{2\pi}\left(\frac{3}{2}-\frac{1}{3}\cos\theta-\frac{1}{3}\sin\theta\right)d\theta$$

$$= \left[\frac{3}{2}\theta-\frac{1}{3}\sin\theta+\frac{1}{3}\cos\theta\right]_0^{2\pi} = 3\pi-\frac{1}{3}\sin 2\pi+\frac{1}{3}\cos 2\pi-\frac{1}{3} = 3\pi$$ **Ans.**

Example 49. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*



Solution. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2}$

$y + z = 4 \quad\Rightarrow\quad z = 4 - y$ and $z = 0$

x varies from -2 to $+2$.

$$V = \iiint dx\,dy\,dz = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{4-y} dz$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,[z]_0^{4-y}$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,(4-y) = \int_{-2}^{2} dx\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

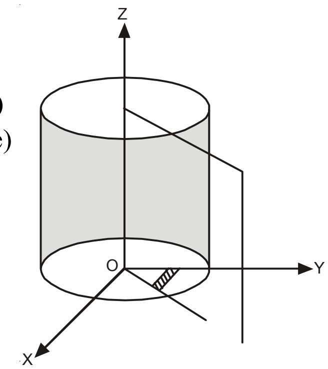
$$= \int_{-2}^{2} dx\left[4\sqrt{4-x^2}-\frac{1}{2}(4-x^2)+4\sqrt{4-x^2}+\frac{1}{2}(4-x^2)\right]$$

$$= 8\int_{-2}^{2}\sqrt{4-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 16\pi$$ **Ans.**

Example 50. *Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$ and $x = 0$.* (*M.U. II Semester 2005*)

Solution. Here, we have the solid bounded by

$$x^2 + y^2 = 2 \text{ (cylinder)}$$
$$(\text{or} \quad r^2 = 2)$$
$$z = x + y \quad \Rightarrow \quad z = r(\cos\theta + \sin\theta) \qquad \text{(plane)}$$
$$y = x \quad \Rightarrow \quad r\sin\theta = r\cos\theta \qquad \text{(plane)}$$



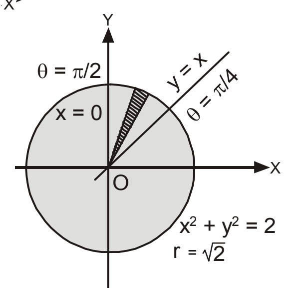
$$\Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad q = \frac{\pi}{4}$$

$$x = 0 \Rightarrow r\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

z varies from 0 to $r(\cos\theta + \sin\theta)$

r varies from 0 to $\sqrt{2}$

θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$



$$\therefore \quad V = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} \int_{z=0}^{r(\cos\theta+\sin\theta)} r\, dr\, d\theta\, dz$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} r[z]_0^{r(\cos\theta+\sin\theta)}\, dr\, d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} r^2(\cos\theta + \sin\theta)\, dr\, d\theta$$

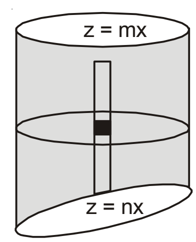
$$= \int_{\theta=\pi/4}^{\pi/2} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta = \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2} (\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{2\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}}{3}\left[(1-0) - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3} \qquad \textbf{Ans.}$$

Example 51. *Show that the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and planes $z = mx$, $z = nx$ is $\pi(m - n)\, a^3$.* (*M.U. II Semester, 2000*)

Solution. The equation of the cylinder is $x^2 + y^2 = 2\,a\,x$

we convert the cartesian coordinates into cylindrical coordinates.



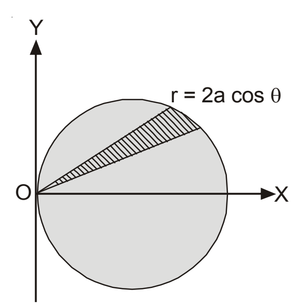
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = 2ax \quad \Rightarrow \quad r^2 = 2ar\cos\theta$$
$$\Rightarrow \quad r = 2a\cos\theta$$

r varies from 0 to $2a\cos\theta$

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

and $\quad z$ varies from $z = nx$ $(z = nr\cos\theta)$ to $z = m\,x$ $(z = m\,r\cos\theta)$

$$V = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \int_{z=nr\cos\theta}^{mr\cos\theta} r\, dr\, d\theta\, dz$$



$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r[z]_{nr\cos\theta}^{mr\cos\theta}\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r\,.(m-n)\, r\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}\cos\theta\,d\theta = 2(m-n)\int_{\theta=0}^{\pi/2}\frac{8a^3}{3}\cos^3\theta\cos\theta\,d\theta$$

$$= \frac{16(m-n)}{3}a^3\int_{\theta=0}^{\pi/2}\cos^4\theta\,d\theta = \frac{16(m-n)}{3}.a^3.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = (m-n)\pi a^3 \quad \textbf{Ans.}$$

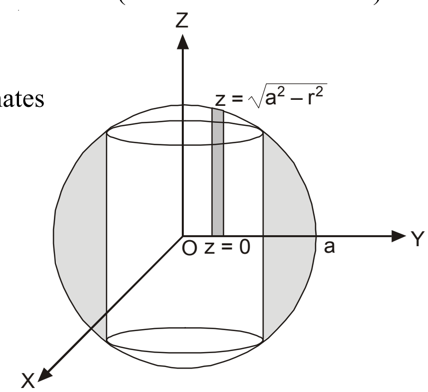
Example 52. *A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid.* (*M.U. II Semester 2004*)

Solution. Let the equation of the sphere be

$$x^2+y^2+z^2 = a^2$$

Now, we will solve this problem using cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



Limits of z are 0 and $\sqrt{a^2-(x^2+y^2)}$ *i.e.*, $\sqrt{a^2-r^2}$

Limits of r are a and b.

and the limits of θ are 0 and $\frac{\pi}{2}$

$$V = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}[z]_0^{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}(a^2-r^2)^{1/2}.r\,dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}.\left(-\frac{1}{2}\right)\right]_b^a d\theta = -\frac{8}{3}\int_0^{\pi/2}-(a^2-b^2)^{\frac{3}{2}}\,d\theta$$

$$= \frac{8}{3}(a^2-b^2)^{\frac{3}{2}}[\theta]_0^{\pi/2} = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}} \quad \textbf{Ans.}$$

Example 53. *Find the volume cut off from the paraboloid*

$x^2+\dfrac{y^2}{4}+z = 1$ *by the plane* $z = 0.$ (*M.U. II Semester 2005*)

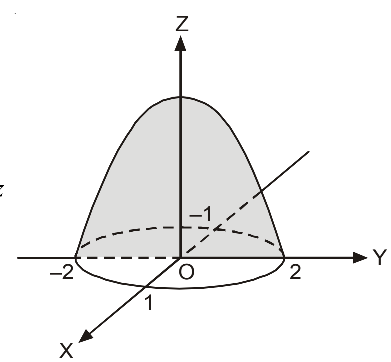
Solution. We have

$$x^2+\frac{y^2}{4}+z = 1 \quad \text{(Paraboloid)} \quad ...(1)$$
$$z = 0 \quad (x\text{-}y \text{ plane}) \quad ...(2)$$

z varies from 0 to $1-x^2-\dfrac{y^2}{4}$

y varies from $-2\sqrt{1-x^2}$ to $2\sqrt{1-x^2}$

x varies from –1 to 1.



$$V = \iiint dx\,dy\,dz = \int_{-1}^{1}dx\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}dy\int_0^{\left(1-x^2-\frac{y^2}{4}\right)}dz$$

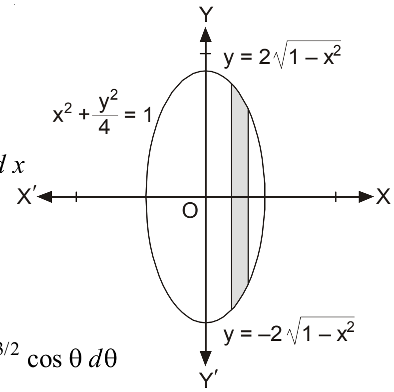
$$= \int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1\int_0^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1 \left[(1-x^2)\,y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} .\,dx$$

$$= 4\int_0^1\left[(1-x^2)\,.\,2\sqrt{1-x^2} - \frac{8}{12}(1-x^2)^{3/2}\right]dx$$

$$= 4\int_0^1\left[2(1-x^2)^{3/2} - \frac{2}{3}(1-x^2)^{3/2}\right]dx$$



On putting $x = \sin\theta$, we get

$$V = 4\int_0^1 \frac{4}{3}(1-x^2)^{3/2}\,dx = \frac{16}{3}\int_0^{\pi/2}(-\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{16}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi \qquad \textbf{Ans.}$$

Example 54. *Find the volume enclosed between the cylinders $x^2 + y^2 = a\,x$, and $z^2 = a\,x$.*

Solution. Here, we have $x^2 + y^2 = ax$ ...(1)

$z^2 = ax$ ...(2)

$$V = \iiint dx\,dy\,dz$$

$$= \int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\int_{-\sqrt{ax}}^{\sqrt{ax}} dz = 2\int_0^a dx\int_{-\sqrt{a^2-x^2}}^{\sqrt{ax-x^2}} dy\int_0^{\sqrt{ax}} dz$$

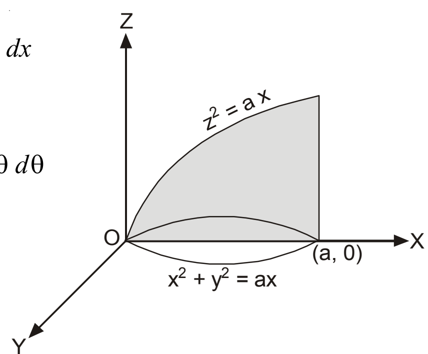
$$= 2\int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,(z)_0^{\sqrt{ax}} = 2\int_0^a dx\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,\sqrt{ax} = 2\int_0^a \sqrt{ax}\,dx\,[y]_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}}$$

$$= 2\int_0^a \sqrt{ax}\,dx\left(2\sqrt{ax-x^2}\right) = 4\sqrt{a}\int_0^a x\sqrt{a-x}\,dx$$

Putting $x = a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$, we get

$$V = 4\sqrt{a}\int_0^{\pi/2} a\sin^2\theta\sqrt{a - a\sin^2\theta}\,.\,2a\sin\theta\cos\theta\,d\theta$$

$$= 8a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$



$$= 8a^3\,\frac{\left|\overline{2}\right.\left|\overline{\tfrac{3}{2}}\right.}{2\left|\overline{\tfrac{7}{2}}\right.} = 4a^3\,\frac{\left|\overline{\tfrac{3}{2}}\right.}{\frac{5}{2}\cdot\frac{3}{2}\left|\overline{\tfrac{3}{2}}\right.} = \frac{16a^3}{15} \qquad \textbf{Ans.}$$

EXERCISE 2.10

1. Find the volume bounded by the coordinate planes and the plane. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **Ans.** $\frac{abc}{6}$

2. Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$ between the planes $z = 0$ and $x + y + z = 2$. **Ans.** $\frac{11}{30}$

3. Find the volume bounded by the co-ordinate planes and the plane.
   $l\,x + m\,y + n\,z = 1$ (*A.M.I.E.T.E. Winter 2001*) **Ans.** $\frac{1}{6\,l\,m\,n}$

4. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration. (*AMIETE, June 2009*) **Ans.** $\frac{4}{3}\pi a^3$

5. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ **Ans.** $\frac{4\pi abc}{3}$

6. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y + z = 2a$ and $z = 0$. (*M.U. II Semester 2000, 02, 06*) **Ans.** $2\pi a^3$

7. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $y + z = b$. **Ans.** $\pi a^2 b$

8. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$. **Ans.** $\frac{8}{3}a^4$

9. Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $x + z = 5$ and $z = 0$. **Ans.** $45\pi - 36$

10. Compute the volume of the solid bounded by $x^2 + y^2 = z$, $z = 2x$.(*A.M.I.E., Summer 2000*) **Ans.** 2π

11. Find the volume cut from the paraboloid $4z = x^2 + y^2$ by plane $z = 4$. (*U.P. I Semester, Dec. 2005*) **Ans.** 32π

12. By using triple integration find the volume cut off from the sphere $x^2 + y^2 + z^2 = 16$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = 4x$. **Ans.** $\frac{64}{9}(3\pi - 4)$

13. The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $x^2 + y^2 = a^2\,(x^2 - y^2)$.

Prove that the volume of the sphere that lies inside the cylinder is $\frac{8}{3}\left[\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right]a^3$.

14. Find the volume of the solid bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and $x^2 + y^2 = 9$. (*A.M.I.E.T.E., Summer 2005*) **Ans.** $\frac{27\pi}{2}$

15. Obtain the volume bounded by the surface $z = c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)$ and a quadrant of the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z > 0$ and where $a, b > 0$. **Ans.** πabc (*A.M.I.E.T.E., Dec. 2005*)

16. Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. **Ans.** 32π

17. Find the volume bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$. **Ans.** $\frac{\pi}{6}$

18. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. **Ans.** $\frac{128a^3}{15}$

19. Find the volume of the solid bounded by the plane $z = 0$, the paraboloid $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$. **Ans.** 16π

20. The triple integral $\iiint dx\,dy\,dz$ gives

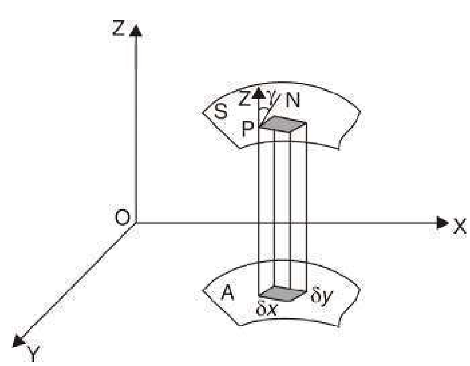
(*a*) Volume of region (*b*) Surface area of region T
(*c*) Area of region *T* (*d*) Density of region *T*. (*A.M.I.E.T.E., Dec. 2006, 2002*) **Ans.** (*a*)

2.16 SURFACE AREA

Let $z = f(x,y)$ be the surface S. Let its projection on the x-y plane be the region A. Consider an element $8x.\ \delta y$ in the region A. Erect a cylinder on the element $\delta x.\ \delta y$ having its generator parallel to OZ and meeting the surface S in an element of area δs.

$$\therefore \qquad \delta x\ \delta y = \delta s \cos\gamma,$$

Where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.,* it is the angle between the Z-axis and the normal to S at P.



The direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$$

$\therefore$ The direction of the normal to $S\,[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the Z-axis are 0, 0 , 1.

$$\text{Direction cosines} = \frac{-\frac{\partial z}{\partial x}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{-\frac{\partial z}{\partial y}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}},$$

Hence
$$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad (\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$\delta S = \frac{\delta x\,\delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\,\delta x\,\delta y\,; \quad S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\,dx\,dy$$

Example 55. *Find the surface area of the cylinder $x^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 4$.*

Solution. $x^2 + y^2 = 4$

$$2x + 2z\frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2}{z^2} + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}$$

$$\text{Hence, the required surface area} = 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\,dx\,dy$$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4 - x^2}}\,dx\,dy = 16\int_0^2 \frac{1}{\sqrt{4 - x^2}}[y]_0^{\sqrt{4-x^2}}\,dx = 16\int_0^2 \frac{1}{\sqrt{4 - x^2}}[\sqrt{4 - x^2}]\,dx$$

$$= 16\int_0^2 dx = 16\,(x)_0^2 = 32$$

Ans.

Example 56. *Find the surface area of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution.
$$x^2 + y^2 + z^2 = 9$$
$$2x + 2z\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$$
$$2x + 2z\frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = \frac{x^2 + y^2 + z^2}{z^2} = \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \quad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

$$x^2 + y^2 = 3y \text{ or } r^2 = 3r\sin\theta \text{ or } r = 3\sin\theta.$$

Hence, the required surface area

$$= \iint \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\,dx\,dy = 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{9 - r^2}}\,r\,d\theta\,dr = 12\int_0^{\pi/2} d\theta \int_0^{3\sin\theta} \frac{r\,dr}{\sqrt{9 - r^2}}$$

$$= 12\int_0^{\pi/2} d\theta\,[-\sqrt{9 - r^2}]_0^{3\sin\theta} = 12\int_0^{\pi/2} [-\sqrt{9 - 9\sin^2\theta} + 3]\,d\theta$$

$$= 36\int_0^{\pi/2}(-\cos\theta+1)\,d\theta = 36\,(-\sin\theta+\theta)_0^{\pi/2} = 36\left(-1+\frac{\pi}{2}\right) = 18\,(\pi-2) \qquad \textbf{Ans.}$$

Example 57. *Find the surface area of the section of the cylinder $x^2 + y^2 = a^2$ made by the plane $x + y + z = a$.*

Solution.

$$x^2 + y^2 = a^2 \qquad \ldots (1)$$

$$x + y + z = a \qquad \ldots (2)$$

The projection of the surface area on xy-plane is a circle

$x^2 + y^2 = a^2$

$$1+\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1$$

$$1+\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -1$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} = \sqrt{(-1)^2+(-1)^2+1} = \sqrt{3}$$

Hence the required surface area

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy = 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{3}\;dx\cdot dy$$

$$= 4\sqrt{3}\int_0^a[y]_0^{\sqrt{a^2-x^2}}\,dx = 4\sqrt{3}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$= 4\sqrt{3}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4\sqrt{3}\left[0+\frac{a^2}{2}\,\frac{\pi}{2}\right] = 4\sqrt{3}\left(\frac{a^2\pi}{4}\right) = \sqrt{3}\,\pi\,a^2 \quad \textbf{Ans.}$$

Example 58. *Find the area of that part of the surface of the paraboloid of the paraboloid $y^2 + z^2 = 2\,ax$, which lies between the cylinder, $y^2 = ax$ and the plane $x = a$.*

Solution.

$$y^2 + z^2 = 2\,ax \qquad \ldots (1)$$

$$y^2 = ax \qquad \ldots (2)$$

$$x = a \qquad \ldots (3)$$

Differentiating (1), we get

$$2\,z\frac{\partial z}{\partial x} = 2\,a,\quad \frac{\partial z}{\partial x} = \frac{a}{z}$$

$$2y + 2\,z\frac{\partial z}{\partial y} = 0,\quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1 = \frac{a^2}{z^2}+\frac{y^2}{z^2}+1 = \frac{a^2+y^2}{z^2}+1 \qquad \left[\begin{matrix} y^2+z^2 = 2\,ax \\ z^2 = 2\,ax-y^2\end{matrix}\right]$$

$$= \frac{a^2+y^2}{2\,a\,x-y^2}+1 = \frac{a^2+y^2+2\,a\,x-y^2}{2\,a\,x-y^2} = \frac{a^2+2\,a\,x}{2\,a\,x-y^2}$$

$$S = \int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy = \int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\frac{a^2+2\,ax}{2\,ax-y^2}}\;dx\,dy \qquad \left[\begin{matrix} y^2 = ax \\ y = \pm\sqrt{ax}\end{matrix}\right]$$

$$= \sqrt{a}\int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\frac{a+2\,x}{2\,ax-y^2}}\;dx\,dy = \sqrt{a}\int_0^a\sqrt{a+2\,x}\;dx\int_{-\sqrt{ax}}^{\sqrt{ax}}\frac{1}{\sqrt{2\,ax+y^2}}\,dy$$

$$= \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\sin^{-1}\frac{y}{\sqrt{2ax}}\right]_{-\sqrt{ax}}^{\sqrt{ax}}$$

$$= \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\sin^{-1}\frac{1}{\sqrt{2}}-\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right] = \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\frac{\pi}{4}-\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{a}\,\frac{\pi}{2}\int_0^a \sqrt{a+2x}\,dx = \frac{\pi}{2}\cdot\frac{\sqrt{a}}{2}\cdot\frac{2}{3}[(a+2x)^{3/2}]_0^a$$

$$= \frac{\pi\sqrt{a}}{6}[(3a)^{3/2}-a^{3/2}] = \frac{\pi a^2}{6}[3\sqrt{3}-1]$$ **Ans.**

EXERCISE 2.11

1. Find the surface area of sphere $x^2 + y^2 + z^2 = 16$. **Ans.** 64π
2. Find the surface area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$. **Ans.** 64.
3. Show that the area of surfaces cz = xy intercepted by the cylinder $x^2 + y^2 = b^2$ is $\iint_A \frac{\sqrt{c^2+x^2+y^2}}{c}\,dx\,dy$, where A is the area of the circle $x^2 + y^2 = b^2$, $z = 0$

   Ans. $\frac{2}{3}\frac{\pi}{c}\left[(c^2+b^2)^{\frac{1}{2}}-c^2\right]$
4. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$. **Ans.** $2(\pi - 2)a^2$
5. Find the area of the surface of the cone $z^2 = 3(x^2 + y^2)$ cut out by the paraboloid $z = x^2 + y^2$ using surface integral. **Ans.** 6π

2.17 CALCULATION OF MASS

We have,

Volume $= \iint_V dx\,dy\,dz$ Density $= \rho = f(x, y, z)$

[Density = Mass per unit volume] Mass = Volume × Density

Mass $= \iint_V dx\,dy\,dz$ $\boxed{Mass = \iint_V f(x,y,z)\,dx\,dy\,dz}$

Example 59. Find the mass of a plate which is formed by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the density is given by $\rho = k\,x\,y\,z$. (*U.P., I Semester*, *Dec*., 2003)

Solution. The plate is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$$\text{Mass} = \iiint dx\,dy\,dz\,\rho = \int_0^c\int_0^{b\left(1-\frac{z}{c}\right)}\int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\,dy\,dz\,(k\,xyz)$$

$$= k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} x\,dx = k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\left(\frac{x^2}{2}\right)_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)}$$

$$= k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\,\frac{a^2}{2}\left(1-\frac{y}{b}-\frac{z}{c}\right)^2 = \frac{k\,a^2}{2}\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\left[\left(1-\frac{z}{c}\right)-\frac{y}{b}\right]^2 dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)}\left[y\left(1-\frac{z}{c}\right)^2+\frac{y^3}{b^2}-\frac{2y^2}{b}\left(1-\frac{z}{c}\right)\right]dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\left[\frac{y^2}{2}\left(1-\frac{z}{c}\right)^2 + \frac{y^4}{4\,b^2} - \frac{2\,y^3}{3\,b}\left(1-\frac{z}{c}\right)\right]_0^{b\left(1-\frac{z}{c}\right)}$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\left[\frac{b^2}{2}\left(1-\frac{z}{c}\right)^4 + \frac{b^4}{4\,b^2}\left(1-\frac{z}{c}\right)^4 - \frac{2}{3}\cdot\frac{b^3}{b}\left(1-\frac{z}{c}\right)^4\right]$$

$$= \frac{k\,a^2}{2}\int_0^c z\left[\frac{b^2}{2}+\frac{b^2}{4}-\frac{2b^2}{3}\right]\left(1-\frac{z}{c}\right)^4 dz = \frac{k\,a^2}{2}\,\frac{b^2}{12}\int_0^c\left(1-\frac{z}{c}\right)^4 dz \qquad [\text{Put } z = c\sin^2\theta]$$

$$= \frac{k\,a^2\,b^2\,c^2}{12}\int_0^{\frac{\pi}{2}} c\sin^2\theta\,(1-\sin^2\theta)^4\,(2\,c\sin\theta\cos\theta\,d\theta)$$

$$= \frac{k^2\,a^2\,b^2\,c^2}{12}\int_0^{\pi/2}\sin^2\theta\,(\cos^8\theta)\sin\theta\cos\theta\,d\theta = \frac{k^2\,a^2\,b^2\,c^2}{12}\int_0^{\pi/2}\sin^3\theta\cos^9\theta\,d\theta$$

$$= \frac{k^2\,a^2\,b^2\,c^2}{12}\,\frac{\left|\overline{\frac{3+1}{2}}\right.\left|\overline{\frac{9+1}{2}}\right.}{2\left|\overline{\frac{3+9+2}{2}}\right.} = \frac{k\,a^2\,b^2\,c^2}{12}\cdot\frac{|\overline{2}\,|\overline{5}}{2\,|\overline{7}} = \frac{k\,a^2\,b^2\,c^2}{12}\,\frac{(1)\,(|\overline{5})}{2\times 6\times 5\,|\overline{5}} = \frac{k\,a^2\,b^2\,c^2}{720} \quad \textbf{Ans.}$$

2.18 CENTRE OF GRAVITY

$$\bar{x} = \frac{\iiint x\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\ \bar{y} = \frac{\iiint y\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\ \bar{z} = \frac{\iiint z\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}$$

Example 60. *Find the co-ordinates of the centre of gravity of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$, density being given = k xyz.*

Solution. $$\bar{x} = \frac{\iiint_V x\,\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz} = \frac{\iiint z\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz} = \frac{\iiint_V x^2yz\,dx\,dy\,dz}{\iiint_V xyz\,dx\,dy\,dz}$$

Converting into polar co-ordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)^2\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)}$$

$$= \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^6\sin^4\theta\cos\theta\sin\phi\cos^2\phi\,dr\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^5\sin^3\theta\cos\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi}$$

$$= \frac{\int_0^{\pi/2}\sin\phi\cos^2\phi\,d\phi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta\int_0^a r^6\,dr}{\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\int_0^a r^5\,dr}$$

$$= \frac{\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}\left[\frac{\sin^5\theta}{5}\right]_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^a}{-\left[\frac{\cos^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^a} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)\left(\frac{a^7}{7}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{a^6}{6}\right)} = \frac{16\,a}{35}$$

Similarly, $\bar{y} = \bar{z} = \frac{16\,a}{35}$; Hence, C.G. is $\left(\frac{16\,a}{35}, \frac{16\,a}{35}, \frac{16\,a}{35}\right)$ **Ans.**

2.19 MOMENT OF INERTIA OF A SOLID

Let the mass of an element of a solid of volume V be $\rho\ \delta x\ \delta y\ \delta z$.

Perpendicular distance of this element from the x-axis = $\sqrt{y^2 + z^2}$

M.I. of this element about the x-axis = $\rho\ \delta x\ \delta y\ \delta z\ \sqrt{y^2 + z^2}$

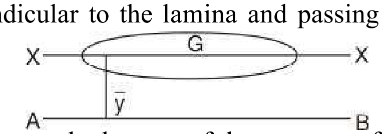
M.I. of the solid about x-axis = $\iiint_V \rho\,(y^2 + z^2)\,dx\,dy\,dz$

M.I. of the solid about y-axis = $\iiint_V \rho\,(x^2 + z^2)\,dx\,dy\,dz$

M.I. of the solid about z-axis = $\iiint_V \rho\,(x^2 + y^2)\,dx\,dy\,dz$

The Perpendicular Axes Theorem

If I_{ox} and I_{oy} be the moments of inertia of a lamina about x-axis and y-axis respectively and I_{oz} be the moment of inertia of the lamina about an axis perpendicular to the lamina and passing through the point of intersection of the axes OX and OY.



$$I_{OZ} = I_{OX} + I_{OY}$$

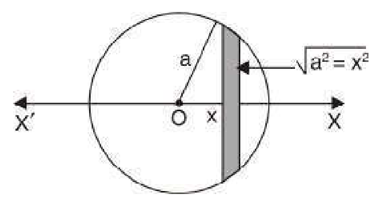
The Parallel Axes Theorem

M.I. of a lamina about an axis in the plane of the lamina equals the sum of the moment of inertia about a parallel centroidal axis in the plane of lamina together with the product of the mass of the lamin *a* and square of the distance between the two axes.

$$I_{AB} = I_{XX} + My^{-2}$$

Example 61. *Find M.I. of a sphere about diameter.*

Solution. Let a circular disc of $\delta\ x$ thickness be perpendicular to the given diameter XX' at a distance x from it.



The radius of the disc = $\sqrt{a^2 - x^2}$

Mass of the disc = $\rho\,\pi\,(a^2 - x^2)$

Moment of inertia of the disc about a diameter perpendicular on it

$$= \frac{1}{2} MR^2 = \frac{1}{2}[\rho\,\pi\,(a^2 - x^2)]\,(a^2 - x^2) = \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2$$

$$\text{M.I. of the sphere} = \int_{-a}^{a} \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2\,dx = 2\left(\frac{1}{2}\rho\,\pi\right)\int_0^a [a^4 - 2\,a^2\,x^2 + x^4]\,dx$$

$$= \rho\,\pi\left[a^4 x - \frac{2\,a^2\,x^3}{3} + \frac{x^5}{5}\right]_0^a = \rho\,\pi\left[a^5 - \frac{2\,a^5}{3} + \frac{a^5}{5}\right]$$

$$= \frac{8}{15}\pi\,\rho\,a^5 = \frac{2}{5}\left(\frac{4\,\pi}{3}a^3\,\rho\right)a^2 = \frac{2}{5}M\,a^2$$ **Ans.**

Example 62. *The mass of a solid right circular cylinder of radius a and height h* is *M. Find the moment of inertia of the cylinder about* (*i*) *its axis* (*ii*) *a line through its centre of gravity perpendicular to its axis* (*iii*) *any diameter through its base.*

Solution. *To find M.I. about OX. Consider a disc at a distance x from O at the base.*

M.I. *of the about OX*, = $\dfrac{(\pi\,a^2\,\rho\,dx)\,a^2}{2} = \dfrac{\pi\,\rho\,a^4\,dx}{2}$

(*i*) M.I. of the cylinder about OX

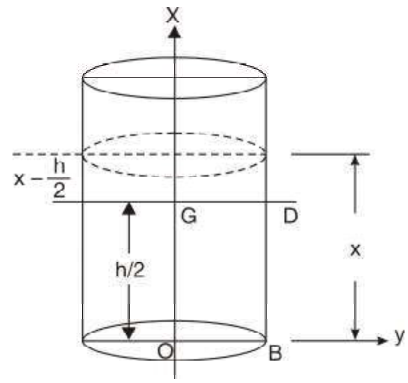
$$\int_0^h \frac{\pi\,\rho\,a^4 dx}{2} = \frac{\pi\,\rho\,a^4}{2}(x)_0^h = \frac{\pi\,\rho\,a^4\,h}{2} = (\pi\,a^2\,h)\,\rho\cdot\frac{a^2}{2} = \frac{M\,a^2}{2}$$

(*ii*) *M.I.* of the disc about a line through *C.G.* and perpendicular to *OX*.

$I_{OX} + I_{OY} = I_{OZ}$

$I_{OX} + I_{OX} = I_{OZ}$

$I_{OX} = \dfrac{1}{2} I_{OZ}$



M.I. of the disc about a line through

$$C.G. = \frac{1}{2}\left(\frac{M\,a^2}{2}\right) = \frac{M\,a^2}{4}$$

M.I. of the disc about the diameter $= \left(\dfrac{\pi\, a^2\, \rho\, dx}{4}\right) a^2$

M.I. of the disc about line $GD = \dfrac{\pi\, a^2\, \rho\, dx}{4} + (\pi\, a^2\, \rho\, dx)\left(x - \dfrac{h}{2}\right)^2$

Hence, *M.I.* of cylinder about $GD = \int_0^h \dfrac{\pi\, a^2\, \rho}{4}\, dx + \int_0^h (\pi\, a^2\, \rho\, dx)\left(x - \dfrac{h}{2}\right)^2$

$$= \frac{\pi\, a^2\, \rho}{4}(x)_0^h + \left[\frac{\pi\, a^2\, \rho}{4}\left(x - \frac{h}{2}\right)^3\right]_0^h = \frac{\pi\, a^2\, \rho\, h}{4} + \left[\frac{\pi\, a^2\, \rho}{3}\left(\frac{h}{2}\right)^3 + \frac{\pi\, a^2\, \rho}{3}\left(\frac{h}{2}\right)^3\right]$$

$$= \frac{\pi\, a^2\, \rho\, h}{4} + \frac{\pi\, a^2\, \rho\, h^3}{12} \qquad \frac{M\, a^2}{4} + \frac{M\, h^2}{12}$$

(*iii*) *M.I.* of cylinder about line *OB* (through) base

$$I_{OB} = I_G + M\left(\frac{h}{2}\right)^2 = \frac{M\, a^2}{4} + \frac{M\, h^2}{12} + \frac{M\, h^2}{4} = \frac{M\, a^2}{4} + \frac{M\, h^2}{3}$$ **Ans.**

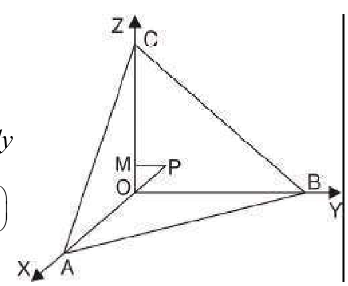
Example 63. *Find the moment of inertia and radius of gyration about z-axis of the region in the first octant bounded by* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Solution. Let *r* be the density. M.I. of tetrahedron about *z*-axis

$$= \iiint (\rho\, dx\, dy\, dz)\,(x^2 + y^2)$$

$$= \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz = \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy\, (z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}$$

$$= \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy\; c\, c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$



$$= \rho c \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2\left(1 - \frac{x}{a}\right) - \frac{x^2 y}{b} + y^2\left(1 - \frac{x}{a}\right) - \frac{y^3}{b}\right] dy$$

$$= \rho c \int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) y - \frac{x^2 y^2}{2\,b} + \frac{y^3}{3}\left(1 - \frac{x}{a}\right) - \frac{y^4}{4\,b}\right]_0^{b\left(1-\frac{x}{a}\right)}$$

$$= \rho c \int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) b\left(1 - \frac{x}{a}\right) - \frac{x^2}{2\,b}\, b^2\left(1 - \frac{x}{a}\right)^2 + \frac{b^3}{3}\left(1 - \frac{x}{a}\right)^3\left(1 - \frac{x}{a}\right) - \frac{b^4}{4\,b}\left(1 - \frac{x}{a}\right)^4\right]$$

$$= b\rho c \int_0^a \left[x^2\left(1 - \frac{x}{a}\right)^2 - \frac{x^2}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{b^2}{3}\left(1 - \frac{x}{a}\right)^4 - \frac{b^2}{4}\left(1 - \frac{x}{a}\right)^4\right] dx$$

$$= \rho bc \int_0^a \left[\frac{x^2}{2}\left(1-\frac{x}{a}\right)^2 + \frac{b^2}{12}\left(1-\frac{x}{a}\right)^4 \right] dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(x^2 - \frac{2\,x^3}{a} + \frac{x^4}{a^2}\right) + \frac{b^2}{12}\left(1 - \frac{4\,x}{a} + \frac{6\,x^2}{a^2} - \frac{4\,x^3}{a^3} + \frac{x^4}{a^4}\right)\right] dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(\frac{x^3}{3} - \frac{x^4}{2\,a} + \frac{x^5}{5\,a^2}\right) + \frac{b^2}{12}\left(x - \frac{2\,x^2}{a} + \frac{6\,x^2}{a^2} - \frac{4\,x^3}{a^3} + \frac{x^4}{a^4}\right)\right]_0^a dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(\frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{5}\right) + \frac{b^2}{12}\left(a - 2a + 2a - a + \frac{a}{5}\right)\right]$$

$$= \rho bc \left[\frac{a^3}{60} + \frac{ab^2}{60}\right] = \rho\,\frac{abc}{60}\,(a^2 + b^2)$$

Radius of gyration $= \sqrt{\dfrac{M.I.}{Mass}} = \dfrac{\sqrt{\dfrac{\rho abc}{60}\,(a^2+b^2)}}{\dfrac{\rho abc}{6}} = \sqrt{\dfrac{1}{10}\,(a^2+b^2)}$ **Ans.**

2.20 CENTRE OF PRESSURE

The centre of pressure of a plane area immersed in a fluid is the point at which the resultant force acts on the area.

Consider a plane area A immersed vertically in a homogeneous liquid. Let x-axis be the line of intersection of the plane with the free surface. Any line in this plane and perpendicular to x-axis is the *y-axis.*

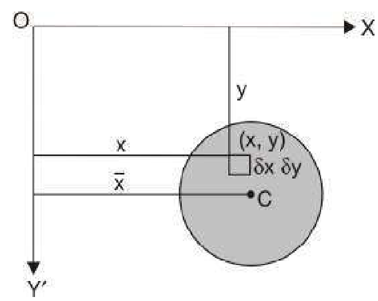
Let P be the pressure at the point (x, y). Then the pressure on elementary area $\delta x\ \delta y$ is $P\ \delta x\ \delta y$.

Let $(\bar{x}, \bar{y})$ be the centre of pressure. Taking moment about y-axis.

$$\bar{x} \cdot \iint_A P\,dx\,dy = \iint_A Px\,dx\,dy$$

$$\bar{x} = \frac{\iint_A Px\,dx\,dy}{\iint_A P\,dx\,dy}$$

Similarly, $$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy}$$



Example 64. *A uniform semi-circular lamina is immersed in a fluid with its plane vertical and its bounding diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.*

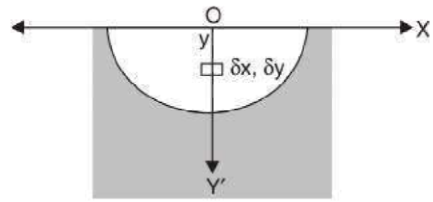
Solution. Let the semi-circular lamina be

$$x^2 + y^2 = a^2$$

By symmetry its centre of pressure lies on OY. Let ky be the density of the fluid.

$$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy} = \frac{\iint_A (\rho y)\,y\,dx\,dy}{\iint_A (\rho y)\,dx\,dy} \qquad (\because \rho = ky)$$

$$= \frac{\iint_A (ky\,.\,y)\,y\,dx\,dy}{\iint_A (ky\,.\,y)\,dx\,dy} = \frac{\iint_A y^3\,dx\,dy}{\iint_A y^2\,dx\,dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\,dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\,dy}$$



$$= \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left[\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}}} = \frac{3}{4}\frac{\int_{-a}^{a} dx\,(a^2-x^2)^2}{\int_{-a}^{a} dx\,(a^2-x^2)^{3/2}}$$

$$= \frac{3}{4}\frac{\int_{-\pi/2}^{\pi/2}(a\cos\theta\, d\theta)\,(a^2-a^2\sin^2\theta)^2}{\int_{-\pi/2}^{\pi/2}(a\cos\theta\, d\theta)\,(a^2-a^2\sin^2\theta)^{3/2}} \qquad \text{(Put } x = a\sin\theta\text{)}$$

$$= \frac{3\,a}{4}\frac{\int_{-\pi/2}^{\pi/2}\cos^5\theta\, d\theta}{\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta} = \frac{3\,a}{4}\frac{2\int_{0}^{\pi/2}\cos^5\theta\, d\theta}{2\int_{0}^{\pi/2}\cos^4\theta\, d\theta} = \frac{3\,a}{4}\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\frac{\pi}{2}} = \frac{32\,a}{15\,\pi} \qquad \textbf{Ans.}$$

EXERCISE 2.12

1. Find the mass of the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the co-ordinate planes, where the density at any point $P\,(x, y, z)$ is $k\,xyz$. **Ans.** P

2. If the density at a point varies as the square of the distance of the point from XOY plane, find the mass of the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.

Ans. $\frac{4\,k}{15}\,a^5\left(\frac{\pi}{2}-\frac{8}{15}\right)$

3. Find the mass of the plate in the form of one loop of leminscate $r^2 = a^2 \sin 2\,\theta$, where $\rho = k\,r^2$.

Ans. $\frac{k\,\pi\,a^4}{16}$

4. Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$, if the density varies as the distance from the pole.

5. Find the mass of a lamina in the form of the cardioid $r = a\,(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line. **Ans.** $\frac{21\,\pi\,\mathrm{k}\,a^4}{32}$

6. Find the centroid of the region in the first octant bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. **Ans.** $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$

7. Find the centroid of the region bounded by $z = 4 - x^2 - y^2$ and xy-plane. **Ans.** $\left(0, 0, \frac{4}{3}\right)$

8. Find the position of $C.G.$ of the volume intercepted between the parallelepiped $x^2+y^2 = a(a - z)$ and the plane $z = 0$. **Ans.** $\left(0, 0, \frac{a}{3}\right)$

9. A solid is cut off the cylinder $x^2 + y^2 = a^2$ by the plane $z = 0$ and that part of the olane $z = mx$ for which z is positive. The density of the solid cut off at any point varies as the height of the point above plane $z = 0$. Find $C.G.$ of the solid. **Ans.** $\bar{z} = \frac{64\,ma}{45\,\pi}$

10. If an area is bounded by two concentric semi-circles with their common bounding diameter in a free surface, prove that the depth of the centre of pressure is $\frac{3\,\pi}{16}\frac{(a+b)\,(a^2+b^2)}{a^2+ab+b^2}$

11. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is immersed vertically in a fluid with its major axis horizontal. If its centre be at depth h, find the depth of its centre of pressure. **Ans.** $h+\frac{b^2}{4\,h}$

12. A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of centre of pressure of either end is 0.7 × total depth approximately.

13. A quadrant of a circle of radius a is just immersed vertically in a homogeneous liquid with one edge in the surface. Determine the co-ordinates of the centre of pressure. **Ans.** $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$

14. Find the product of inertia of an equilateral triangle about two perpendicular axes in its plane at a vertex, one of the axes being along a side.

15. Find the *M.I.* of a right circular cylinder of radius a and height h about axis if density varies as distance from the axis. **Ans.** $\frac{2}{5} k \pi a^5 h$

16. Compute the moment of inertia of a right circular cone whose altitude is h and base radius r, about (i) the axis of symmetry (ii) the diameter of the base. **Ans.** (i) $\frac{\pi h r^4}{10}$ (ii) $\frac{\pi h r^2}{60}(2h^2 + 3r^2)$

17. Find the moment of inertia for the area of the cardioid $r = a(1 - \cos\theta)$ relative to the pole.

Ans. $\frac{35\pi a^4}{16}$

18. Find the M.I. about the line $\theta = \frac{\pi}{2}$ of the area enclosed by $r = a(1 + \cos\theta)$.

19. Find the moment of inertia of the uniform solid in the form of octant of the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about OX **Ans.** $\frac{M}{5}(b^2 + c^2)$

20. Prove that the moment of inertia of the area included between the curves $y^2 = 4ax$ and $x^2 = 4ay$ about the x-axis is $\frac{144}{35} M a^2$, where M is the mass of area included between the curves.

21. A solid body of density p is the shape of solid formed by revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\left(\frac{352}{105}\right)\pi l a^5$. (*U. P. II Semester, Summer 2001*)

22. Find the product of inertia of a disc in the form of a quadrant of a circle of radius 'a' about bounding radii. (*U. P. II Semester, Summer 2002*) **Ans.** $\rho \frac{a^4}{4}$

23. Show that the principal axes at the origin of the triangle enclosed by $x = 0$, $y = 0$, $\frac{x}{a} + \frac{y}{b} = 1$ are inclined at angles α and $\alpha + \frac{\pi}{2}$ to the x-axis, where $a = \frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2 - b^2}\right)$ (*U.P. II Semester Summer* 2001)

Choose the correct answer:

24. The triple integral $\iiint_T dx\,dy\,dz$ gives

(i) Volume of region T (ii) Surface area of region T
(ii) Area of region T (iv) Density of region T. (*A.M.I.E.T.E. 2002*)

Ans. (i)

25. The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

(i) $\frac{\pi}{2a}$ (ii) $\frac{\pi a^3}{2}$ **Ans.** (ii)
(iii) $\frac{4}{3}\pi a^3$ (iv) None of the above. [*U.P., I. Sem. Dec. 2008*]

UNIT-3

Gamma, Beta Functions

21.1 GAMMA FUNCTION

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that* $\Gamma 1 = 1$

Solution. $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Put $n = 1$, $\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\dfrac{e^{-x}}{-1}\right]_0^\infty = 1$ **Proved**

Example 2. *Prove that*

(i) $\Gamma\overline{n+1} = n\,\Gamma n$ (ii) $\Gamma\overline{n+1} = \underline{|n}$ (*Reduction formula*)

Solution.

(i) $\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx$ ...(1)

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0}\ 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$\therefore$ $\Gamma n = (n-1)\,\Gamma\overline{n-1}$ ...(2)

$\Gamma\overline{n+1} = n\,\Gamma n$ Replacing n by $(n+1)$ **Proved**

(ii) Replace n by $n-1$ in (2), we get

$$\overline{|n-1} = (n-2)\,\overline{|n-2}$$

Putting the value $\Gamma n-1$ in (2), we get

$$\Gamma n = (n-1)(n-2)\overline{|n-2}$$

Similarly $$\Gamma n = (n-1)(n-2)\ldots 3.2.1\,\Gamma 1 \qquad \ldots (3)$$

Putting the value of $\overline{|1}$ in (3), we have

$$\Gamma n = (n-1)(n-2)\ldots 3.2.1.1$$

$$\Gamma n = \underline{|n-1}$$

Replacing n by $n+1$, we have

$$\overline{|n+1} = \underline{|n}$$ **Proved**

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$ ...(1)

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2}\, e^{-t}\, 2\,t\,dt = 2\int_0^\infty t^{3/2}\, e^{-t}\, dt$$

$$= 2\Gamma\frac{5}{2} \qquad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\Gamma\frac{3}{2} = 2\,.\,\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{2}\sqrt{\pi}$$ **Ans.**

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$ ...(1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3\,t^2\,dt$ in (1) we get

$$I = \int_0^\infty t^{3/2}\, e^{-t}\, 3\,t^2\,dt = 3\int_0^\infty t^{7/2}\, e^{-t}\, dt = 3\Gamma\frac{9}{2} = 3\,.\,\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$$ **Ans.**

Example 5. *Evaluate* $\int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx.$

Solution. Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx$ ...(1)

Putting $t = h^2x^2$ or $x = \dfrac{\sqrt{t}}{h}$ or $dx = \dfrac{dt}{2\,h\sqrt{t}}$,

(1) becomes $$I = \int_o^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\,\frac{dt}{2\,h\,\sqrt{t}}$$

$$= \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-1}{2}}\, e^{-t}\,\frac{dt}{\sqrt{t}} = \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-2}{2}}\, e^{-t}\, dt$$

$$= \frac{1}{2\,h^n}\Gamma\frac{n}{2} \qquad \textbf{Ans.}$$

Example 6. *Evaluate* $\int_0^\infty \frac{x^a}{a^x}\,dx.$ $(a > 1)$

Solution: $I = \int_0^\infty \frac{x^a}{a^x}\,dx$ ...(1)

Putting $a^x = e^t$ or $x \log a = t$, $x = \frac{t}{\log a}$, $dx = \frac{dt}{\log a}$ in (1), we have

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\,t^a\,dt$$

$$= \frac{1}{(\log a)^{a+1}}\,\Gamma(a+1) \qquad \textbf{Ans.}$$

Example 7. *Evaluate* $\int_0^1 x^{n-1}\cdot\left[\log_e\cdot\left(\frac{1}{x}\right)\right]^{m-1}\cdot dx$

Solution: Put $\log_e \frac{1}{x} = t$ or $x = e^{-t}$ $\therefore dx = -e^{-t}\,dt$

$$\int_0^1 x^{n-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} dx = \int_\infty^0 (e^{-t})^{n-1}\,[t]^{m-1}\,(-e^{-t}\,dt) = \int_0^\infty e^{-nt}\,t^{m-1}\,dt$$

Put $nt = u$ or $t = \frac{u}{n}$ $\therefore$ $dt = \frac{du}{n}$

$$= \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1}\frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}\,u^{m-1}\,du = \frac{1}{n^m}\Gamma m \qquad \textbf{Ans.}$$

21.2 TRANSFORMATION OF GAMA FUNCTION

Prove that (1) $\int_0^\infty e^{-ky}y^{n-1}\,dy = \frac{\Gamma n}{k^n}$ (2) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (3) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1} dy = \Gamma n$

Solution: We know that $\Gamma n = \int_0^\infty x^{n-1}\,e^{-x}\,dx$ ...(1)

(i) Replace x by $k\,y$, so that $dx = kdy$; then

(1) becomes $\Gamma n = \int_0^\infty (ky)^{n-1}\,e^{-ky}\,k\,dy.$

$$\Gamma n = k^n\int_0^\infty e^{-ky}\,y^{n-1}\,dy$$

$$\therefore \qquad \int_0^\infty e^{-ky}\,y^{n-1}\,dy = \frac{\Gamma n}{K^n} \qquad \text{...(2)} \quad \textbf{Proved}$$

(*ii*) Replace x^n by y, $n\,x^{n-1}\,dx = dy$ in (1), then

$$\Gamma n = \int_0^\infty y^{\frac{n-1}{n}}\,e^{-y^{1/n}}\,\frac{dy}{nx^{n-1}}$$

$$= \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{n y^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\, dy$$

When $n = \frac{1}{2},$ $$\Gamma\frac{1}{2} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}\, dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$$

Proved $$\Gamma\frac{1}{2} = \sqrt{\pi}$$

(*iii*) Substitute e^{-x} by $y, -e^{-x}\,dx = dy$

$$-x = \log y,\ x = \log\frac{1}{y},\quad \text{Then (1) becomes}$$

$$\Gamma n = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} y\,.\,\frac{dy}{e^{-x}}$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} y\,.\,\frac{dy}{y} = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy. \quad \textbf{Proved}$$

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma -\frac{1}{2}$ (*ii*) $\Gamma\frac{-3}{2}$ (*iii*) $\Gamma\frac{-15}{2}$ (*iv*) $\Gamma\frac{7}{2}$ (*v*) $\Gamma 0$

Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\frac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\, e^{-x}\, dx$ **Ans.** $\Gamma\frac{3}{2}$

3. $\int_0^\infty x^4 e^{-x^2}\, dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}\, dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

5. $\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy,\ a, b, m, n > 0$ **Ans.** $\frac{\Gamma m\,\Gamma n}{4a^m b^n}$

6. $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy,\quad n > 0$ **Ans.** Γn

7. $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ **Ans.** $\sqrt{\pi}$

8. $\int_0^1 (x\log x)^3\, dx$ **Ans.** $-\frac{3}{128}$

9. $\int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}}$ **Ans.** $\sqrt{2\pi}$

10. Prove that $1.3.5.....(2n-1) = \dfrac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

11. $\int_0^\infty e^{-y^{1/m}}\, dy = m\Gamma m.$

21.3 BETA FUNCTION

$$\int_0^{\infty} x^{l-1}(1-x)^{m-1}\,dx \qquad (l > 0,\ m > 0)$$

is called the Beta function of *l, m.* It is also written as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

21.4 EVALUATION OF BETA FUNCTION

$$\beta(l\,, m) = \frac{\Gamma l\ \Gamma m}{\Gamma l+m}$$

Solution. We have $\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx = \int_0^1 (1-x)^{m-1}x^{l-1}\,dx$

Integrating by parts, we have

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right)dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2}x^l\,dx$$

Again integrating by parts

$$= \frac{(m-1)(m-2)}{l\,(l+1)}\int_0^1 (1-x)^{m-3}x^{l+1}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l\,(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l\,(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l\,(l+1)\ldots.(l+m-2)(l+m-1)}$$

$$= \frac{\underline{|m-1}}{l\,(l+1)\ldots(l+m-2)(l+m-1)} \times \frac{(l-1)(l-2)\ldots 1}{(l-1)(l-2\ldots 1}$$

$$= \frac{\underline{|m-1}\ \underline{|l-1}}{1.2\ldots(l-2)(l-1)\,.\,l\,(l+1)\ldots(l+m-2)(l+m-1)}$$

$$= \frac{\underline{|l-1}\ \ \underline{|m-1}}{\underline{|l+m-1}}$$

$$= \frac{\Gamma l\ \Gamma m}{\Gamma l+m}$$

And if only l is positive integer and not m then

$$\beta(l, m) = \frac{\underline{|l-1}}{m\,(m+1)\ldots(m+l-1)} \qquad \textbf{Ans.}$$

21.5 A PROPERTY OF BETA FUNCTION

$$\beta(l, m) = \beta(m, l)$$

Solution. We have

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{l-1}\,[1-(1-x)]^{m-1}\,dx$$

$$= \int_0^1 (1-x)^{l-1}\,x^{m-1}\,dx$$

$$= \int_0^1 x^{m-1}\,(1-x)^{l-1}\,dx = \beta\,(m, l)$$ l and m are interchanged. **Proved**

Example 8. *Evaluate* $\int_0^1 x^4\,(1-\sqrt{x})^5\,dx$

Solution. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,tdt$

$$\int_0^1 x^4\,(1-\sqrt{x})^5\,dx = \int_0^1 (t^2)^4\,(1-t)^5\,(2\,tdt)$$

$$= 2\int_0^1 t^9\,(1-t)^5\,dt = 2\,\beta\,(10, 6) = 2\,\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\,\frac{\lfloor 9\,\lfloor 5}{\lfloor 15}$$

$$= 2\,.\,\frac{\lfloor 5}{10\times 11\times 12\times 13\times 14\times 15} = \frac{2\times 1\times 2\times 3\times 4\times 5}{10\times 11\times 12\times 13\times 14\times 15}$$

$$= \frac{1}{11\times 13\times 7\times 15} = \frac{1}{15015}$$ **Ans.**

Example 9. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx$

Solution. Let $x^3 = y$ or $x = y^{1/3}$ or $dx = \frac{1}{3}\,y^{-\frac{2}{3}}\,dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx = \int_0^1 (1-y)^{-\frac{1}{2}}\left(\frac{1}{3}\,y^{-\frac{2}{3}}\,dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}}\,(1-y)^{-\frac{1}{2}}\,dy = \frac{1}{3}\,\beta\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

21.6 TRANSFORMATION OF BETA FUNCTION

We know that

$$\beta\,(l, m) = \int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx$$

Putting $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2}\,dy$ and $1-x = \frac{y}{1+y}$.

$$\beta\,(l, m) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}\left[-\frac{1}{(1+y)^2}\,dy\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}\,dy$$

Since l, m can be interchanged in $\beta\,(l, m)$,

$$\beta(l, m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}}\, dy \quad \text{or} \quad \beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}}\, dx$$

Example 10. *Evaluate* $\displaystyle\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$

Solution. We know that

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta(m, n)$$

$$\Rightarrow \qquad \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m, n) \qquad ...(1)$$

Consider $\displaystyle\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ $\qquad \left(\text{Put } x = \frac{1}{t}\right)$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\, dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{m-1} \frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n} (t+1)^{m+n}}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting the value of $\displaystyle\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ in (1) we get

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \beta(m, n)$$

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = \beta(m, n) \qquad \textbf{Ans.}$$

21.7 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that

$$\Gamma l = \int_0^\infty e^{-x} x^{l-1}\, dx, \qquad \frac{\Gamma l}{z^l} = \int_0^\infty e^{-zx} x^{l-1}\, dx$$

$$\Gamma l = \int_0^\infty z^l e^{-zx} x^{l-1}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^\infty e^{-z} \,.\, z^{m-1} \,.\, z^l \,.\, e^{-zx} x^{l-1}\, dx$$

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx$$

Integrating both sides w.r.t. 'x' we get

$$\int_0^\infty \Gamma l\, e^{-}\ z^{m-1}\, dz = \int_0^\infty \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx\, dz$$

$$\Gamma l\, \Gamma m = \int_0^\infty x^{l-1}\, dx \int_0^\infty e^{-(1+x)z} z^{l+m-1}\, dz$$

$$= \int_0^\infty x^{l-1}\,dx \,.\, \frac{\overline{|l+m}}{(1+x)^{l+m}}$$

$$\Gamma l\,\Gamma m = \overline{|l+m} \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx = \overline{|l+m}\,.\,\beta(l,m)$$

\ $$\beta(l,m) = \frac{\Gamma l\,\Gamma m}{\overline{|l+m}}$$

This is the required relation.

Example 11. *Show that*

$$\int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta\, d\theta = \frac{\overline{\left|\left(\frac{P+1}{2}\right)\right.}\;\overline{\left|\left(\frac{q+1}{2}\right)\right.}}{2\,\overline{\left|\left(\frac{P+q+2}{2}\right)\right.}}$$

Solution. We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{...(1)}$$

Putting $x = \sin^2\theta,\quad dx = 2\sin\theta\cos\theta\,d\theta$

and $1 - x = 1 - \sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta\; 2\sin\theta\cos\theta\,d\theta$$

or $$\frac{\Gamma m\,\Gamma n}{\Gamma m+n} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Putting $2m - 1 = p,\quad i.e.\quad m = \dfrac{p+1}{2}$

and $2n - 1 = q,\quad i.e.\quad n = \dfrac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{P+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^P\theta\,.\,\cos^q\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^P\theta\,.\,\cos^q\theta\,d\theta = \frac{\overline{\left|\frac{P+1}{2}\right.}\;\overline{\left|\frac{q+1}{2}\right.}}{2\,\overline{\left|\frac{P+q+2}{2}\right.}} \qquad \textbf{Proved}$$

Example 12. *Find the value of* $\overline{\left|\frac{1}{2}\right.}$.

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

Putting $P = q = 0$ $\quad \displaystyle\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(1)}$

or $\quad \displaystyle [\,\theta\,]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad \text{or} \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2$

or $\quad \displaystyle \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **Ans.**

Example 13. *Show that*

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cot \theta)}\, d\theta = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P x \cos^q x \, dx = \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)} \qquad ...(1)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2} \theta}{\sin^{1/2} \theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2} \theta \cos^{1/2} \theta \, d\theta$$

On applying formula (1), we have

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right) \Gamma\left(\frac{1/2+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+1/2+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)} = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \qquad \textbf{Proved}$$

Example 14. *Evaluate* $\displaystyle\int_{-1}^{+1} (1+x)^{P-1} (1-x)^{q-1}\, dx.$

Solution. Put $x = \cos 2\theta$, then $dx = -2 \sin 2\theta \, d\theta$

$$\int_{-1}^{+1} (1+x)^{P-1} (1-x)^{q-1}\, dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{P-1} (1-\cos 2\theta)^{q-1} (-2 \sin 2\theta \, d\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} (1 + 2\cos^2\theta - 1)^{P-1} (1 - 1 + 2\sin^2\theta)^{q-1} (-4 \sin\theta \cos\theta \, d\theta)$$

$$= 4\int_0^{\frac{\pi}{2}} 2^{P-1}\cos^{2P-2}\theta \,.\, 2^{q-1}\sin^{2q-2}\theta \,.\, \sin\theta\cos\theta\, d\theta$$

$$= 2^{P+q}\int_0^{\pi} \sin^{2q-1}\theta\cos^{2P-1}\theta\, d\theta$$

$$= 2^{P+q}\,\frac{\Gamma\left(\frac{2q}{2}\right)\Gamma\left(\frac{2P}{2}\right)}{2\,\Gamma\left(\frac{2P+2q}{2}\right)} = 2^{P+q-1}\,\frac{\Gamma(P)\,\Gamma(q)}{\Gamma(P+q)} \qquad \textbf{Ans.}$$

Example 15. *Show that* $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ $\qquad$ $(0 < n < 1)$

Solution. We know that

$$\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Putting $m + n = 1$ or $m = 1 - n$

$$\frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{1}}\,dx$$

$$\Gamma(1-n)\,\Gamma(n) = \int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx \qquad \left[\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sin n\pi} \qquad \textbf{Proved}$$

Example 16. *Evaluate* $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$.

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $\qquad dx = \dfrac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta}{(1-\sin^2\theta)^{1/n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta\cos\theta\, d\theta}{(\cos^2\theta)^{1/n}}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta\cos^{1-2/n}\theta\, d\theta$$

$$= \frac{2}{n}\,\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right)}$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{\Gamma(1)} \qquad \left(\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right)$$

$$= \frac{\pi}{n\sin\frac{\pi}{n}}$$ **Ans.**

Example 17. Show that $\int_0^{\frac{\pi}{2}} \tan^P\theta\, d\theta = \frac{\pi}{2}\sec\frac{P\pi}{2}$ *and indicate the restriction on the values of P.*

Solution. $\int_0^{\frac{\pi}{2}} \tan^P\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^{-P}\theta\, d\theta$

$$= \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{-P+1}{2}\right)}{2\,\Gamma\left(\frac{P+1\;-P+1}{2}\right)} \qquad \left[\begin{array}{r} 1-P > 0 \\ 1 > P \end{array}\right]$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{-p+1}{2}\right)}{2\,\Gamma(1)} \qquad \left[\begin{array}{r} 1+P>0 \\ P>-1 \end{array}\right]$$

$$= \frac{1}{2}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{-p+1}{2}\right) \qquad \therefore 1 > P > -1$$

$$= \frac{1}{2}\frac{\pi}{\sin\frac{p+1}{2}\pi} = \frac{1}{2}\frac{\pi}{\cos\frac{p\pi}{2}} = \frac{\pi}{2}\sec\frac{p\pi}{2}$$ **Proved**

Example 18. *Prove Duplication Formula*

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m).$$

Hence show that $\beta(m, m) = 2^{1-2m}\beta\left(m, \frac{1}{2}\right)$ (*U.P., II Semeter, Summer 2001*)

Solution. We know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^q\theta\, d\theta$$

Putting $q = p$ we get

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma(p+1)} = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^P\theta\, d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^P\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^P} (2 \sin \theta \cos \theta)^P \, d\theta = \frac{1}{2^P} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^P \, d\theta$$

Putting $2\theta = t$, we have

$$= \frac{1}{2^p} \int_0^{\pi} \sin^P t \, \frac{dt}{2}$$

$$= \frac{1}{2^P} \cdot \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^P t \, dt = \frac{1}{2^P} \int_0^{\frac{\pi}{2}} \sin^P t \cos^0 t \, dt$$

$$= \frac{1}{2^P} \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2 \, \Gamma\left(\frac{P+2}{2}\right)}$$

or $$\frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{P+1}{2}\right)}{2 \, \Gamma(P+1)} = \frac{1}{2^P} \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2 \, \Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P} \frac{\sqrt{\pi}}{\Gamma\left(\frac{P+2}{2}\right)}$$

Take $\frac{P+1}{2} = m$ or $P = 2m - 1$

or $$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \quad ...(1)$$

$$\Gamma(m) \, \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$ **Proved**

Multiplying both sides of (1) by Γm, we have

$$\frac{\Gamma(m) \, \Gamma(m)}{\Gamma(2m)} = 2^{1-2m} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(m)}{\Gamma\left(m + \frac{1}{2}\right)}$$ **Proved**

$$\beta(m, m) = 2^{1-2m} \, \beta\left(m, \frac{1}{2}\right)$$

Example 19. *Evaluate* $\iint_A \frac{dx\,dy}{\sqrt{xy}}$, *using the substitutions*

$$x = \frac{u}{1+v^2}, \quad y = \frac{uv}{1+v^2}$$

where A is bounded by $x^2 + y^2 - x = 0,\ y = 0,\ y > 0.$

Solution. Here $\sqrt{xy} = \sqrt{\left(\frac{u}{1+v^2}\right)\left(\frac{uv}{(1+v^2)}\right)} = \frac{u\sqrt{v}}{1+v^2}$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

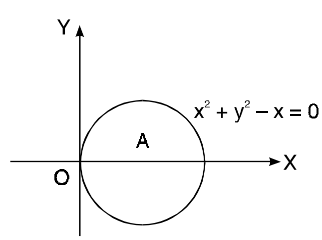
$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} \frac{1}{1+v^2} & -\frac{2uv}{(1+v^2)^2} \\ \frac{v}{1+v^2} & \frac{u(1-v^2)}{(1+v^2)^2} \end{vmatrix} du\,dv$$

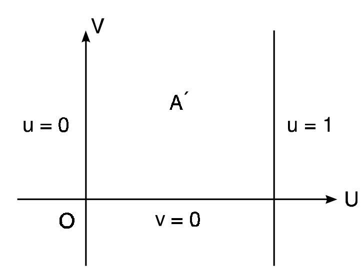
$$= \left[\frac{u(1-v^2)}{(1+v^2)^3} + \frac{2uv^2}{(1+v^2)^3}\right] du\,dv = \left[\frac{u - uv^2 + 2uv^2}{(1+v^2)^3}\right] du\,dv$$

$$= \frac{u(1+v^2)}{(1+v^2)^3}\,du\,dv = \frac{u}{(1+v^2)^2}\,du\,dv$$

Also the circle $x^2 + y^2 - x = 0$ is transformed into

$$\frac{u^2}{(1+v^2)^2} + \frac{u^2 v^2}{(1+v^2)^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad \frac{u^2(1+v^2)}{(1+v^2)^2} - \frac{u}{1+v^2} = 0$$





$$\frac{u^2}{1+v^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad u^2 - u = 0 \quad \text{or} \quad u(u-1) = 0 \ \Rightarrow\ u = 0,\ u = 1$$

Further $\qquad y = 0 \ \Rightarrow\ \frac{uv}{1+v^2} = 0 \ \Rightarrow\ u = 0,\ v = 0$

and $y > 0 \ \Rightarrow\ uv > 0$ either both u and v are positive or both negative.

The area A, *i.e.*, $x^2 + y^2 - x = 0$ is transformed into A' bounded by $u = 0$, $v = 0$ and $u = 1$ *and* $v = \infty$.

$$\iint \frac{dx\,dy}{\sqrt{x}} = \int_0^1 \int_0^\infty \frac{\frac{u}{(1+v^2)^2}\,du\,dv}{\frac{u\sqrt{v}}{1+v^2}} = \int_0^1 \int_0^\infty \frac{1}{\sqrt{v}\,(1+v^2)}\,dv\,du$$

On putting $v = \tan\theta$, dv $= sec^2\theta\, d\theta$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{sec^2\,\theta\, d\,\theta\, du}{\sqrt{\tan\theta}\,(1+\tan^2\theta)} = \int_0^1 du \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\,\theta = \int_0^1 du \int_0^{\frac{\pi}{2}} \sin\theta^{-\frac{1}{2}} \cos\theta^{\frac{1}{2}}\, d\,\theta$$

$$= \int_0^1 du\, \frac{\left|\overline{-\frac{1}{2}+1}\right.}{\Gamma 2}\, \frac{\left|\overline{\frac{1}{2}+1}\right.}{\Gamma 2} = \frac{1}{2}\int_0^1 du \left|\overline{\frac{1}{4}}\right. \left|\overline{\frac{3}{4}}\right. = \frac{1}{2}\int_0^1 du \left[\frac{\sqrt{\pi}}{2^{\frac{-1}{2}}} \left|\overline{\frac{1}{2}}\right.\right]$$

$$= \frac{1}{2}\int_0^1 du\ \sqrt{2}\,\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi}{\sqrt{2}}\,[\,u\,]_0^1 = \frac{\pi}{\sqrt{2}}$$ **Ans.**

Example 20. *Prove that*

$$\iint_D x^{l-1}\, y^{m-1}\ dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}\ h^{l+m}$$

where D is the domain $x \ge 0,\quad y \ge 0$ *and* $x + y \le h.$

Solution. Putting $x = Xh$ and $y = Yh$, $dx\,dy = h^2\, dX\, dY$

$$\iint_D x^{l-1}\, y^{m-1}\, dx\, dy = \iint_{D'} (Xh)^{l-1}\,(Yh)^{m-1}\, h^2\, dX\, dY$$

where D' is the domain

$X \ge 0,\quad Y \ge 0,\quad X + Y \le 1$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}\, Y^{m-1}\, dX\, dY = h^{l+m}\int_0^1 X^{l-1}\, dX \int_0^{1-X} y^{m-1}\, dY$$

$$= h^{l+m}\int_0^1 X^{l-1}\, dX \left[\frac{Y^m}{m}\right]_0^{1-X} = \frac{h^{l+m}}{m}\int_0^1 X^{l-1}\,(1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m}\,\beta\,(l,\ m+1) = \frac{h^{l+m}}{m}\,\frac{\Gamma l\ \left|\overline{m+1}\right.}{\left|\overline{l+m+1}\right.}$$

$$= \frac{h^{l+m}}{m}\,\frac{m\,\Gamma l\,\Gamma m}{\left|\overline{l+m+1}\right.} = h^{l+m}\,\frac{\Gamma l\,\Gamma m}{\left|\overline{l+m+1}\right.}.$$ **Proved.**

Example 21. *Establish* ***Dirichlet's integral***

$$\iiint_V x^{l-1}\, y^{m-l}\, z^{n-l}\ dx\, dy\, dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\left|\overline{l+m+n+1}\right.}$$

where V is the region $x \ge 0,\ y \ge 0,\ z \ge 0$ *and* $x + y + z \le 1.$

Solution. Putting $y + z \le 1 - x = h$. Then $z \le h - y$

$$\iiint_V x^{l-1}\, y^{m-1}\, z^{n-1}\ dx\, dy\, dz = \int_0^1 x^{l-1}\, dx \int_0^{1-x} y^{m-1}\, dy \int_0^{1-x-y} z^{n-1}\, dz$$

$$= \int_0^1 x^{l-1}\, dx \left[\int_0^h \int_0^{h-y} y^{m-1}\, z^{n-1}\, dy\, dz\right]$$

$$= \int_0^1 x^{l-1}\, dx \left[\frac{\Gamma m\,\Gamma n}{\left|\overline{m+n+1}\right.}\ h^{m+n}\right]$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n} \, dx$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \beta(l, m+n+1)$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \frac{\Gamma l \, \Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)}$$ **Proved.**

Note. $$\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)} h^{l+m+n}$$

where V is the domain, $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq h$.

Example 22. *Find the mass of an octant of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *the density at any point being* $\rho = k\,x\,y\,z$.

Solution. $$\text{Mass} = \iiint \rho \, dv = \iiint (k\,x\,y\,z) \, dx \, dy \, dz$$

$$= k \iiint (x dx)(y dy)(z dz) \qquad \ldots (1)$$

Putting $\frac{x^2}{a^2} = u$, $\frac{y^2}{b^2} = v$, $\frac{z^2}{c^2} = w$ and $u + v + w = 1$

so that $\frac{2x dx}{a^2} = du$, $\frac{2\,y\,dy}{b^2} = dv$, $\frac{2\,z\,dz}{c^2} = dw$

$$\text{Mass} = k \iiint \left(\frac{a^2\,du}{2}\right)\left(\frac{b^2\,dv}{2}\right)\left(\frac{c^2\,dw}{2}\right)$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint du\,dv\,dw \quad \text{where} \quad u + v + w \leq 1$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1} \, du\,dv\,dw$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \frac{\Gamma 1 \, \Gamma 1 \, \Gamma 1}{\Gamma(3+1)} = \frac{k\,a^2\,b^2\,c^2}{8 \times 6}$$

$$= \frac{k\,a^2\,b^2\,c^2}{48}$$ **Ans.**

Example 23. *Show that*

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\beta(m, n)}{a^n (1+a)^m}$$

Solution: Put $$\frac{x}{a+x} = \frac{t}{a+1}$$

$$(a+1)\,x = t\,(a+x) \quad \text{or} \quad x = \frac{at}{a+1-t}$$

$$dx = \frac{(a+1-t)\,a\,dt - at\,(-dt)}{(a+1-t)^2}$$

$$= \frac{(a^2 + a - at + at)}{(a + 1 - t)^2}\, dt \;=\; \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}}\, dx \;=\; \int_0^1 \frac{\left(\dfrac{at}{a+1-t}\right)^{m-1} \cdot \left(1 - \dfrac{at}{a+1-t}\right)^{n-1}}{\left(a + \dfrac{at}{a+1-t}\right)^{m+n}} \; \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$= \int_0^1 \frac{(at)^{m-1}\,(a+1-t-at)^{n-1}}{(a^2 + a - at + at)^{m+n}}\, a\,(a+1)\, dt$$

$$= \int_0^1 \frac{a^{m-1}\, t^{m-1}\,(a+1)^{n-1}\,(1-t)^{n-1}}{a^{m+n}\,(a+1)^{m+n}}\, a\,(a+1)\, dt$$

$$= \frac{1}{a^n\,(a+1)^m} \int_0^1 t^{m-1}\,(1-t)^{n-1}\, dt$$

$$= \frac{1}{a^n\,(a+1)^m}\, \beta\,(m, n)$$ **Proved**

Exercise 21.2

Prove that

1. (*a*) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$ (*b*) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5\pi}{32}$

2. (*a*) $\beta\,(m+1, n) = \dfrac{m}{m+n}\,\beta\,(m, n)$ (*b*) $\beta\,(m, n+1) = \dfrac{n}{m+n}\,\beta\,(m, n)$

(*c*) $\beta\,(m+1, n) + \beta\,(m, n+1) = \beta\,(m, n)$

3. $\displaystyle\int_0^1 \sqrt{x}\,\sqrt[3]{1-x^2}\, dx = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{2\,\Gamma\left(\frac{7}{12}\right)}$

4. $\displaystyle\int_0^1 (1-x^n)^{-\frac{1}{2}}\, dx = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{n\,\Gamma\left(\frac{n+2}{2n}\right)}$

5. $\displaystyle\int_0^1 (1 - x^{1/n})^m\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

6. $\displaystyle\int_1^\infty \frac{dx}{x^{p+1}\,(x-1)^q} = \beta\,(p+q, 1-q)$ if $-p < q < 1$

7. $\displaystyle\int_0^1 x^m\,(1-x^n)^p\, dx = \frac{1}{n}\,\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)}$

8. $\displaystyle\int_0^b (x-a)^m\,(b-x)^n\, dx = (b-a)^{m+n+1}\,.\,\beta\,(m+1, n+1)$

9. $\displaystyle\int_3^7 \sqrt[4]{(x-3)\,(7-x)}\, dx = \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$ Put $x = 4t + 3$

10. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{4\sqrt{\pi}}$

11. If $\int_0^\infty e^{-x} x^{n-1}\, dx$ = In for $n >$ 0 find $\frac{I_{n+1}}{I_n}$ (*A.M.I.E., Summer 2000*) **Ans.** n

21.8 LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables ***x,y,z* are all positive such that $h_1 < x + y + z < h_2$, then**

$$\iiint f(x \quad y + z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l \Gamma m \Gamma n}{\Gamma l \quad m \quad n} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

Proof Let $I = \iiint x^{l-1} \; y^{m-1} \; z^{n-1}\, dx\, dy\, dz$

under the condition $x + y + z \le u$ then

$$I = u^{l+m+n} \frac{\Gamma l + \Gamma m + \Gamma n}{\Gamma 1 + l + m + n} \qquad \text{... (1)} \quad \text{(By Dirichlet Th.)}$$

If $x + y + z \le u + \delta u$, then

$$I = (u + \delta u)^{l+m+n} \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} \qquad \text{... (2)}$$

If $u < x + y + z + < u + \delta u$, then

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} \left[(u + \delta u)^{p+m+n} - u^{l+m+n}\right]$$

$$= \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} u^{l+m+n} \left[1 + (\frac{\delta u}{u})^{l+m+n} - 1\right]$$

$$= \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} u^{l+m+n} \left[1 + (l + m + n) \frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} u^{l+m+n} (l + m + n) \frac{\delta u}{u} = \frac{\Gamma l \Gamma m \Gamma n}{\Gamma l + m + n} u^{l+m+n-1}\, \delta u$$

Let us consider $\iiint f(x + y + z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

under the condition $h_1 \le x + y + z \le h_2$

When $x + y + z$ lies between u and $u + \delta u$, the value of $f(x + y + z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence, neglecting square of δu, the part of the integral

$$\iiint f(x + y + z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma l \Gamma m \Gamma n}{\Gamma 1 + l + m + n} f(u) \; u^{l+m+n-1} \; \delta u$$

(supposing the sum of variables to be between u and $u + \delta u$)

So $\iiint f(x + y + z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l \Gamma m \Gamma n}{(1 + l + m + n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$

Example 24. Show that $\iiint \frac{dx\, dy\, dz}{(x + y + z)^3} = \frac{1}{2} \log 2 - \frac{5}{16}$, the integral being taken throughout the volume bounded by planes $x + 1 = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz}{(x+y+z+1)^3} \qquad (0 \le x+y+z \le 1)$$

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3}\,u^{3-1}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^3}\,du$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{u+1} - \frac{2}{(u+1)^2} - \frac{1}{(u+1)^3}\right] du \quad \text{(Partial fractions)}$$

$$= \frac{1}{2}\left[\log(u+1) + \frac{2}{u+1} + \frac{1}{2(u+1)^2}\right]_0^1$$

$$= \left[\log 2 + 2\left(\frac{1}{2} - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right] = \frac{1}{2}\log 2 - \frac{5}{16} \quad \textbf{Proved}$$

Example 25. Find the value of $\iiint \log(x+y+z)\,dx\,dy\,dz$ the integral extending over all positive values of x,y,z subject to the condition $x + y + z < 1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \log(x+y+z)\,dx\,dy\,dz$$

$$= \iiint \log(x+y+z)\,x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz$$

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1-1)} \int_0^1 \log u\;u^{1+1+1}\,du$$

$$= \frac{1}{\Gamma 3}\int u^2 \log u\,du = \frac{1}{2}\left[\log u\,\frac{u^3}{3} - \frac{1}{3}\,\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(-\frac{1}{9}\right) = -\frac{1}{18} \qquad \textbf{Ans.}$$

Exercise 21.3.

Evaluate:

1. $\iiint e^{x+y+z}\,dx\,dy\,dz$ taken over the positive octant such that $x + y + z \le 1$. **Ans.** $\dfrac{e-2}{2}$

2. $\iiint \dfrac{dx\,dy\,dz}{(a^2 - x^2 - y^2 - z^2)}$ for all positive values of the variables for which the expression is real.

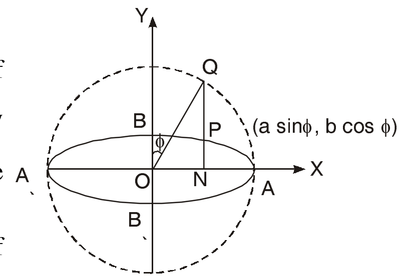
Hint. $a^2 - x^2 - y^2 - z^2 > 0 \Rightarrow 0 < x^2 + y^2 + z^2 < a^2$ **Ans.** $\dfrac{\pi^2 a^2}{8}$

3. $\iiint_R (x + y\,z + 1)^2\,dx\,dy\,dz$ where R is defined by $x \ge 0 \quad y \ge 0 \quad z \ge 0, \quad x + y + z \le 1$ **Ans.** $\dfrac{31}{60}$

4. $\iiint x^{-\frac{1}{2}}\;y^{-\frac{1}{2}}\;z^{-\frac{1}{2}}\;(1 - x - y - z)^{\frac{1}{2}}\,dx\,dy\,dz,\; x + y + z \le 1 \quad x > 0,\; y > 0,\; z > 0$ **Ans.** $\dfrac{\pi^2}{4}$

5. Evaluate $\int\int\int \dfrac{dx_1\,dx_2 \ldots \ldots dx_n}{\sqrt{1 - x_1^2 - x_2^2 \ldots \ldots - x_n^2}}$, integral being extended to all positive values of the variables for which the expression is real. (*U.P.., II Semester, Summer 2001*)

21.9 ELLIPTIC INTEGRALS



Draw a circle with AA' (diameter) the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. This circle is called the auxiliary circle $x^2+y^2 = a^2$. The co-ordinates of a point P on the ellipse are $(a \sin\phi,\ b\cos\phi)$.

$x = a\sin\phi,\ y = b\cos\phi$ is the parametric equation of the ellipse.

Now the length of the arc BP of the ellipse

$$= \int_0^{\phi} \sqrt{\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}}\, d\phi = \int_0^{\phi} \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}\, d\phi \qquad \{b^2 = a^2(1-e^2)\}$$

$$= \int_0^{\phi} \sqrt{\{a^2\cos^2\phi + (a^2 - a^2e^2)\sin^2\phi\}}\, d\phi = \boldsymbol{a\int_0^{\phi} \sqrt{(1 - e^2\sin^2\phi)}\, d\phi}$$

where e is the eccentricity of the ellipse.

This integral cannot be evaluated in the form of the elementary function. It defines a new function, called *elliptic function*. This integral is called the elliptic integrals as it is derived from the determination of the Perimeter of the ellipse. This integral cannot be evaluated by standard methods of integration. First the integrand $\sqrt{1-e^2\sin^2\phi}$ is expanded as power series and then is integrated term by term.

21.10 DEFINITION AND PROPERTY

Elliptic integral of first kind $= F(k,\ \phi) = \int_0^{\phi} \frac{1}{\sqrt{(1-k^2\sin^2\phi)}}\, d\phi \qquad k^2 < 1$

Elliptic integral of second kind $= E(k,\ \phi) = \int_0^{\phi} \sqrt{(1-k^2\sin^2\phi)}\, d\phi \qquad k^2 < 1$

Here k is known as modulus and ϕ amplitude.

The following results are easy to prove

$$F(0,\ \phi) = E(0,\ \phi) = \phi$$
$$F(1,\ \phi) = \log(\tan\phi + \sec\phi)$$
$$E(1,\ \phi) = \sin\phi$$

If $\phi = \frac{\pi}{2}$ is the upper limit of the integral then the integral is called *complete elliptic integral* as under:

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad \ldots (1)$$

and

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\phi}\, d\phi \qquad \ldots (2)$$

These integrals can be evaluated by expanding the integrand in binomial series and integrating term by term.

$$(1-k^2\sin^2\phi)^{-1/2} = 1 + \frac{k^2}{2}\sin^2\phi + \frac{1.3}{2.4}k^4\sin^4\phi + \ldots\ldots$$

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = \phi + \frac{k^2}{2} \int_0^{\phi} \sin^2 \phi \; d\phi + \frac{1.3}{2.4} k^4 \int_0^{\phi} \sin^4 \phi \, d\phi + \ldots \quad \ldots (3)$$

which can be evaluated by the *Reduction Formula*

$$\int_0^{\phi} \sin^n \phi \, d\phi = -\frac{\sin^{n-1} \phi \cos \phi}{n} + \frac{n-1}{n} \int_0^{\phi} \frac{\sin^{n-2}}{} \phi \, d\phi$$

From (3), we get

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = \frac{\pi}{2} + \frac{k^2}{2}\left(\frac{1}{2}\ \frac{\pi}{2}\right) + \frac{1.3}{2.4} k^4 \left(\frac{3.1}{4.2}\ \frac{\pi}{2}\right) + \ldots\ldots$$

or

$$K(k) = \frac{\pi}{2}\left[1 + \frac{k^2}{4} + \frac{9\,k^4}{64} + \ldots\ldots\right]$$

If $k = \sin 10'$

$$K = \frac{\pi}{2}[1 + 0.00754 + 0.00012 + \ldots\ldots] = 1.5828$$

The elliptic integrals are periodic functions with a period π.

$$F(k, \phi + P\pi) = PF(k, \pi) + F(k, \phi), \qquad P = 0, 1, 2, \ldots\ldots$$
$$E(k, \phi + P\pi) = PE(k, \pi) + E(k, \phi), \qquad P = 0, 1, 2, \ldots\ldots$$
$$F(k, \phi + P\pi) = 2\,PF(k) + F(k, \phi)$$
$$E(k, \phi + P\pi) = 2\,PE(k) + E(k, \phi)$$

If we substitute $\sin \phi = x$, $d\phi = \dfrac{dx}{\sqrt{(1 - x^2)}}$ in (1) and (2), we have

$$F_1(k, x) = \int_0^x \frac{dx}{\sqrt{\{(1 - x^2)(1 - k^2 x^2)\}}}$$

$$E_1(k, x) = \int_0^x \sqrt{\left(\frac{1 - k^2 x^2}{1 - x^2}\right)}\; dx$$

These are known as Jacobi's form of elliptic integrals.

Example 26. *Express* $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ *in terms of elliptic integrals.*

Solution. Substitute $\cos x = \cos^2 \phi$

so that $\quad x = \cos^{-1} \cos^2 \phi,\ dx = \dfrac{2 \cos \phi \sin \phi \, d\phi}{\sqrt{1 - \cos^4 \phi}} = \dfrac{2 \cos \phi \, d\phi}{\sqrt{1 + \cos^2 \phi}}$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{2 \cos^2 \phi \, d\phi}{\sqrt{1 + \cos^2 \phi}} = 2\int_0^{\frac{\pi}{2}} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}}\; d\phi$$

$$= 2\left\{\int_0^{\frac{\pi}{2}} \sqrt{(1 + \cos^2 \phi)}\; d\phi - \int_0^{\pi/2} \frac{1}{\sqrt{1 + \cos^2 \phi}}\; d\phi\right\}$$

$$= 2\left\{\int_0^{\frac{\pi}{2}} \sqrt{(2 - \sin^2 \phi)}\; d\phi - \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(2 - \sin^2 \phi)}}\; d\phi\right\}$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{2}}\sqrt{\left(1-\frac{1}{2}\sin^2\phi\right)}\,d\phi - \frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\left(1-\frac{1}{2}\sin^2\phi\right)}}\,d\phi$$

$$= 2\sqrt{2}\,E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\,K\left(\frac{1}{\sqrt{2}}\right)$$ **Ans.**

Example 27. *Express* $\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{(2-\cos x)}}$ *in terms of elliptic integrals.*

Solution. $$\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{(2-\cos x)}} = \int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\left\{2-\left(2\cos^2\frac{x}{2}-1\right)\right\}}}$$

$$= \int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\left(3-2\cos^2\frac{x}{2}\right)}} = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\left(1-\frac{2}{3}\cos^2\frac{x}{2}\right)}}$$

On putting $x = \pi - 2\phi$, so that $dx = -2d\phi$

$$= \frac{1}{\sqrt{3}}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\frac{-2d\phi}{\sqrt{1-\frac{2}{3}\cos^2\left(\frac{\pi}{2}-\phi\right)}} = \frac{-2}{\sqrt{3}}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{\left(1-\frac{2}{3}\sin^2\phi\right)}}$$

$$= \frac{2}{\sqrt{3}}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\left(1-\frac{2}{3}\sin^2\phi\right)}} - \int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{\left(1-\frac{2}{3}\sin^2\phi\right)}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[K\left(\sqrt{\frac{2}{3}}\right) - F\left(\sqrt{\frac{2}{3}},\ \frac{\pi}{4}\right)\right]$$ **Ans.**

Example 28. *Show that* $\int_0^{\frac{a}{2}}\frac{dx}{\sqrt{(2ax-x^2)(a^2-x^2)}} = \frac{2}{3a}K\left(\frac{1}{3}\right)$

Solution. On substituting $x = \frac{a}{2}(1-\sin\theta)$ so that $dx = -\frac{a}{2}\cos\theta\,d\theta$

Upper limit, $x = \frac{a}{2},\ \frac{a}{2} = \frac{a}{2}(1-\sin\theta) \Rightarrow \theta = 0$

Lower limit, $x = 0,\ 0 = \frac{a}{2}(1-\sin\theta) \Rightarrow \theta = \frac{\pi}{2}$

$$2ax - x^2 = (2a)\frac{a}{2}(1-\sin\theta) - \frac{a^2}{4}(1-\sin\theta)^2 = \frac{a^2}{4}[4-4\sin\theta-1+2\sin\theta-\sin^2\theta]$$

$$= \frac{a^2}{4}(3-2\sin\theta-\sin^2\theta) = \frac{a^2}{4}(1-\sin\theta)(3+\sin\theta)$$

$$a^2 - x^2 = a^2 - \frac{a^2}{4}(1-\sin\theta)^2 = \frac{a^2}{4}[4-1-\sin^2\theta+2\sin\theta] = \frac{a^2}{4}[3+2\sin\theta-\sin^2\theta]$$

$$= \frac{a^2}{4}(1+\sin\theta)(3-\sin\theta)$$

$$\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{(2ax - x^2)(a^2 - x^2)}} = \int_{\frac{\pi}{2}}^{0} \frac{-\frac{a}{2}\cos\theta\, d\theta}{\sqrt{\frac{a^2}{4}(1-\sin\theta)(3+\sin\theta)\frac{a^2}{4}(1+\sin\theta)(3-\sin\theta)}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{\frac{a}{2}\sqrt{(1-\sin^2\theta)(9-\sin^2\theta)}} = \frac{2}{a}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(9-\sin^2\theta)}}$$

$$= \frac{2}{3a}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\left(1-\left(\frac{1}{3}\right)^2 \sin^2\theta\right)}} = \frac{2}{3a} K\left(\frac{1}{3}\right) \qquad \textbf{Proved.}$$

Exercise 21.4

Show that

1. $\displaystyle\int_0^{\pi} \frac{dx}{\sqrt{(1-k^2\sin^2\phi)}} = \frac{1}{k} F\left(\frac{1}{k}, \pi\right)$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1+3\sin^2 x)}} = \frac{1}{2} K\left(\frac{\sqrt{3}}{2}\right)$

3. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{dx}{\sqrt{\sin x}} = \sqrt{2}\left[K\left(\frac{1}{\sqrt{2}}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)\right]$

4. $\displaystyle\int_0^{\phi} \frac{\sin^2\phi}{\sqrt{(1-k^2\sin^2\phi)}}\, d\theta = \frac{1}{k^2}[F(k,\phi) - E(k,\phi)]$

5. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{(1-x^4)}} = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right)$

6. $\displaystyle\int_0^{\phi} \sqrt{(1-k^2\sin^2\phi)}\, d\phi = \left(\frac{1}{k} - k\right) F\left(\frac{1}{k}, x\right) + KE\left(\frac{1}{k}, x\right)$ and $k\sin\phi < 1$ [**Hint.** Put $\sin x = k\sin\phi$]

21.11 ERROR FUNCTION

1. $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$ is called error function of x and is also written as *erf (x).*

2. $\displaystyle\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\, dt$ is called complementary error function of x and is also written as $erf_c(x)$.

3. Important formula.

$$\int_0^{\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$$

Example 29. *Prove that* $erf(0) = 0$

Solution.

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

$$erf(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}\, dt = 0 \qquad \textbf{Proved}$$

Example 30. *Prove that* $erf(\infty) = 1$

Solution. $\displaystyle erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \qquad \textbf{Proved}$$

Example 31. *Prove that* $erf(x) + erf_c(x) = 1$

Solution.

$$erf(x) + erf_c(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt + \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^x e^{-t^2}\, dt + \int_x^{\infty} e^{-t^2}\, dt \right]$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1$$

Proved

Example 32. *Prove that* $erf(-x) = -erf(x)$

Solution.

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt \qquad \text{Put } t = -\mu$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} (-du) = -\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du = -erf(x)$$

Proved

Example 33. *Show that*

$$\int_a^b e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} [erf(b) - erf(a)]$$

Solution. $\frac{\sqrt{\pi}}{2} [erf(b) - erf(a)]$

$$= \frac{\sqrt{\pi}}{2} \left[\frac{2}{\sqrt{\pi}} \int_0^b e^{-t^2}\, dt - \frac{2}{\sqrt{\pi}} \int_0^a e^{-t^2}\, dt \right] = \int_0^b e^{-t^2}\, dt - \int_0^a e^{-t^2}\, dt$$

$$= \int_0^b e^{-t^2}\, dt + \int_a^0 e^{-t^2}\, dt = \int_a^b e^{-t^2}\, dt = \int_a^b e^{-x^2}\, dx$$

Proved

Example 34. *Show that*

$$\int_0^{\infty} e^{-x^2 - 2bx}\, dx = \frac{\sqrt{\pi}}{2} e^{b^2} [1 - erf(b)]$$

Solution.

$$\int_0^{\infty} e^{-x^2 - 2bx}\, dx = \int_0^{\infty} e^{-x^2 - 2bx - b^2 + b^2}\, dx = \int_0^{\infty} e^{-(x+b)^2} . e^{b^2}\, dx$$

$$= e^{b^2} \left[\int_b^{\infty} e^{-t^2}\, dt + \int_0^b e^{-t^2}\, dt \right] \qquad [\text{Put } x + b = t]$$

$$= e^{b^2} \left[-\int_b^{o} e^{-t^2}\, dt + erf(\infty) \right]$$

$$= e^{b^2} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} erf(b) \right] = e^{b^2} \frac{\sqrt{\pi}}{2} [1 - erf(b)]$$

Proved

Example 35. *Prove that*

$$\frac{d}{dx} [erf_c(\alpha x)] = \frac{-2a}{\sqrt{\pi}} e^{-\alpha^2 x^2}$$

Solution.

$$\frac{d}{dx} \left[erf_c(\alpha x) \right] = \frac{d}{dx} \left[\frac{2}{\sqrt{\pi}} \int_{\alpha x}^{\infty} e^{-t^2}\, dt \right]$$

On applying the rule of differentiation under integral sign, we get

$$= \frac{2}{\sqrt{\pi}}\left[\int_{\alpha x}^{\infty}\left(\frac{\partial}{\partial x} e^{-t^2}\right) dt + \frac{d}{dx}(\infty)\, e^{-\infty} - \frac{d}{dx}(\alpha\, x)\, e^{-\alpha^2 x^2}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + 0 - \alpha \,.\, e^{-\alpha^2 x^2}\right] = -\frac{2\,\alpha}{\sqrt{\pi}}\, e^{-\alpha^2 x^2} \qquad \textbf{Proved}$$

Exercise 21.5

Prove that

1. $erf_c(x) + erf_c(-x) = 2$

2. $erf_c(-x) = 1 + erf(x)$

3. $erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{1}{2\,!}\cdot\frac{x^5}{5} - \frac{1}{3\,!}\cdot\frac{x^7}{7} + \ldots\right]$

4. $\int_0^{\infty} e^{-(x+a)^2}\, dx = \frac{\sqrt{\pi}}{2}[1 - erf(a)]$

5. $\int_0^t erf_c(ax)\, dx = t\, erf_c(at) - \frac{e^{-a^2 t^2}}{a\sqrt{\pi}} + \frac{1}{a\sqrt{\pi}}$

6. $\frac{d}{dx}[erf(ax)] = \frac{2\,a}{\sqrt{\pi}}\, e^{-a^2 x^2}$

7. $\frac{d}{dx}[erf(\sqrt{x})] = \frac{e^{-x}}{\sqrt{\pi x}}$

8. $\frac{d}{dx}[erf(x)] = \frac{2}{\sqrt{\pi}}\, e^{-x^2}$

21.12 DIFFERENTIATION UNDER THE INTEGRAL SIGN

The value of a definite integral $\int_a^b f(x, \alpha)\, dx$ is a function of α (parameter), $F(\alpha)$ say. To find $F'(\alpha)$, first we have to evaluate the integral $\int_a^b f(x, \alpha)\, dx$ and then differentiate $F(\alpha)$ w.r.t. α. However, it is not always possible to evaluate the integral and then to find its derivative. Such problems are solved by reversing the order of the integration and differentiation *i.e.*, first differentiate $f(x, \alpha)$ partially w.r.t. "α" and then integrate it.

21.13 LEIBNITZ'S RULE

If $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ be continuous functions of x and α, then

$$\frac{d}{dx}\left[\int_a^b f(x, \alpha)\, dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx\,.$$

Proof. Let
$$\int_a^b f(x, \alpha) = F(\alpha)$$

then
$$F(\alpha + \delta\,\alpha) = \int_a^b f(x, \alpha + \delta\alpha)\, dx$$

Hence
$$F(\alpha + \delta) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx$$

$$\frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{f(x, \alpha + \delta\alpha) - f(x, \alpha)}{\delta\alpha}\, dx$$

Taking limits of both sides as $\delta\alpha \to 0$, we have

$$\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial\alpha}\, dx$$

The above formula is useful for evaluating definite integrals which are otherwise impossible to evaluate.

Example 36. *Evaluate* $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$

Solution. Let
$$I = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx \qquad \ldots (1)$$

$$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$$

$$= \int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{x}{1+a^2x^2}\,dx = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

Breaking the integrand into partial fractions,

$$= \int_0^\infty \frac{1}{1-a^2}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right]dx = \frac{1}{1-a^2}\left[\tan^{-1}x - a\tan^{-1}ax\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\,\frac{\pi}{2}\right] = \frac{\pi}{2}\,\frac{1-a}{1-a^2} = \frac{\pi}{2}\cdot\frac{1}{1+a}$$

Now, integrating with respect to "a", we get

$$I = \frac{\pi}{2}\log(1+a) + c \qquad \ldots (2)$$

From (1), when $a = 0$, then $I = 0$

Putting $a = 0$ and $I = 0$ in (2), we get

$$c = 0$$

Hence (2) gives $\quad I = \frac{\pi}{2}\log(1+a)$ **Ans.**

Example 37. *Evaluate* $\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx,\ \ \alpha \ge 0$.

Solution. Let
$$I = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx \qquad \ldots (1)$$

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\log x}\right)dx = \int_0^1 \frac{x^\alpha\log x - 0}{\log x}\,dx = \int_0^1 x^\alpha\,dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

Now integrating both sides w.r.t. "α", we get

$$I = \log(\alpha+1) + c \qquad \ldots (2)$$

From (1), when $\alpha = 0$, $\quad I = 0$

Putting $\alpha = 0$, $I = 0$ in (2), we get

$$c = 0$$

Hence (2) gives $\ I = \log(\alpha+1)$ **Ans.**

Example 38. *Evaluate* $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$

using the rule of differentiation under the sign of integration.

Solution. Let
$$I = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx \qquad \ldots (1)$$

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-2ax}\right)\right] dx \;=\; \int_0^\infty \frac{e^{-x}}{x}\left(1 - 0 - \frac{x}{x}e^{-ax}\right) dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\, dx \qquad \ldots (2)$$

$$\frac{d^2I}{da^2} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}\left(1 - e^{-ax}\right)\right] dx = \int_0^\infty \frac{e^{-x}}{x}\left(x\, e^{-ax}\right) dx = \int_0^\infty e^{-(a+1)x}\, dx$$

$$= \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^\infty = \left[0 + \frac{1}{a+1}\right] = \frac{1}{a+1} \qquad \ldots (3)$$

Integrating w.r.t. a, we have

$$\frac{dI}{da} = \log(a+1) + c_1 \qquad \ldots (4)$$

Putting $a = 0$ in (2), we get

$$\frac{dI}{da} = 0$$

Putting $a = 0$ and $\frac{dI}{da} = 0$ in (4), we get $\quad c_1 = 0$

From (4), $$\frac{dI}{da} = \log(a+1)$$

$$I = \int \log(a+1)\; da = \log(a+1)\cdot a - \int \frac{a}{a+1}\; da \;=\; a\log(a+1) - \int\left(1 - \frac{1}{a+1}\right) da$$

$$I = a\log(a+1) - a + \log(a+1) + c_2$$

$$I = (a+I)\log(a+1) - a + c_2 \qquad \ldots (5)$$

Putting $a = 0$ in (1), we get $I = 0$

Putting $a = 0$, $I = 0$ in (5), we get

$$0 = c_2$$

Hence (5) gives

$$I = (a+1)\log(a+1) - a \qquad \textbf{Ans.}$$

Example 39. *Evaluate the integral*

$$\int_0^\infty \frac{e^{-x}\sin bx}{x}\; dx$$

Solution. Let $$I = \int_0^\infty \frac{e^{-x}\sin bx}{x}\; dx \qquad .. (1)$$

$$\frac{dI}{db} = \int_0^\infty \frac{\partial}{\partial b}\left(\frac{e^{-x}\sin bx}{x}\right) dx \;=\; \int_0^\infty \frac{e^{-x}\cdot x\cos bx}{x}\; dx = \int_0^\infty e^{-x}\cos bx\, dx$$

[We know that $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\;(a\cos bx + b\sin bx)$]

$$= \left[\frac{e^{-x}}{1+b^2}(-\cos bx + b\sin bx)\right]_0^\infty$$

$$\frac{dI}{db} = \frac{1}{1+b^2} \qquad \ldots (2)$$

Integrating both sides of (2) w.r.t. 'b', we have

$$I = \tan^{-1} b + c \qquad \text{... (3)}$$

On putting $b = 0$ in (1), we have $I = 0$

On putting $b = 0$, $I = 0$ in (3), we get

$$c = 0$$

Hence (3) gives

$$I = \tan^{-1} b$$

or $$\int_0^\infty \frac{e^{-x} \sin bx}{x}\, dx = \tan^{-1} b\,. \qquad \textbf{Ans.}$$

Example 40. *Find the value of*

$$\int_0^\pi \frac{dx}{a + b\cos x} \qquad (when\ a > 0,\ |b| < a)$$

and deduce that $$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Solution. Let $$I = \int_0^\pi \frac{dx}{a + b\cos x} = \int_0^\pi \frac{dx}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{\pi}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int_0^\pi \frac{dx}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}} = \frac{1}{a-b}\int_0^\pi \frac{sec^2\frac{x}{2}\, dx}{\frac{a+b}{a-b} + \tan^2\frac{x}{2}}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\left\{\tan\frac{x}{2}\cdot\sqrt{\frac{a-b}{a+b}}\right\}\right]_0^\pi = \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}} \qquad \textbf{Proved.}$$

Now differentiating both sides w.r.t. 'a', we get

$$\frac{dI}{da} = -\frac{1}{2}\,\frac{2\pi a}{(a^2-b^2)^{3/2}} \quad \text{or} \quad \int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right) dx = -\frac{1}{2}\,\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

or $$\int_0^\pi \frac{-1}{(a + b\cos x)^2}\, dx = \frac{-\pi a}{(a^2-b^2)^{3/2}}$$

or $$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \qquad \textbf{Ans.}$$

Exercise 21.6

Prove that

1. $\displaystyle\int_0^\infty \frac{1 - e^{-ax}}{x}\, e^{-x}\, dx = \log(1 + a)$ **2.** $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\frac{b}{a}$

3. $\displaystyle\int_0^\infty \frac{e^{-ax}\sin x}{x}\, dx = \cot^{-1} a$ and hence deduce that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

4. $\displaystyle\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\frac{a+1}{b+1}$

5. $\displaystyle\int_0^\infty \frac{\cos \lambda x}{x}\left(e^{-ax}-e^{-bx}\right) dx = \frac{1}{2}\log \frac{b^2+\lambda^2}{a^2+\lambda^2}, \quad (a>0,\ b>0)$

6. $\displaystyle\int_0^\infty e^{-bx^2}\cos 2ax\, dx = \frac{1}{2}\sqrt{\frac{\pi}{b}}\; e^{-a^2/b} \quad (b>0).$ Assume $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Evaluate the following.

7. $\displaystyle\int_0^{\frac{\pi}{2}} \log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\, d\theta \quad (\alpha>0,\quad \beta>0)$ **Ans.** $\pi \log \dfrac{\alpha+\beta}{2}$

8. $\displaystyle\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\, dx$ **Ans.** $\dfrac{\pi}{l}\log\dfrac{a+b}{b}$

9. $\displaystyle\int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\cdot\frac{d\theta}{\sin\theta}$ **Ans.** $\pi\sin^{-1}\dfrac{b}{a}$

10. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos\alpha\cdot\cos x)}{\cos x}\, dx$ **Ans.** $\dfrac{1}{2}\left(\dfrac{\pi^2}{4}-\alpha^2\right)$

Prove that

11. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\, dx = \pi\,[\sqrt{(1+y)}-1]$ when $y>1$.

12. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$

21.14 RULE OF DIFFERENTIATION UNDER THE INTEGRAL SIGN WHEN THE LIMITS OF INTEGRATION ARE FUNCTIONS OF THE PARAMETER

If $f(x,\alpha)$, $\dfrac{\partial f(x,\alpha)}{\partial\alpha}$ be continuous functions of x and α, then

$$\frac{d}{d\alpha}\left\{\int_{\phi\ \alpha)}^{(\alpha)} f(x,\alpha)\ dx\right\} = \int_{\phi\ \alpha)}^{\ )} \frac{\partial f(x,\alpha)}{\partial\alpha}\ dx - \frac{d\phi}{d\alpha}\ f[\phi(\alpha),\alpha]\quad \frac{d}{d\alpha}\ f[\quad(\alpha),\alpha]$$

Example 41. *Verify the rule of differentiation under the sign of integration for*

$$\int_0^{a^2}\tan^{-1}\frac{x}{a}\ dx.$$

Solution. Let $\qquad I = \displaystyle\int_0^{a^2}\tan^{-1}\frac{x}{a}\ dx$

$$\left[\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi\ \alpha)}^{\psi\ \alpha)}\frac{\partial f(x,\alpha)}{\partial\alpha}\ dx - \frac{d\phi}{d\alpha}\ f[\phi(\alpha),\alpha] + \frac{d\psi}{d\alpha} f[\psi(\alpha),\alpha]\right]$$

$$\frac{dI}{da} = \int_0^{a^2}\left[\frac{\partial}{\partial a}\left(\tan^{-1}\frac{x}{a}\right)\right] dx - 0 + 2a\left[\tan^{-1}\frac{a^2}{a}\right]$$

$$= \int_0^{a^2}\frac{1}{1+\frac{x^2}{a^2}}\left(-\frac{x}{a^2}\right) dx + 2a\tan^{-1}a = \int_0^{a^2} -\frac{x}{a^2+x^2}\ dx + 2a\tan^{-1}a$$

$$= \left[-\frac{1}{2}\log(a^2+x^2)\right]_0^{a^2} + 2a\tan^{-1}a = -\frac{1}{2}\log(a^2+a^4)+\frac{1}{2}\log a^2 + 2a\tan^{-1}a$$

$$= -\frac{1}{2}\log\frac{a^2+a^4}{a^2} + 2a\tan^{-1}a \;=\; -\frac{1}{2}\log(a^2+1) + 2a\tan^{-1}a \qquad \text{... (1)}$$

Now integration by parts

$$I = \int_0^{a^2} \tan^{-1}\frac{x}{a}\cdot 1\,dx = \left[\left(\tan^{-1}\frac{x}{a}\right)\cdot x\right]_0^{a^2} - \int_0^{a^2} \frac{1}{1+\dfrac{x^2}{a^2}}\,\frac{1}{a}\cdot x\,dx$$

$$= \left[a^2\tan^{-1}\frac{a^2}{a}\right] - \int_0^{a^2}\frac{ax}{a^2+x^2}\,dx \;=\; a^2\tan^{-1}a - \frac{a}{2}\,[\log(x^2+a^2)]_0^{a^2}$$

$$= a^2\tan^{-1}a - \frac{a}{2}\,[\log(a^4+a^2) - \log a^2] = a^2\tan^{-1}a - \frac{a}{2}\,\log\frac{a^4+a^2}{a^2}$$

$$= a^2\tan^{-1}a - \frac{a}{2}\,\log(a^2+1)$$

$$\frac{dI}{da} = \left[a^2\frac{1}{1+a^2} + 2a\tan^{-1}a\right] - \left[\frac{a}{2}\,\frac{2a}{a^2+1} + \frac{1}{2}\,\log\,(a^2+1)\right]$$

$$= \frac{a^2}{1+a^2} + 2a\tan^{-1}a - \frac{a^2}{a^2+1} - \frac{1}{2}\,\log\,(a^2+1)$$

$$= 2a\tan^{-1}a - \frac{1}{2}\,\log\,(a^2+1) \qquad \text{... (2)}$$

From (1) and (2), the rule is verified.

Example 42. *Evaluate* $\displaystyle\int_0^{\alpha}\frac{\log(1+\alpha x)}{1+x^2}\,dx$ *and hence show that*

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log_e 2$$

Solution. Let $\qquad I = \displaystyle\int_0^{\alpha}\frac{\log(1+\alpha x)}{1+x^2}\,dx \qquad \text{... (1)}$

$$\left[\frac{d}{d\alpha}\left\{\int_{\phi\;\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)}\frac{\partial f(x,\alpha)}{\partial\alpha}\,dx - \frac{d\phi}{d\alpha}\,f[\phi(\alpha),\alpha] + \frac{d\psi}{d\alpha}\,f[\quad(\alpha),\alpha]\right]$$

$$\frac{dI}{d\alpha} = \int_0^{\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{\log(1+\alpha x)}{1+x^2}\right\}dx + \frac{d\alpha}{d\alpha}\,f(\alpha,\alpha) = \int_0^{\alpha}\frac{x}{(1+x^2)(1+\alpha x)}\,dx + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

Converting into partial fractions,

$$= -\frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+\alpha x} + \frac{1}{2(1+\alpha^2)}\int_0^{\alpha}\frac{2x}{1+x^2}\,dx + \frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+x^2} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= -\frac{\alpha}{1+\alpha^2}\left[\frac{1}{\alpha}\log(1+\alpha x)\right]_0^{\alpha} + \frac{1}{2(1+\alpha^2)}\,[\log(1+x^2)]_0^{\alpha} + \frac{\alpha}{1+\alpha^2}\left[\tan^{-1}x\right]_0^{\alpha} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= -\frac{1}{1+\alpha^2}\log(1+\alpha^2) + \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha$$

On integrating both sides w.r.t. α, we have

$$I = \frac{1}{2}\int \log(1+\alpha^2)\cdot\frac{1}{1+\alpha^2}\,d\alpha + \int \frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha + c$$

$$I = \frac{1}{2}\log(1+\alpha^2)\cdot\tan^{-1}\alpha - \frac{1}{2}\int \frac{2\alpha}{1+\alpha^2}\cdot\tan^{-1}\alpha\,d\alpha + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}\,d\alpha$$

$$I = \frac{1}{2}\log(1+\alpha^2)\cdot\tan^{-1}\alpha + c \qquad \text{... (2)}$$

From (1), when $\alpha = 0$, then $I = 0$. From (2), when $\alpha = 0$, $I = 0$, then $c = 0$

Hence (2) gives

$$I = \frac{1}{2}\log(1+\alpha^2)\cdot\tan^{-1}\alpha \qquad \text{... (3)}$$

On putting $\alpha = 1$ in (3), we have

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{1}{2}\log(1+1)\cdot\tan^{-1}(1) = \frac{1}{2}(\log_e 2)\,\frac{\pi}{4}$$

$$= \frac{\pi}{8}\log_e 2$$ **Ans.**

Example 43. *Evaluate* $\int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{\sin ax}{x}\,dx$.

Solution. Let
$$I = \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{\sin ax}{x}\,dx. \qquad \text{... (1)}$$

$$\left[\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx - \frac{d\phi}{d\alpha}\,f[\phi(\alpha),\alpha] + \frac{d}{d\alpha}\,f[\quad(\alpha),\alpha]\right]$$

$$\frac{dI}{da} = \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{\partial}{\partial a}\left(\frac{\sin ax}{x}\right)dx - \frac{d}{da}\left(\frac{\pi}{6a}\right)\cdot\frac{\left[\sin a\cdot\frac{\pi}{6a}\right]}{\frac{\pi}{6a}} + \frac{d}{da}\left(\frac{\pi}{2a}\right)\cdot\frac{\sin a\cdot\frac{\pi}{2a}}{\frac{\pi}{2a}}$$

$$= \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \frac{x\cos ax}{x}\,dx + \frac{\pi}{6a^2}\,\frac{6a}{\pi}\sin\frac{\pi}{6} - \frac{\pi}{2a^2}\cdot\frac{2a}{\pi}\sin\frac{\pi}{2}$$

$$= \int_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} \cos ax\,dx + \frac{1}{2a} - \frac{1}{a} = \left[\frac{\sin ax}{a}\right]_{\frac{\pi}{6a}}^{\frac{\pi}{2a}} - \frac{1}{2a}$$

$$= \frac{1}{a}\left[\sin a\cdot\frac{\pi}{2a} - \sin a\cdot\frac{\pi}{6a}\right] - \frac{1}{2a}$$

$$= \frac{1}{a}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right] - \frac{1}{2a} = \frac{1}{a}\left[1-\frac{1}{2}\right] - \frac{1}{2a} = \frac{1}{2a} - \frac{1}{2a} = 0$$

Integrating we have $I = \text{Constant}$ **Ans.**

Example 44. *If* $y = \int_0^x f(t)\sin[k(x-t)]\,dt$, *prove that* y *satisfies the differential equation*

$$\frac{d^2y}{dx^2} + k^2 y = k f(x)$$

Solution.

$$y = \int_0^x f(t) \sin [k(x-t)]\, dt$$

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x} [f(t) \sin \{k(x-t)\}]\, dt - 0 + \frac{d}{dx}(x) \cdot f(x) \sin \{k(x-x)\}$$

$$= \int_0^x f(t)\, k \cos \{k(x-t)\} \cdot dt$$

$$= k \int_0^x f(t) \cos \{k(x-t)\}\, dt$$

Again applying the same rule

$$\frac{d^2y}{dx^2} = k\left[\int_0^x \frac{\partial}{\partial x} \{f(t) \cos k(x-t)\}\, dt - 0 + \frac{d}{dx}(x) \cdot f(x) \cos k(x-x)\right]$$

$$= -k^2 \int_0^x f(t) \sin [k(x-t)]\, dt + kf(x)$$

Proved.

UNIT-4

Vector Differentiation

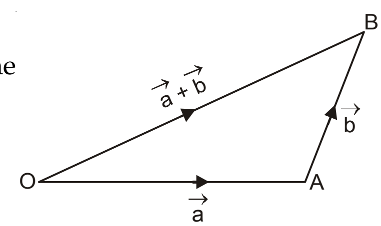
5.1 VECTORS

A vector is a quantity having both magnitude and direction such as force, velocity acceleration, displacement etc.

5.2 ADDITION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two given vectors

$\vec{OA} = \vec{a}$ and $\vec{AB} = \vec{b}$ then vector $\vec{OB}$ is called the sum of $\vec{a}$ and $\vec{b}$.



Symbolically

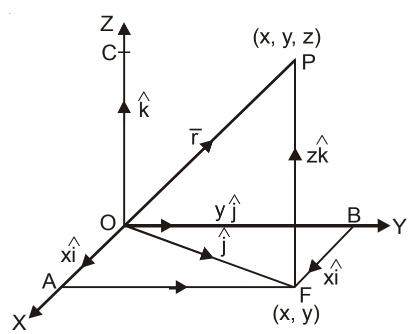
$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\vec{a} + \vec{b} = \overrightarrow{OB}$$

5.3 RECTANGULAR RESOLUTION OF A VECTOR

Let OX, OY, OZ be the three rectangular axes. Let $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors and parallel to three axes.

If $\vec{OP} = \hat{n}$ and the co-ordinates of P be (x, y, z)



$$\vec{OA} = x\hat{i}, \quad \vec{OB} = y\hat{j} \quad \text{and} \quad \vec{OC} = z\hat{k}$$

$$\vec{OP} = \vec{OF} + \vec{FP}$$

$$\Rightarrow \quad \vec{OP} = (\vec{OA} + \vec{AF}) + \vec{FP}$$

$$\Rightarrow \quad \vec{OP} = \vec{OA} + \vec{OB} + \vec{OC}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \quad OP^2 = OF^2 + FP^2$$

$$= (OA^2 + AF^2) + FP^2 = OA^2 + OB^2 + OC^2 = x^2 + y^2 + z^2$$

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

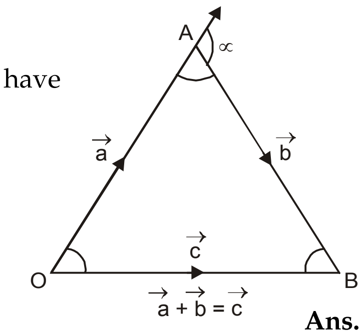
5.4 UNIT VECTOR

Let a vector be $x\hat{i} + y\hat{j} + z\hat{k}$.

$$\text{Unit vector} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Example 1. *If* $\vec{a}$ *and* $\vec{b}$ *be two unit vectors and* α *be the angle between them, then the value of* α *such that* $\vec{a} + \vec{b}$ *is a unit vector.* (*Nagpur, University, Winter 2001*)

Solution. Let $\vec{OA} = \vec{a}$ be a unit vector and $\vec{AB} = \vec{b}$ is another unit vector and α be the angle between $\vec{a}$ and $\vec{b}$.



If $\vec{OB} = \vec{c} = \vec{a} + \vec{b}$ is also a unit vector then, we have

$$|\vec{OA}| = 1$$

$$|\vec{AB}| = 1$$

$$|\vec{OB}| = 1$$

OAB is an equilateral triangle.

So, each angle of $\Delta\, OAB$ is $\dfrac{\pi}{3}$

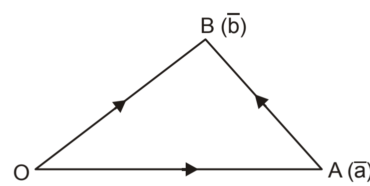
Hence $\propto = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$ **Ans.**

5.5 POSITION VECTOR OF A POINT

The position vector of a point *A* with respect to origin *O* is the vector $\vec{OA}$ which is used to specify the position of *A* w.r.t. *O*.

To find $\overrightarrow{AB}$ if the position vectors of the point *A* and point *B* are given.

If the position vectors of *A* and *B* are $\vec{a}$ and $\vec{b}$. Let the origin be *O*.



$$\text{Then} \quad \vec{OA} = \vec{a}, \quad \vec{OB} = \vec{b}$$

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\Rightarrow \quad \vec{AB} = \vec{b} - \vec{a}$$

$$\vec{AB} = \textbf{Position vector of } B - \textbf{Position vector of } A$$

Example 2. *If A and* B *are* (3, 4, 5) *and* (6, 8, 9), *find* $\vec{AB}$.

Solution. $\vec{AB}$ = Position vector of B – Position vector of A

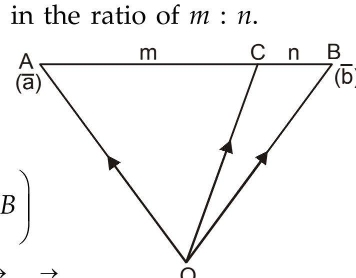
$$= (6\hat{i} + 8\hat{j} + 9\hat{k}) - (3\hat{i} + 4\hat{j} + 5\hat{k}) = 3\hat{i} + 4\hat{j} + 4\hat{k}$$ **Ans.**

5.6 RATIO FORMULA

To find the position vector of the point which divides the line joining two given points.

Let *A* and *B* be two points and a point *C* divides *AB* in the ratio of *m* : *n*.

Let *O* be the origin, then



$$\vec{OA} = \vec{a}, \quad \text{and} \quad \vec{OB} = \bar{b}, \quad \vec{OC} = ?$$

$$\vec{OC} = \vec{OA} + \vec{AC}$$

$$= \vec{OA} + \frac{m}{m+n}\vec{AB} \qquad \left(\because AC = \frac{m}{m+n} AB\right)$$

$$= \vec{a} + \frac{m}{m+n}.(\vec{b} - \vec{a}) \qquad (\because \vec{AB} = \vec{b} - \vec{a})$$

$$\vec{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Cor. If $m = n = 1$, then C will be the mid-point, and

$$\vec{OC} = \frac{\vec{a} + \vec{b}}{2}$$

5.7 PRODUCT OF TWO VECTORS

The product of two vectors results in two different ways, the one is a number and the other is vector. So, there are two types of product of two vectors, namely scalar product and vector product. They are written as $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$.

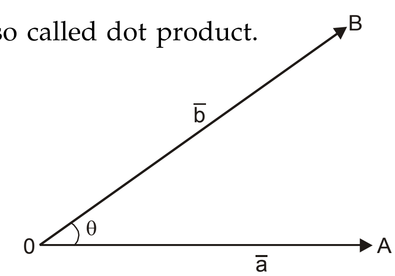
5.8 SCALAR, OR DOT PRODUCT

The scalar, or dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be $\left|\vec{a}\right|\left|\vec{b}\right| \cos\theta$ *i.e.,* scalar where θ is the angle between $\vec{a}$ and $\vec{b}$.

Symbolically, $\boldsymbol{\vec{a} \cdot \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \cos\theta}$

Due to a dot between $\vec{a}$ and $\vec{b}$ this product is also called dot product.

The scalar product is commutative

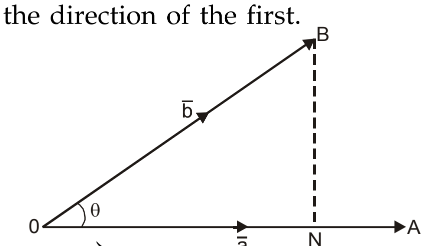


To Prove. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Proof.

$$\begin{aligned} \vec{b} \cdot \vec{a} &= \left|\vec{b}\right|\left|\vec{a}\right| \cos(-\theta) \\ &= \left|\vec{a}\right|\left|\vec{b}\right| \cos\theta \\ &= \vec{a} \cdot \vec{b} \qquad \textbf{Proved.} \end{aligned}$$

Geometrical interpretation. The scalar product of two vectors is the product of one vector and the length of the projection of the other in the direction of the first.



Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

then

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (OA)\,.\,(OB) \cos\theta \\ &= OA\,.\,OB\,.\,\frac{ON}{OB} \\ &= OA\,.\,ON \\ &= (\text{Length of } \vec{a})\ (\text{projection of } \vec{b} \text{ along } \vec{a}) \end{aligned}$$

5.9 USEFUL RESULTS

$\hat{i} \cdot \hat{i} = (1)\,(1) \cos 0° = 1$ Similarly, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = (1)\,(1) \cos 90° = 0$ Similarly, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

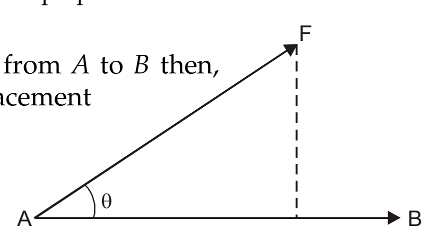
Note. If the dot product of two vectors is zero then vectors are prependicular to each other.

5.10 WORK DONE AS A SCALAR PRODUCT

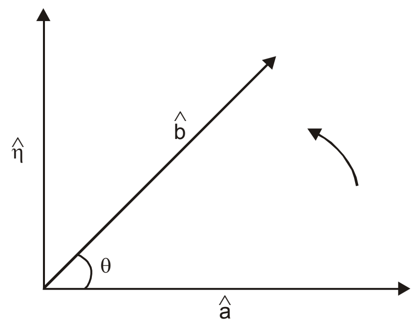
If a constant force F acting on a particle displaces it from A to B then,

$$\begin{aligned} \text{Work done} &= (\text{component of } F \text{ along } AB).\ \text{Displacement} \\ &= F\cos\theta.\ AB \\ &= \vec{F} \cdot \vec{AB} \end{aligned}$$

$$\boxed{\textbf{Work done} = \textbf{Force . Displacement}}$$



5.11 VECTOR PRODUCT OR CROSS PRODUCT



1. The vector, or cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be a vector such that

(*i*) Its magnitude is $|\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

(*ii*) Its direction is perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

(*iii*) It forms with a right handed system.

Let $\hat{\eta}$ be a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta \,.\, \hat{\eta}$$

2. Useful results

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{i} \times \hat{j}$$

$$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \qquad \text{and} \qquad \hat{k} \times \hat{j} = -\hat{j} \times \hat{k}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{k} \times \hat{i}$$

5.12 VECTOR PRODUCT EXPRESSED AS A DETERMINANT

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) + a_2b_2(\hat{j} \times \hat{j}) \\
&\quad + a_2b_3(\hat{j} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k}) \\
&= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i} \\
&= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

5.13 AREA OF PARALLELOGRAM

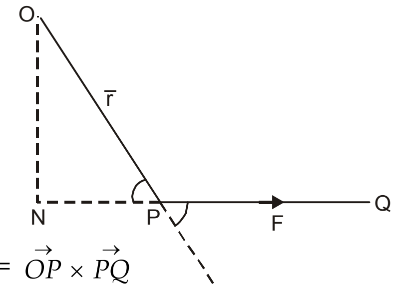
Example 3. *Find the area of a parallelogram whose adjacent sides are i – 2j + 3 k and 2i + j – 4k.*

Solution. Vector area of || gm $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}$

$$= (8-3)\hat{i} - (-4-6)\hat{j} + (1+4)\hat{k} = 5\hat{i} + 10\hat{j} + 5\hat{k}$$

Area of parallelogram $= \sqrt{(5)^2 + (10)^2 + (5)^2} = 5\sqrt{6}$ **Ans.**

5.14 MOMENT OF A FORCE



Let a force F $(\overrightarrow{PQ})$ act at a point P.

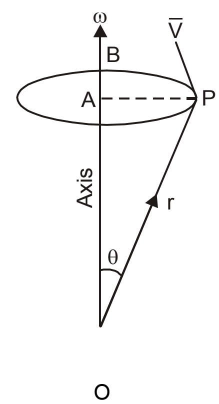
Moment of $\vec{F}$ about O

= Product of force F and perpendicular distance $(ON.\hat{\eta})$

$= (PQ)\ (ON)(\hat{\eta}) = (PQ)\ (OP)\sin\theta\ (\hat{\eta}) = \overrightarrow{OP} \times \overrightarrow{PQ}$

$\Rightarrow \quad \vec{M} = \vec{r} \times \vec{F}$

5.15 ANGULAR VELOCITY



Let a rigid body be rotating about the axis OA with the angular velocity ω which is a vector and its magnitude is ω radians per second and its direction is parallel to the axis of rotation OA.

Let P be any point on the body such that $\overline{OP} = \vec{r}$ and $\angle AOP = \theta$ and $AP \perp OA$. Let the velocity of P be V.

Let $\hat{\eta}$ be a unit vector perpendicular to $\vec{\omega}$ and $\vec{r}$.

$$\vec{\omega} \times \vec{r} = (\omega\, r \sin\theta)\ \hat{\eta} = (\omega\, AP)\ \hat{\eta} = (\text{Speed of } P)\ \hat{\eta}$$

$$= \text{Velocity of } P \perp \text{to } \vec{\omega} \text{ and } r$$

Hence $\boxed{\vec{V} = \vec{\omega} \times \vec{r}}$

5.16 SCALAR TRIPLE PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors then their dot product is written as $\vec{a}\,.\,(\vec{b} \times \vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{a}\,.(\vec{b} \times \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).[(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})]$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).[(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}]$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Similarly, $\vec{b}\,.\,(\vec{c} \times \vec{a})$ and $\vec{c}\,.\,(\vec{a} \times \vec{b})$ have the same value.

$\therefore \quad \vec{a}\,.\,(\vec{b} \times \vec{c}) = \vec{b}\,.\,(\vec{c} \times \vec{a}) = \vec{c}\,.\,(\vec{a} \times \vec{b})$

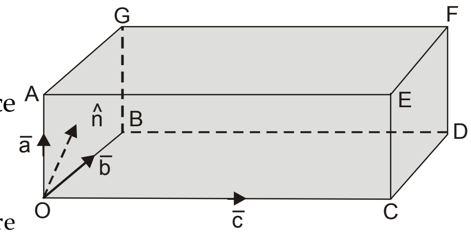
The value of the product depends upon the cyclic order of the vector, but is independent of the position of the dot and cross. These may be interchanged.

The value of the product changes if the order is non-cyclic.

Note. $\vec{a} \times (\vec{b}\,.\,\vec{c})$ and $(\vec{a}\,.\,\vec{b}) \times \vec{c}$ are meaningless.

5.17 GEOMETRICAL INTERPRETATION

The scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$ represents the volume of the parallelopiped having $\vec{a}$, $\vec{b}$, $\vec{c}$ as its co-terminous edges.



$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a}$. Area of || gm $OBDC$ $\hat{\eta}$

= Area of || gm $OBDC$ × perpendicular distance between the parallel faces $OBDC$ and $AEFG$.

= Volume of the parallelopiped

Note. (1) If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

(2) Volume of tetrahedron $\frac{1}{6}(\vec{a}\ \vec{b}\ \vec{c})$.

Example 4. *Find the volume of parallelopiped if*

$$\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}, \quad \vec{b} = -3\hat{i} + 7\hat{j} - 3\hat{k}, \text{ and } \vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$$

are the three co-terminous edges of the parallelopiped.

Solution.

$$\text{Volume} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3(-21 - 15) - 7(9 + 21) + 5(15 - 49)$$

$$= 108 - 210 - 170 = -272$$

Volume = 272 cube units. **Ans.**

Example 5. *Show that the volume of the tetrahedron having* $\vec{A} + \vec{B}, \vec{B} + \vec{C}, \vec{C} + \vec{A}$ *as concurrent edges is twice the volume of the tetrahendron having* $\vec{A}, \vec{B}, \vec{C}$ *as concurrent edges.*

Solution. Volume of tetrahendron $= \frac{1}{6}(\vec{A} + \vec{B}) \cdot [(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})]$

$$= \frac{1}{6}(\vec{A} + \vec{B}) \cdot [\vec{B} \times \vec{C} + \vec{B} \times \vec{A} + \vec{C} \times \vec{C} + \vec{C} \times \vec{A}] \qquad [\vec{C} \times \vec{C} = 0]$$

$$= \frac{1}{6}(\vec{A} + \vec{B}) \cdot (\vec{B} \times \vec{C} + \vec{B} \times \vec{A} + \vec{C} \times \vec{A})$$

$$= \frac{1}{6}[\vec{A} \cdot (\vec{B} \times \vec{C}) + \vec{A} \cdot (\vec{B} \times \vec{A}) + \vec{A} \cdot (\vec{C} \times \vec{A}) + \vec{B} \cdot (\vec{B} \times \vec{C}) + \vec{B} \cdot (\vec{B} \times \vec{A}) + \bar{B} \cdot (\vec{C} \times \vec{A})]$$

$$= \frac{1}{6}[\vec{A} \cdot (\vec{B} \times \vec{C}) + \vec{B} \cdot (\vec{C} \times \vec{A})] = \frac{1}{3}\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$= 2 \times \frac{1}{6}[\vec{A}\ \vec{B}\ \vec{C}]$$

= 2 Volume of tetrahedron having $\vec{A}, \vec{B}, \vec{C}$, as concurrent edges. **Proved.**

EXERCISE 5.1

1. Find the volume of the parallelopiped with adjacent sides.

$$\overline{OA} = 3\hat{i} - \hat{j}, \quad \overline{OB} = \hat{j} + 2\hat{k}, \quad \text{and} \quad \overline{OC} = \hat{i} + 5\hat{j} + 4\hat{k}$$

extending from the origin of co-ordinates O. **Ans.** 20

2. Find the volume of the tetrahedron whose vertices are the points A (2, –1, –3), B (4, 1, 3) C (3, 2, –1) and D (1, 4, 2). **Ans.** $7\frac{1}{3}$

3. Choose y in order that the vectors $\vec{a} = 7\hat{i} + y\hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = 5\hat{i} + 3\hat{j} + \hat{k}$ are linearly dependent. **Ans.** $y = 4$
4. Prove that

$$[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

5.18 COPLANARITY QUESTIONS

Example 6. *Find the volume of tetrahedron having vertices*

$$(-\hat{j} - \hat{k}),\quad (4\hat{i} + 5\hat{j} + q\hat{k}),\quad (3\hat{i} + 9\hat{j} + 4\hat{k}) \text{ and } 4(-\hat{i} + \hat{j} + \hat{k}).$$

Also find the value of q for which these four points are coplanar.

(*Nagpur University, Summer 2004, 2003, 2002*)

Solution. Let $\vec{A} = -\hat{j} - \hat{k},\ \bar{B} = 4\hat{i} + 5\hat{j} + q\hat{k},\ \bar{C} = 3\hat{i} + 9\hat{j} + 4\hat{k},\ \bar{D} = 4(-\hat{i} + \hat{j} + \hat{k})$

$$\overline{AB} = \vec{B} - \vec{A} = 4\hat{i} + 5\hat{j} + q\hat{k} - (-\hat{j} - \hat{k}) = 4\hat{i} + 6\hat{j} + (q+1)\hat{k}$$

$$\overline{AC} = \vec{C} - \vec{A} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (-\hat{j} - \hat{k}) = 3\hat{i} + 10\hat{j} + 5\hat{k}$$

$$\overline{AD} = \vec{D} - \vec{A} = 4(-\hat{i} + \hat{j} + \hat{k}) - (-\hat{j} - \hat{k}) = -4\hat{i} + 5\hat{j} + 5\hat{k}$$

$$\text{Volume of the tetrahedron} = \frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$$

$$= \frac{1}{6}\begin{vmatrix} 4 & 6 & q+1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = \frac{1}{6}\{4(50-25) - 6(15+20) + (q+1)(15+40)\}$$

$$= \frac{1}{6}\{100 - 210 + 55(q+1)\} = \frac{1}{6}(-110 + 55 + 55q)$$

$$= \frac{1}{6}(-55 + 55q) = \frac{55}{6}(q-1)$$

If four points A, B, C and D are coplanar, then $(\overline{AB}\ \overline{AC}\ \overline{AD}) = 0$

i.e., Volume of the tetrahedron = 0

$$\Rightarrow \quad \frac{55}{6}(q-1) = 0 \quad \Rightarrow \quad q = 1$$ **Ans.**

Example 7. *If four points whose position vectors are* $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ *are coplanar, show that*

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{d}\ \vec{b}] + [\vec{a}\ \vec{d}\ \vec{c}] + [\vec{d}\ \vec{b}\ \vec{c}]$$ (*Nagpur University, Summer 2005*)

Solution. Let A, B, C, D be four points whose position vectors are $\vec{a}, \vec{b}, \vec{c}, \vec{d}$.

$$\vec{AD} = \vec{d} - \vec{a},\quad \vec{BD} = \vec{d} - \vec{b} \quad \text{and} \quad \vec{CD} = \vec{d} - \vec{c}$$

If $\vec{AD}, \vec{BD}, \vec{CD}$ are coplanar, then

$$\vec{AD}\,.\,(\vec{BD} \times \vec{CD}) = 0$$

$$\Rightarrow \quad (\vec{d} - \vec{a})\,.\,[(\vec{d} - \vec{b}) \times (\vec{d} - \vec{c})] = 0$$

$$\Rightarrow \quad (\vec{d} - \vec{a})\,.\,[\vec{d} \times \vec{d} - \vec{d} \times \vec{c} - \vec{b} \times \vec{d} + \vec{b} \times \vec{c}] = 0$$

$$\Rightarrow \quad (\vec{d} - \vec{a})\,.\,[-\vec{d} \times \vec{c} - \vec{b} \times \vec{d} + \vec{b} \times \vec{c}] = 0$$

$$\Rightarrow \quad -\vec{d}\,.(\vec{d} \times \vec{c}) - \vec{d}\,.(\vec{b} \times \vec{d}) + \vec{d}\,.(\vec{b} \times \vec{c}) + \vec{a}\,.(\vec{d} \times \vec{c}) + \vec{a}\,.(\vec{b} \times \vec{a}) - \vec{a}\,.(\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow -0 + 0 + [\vec{d}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{d}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{d}\ \vec{c}] + [\vec{d}\ \vec{b}\ \vec{c}]$$ **Proved.**

EXERCISE 5.2

1. Determine λ such that

$\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - 4\hat{k}$, and $\bar{c} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are coplanar. **Ans.** $\lambda = 5/3$

2. Show that the four points

$-6\hat{i} + 3\hat{j} + 2\hat{k}$, $3\hat{i} - 2\hat{j} + 4\hat{k}$, $5\hat{i} + 7\hat{j} + 3\hat{k}$ and $-13\hat{i} + 17\hat{j} - \hat{k}$ are coplanar.

3. Find the constant a such that the vectors

$2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$, and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar. **Ans.** -4

4. Prove that four points

$4\hat{i} + 5\hat{j} + \hat{k}$, $-(\hat{j} + \hat{k})$, $3\hat{i} + 9\hat{j} + 4\hat{k}$, $4(-\hat{i} + \hat{j} + \hat{k})$ are coplanar.

5. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, show that

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = 0$$

5.19 VECTOR PRODUCT OF THREE VECTORS (*A.M.I.E.T.E., Summer, 2004, 2000*)

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors then their vector product is written as $\vec{a} \times (\vec{b} \times \vec{c})$.

Let
$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\
&= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times [(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}] \\
&= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\hat{j} \\
&\qquad + [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\hat{k}] \\
&= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\
&= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.
\end{aligned}$$
Ans.

Example 8. *Prove that :*

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$ (*Nagpur University, Winter 2008*)

Solution. Here, we have

$$\begin{aligned}
&\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\
&= [(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}] + [(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\bar{c})\vec{a}] + [(\vec{c}\cdot\bar{b})\vec{b} - (\vec{c}\cdot\vec{a})\vec{b}] \\
&= [(\vec{b}\cdot\vec{a})\vec{c} - (\vec{a}\cdot\vec{b})\vec{c}] + [(\vec{c}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{c})\vec{a}] + [(\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{a})\vec{b}] \\
&= [(\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{b})\vec{c}] + [(\vec{b}\cdot\vec{c})\vec{a} - (\vec{b}\cdot\vec{c})\vec{a}] + [(\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{a})\vec{b}] \\
&= 0 + 0 + 0 = 0
\end{aligned}$$
Proved.

Example 9. *Prove that :*

$$\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$$ (*Nagpur University, Winter 2003*)

Solution. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Now, L.H.S. $= \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$

$$= \hat{i} \times \left[(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{i}\right] + \hat{j} \times \left[(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{j}\right] + \hat{k} \times \left[(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{k}\right]$$

$$= \hat{i} \times \left[a_1(\hat{i} \times \hat{i}) + a_2(\hat{j} \times \hat{i}) + a_3(\hat{k} \times \hat{i})\right] + \hat{j} \times \left[a_1(\hat{i} \times \hat{j}) + a_2(\hat{j} \times \hat{j}) + a_3(\hat{k} \times \hat{j})\right] + \hat{k} \times \left[a_1(\hat{i} \times \hat{k}) + a_2(\hat{j} \times \hat{k}) + a_3(\hat{k} \times \hat{k})\right]$$

$$= \hat{i} \times \left[0 - a_2 \hat{k} + a_3 \hat{j}\right] + \hat{j} \times \left[a_1 \hat{k} + 0 - a_3 \hat{i}\right] + \hat{k} \times \left[-a_1 \hat{j} + a_2 \hat{i} + 0\right]$$

$$= -a_2(\hat{i} \times \hat{k}) + a_3(\hat{i} \times \hat{j}) + a_1(\hat{j} \times \hat{k}) - a_3(\hat{j} \times \hat{i}) - a_1(\hat{k} \times \hat{j}) + a_2(\hat{k} \times \hat{i})$$

$$= a_2 \hat{j} + a_3 \hat{k} + a_1 \hat{i} + a_3 \hat{k} + a_1 \hat{i} + a_2 \hat{j} = 2a_1 \hat{i} + 2a_2 \hat{j} + 2a_3 \hat{k}$$

$$= 2 (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = 2 \vec{a}$$ **Proved.**

Example 10. *Show that for any scalar* λ, *the vectors* $\vec{x}, \vec{y}$ *given by*

$\vec{x} = \lambda \vec{a} + \dfrac{q (\vec{a} \times \vec{b})}{a^2}, \vec{y} = \dfrac{(1 - p \lambda)}{q} \vec{a} - \dfrac{p (\vec{a} \times \vec{b})}{a^2}$ *satisfy the equations*

$p \vec{x} + q \vec{y} = \vec{a}$ *and* $\vec{x} \times \vec{y} = \vec{b}$. (*Nagpur University, Winter 2004*)

Solution. The given equations are

$$p\vec{x} + q\vec{y} = \vec{a} \qquad ...(1)$$

$$\vec{x} \times \vec{y} = \vec{b} \qquad ...(2)$$

Multiplying equation (1) vectorially by $\vec{x}$, we get

$$\vec{x} \times (p\vec{x} + q\vec{y}) = \vec{x} \times \vec{a}$$

$$p (\vec{x} \times \vec{x}) + q (\vec{x} \times \vec{y}) = \vec{x} \times \vec{a}$$

$$q \times (\vec{x} \times \vec{y}) = \vec{x} \times \vec{a}, \qquad \text{as } \vec{x} \times \vec{x} = 0$$

$$\vec{x} \times \vec{a} = q\vec{b}, \qquad [\text{From (2) } \vec{x} \times \vec{y} = \vec{b}] \quad ...(3)$$

Multiplying (3) vectorially by $\vec{a}$, we have

$$\vec{a} \times (\vec{x} \times \vec{a}) = \vec{a} \times q \vec{b}$$

$$(\vec{a} \cdot \vec{a}) \vec{x} - (\vec{a} \cdot \vec{x}) \vec{a} = q (\vec{a} \times \vec{b})$$

$$a^2 \vec{x} - (\vec{a} \cdot \vec{x}) \vec{a} = q (\vec{a} \times \vec{b}) \Rightarrow a^2 \vec{x} = (\vec{a} \cdot \vec{x}) \vec{a} + q (\vec{a} \times \vec{b})$$

$$\vec{x} = \frac{(\vec{a} \cdot \vec{x}) \vec{a}}{a^2} + \frac{q (\vec{a} \times \vec{b})}{a^2}$$

$$\vec{x} = \lambda a + \frac{q(\vec{a} \times \vec{b})}{a^2} \qquad \text{where } \lambda = \frac{\vec{a} \cdot \vec{x}}{a^2}$$

Substituting the value of $\vec{x}$ in (1), we get $p\left\{\lambda \vec{a} + \frac{q(\vec{a} \times \vec{b})}{a^2}\right\} + q\vec{y} = \vec{a}$

$$q\vec{y} = \vec{a} - p\left\{\lambda \vec{a} + \frac{q(\vec{a} \times \vec{b})}{a^2}\right\}$$

$$\vec{y} = \frac{(1 - p\lambda)\vec{a}}{q} - \frac{p(\vec{a} \times \vec{b})}{a^2}$$ **Ans.**

EXERCISE 5.3

1. Show that $\vec{a} \times (\vec{b} \times \vec{a}) = (\vec{a} \times \vec{b}) \times \vec{a}$
2. Write the correct answer

   (*a*) $(\vec{A} \times \vec{B}) \times \vec{C}$ lies in the plane of

   (*i*) $\vec{A}$ and $\vec{B}$ (*ii*) $\vec{B}$ and $\vec{C}$ (*iii*) $\vec{C}$ and $\vec{A}$ **Ans.** (*ii*)

   (*b*) The value of $\vec{a} \cdot (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c})$ is

   (*i*) Zero (*ii*) $[\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$ (*iii*) $[\vec{a}, \vec{b}, \vec{c}]$ (*iv*) None of these

   Ans. (*ii*)

5.20 SCALAR PRODUCT OF FOUR VECTORS

Prove the identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

Proof. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \cdot \vec{r}$

$= \vec{a} \cdot (\vec{b} \times \vec{r})$ dot and cross can be interchanged. Put $\vec{c} \times \vec{d} = \vec{r}$

$= \vec{a} \cdot [\vec{b} \times (\vec{c} \times \vec{d})] = \vec{a} \cdot [(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}]$

$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$ **Proved.**

EXERCISE 5.4

1. If $\vec{a} = 2i + 3j - k$, $\vec{b} = -i + 2j - 4k$, $\vec{c} = i + j + k$, find $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$. **Ans.** –74
2. Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = a^2(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \bar{b})(\vec{a} \cdot \vec{c})$.

5.21 VECTOR PRODUCT OF FOUR VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be four vectors then their vector product is written as

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$$

Now, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{r} \times (\vec{c} \times \vec{d})$ [Put $\vec{a} \times \vec{b} = \vec{r}$]

$= (\vec{r} \cdot \vec{d})\vec{c} - (\vec{r} \cdot \vec{c})\vec{d}$

$$= [(\vec{a} \times \vec{b}) \cdot \vec{d}]\, \vec{c} - [(\vec{a} \times \vec{b}) \cdot \vec{c}]\, \vec{d}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\, \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\, \vec{d}$$

$\therefore$ $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ lies in the plane of $\vec{c}$ and $\vec{d}$. ...(1)

Again, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{s}$ [Put = $\vec{c} \times \vec{d} = \vec{s}$]

$$= -\vec{s} \times (\vec{a} \times \vec{b}) = -(\vec{s} \cdot \vec{b})\, \vec{a} + (\vec{s} \cdot \vec{a})\, \vec{b}$$

$$= -[(\vec{c} \times \vec{d}) \cdot \vec{b}]\, \vec{a} + [(\vec{c} \times \vec{d}) \cdot \vec{a}]\, \vec{b} = -[(\vec{b}\ \vec{c}\ \vec{d}]\, \vec{a} + [\vec{a}\ \vec{c}\ \vec{d}]\, \vec{b}$$

$\therefore$ $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ lies in the plane of $\vec{a}$ and $\vec{b}$. ...(2)

Geometrical interpretation : From (1) and (2) we conclude that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector parallel to the line of intersection of the plane containing $\vec{a}$, $\vec{b}$ and plane containing $\vec{c}$, $\vec{d}$.

Example 11. *Show that*

$$(\vec{B} \times \vec{C}) \times (\vec{A} \times \vec{D}) + (\vec{C} \times \vec{A}) \times (\vec{B} \times \vec{D}) + (\overline{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = -2\,(\vec{A}\ \vec{B}\ \vec{C})\, \vec{D}$$

Solution. L.H.S. $= (\vec{B} \times \vec{C}) \times (\vec{A} \times \vec{D}) + (\vec{C} \times \vec{A}) \times (\vec{B} \times \vec{D}) + (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$

$$= [(\vec{B}\ \vec{C}\ \vec{D})\, \vec{A} - (\vec{B}\ \vec{C}\ \vec{A})\, \vec{D}] + [(\vec{C}\ \vec{A}\ \vec{D})\, \vec{B} - (\vec{C}\ \vec{A}\ \vec{B})\, \vec{D}] + [(-\vec{B}\ \vec{C}\ \vec{D})\, \vec{A} + (\vec{A}\ \vec{C}\ \vec{D})\, \vec{B}]$$

$$= (\vec{B}\ \vec{C}\ \vec{D})\, \vec{A} - (\vec{B}\ \vec{C}\ \vec{D})\, \vec{A} + (\vec{C}\ \vec{A}\ \vec{D})\, \vec{B} + (\vec{A}\ \vec{C}\ \vec{D})\, \overline{B} - (\vec{B}\ \vec{C}\ \vec{A})\, \vec{D} - (\vec{C}\ \vec{A}\ \vec{B})\, \vec{D}$$

$$= -(\vec{A}\ \vec{C}\ \vec{D})\, \vec{B} + (\vec{A}\ \vec{C}\ \vec{D})\, \vec{B} - (\vec{A}\ \vec{B}\ \vec{C})\, \vec{D} - (\vec{A}\ \vec{B}\ \vec{C})\, \vec{D}$$

$$= -2\,(\vec{A}\ \vec{B}\ \vec{C})\, \vec{D} = \text{R.H.S.}$$ **Proved.**

EXERCISE 5.5

Show that:

1. $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{c}\ (\vec{a}\ \vec{b}\ \vec{c})$ when $(\vec{a}\ \vec{b}\ \vec{c})$ stands for scalar triple product.
2. $[\vec{b} \times \vec{c},\quad \vec{c} \times \vec{a},\quad \vec{a} \times \vec{b})] = [\vec{a}\ \vec{v}\ \vec{c}]^2$
3. $\vec{d}\ [\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\}] = [(\vec{b} \cdot \vec{d})[\vec{a} \cdot (\vec{c} \times \vec{d})]$
4. $\vec{a}\ [\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = a^2\ (\vec{b} \times \vec{a})$
5. $[(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})] \cdot \vec{d} = (\vec{a} \cdot \vec{d})\, [\vec{a}\ \vec{b}\ \vec{c}]$
6. $2a^2 = \left|\vec{a} \times \hat{i}\right|^2 + \left|\vec{a} \times \hat{j}\right|^2 + \left|\vec{a} \times \hat{k}\right|^2$
7. $\vec{a} \times \vec{b} = [(\hat{i} \times \vec{a}) \cdot \bar{b}]\, \hat{i} + [(\hat{j} \times \vec{a}) \cdot \vec{b}]\, \hat{j} + [(\hat{k} \times \vec{a}) \cdot \vec{b}]\, \hat{k}$
8. $\vec{p} \times [(\vec{a} \times \vec{q}) \times (\vec{b} \times \vec{r})] + \vec{q} \times [(\vec{a} \times \vec{r}) \times (\vec{b} \times \vec{p})] + \vec{r} \times [(\vec{a} \times \vec{p}) \times (\vec{b} \times \vec{q})] = 0$

5.22 VECTOR FUNCTION

If vector r is a function of a scalar variable t, then we write

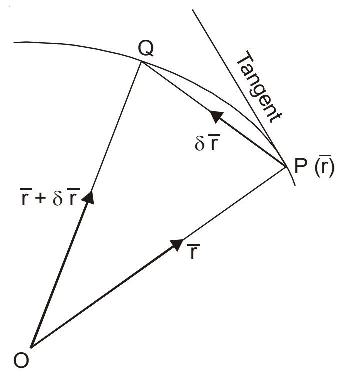
$$\vec{r} = \vec{r}(t)$$

If a particle is moving along a curved path then the position vector $\vec{r}$ of the particle is a function of t. If the component of $f(t)$ along x-axis, y-axis, z-axis are $f_1(t), f_2(t), f_3(t)$ respectively. Then,

$$\overrightarrow{f(t)} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

5.23 DIFFERENTIATION OF VECTORS

Let O be the origin and P be the position of a moving particle at time t.



Let $\quad \overrightarrow{OP} = \vec{r}$

Let Q be the position of the particle at the time $t + \delta t$ and the position vector of Q is $\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (\vec{r} + \delta\vec{r}) - \vec{r} = \delta\vec{r}$$

$\dfrac{\delta\vec{r}}{\delta t}$ is a vector. As $\delta t \to 0$, Q tends to P and the chord becomes the tangent at P.

We define $\quad \dfrac{d\vec{r}}{dt} = \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{r}}{\delta t}$, then

$\dfrac{d\vec{r}}{dt}$ is a vector in the direction of the *tangent* at P.

$\dfrac{d\vec{r}}{dt}$ is also called the differential coefficient of $\vec{r}$ with respect to 't'.

Similarly, $\dfrac{d^2\vec{r}}{dt^2}$ is the second order derivative of $\vec{r}$.

$\dfrac{d\vec{r}}{dt}$ gives the velocity of the particle at P, which is along the tangent to its path. Also $\dfrac{d^2\vec{r}}{dt^2}$ gives the *acceleration* of the particle at P.

5.24 FORMULAE OF DIFFERENTIATION

(*i*) $\dfrac{d}{dt}(\vec{F} + \vec{G}) = \dfrac{d\vec{F}}{dt} + \dfrac{d\vec{G}}{dt}$ $\qquad$ (*ii*) $\dfrac{d}{dt}(\vec{F}\phi) = \dfrac{d\vec{F}}{dt}\phi + \vec{F}\dfrac{d\phi}{dt}$ $\qquad$ (*U.P. I semester, Dec. 2005*)

(*iii*) $\dfrac{d}{dt}(\vec{F}.\vec{G}) = \vec{F}.\dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt}.\bar{G}$ $\qquad$ (*iv*) $\dfrac{d}{dt}(\vec{F} \times \vec{G}) = \vec{F} \times \dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt} \times \vec{G}$

(*v*) $\dfrac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\dfrac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \dfrac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \dfrac{d\vec{c}}{dt}\right]$

(*vi*) $\dfrac{d}{dt}[\vec{a} \times (\vec{b} \times \vec{c})] = \dfrac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\dfrac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \dfrac{d\vec{c}}{dt}\right)$

The order of the functions $\vec{F}, \vec{G}$ is not to be changed.

Example 12. *A particle moves along the curve* $\overline{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, *where t is the time. Find the magnitude of the tangential components of its acceleration at t = 2.*

(*Nagpur University, Summer 2005*)

Solution. We have, $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$

$$\text{Velocity} = \frac{d\vec{r}}{dt} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$$

At $t = 2$, Velocity $= 8\hat{i} + 8\hat{j} - 4\hat{k}$

$$\text{Acceleration} = \vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\,\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$$

At $t = 2$ $\quad \vec{a} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

The direction of velocity is along tangent.
So the tangent vector is velocity.

Unit tangent vector, $\hat{T} = \dfrac{\vec{v}}{|v|} = \dfrac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64 + 64 + 16}} = \dfrac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} = \dfrac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$

Tangential component of acceleration, $a_t = \overline{a}.\hat{T}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}).\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{24 + 4 + 20}{3} = \frac{48}{3} = 16 \textbf{ Ans.}$$

Example 13. *If* $\dfrac{d\vec{a}}{dt} = \vec{u} \times \vec{a}$ *and* $\dfrac{d\vec{b}}{dt} = \vec{u} \times \vec{b}$ *then prove that* $\dfrac{d}{dt}[\vec{a} \times \vec{b}] = \vec{u} \times (\vec{a} \times \vec{b})$

(*M.U. 2009*)

Solution. We have,

$$\frac{d}{dt}[\vec{a} \times \vec{b}] = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b} = \vec{a} \times (\vec{u} \times \vec{b}) + (\vec{u} \times \vec{a}) \times \vec{b}$$

$$= \vec{a} \times (\vec{u} \times \vec{b}) - \vec{b} \times (\vec{u} \times \vec{a})$$

$$= (\vec{a}\,.\vec{b})\,\vec{u} - (\vec{a}\,.\vec{u})\,\vec{b} - [(\vec{b}\,.\vec{a})\,\vec{u} - (\overline{b}.\vec{u})\,\vec{a}]$$

(Vector triple product)

$$= (\vec{a}\,.\vec{b})\,\vec{u} - (\vec{u}\,.\vec{a})\,\vec{b} - (\vec{a}\,.\vec{b})\,\vec{u} + (\vec{u}\,.\vec{b})\,\vec{a}$$

$$= (\vec{u}\,.\vec{b})\,\vec{a} - (\vec{u}\,.\vec{a})\,\vec{b}$$

$$= \vec{u} \times (\vec{a} \times \vec{b})$$

Proved.

Example 14. *Find the angle between the surface* $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at (2, –1, 2). (*M.D.U. Dec. 2009*)

Solution. Here, we have

$$x^2 + y^2 + z^2 = 9 \qquad ...(1)$$

$$z = x^2 + y^2 - 3 \qquad ...(2)$$

Normal to (1) $\eta_1 = \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal to (1) at (2, – 1, 2), $\eta_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(3)

Normal to (2), $\eta_2 = \nabla(z - x^2 - y^2 + 3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(z - x^2 - y^2 + 3) = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

Normal to (2) at (2, – 1, 2), $\eta_2 = -4\hat{i} + 2\hat{j} + \hat{k}$ ...(4)

$$\eta_1.\eta_2 = |\eta_1||\eta_2|\cos\theta$$

$$\cos\theta = \frac{\eta_1.\eta_2}{|\eta_1||\eta_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}).(-4\hat{i} + 2\hat{j} + \hat{k})}{|4\hat{i} - 2\hat{j} + 4\hat{k}||-4\hat{i} + 2\hat{j} + \hat{k}|} = \frac{-16 - 4 + 4}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{-16}{6\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$$

Hence the angle between (1) and (2) $\cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$ **Ans**

EXERCISE 5.6

1. The coordinates of a moving particle are given by $x = 4t - \frac{t^2}{2}$ and $y = 3 + 6t - \frac{t^3}{6}$. Find the velocity and acceleration of the particle when $t = 2$ secs. **Ans.** 4.47, 2.24

2. A particle moves along the curve

$$x = 2t^2, \quad y = t^2 - 4t \quad \text{and} \quad z = 3t - 5$$

where t is the time. Find the components of its velocity and acceleration at time $t = 1$, in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$. (*Nagpur, Summer 2001*) **Ans.** $\frac{8\sqrt{14}}{7}, -\frac{\sqrt{14}}{7}$

3. Find the unit tangent and unit normal vector at $t = 2$ on the curve $x = t^2 - 1$, $y = 4t - 3$, $z = 2t^2 - 6t$ where t is any variable. **Ans.** $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$, $\frac{1}{3\sqrt{5}}(2\hat{i} + 2\hat{k})$

4. Prove that $\frac{d}{dt}(\vec{F} \times \vec{G}) = \vec{F} \times \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \times \vec{G}$

5. Find the angle between the tangents to the curve $\bar{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$, at the points $t = \pm 1$.

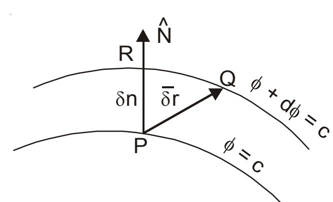
Ans. $\cos^{-1}\left(\frac{9}{17}\right)$

6. If the surface $5x^2 - 2byz = 9x$ be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2) then b is equal to

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3 (*AMIETE, Dec. 2009*) **Ans.** (*b*)

5.25 SCALAR AND VECTOR POINT FUNCTIONS

Point function. A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.



(*i*) **Scalar point function.** If to each point P (x, y, z) of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function. *For example*, the temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

(*ii*) **Vector point function.** If to each point P (x, y, z) of a region R in space there corresponds a unique vector $f(P)$, then f is called a *vector point function*. The velocity of a moving fluid, gravitational force are the examples of vector point function.

(*U.P., I Semester, Winter 2000*)

Vector Differential Operator Del *i.e.* ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

5.26 GRADIENT OF A SCALAR FUNCTION

If $\phi\,(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus,
$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\text{gard } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$\text{gard } \phi = \nabla\,\phi \qquad (\nabla \text{ is read del or nebla})$$

5.27 GEOMETRICAL MEANING OF GRADIENT, NORMAL

(*U.P. Ist Semester, Dec 2006*)

If a surface $\phi(x, y, z) = c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a *level surface* through P. *For example*, If $\phi(x, y, z)$ represents potential at the point P, then *equipotential surface* $\phi\,(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.

Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla\phi.d\bar{r} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi \qquad ...(1)$$

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes $\nabla\phi\,.\,dr = 0$. Then $\overline{\nabla\phi}$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

Let $\nabla\phi = |\nabla\phi|\,\widehat{N}$, where $\widehat{N}$ is a unit normal vector. Let δn be the perpendicular distance between two level surfaces through P and R. Then the rate of change of ϕ in the direction of the normal to the surface through P is $\frac{\partial \phi}{\partial n}$.

$$\frac{d\phi}{dn} = \lim_{\delta n \to 0}\frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0}\frac{\nabla\phi.d\vec{r}}{\delta n}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\phi|\,\hat{N}.d\vec{r}}{\delta n} \qquad \left\{\begin{aligned}\widehat{N}.\delta\vec{r} &= |\vec{N}|\,|\delta\vec{r}|\cos\theta\\ &= |\delta\vec{r}|\cos\theta = \delta n\end{aligned}\right\}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\phi|\,\delta n}{\delta n} = |\nabla\phi|$$

$$\therefore \qquad |\nabla\phi| = \frac{\partial\phi}{\partial n}$$

Hence, gradient ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

5.28 NORMAL AND DIRECTIONAL DERIVATIVE

(*i*) **Normal.** If $\phi(x, y, z) = c$ represents a family of surfaces for different values of the constant c. On differentiating ϕ, we get $d\phi = 0$

But $\qquad d\phi = \nabla\phi \,.\, d\vec{r} \quad$ so $\quad \nabla\phi \,.\, d\,r = 0$

The scalar product of two vectors $\nabla\phi$ and $d\vec{r}$ being zero, $\nabla\phi$ and $d\vec{r}$ are perpendicular to each other. $d\vec{r}$ is in the direction of tangent to the given surface.

Thus $\nabla\phi$ is a vector *normal* to the surface $\phi(x, y, z) = c$.

(*ii*) **Directional derivative.** The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi.\hat{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

$$\frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta\phi}{\delta r} \qquad \text{where, } \delta r = PQ$$

$\dfrac{\partial\phi}{\partial r}$ is called the *directional derivative* of ϕ at P in the direction of PQ.

Let a unit vector along PQ be $\hat{N}'$.

$$\frac{\delta n}{\delta r} = \cos\theta \;\Rightarrow\; \delta r = \frac{\delta n}{\cos\theta} = \frac{\delta n}{\hat{N}\cdot\hat{N}'} \qquad ...(1)$$

Now
$$\frac{\partial\phi}{\partial r} = \lim_{\delta r\to 0}\left[\frac{\delta\phi}{\dfrac{\delta n}{\hat{N}.\hat{N}'}}\right] = \hat{N}.\hat{N}'\frac{\partial\phi}{\partial n} \qquad \left[\text{From (1), } \delta r = \frac{\delta n}{\hat{N}.\hat{N}'}\right]$$

$$= \hat{N}'.\hat{N}\,|\nabla\phi| \quad = \hat{N}'.\nabla\phi \qquad (\because \hat{N}\,|\nabla\phi| = \nabla\phi)$$

Hence, $\dfrac{\partial\phi}{\partial r}$, directional derivative is the component of $\nabla\phi$ in the direction $\hat{N}'$.

$$\frac{\partial\phi}{\partial r} = \hat{N}'.\nabla\phi = |\nabla\phi|\cos\theta \le |\nabla\phi|$$

Hence, $\nabla\phi$ is the maximum rate of change of ϕ.

Example 15. *For the vector field* (*i*) $\vec{A} = m\hat{i}$ *and* (*ii*) $\vec{A} = m\,\vec{r}$. *Find* $\nabla.\vec{A}$ *and* $\nabla\times\vec{A}$. *Draw the sketch in each case.* (*Gujarat, I Semester, Jan. 2009*)

Solution. (*i*) Vector $\vec{A} = m\,\hat{i}$ is represented in the figure (*i*).

(*ii*) $\qquad \vec{A} = m\,\vec{r}$ is represented in the figure (*ii*).

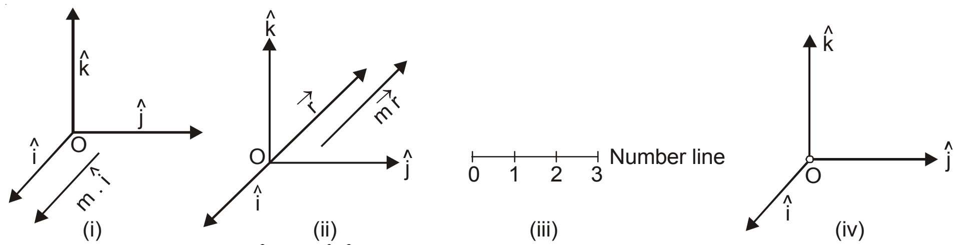
(*iii*)
$$\nabla.\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x\hat{i} + y\hat{j} + z\hat{k}) = 1 + 1 + 1 = 3$$

$\nabla.\vec{A} = 3$ is represented on the number line at 3.

(*iv*)
$$\nabla\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

are represented in the adjoining figure.



Example 16. *If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point $(1, -2, -1)$.*

(*AMIETE, June 2009*, *U.P., I Semester, Dec. 2006*)

Solution.

$$\text{grad } \phi = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\,(6xy) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = \hat{i}\,(6)\,(1)\,(-2) + \hat{j}\,[(3)\,(1) - 3(4)\,(1)] + \hat{k}\,(-2)(-8)\,(-1)$$

$$= -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k}$$ **Ans.**

Example 17. *If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$ prove that grad u, grad v and grad w are coplanar vectors.* [*U.P., I Semester, 2001*]

Solution. We have,

$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy) = \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x)$$

[For vectors to be coplanar, their scalar triple product is 0]

$$\text{Now, grad } u.(\text{grad } v \times \text{grad } w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= 2(x + y + z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Since the scalar product of grad u, grad v and grad w are zero, hence these vectors are coplanar vectors. **Proved.**

Example 18. *Find the directional derivative of $x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.*

(*Nagpur University, Summer 2005*)

Solution. Let $\phi = x^2\, y^2\, z^2$

Directional Derivative of ϕ

$$= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$\nabla\phi = 2xy^2z^2\,\hat{i} + 2yx^2z^2\,\hat{j} + 2zx^2y^2\,\hat{k}$$

Directional Derivative of ϕ at (1, 1, –1)

$$= 2(1)(1)^2(-1)^2\,\hat{i} + 2(1)(1)^2\,(-1)^2\,\hat{j} + 2(-1)(1)^2\,(1)^2\,\hat{k}$$

$$= 2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k} \qquad ...(1)$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = e^t\,\hat{i} + (\sin 2t + 1)\,\hat{j} + (1 - \cos t)\,\hat{k}$$

Tangent vector, $$\vec{T} = \frac{d\vec{r}}{dt} = e^t\,\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$$

$$\text{Tangent(at } t = 0) = e^0\,\hat{i} + 2\,(\cos 0)\,\hat{j} + (\sin 0)\,\hat{k} = \hat{i} + 2\,\hat{j} \qquad ...(2)$$

Required directional derivative along tangent $= (2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k})\,\dfrac{(\hat{i} + 2\,\hat{j})}{\sqrt{1+4}}$

[From (1), (2)]

$$= \frac{2+4+0}{\sqrt{5}} = \frac{6}{\sqrt{5}} \qquad \textbf{Ans.}$$

Example 19. *Find the unit normal to the surface $xy^3z^2 = 4$ at (–1, –1, 2).* (*M.U. 2008*)

Solution. Let $\phi(x, y, z) = xy^3z^2 = 4$

We know that $\nabla\phi$ is the vector normal to the surface $\phi\,(x, y, z) = c$.

$$\text{Normal vector} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Now $$= \hat{i}\frac{\partial}{\partial x}(xy^3z^2) + \hat{j}\frac{\partial}{\partial y}(xy^3z^2) + \hat{k}\frac{\partial}{\partial z}(xy^3z^2)$$

$\Rightarrow$ $$\text{Normal vector} = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

Normal vector at (–1, –1, 2) $= -4\,\hat{i} - 12\,\hat{j} + 4\,\hat{k}$

Unit vector normal to the surface at (–1, –1, 2).

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\,\hat{i} - 12\,\hat{j} + 4\,\hat{k}}{\sqrt{16+144+16}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\,\hat{j} - \hat{k}) \qquad \textbf{Ans.}$$

Example 20. *Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).* (*Nagpur University, Summer 2001*)

Solution. Rate of change of $\phi = \Delta\,\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\,y\,z) = \hat{i}\,yz + \hat{j}\,xz + \hat{k}\,xy$$

Rate of change of ϕ at (1, 1, 1) = $(\hat{i} + \hat{j} + \hat{k})$

Normal to the surface $\Psi = x^2y + y^2x + yz^2 - 3$ is given as -

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2xy + z^2) + \hat{k}\,2yz$$

$$(\nabla\psi)_{(1,\,1,\,1)} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\text{Unit normal} = \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{9 + 16 + 4}}$$

Required rate of change of $\phi = (\hat{i} + \hat{j} + \hat{k}).\dfrac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9 + 16 + 4}} = \dfrac{3 + 4 + 2}{\sqrt{29}} = \dfrac{9}{\sqrt{29}}$ **Ans.**

Example 21. *Find the constants m and n such that the surface $m\,x^2 - 2nyz = (m + 4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).*

(*M.D.U. Dec. 2009, Nagpur University, Summer 2002*)

Solution. The point P (1, –1, 2) lies on both surfaces. As this point lies in

$$mx^2 - 2nyz = (m + 4)x, \text{ so we have}$$

$$m - 2n(-2) = (m + 4)$$

$$\Rightarrow \quad m + 4n = m + 4 \quad \Rightarrow \quad n = 1$$

$$\therefore \quad \text{Let } \phi_1 = mx^2 - 2yz - (m + 4)x \text{ and } \phi_2 = 4x^2y + z^3 - 4$$

$$\text{Normal to } \phi_1 = \nabla\phi_1$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[mx^2 - 2yz - (m + 4)x]$$

$$= \hat{i}(2mx - m - 4) - 2z\,\hat{j} - 2y\,\hat{k}$$

Normal to ϕ_1 at (1, –1, 2) $= \hat{i}(2m - m - 4) - 4\hat{j} + 2\hat{k} = (m - 4)\hat{i} - 4\hat{j} + 2\hat{k}$

Normal to $\phi_2 = \nabla\phi_2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4) = \hat{i}\,8xy + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

Normal to ϕ_2 at (1, –1, 2) $= -8\hat{i} + 4\hat{j} + 12\hat{k}$

Sinec ϕ_1 and ϕ_2 are orthogonal, then normals are perpendicular to each other.

$$\nabla\phi_1 \,.\, \nabla\phi_2 = 0$$

$$\Rightarrow \quad [(m - 4)\hat{i} - 4\hat{j} + 2\hat{k}].[-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$\Rightarrow \quad -8(m - 4) - 16 + 24 = 0$$

$$\Rightarrow \quad m - 4 = -2 + 3 \quad \Rightarrow \quad m = 5$$

Hence $m = 5$, $n = 1$ **Ans.**

Example 22. *Find the values of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)\,x$, $4x^2y + z^3 = 4$ intersect orthogonally at the point (1, – 1, 2).*

(*AMIETE, II Sem., Dec. 2010, June 2009*)

Solution. Here, we have

$$\lambda x^2 - \mu yz = (\lambda + 2)\,x \qquad ...(1)$$

$$4x^2\,y + z^3 = 4 \qquad ...(2)$$

Normal to the surface (1), $= \nabla\left[\lambda x^2 - \mu yz - (\lambda+2)x\right]$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[\lambda x^2 - \mu yz - (\lambda+2)x\right]$$

$$= \hat{i}\,(2\lambda x - \lambda - 2) + \hat{j}\,(-\mu z) + \hat{k}\,(-\mu y)$$

Normal at (1, –1, 2) $= \hat{i}\ (2\lambda - \lambda - 2) - \hat{j}\ (-2\mu) + \hat{k}\,\mu$ ...(3)

$$= \hat{i}\ (\lambda - 2) + \hat{j}\,z\ (2\mu) + \hat{k}\,\mu$$

Normal at the surface (2)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2 y + z^3 - 4)$$

$$= \hat{i}\ (8 \times y) + \hat{j}\ (4x^2) + \hat{k}\ (3z^2)$$

Normal at the point (1, –1, 2) $= -8\hat{i} + 4\hat{j} + 12\hat{k}$ ...(4)

Since (3) and (4) are orthogonal so

$$\left[\hat{i}\,(\lambda-2) + \hat{j}\,(2\mu) + \hat{k}\,\mu\right].\left[-8\hat{i} + 4\hat{j} + 12\hat{k}\right] = 0$$

$$-8\,(\lambda-2) + 4(2\mu) + 12\mu = 0 \quad\Rightarrow\quad -8\lambda + 16 + 8\mu + 12\mu = 0$$

$$-8\lambda - 20\mu + 16 = 0 \quad\Rightarrow\quad 4(-2\lambda + 5\mu + 4) = 0$$

$$-2\lambda + 5\mu + 4 = 0 \quad\Rightarrow\quad 2\lambda - 5\mu = 4 \qquad ...(5)$$

Point (1, – 1, 2) will satisfy (1)

$\therefore \qquad \lambda(1)^2 - \mu(-1)(2) = (\lambda+2)\ (1) \Rightarrow \quad \lambda + 2\mu = \lambda + 2 \ \Rightarrow\ \mu = 1$

Putting $\mu = 1$ in (5), we get

$$2\lambda - 5 = 4 \quad\Rightarrow\quad \lambda = \frac{9}{2}$$

Hence $\quad \lambda = \dfrac{9}{2}$ and $\mu = 1$ **Ans.**

Example 23. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, –1, 2).* (*Nagpur University, Summer 2002*)

Solution. Normal on the surface $(x^2 + y^2 + z^2 - 9 = 0)$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = (2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k})$$

Normal at the point (2, –1, 2) $= 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(1)

Normal on the surface $(z = x^2 + y^2 - 3) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3)$

$$= 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

Normal at the point (2, –1, 2) $= 4\hat{i} - 2\hat{j} - \hat{k}$ ...(2)

Let θ be the angle between normals (1) and (2).

$$(4\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{16+4+16}\,\sqrt{16+4+1}\,\cos\theta$$

$$16 + 4 - 4 = 6\sqrt{21}\cos\theta \qquad\Rightarrow\qquad 16 = 6\sqrt{21}\cos\theta$$

$$\Rightarrow \qquad \cos\theta = \frac{8}{3\sqrt{21}} \qquad \Rightarrow \qquad \theta = \cos^{-1}\frac{8}{3\sqrt{21}} \qquad \textbf{Ans.}$$

Example 24. *Find the directional derivative of* $\frac{1}{r}$ *in the direction* $\overline{r}$ *where* $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
(*Nagpur University, Summer 2004, U.P., I Semester, Winter 2005, 2002*)

Solution. Here, $\phi(x, y, z) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

Now $\nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$$= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{i} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{j} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{k}$$

$$= \left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2x\right\}\hat{i} + \left(-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2y\right)\hat{j} + \left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2z\right\}\hat{k}$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} \qquad ...(1)$$

and $\hat{r}$ = unit vector in the direction of $x\hat{i} + y\hat{j} + z\hat{k}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \qquad ...(2)$$

So, the required directional derivative

$$= \nabla\phi.\hat{r} = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} \quad \text{[From (1), (2)]}$$

$$= \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2} \qquad \textbf{Ans.}$$

Example 25. *Find the direction in which the directional derivative of* $\phi(x, y) = \frac{x^2 + y^2}{xy}$ *at* (*1, 1*) *is zero and hence find out component of velocity of the vector* $\overline{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$ *in the same direction at t = 1.* (*Nagpur University, Winter 2000*)

Solution. Directional derivative $= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x^2 + y^2}{xy}\right)$

$$= \hat{i}\left[\frac{xy.2x - (x^2 + y^2)y}{x^2y^2}\right] + \hat{j}\left[\frac{xy.2y - x(y^2 + x^2)}{x^2y^2}\right]$$

$$= \hat{i}\left[\frac{x^2y - y^3}{x^2y^2}\right] + \hat{j}\left[\frac{xy^2 - x^3}{x^2y^2}\right]$$

Directional Derivative at (1, 1) $= \hat{i}\,0 + \hat{j}\,0 = 0$

Since $(\nabla\phi)_{(1, 1)} = 0$, the directional derivative of ϕ at (1, 1) is zero in any direction.

Again $\qquad \overline{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$

Velocity, $\quad \bar{v} = \frac{d\bar{r}}{dt} = 3t^2\,\hat{i} + 2t\,\hat{j}$

Velocity at $\quad t = 1$ is $= 3\hat{i} + 2\hat{j}$

The component of velocity in the same direction of velocity

$$= (3\hat{i} + 2\hat{j}).\left(\frac{3\hat{i} + 2\hat{j}}{\sqrt{9+4}}\right) = \frac{9+4}{\sqrt{13}} = \sqrt{13}$$ **Ans.**

Example 26. *Find the directional derivative of* $\phi(x, y, z) = x^2yz + 4xz^2$ *at* $(1, -2, 1)$ *in the direction of* $2\hat{i} - \hat{j} - 2\hat{k}$. *Find the greatest rate of increase of* ϕ.

(*Uttarakhand, I Semester, Dec. 2006*)

Solution. Here, $\quad \phi(x, y, z) = x^2yz + 4xz^2$

Now,
$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\,\hat{i} + (x^2z)\,\hat{j} + (x^2y + 8xz)\,\hat{k}$$

$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\,\hat{i} + (1\times 1)\,\hat{j} + \{1(-2) + 8(1)(1)\}\,\hat{k}$$

$$= (-4+4)\,\hat{i} + \hat{j} + (-2+8)\,\hat{k} = \hat{j} + 6\hat{k}$$

Let
$$\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

So, the required directional derivative at $(1, -2, 1)$

$$= \nabla\phi.\hat{a} = (\hat{j} + 6\hat{k}).\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left|\hat{j} + 6\hat{k}\right| = \sqrt{1+36}$

$$= \sqrt{37}$$ **Ans.**

Example 27. *Find the directional derivative of the function* $\phi = x^2 - y^2 + 2z^2$ *at the point P* $(1, 2, 3)$ *in the direction of the line PQ where Q is the point* $(5, 0, 4)$.

(*AMIETE, Dec. 20010, Nagpur University, Summer 2008, U.P., I Sem., Winter 2000*)

Solution. Directional derivative $= \bar{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\,\hat{i} - 2y\,\hat{j} + 4z\,\hat{k}$$

Directional Derivative at the point $P\ (1, 2, 3) = 2\hat{i} - 4\hat{j} + 12\hat{k}$ ...(1)

$$\overline{PQ} = \bar{Q} - \bar{P} = (5, 0, 4) - (1, 2, 3) = (4, -2, 1) \quad ...(2)$$

Directional Derivative along $PQ = (2\hat{i} - 4\hat{j} + 12\hat{k}).\frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$ [From (1) and (2)]

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$ **Ans.**

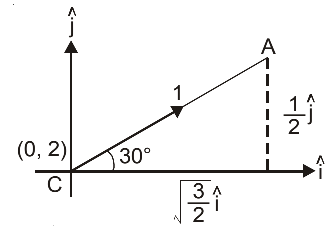
Example 28. *For the function* $\phi(x, y) = \frac{x}{x^2 + y^2}$, *find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at* $(0, 2)$.

(*A.M.I.E.T.E., Winter 2002*)

Solution. Directional derivative $= \vec{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{x}{x^2+y^2} \quad = \hat{i}\left(\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}\right) - \hat{j}\frac{x(2y)}{(x^2+y^2)^2}$$

$$= \hat{i}\frac{y^2-x^2}{(x^2+y^2)^2} - \hat{j}\frac{2xy}{(x^2+y^2)^2}$$



Directional derivative at the point (0, 2)

$$= \hat{i}\frac{4-0}{(0+4)^2} - \hat{j}\frac{2(0)(2)}{(0+4)^2} = \frac{\hat{i}}{4}$$

Directional derivative at the point (0, 2) in the direction $\overrightarrow{CA}$ *i.e.* $\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$$= \frac{\hat{i}}{4}.\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) \qquad \left\{\begin{aligned}\overrightarrow{CA} = \overrightarrow{OB} + \overrightarrow{BA} &= \hat{i}\cos 30° + \hat{j}\sin 30°\\ &= \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)\end{aligned}\right\}$$

$$= \frac{\sqrt{3}}{8}$$ **Ans.**

Example 29. *Find the directional derivative of* $\vec{V}^2$, *where* $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, *at the point (2, 0, 3) in the direction of the outward normal to the sphere* $x^2 + y^2 + z^2 = 14$ *at the point (3, 2, 1).* (*A.M.I.E.T.E., Dec. 2007*)

Solution. $V^2 = \vec{V}.\vec{V}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}).(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}) = x^2y^4 + z^2y^4 + x^2z^4$$

Directional derivative $= \vec{\nabla}V^2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2y^4z + 4x^2z^3)\hat{k}$$

Directional derivative at (2, 0, 3) $= (0 + 2\times 2\times 81)\hat{i} + (0+0)\hat{j} + (0 + 4\times 4\times 27)\hat{k}$

$$= 324\hat{i} + 432\hat{k} = 108(3\hat{i} + 4\hat{k}) \qquad ...(1)$$

Normal to $x^2 + y^2 + z^2 - 14 = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 14)$$

$$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Normal vector at (3, 2, 1) $= 6\hat{i} + 4\hat{j} + 2\hat{k}$ ...(2)

Unit normal vector $= \dfrac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \dfrac{2(3\hat{i} + 2\hat{j} + \hat{k})}{2\sqrt{14}} = \dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$ [From (1), (2)]

Directional derivative along the normal $= 108(3\hat{i} + 4\hat{k}).\dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$.

$$= \frac{108\times(9+4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$ **Ans.**

Example 30. *Find the directional derivative of $\nabla(\nabla f)$ at the point (1, – 2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.* (*U.P., I Semester, Dec 2008*)

Solution. Here, we have

$$f = 2x^3y^2z^4$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4) = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Directional derivative of $\nabla(\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz4 + 48x3yz2)\hat{j}$$

$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Directional derivative at (1, – 2, 1) = $(48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k}$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Normal to$(xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

Normal at(1, – 2, 1) = $\hat{i} - 4\hat{j} + 2\hat{k}$

Unit Normal Vector = $\dfrac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}} = \dfrac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$

Directional derivative in the direction of normal

$$= (348\hat{i} - 144\hat{j} + 400\hat{k})\frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$ **Ans.**

Example 31. *If the directional derivative of $\phi = a\,x^2y + b\,y^2z + c\,z^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$, find the values of a, b and c.* (*U.P. I Semester, June 2007, Winter 2001*)

Solution. Given $\phi = a\,x^2y + b\,y^2z + c\,z^2x$

$$\therefore \quad \bar{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(a\,x^2y + b\,y^2z + c\,z^2x)$$

$$= \hat{i}(2a\,xy + c\,z^2) + \hat{j}(a\,x^2 + 2\,b\,yz) + \hat{k}(b\,y^2 + 2\,c\,zx)$$

$\bar{\nabla}\phi$ at the point (1, 1, 1) $= \hat{i}(2a + c) + \hat{j}(a + 2b) + \hat{k}(b + 2c)$ ...(1)

We know that the maximum value of the directional derivative is in the direction of $\bar{\nabla}\phi$.

i.e. $|\nabla\phi| = 15 \Rightarrow (2a + c)^2 + (2b + a)^2 + (2c + b)^2 = (15)^2$

But, the directional derivative is given to be maximum parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \text{ i.e., parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}. \qquad ...(2)$$

On comparing the coefficients of (1) and (2)

$$\Rightarrow \quad \frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1}$$

$$\Rightarrow \quad 2a + c = -2b - a \quad \Rightarrow \quad 3a + 2b + c = 0 \qquad ...(3)$$

and $\quad 2b + a = -2(2c + b)$

$$\Rightarrow \quad 2b + a = -4c - 2b \quad \Rightarrow \quad a + 4b + 4c = 0 \qquad ...(4)$$

Rewriting (3) and (4), we have

$$\left.\begin{array}{l} 3a+2b+c=0 \\ a+4b+4c=0 \end{array}\right\} \quad \Rightarrow \quad \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \text{ (say)}$$

$$\Rightarrow \quad a = 4k, \quad b = -11k \quad \text{and} \quad c = 10k.$$

Now, we have

$$(2a + c)^2 + (2b + a)^2 + (2c + b)^2 = (15)^2$$

$$\Rightarrow \quad (8k + 10k)^2 + (-22k + 4k)^2 + (20k - 11k)^2 = (15)^2$$

$$\Rightarrow \quad k = \pm\frac{5}{9}$$

$$\Rightarrow \quad a = \pm\frac{20}{9}, \quad b = \pm\frac{55}{9} \quad \text{and} \quad c = \pm\frac{50}{9}$$ **Ans.**

Example 32. *If* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that :*

(i) grad $r = \frac{\vec{r}}{r}$ $\qquad$ *(ii) grad* $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$. $\qquad$ (*Nagpur University, Summer 2002*)

Solution. (*i*) $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\bar{r}}{r}$$ **Proved.**

$$(ii) \text{ grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{z}{r}\right) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = -\frac{\bar{r}}{r^3}$$ **Proved.**

Example 33. *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. $\qquad$ (*K. University, Dec. 2008*)

Solution.

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$\left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\, f'(r)\frac{\partial r}{\partial z} = f'(r)\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= f'(r)\frac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla\,[\nabla f(r)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[f'(r)\frac{xi + yj + zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= \left(f''(r)\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r1 - x\dfrac{\partial r}{\partial x}}{r^2} + \left(f''(r)\frac{\partial r}{\partial y}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y\dfrac{\partial r}{\partial y}}{r^2} +$$

$$\left(f''(r)\frac{\partial r}{\partial z}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r.1 - z\dfrac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r - \dfrac{x^2}{r}}{r^2} + \left(f''(r)\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r - \dfrac{y^2}{r}}{r^2} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r - \dfrac{z^2}{r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r^2 - x^2}{r^3} + \left(f''(r)\frac{y}{r}\right) + f'(r)\frac{r^2 - y^2}{r^3} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r^2 - z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2 + z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2 + z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2 + y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2 + z^2}{r^3} + \frac{z^2 + x^2}{r^3} + \frac{x^2 + y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2 + y^2 + x^2}{r^2} + f'(r)\frac{2(x^2 + y^2 + z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r}$$

Ans.

EXERCISE 5.7

1. Evaluate grad ϕ if $\phi = \log(x^2 + y^2 + z^2)$ **Ans.** $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$
2. Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 5$ at the point (0, 1, 2). **Ans.** $\dfrac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$
(*AMIETE, June 2010*)
3. Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point (1, –1, 1) in the direction of (3, 1, –1) (*A.M.I.E.T.E. Winter 2009, 2000*) **Ans.** $\dfrac{5}{\sqrt{11}}$
4. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero.
(*Nagpur Winter 2000*) **Ans.** $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

5. Find the directional derivative of the scalar function of $(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3). **Ans.** $\frac{27}{\sqrt{11}}$

6. The temperature of the points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move? **Ans.** $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

7. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find grad ϕ at the point $(1, -2, -1)$ **Ans.** $-(16\hat{i} + 9\hat{j} + 4\hat{k})$

8. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Ans. $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

9. What is the greatest rate of increase of the function $u = xyz^2$ at the point (1, 0, 3)? **Ans.** 9

10. If θ is the acute angle between the surfaces $xyz^2 = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$ show that $\cos\theta = 3/7\sqrt{6}$.

11. Find the values of constants a, b, c so that the maximum value of the directional directive of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the axis of z. **Ans.** $a = b$, $b = 24$, $c = -8$

12. Find the values of λ and μ so that surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$. **Ans.** $\lambda = \frac{9}{2}, \mu = 1$

13. The position vector of a particle at time t is $R = \cos(t - 1)\, i + \sinh(t - 1)\, j + at^2k$. If at $t = 1$, the acceleration of the particle be perpendicular to its position vector, then a is equal to

(*a*) 0 (*b*) 1 (*c*) $\frac{1}{2}$ (*d*) $\frac{1}{\sqrt{2}}$ (*AMIETE, Dec. 2009*) **Ans.** (*d*)

5.29 DIVERGENCE OF A VECTOR FUNCTION

The divergence of a vector point function $\vec{F}$ is denoted by *div F* and is defined as below.

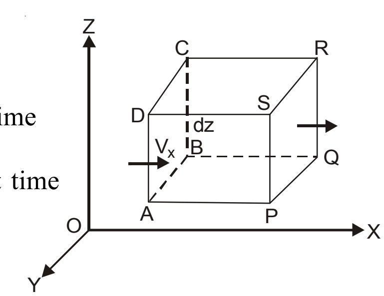
Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\text{div}\vec{F} = \vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

5.30 PHYSICAL INTERPRETATION OF DIVERGENCE

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to x,y and z axes respectively.



Let $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be the velocity of the fluid at $P(x, y, z)$.

$\therefore$ Mass of fluid flowing in through the face *ABCD* in unit time

$= \text{Velocity} \times \text{Area of the face} = V_x\,(dy\, dz)$

Mass of fluid flowing out across the face *PQRS* per unit time

$= V_x\,(x + dx)\,(dy\, dz)$

$= \left(V_x + \frac{\partial V_x}{\partial x}dx\right)(dy\, dz)$

Net decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time

$$= V_x \, dy \, dz - \left(V_x + \frac{\partial V_x}{\partial x} dx\right) dy \, dz$$

$$= -\frac{\partial V_x}{\partial x} dx \, dy \, dz \qquad \text{(Minus sign shows decrease)}$$

Similarly, the decrease in mass of fluid to the flow along y-axis $= \dfrac{\partial V_y}{\partial y} dx\, dy\, dz$

and the decrease in mass of fluid to the flow along z-axis $= \dfrac{\partial V_z}{\partial z} dx\, dy\, dz$

Total decrease of the amount of fluid per unit time $= \left(\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}\right) dx\, dy\, dz$

Thus the rate of loss of fluid per unit volume $= \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{i} V_x + \hat{j} V_y + \hat{k} V_z) = \bar{\nabla}.\bar{V} = \text{div}\, \bar{V}$$

If the fluid is compressible, there can be no gain or loss in the volume element. Hence

$$\text{div}\, \vec{V} = 0 \qquad \text{...(1)}$$

and V is called a *Solenoidal* vector function.

Equation (1) is also called the *equation of continuity or conservation of mass*.

Example 34. *If* $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of div* $\bar{v}$.

(*U.P., I Semester, Winter 2000*)

Solution. We have, $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\text{div}\, \vec{v} = \vec{\nabla}.\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{\left[(x^2 + y^2 + z^2)^{1/2} - x.\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x\right]}{(x^2 + y^2 + z^2)}$$

$$+ \frac{\left[(x^2 + y^2 + z^2)^{\frac{1}{2}} - y.\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}} \times 2y\right]}{(x^2 + y^2 + z^2)} + \frac{\left[(x^2 + y^2 + z^2)^{1/2} - z.\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.2z\right]}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2) - x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - y^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{\sqrt{(x^2 + y^2 + z^2)}} \quad \textbf{Ans.}$$

Example 35. *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.*

(*A.M.I.E.T.E., Summer 2004*)

Solution. div $(u\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})]$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)x\hat{i}+(x^2+y^2+z^2)y\hat{j}+(x^2+y^2+z^2)z\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^3+xy^2+xz^2)+\frac{\partial}{\partial y}(x^2y+y^3+yz^2)+\frac{\partial}{\partial z}(x^2z+y^2z+z^3)$$

$= (3x^2+y^2+z^2)+(x^2+3y^2+z^2)+(x^2+y^2+3z^2) = 5(x^2+y^2+z^2) = 5u$ **Ans.**

Example 36. *Find the value of n for which the vector* $r^n\vec{r}$ *is solenoidal, where* $\overline{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Solution. Divergence $\vec{F} = \vec{\nabla}.\vec{F} = \vec{\nabla}.r^n\vec{r} = \nabla.(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})$

$$= \left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right].[(x^2+y^2+z^2)^{n/2}x\hat{i}+(x^2+y^2+z^2)^{n/2}y\hat{j}+(x^2+y^2+z^2)^{n/2}z\hat{k}]$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x^2)+(x^2+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2y^2)$$

$$+(x^2+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2z^2)+(x^2+y^2+z^2)^{n/2}$$

$= n(x^2+y^2+z^2)^{n/2-1}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{n/2}$
$= n(x^2+y^2+z^2)^{n/2}+3(x^2+y^2+z^2)^{n/2} = (n+3)(x^2+y^2+z^2)^{n/2}$

If $r^n\vec{r}$ is solenoidal, then $(n+3)(x^2+y^2+z^2)^{n/2} = 0$ or $n+3 = 0$ or $n = -3$. **Ans.**

Example 37. *Show that* $\nabla\left[\frac{(\vec{a}.\vec{r})}{r^n}\right] = \frac{\vec{a}}{r^n}-\frac{n(\vec{a}.\vec{r})\vec{r}}{r^{n+2}}$. (*M.U. 2005*)

Solution. We have, $\frac{\vec{a}.\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k})}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

Let $\phi = \frac{\vec{a}.\vec{r}}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

$\therefore$ $\frac{\partial\phi}{\partial x} = \frac{r^n.a_1-(a_1x+a_2y+a_3z)\,n\,r^{n-1}(\partial r/\partial x)}{r^{2n}}$

But $r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore$ $\frac{\partial\phi}{\partial x} = \frac{a_1r^n-(a_1x+a_2y+a_3z).n\,r^{n-2}.x}{r^{2n}} = \frac{a_1}{r^n}-\frac{n(a_1x+a_2y+a_3z)x}{r^{n+2}}$

$\therefore$ $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

$$= \frac{1}{r^n}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})-\frac{n}{r^{n+2}}[(a_1x+a_2y+a_3z)(x\hat{i}+y\hat{j}+z\hat{k})]$$

$$= \frac{\overline{a}}{r^n}-\frac{n}{r^{n+2}}(\overline{a}.\overline{r})\overline{r}$$

Example 38. *Let* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$ *and* $\vec{a}$ *is a constant vector. Find the value of*

$$div\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$$

Solution. Let
$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{a}\times\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z - a_3y)\hat{i} - (a_1z - a_3x)\hat{j} + (a_1y - a_2x)\hat{k}$$
$$\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n} = \frac{(a_2z - a_3y)\hat{i} - (a_1z - a_3x)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$
$$\text{div}\left(\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}\right) = \vec{\nabla}.\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\frac{(a_2z - a_3y)\hat{i} - (a_1z - a_3x)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$
$$= \frac{\partial}{\partial x}\frac{a_2z - a_3y}{(x^2+y^2+z^2)^{n/2}} - \frac{\partial}{\partial y}\frac{a_1z - a_3x}{(x^2+y^2+z^2)^{n/2}} + \frac{\partial}{\partial z}\frac{(a_1y - a_2x)}{(x^2+y^2+z^2)^{n/2}}$$
$$= -\frac{n}{2}\frac{(a_2z - a_3y)\,2x}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} + \frac{n}{2}\frac{(a_1z - a_3x)\,2y}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} - \frac{n}{2}\frac{(a_1y - a_2x)\,2z}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}$$
$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}[(a_2z - a_3y)x - (a_1z - a_3x)y + (a_1y - a_2x)z]$$
$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}[a_2zx - a_3xy - a_1yz + a_3xy + a_1yz - a_2zx] = 0 \qquad \textbf{Ans.}$$

Example 39. *Find the directional derivative of div* $(\vec{u})$ *at the point (1, 2, 2) in the direction of the outer normal of the sphere* $x^2 + y^2 + z^2 = 9$ *for* $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.

Solution. $\text{div}\,(\vec{u}) = \nabla.\vec{u}$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}) = 4x^3 + 4y^3 + 4z^3$$

Outer normal of the sphere $= \nabla(x^2 + y^2 + z^2 - 9)$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Outer normal of the sphere at (1, 2, 2) $= 2\hat{i} + 4\hat{j} + 4\hat{k}$ ...(1)

Directional derivative $= \vec{\nabla}\,(4x^3 + 4y^3 + 4z^3)$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^3 + 4y^3 + 4z^3) = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$$

Directional derivative at (1, 2, 2) $= 12\hat{i} + 48\hat{j} + 48\hat{k}$ ...(2)

Directional derivative along the outer normal $= (12\hat{i} + 48\hat{j} + 48\hat{k}).\dfrac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}}$ [From (1), (2)]

$$= \frac{24+192+192}{6} = 68$$ **Ans.**

Example 40. *Show that div (grad r^n) = $n(n+1)r^{n-2}$, where*

$$r = \sqrt{x^2+y^2+z^2}$$

Hence, show that $\Delta^2\left(\dfrac{1}{r}\right) = 0.$ (*U.P. I Semester, Dec. 2004, Winter 2002*)

Solution. $\text{grad}\,(r^n) = \hat{i}\dfrac{\partial}{\partial x}r^n + \hat{j}\dfrac{\partial}{\partial y}r^n + \hat{k}\dfrac{\partial}{\partial z}r^n$ by definition

$$= \hat{i}\,n\,r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,n\,r^{n-1}\frac{\partial r}{\partial y} + \hat{k}.n\,r^{n-1}\frac{\partial r}{\partial z} = n\,r^{n-1}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

$$= n\,r^{n-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right] = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\overline{r}.$$

$$\left[\because r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}\right]$$

Thus, $\text{grad}\,(r^n) = n\,r^{n-2}\,x\hat{i} + n\,r^{n-2}\,y\hat{j} + n\,r^{n-2}\,z\hat{k}$ ...(1)

$\therefore$ $\text{div grad}\,r^n = \text{div}\,[n\,r^{n-2}\,x\hat{i} + n\,r^{n-2}\,y\hat{j} + n\,r^{n-2}\,z\hat{k}]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(nr^{n-2}\,x\hat{i} + nr^{n-2}\,y\hat{j} + nr^{n-2}\,z\hat{k})$$ [From (1)]

$$= \frac{\partial}{\partial x}(n\,r^{n-2}x) + \frac{\partial}{\partial y}(n\,r^{n-2}y) + \frac{\partial}{\partial z}(n\,r^{n-2}z)$$ (By definition)

$$= \left(n\,r^{n-2} + nx\,(n-2)\,r^{n-3}\frac{\partial r}{\partial x}\right) + \left(n\,r^{n-2} + ny\,(n-2)\,r^{n-3}\frac{\partial r}{\partial y}\right) + \left(n\,r^{n-2} + nz\,(n-2)\,r^{n-3}\frac{\partial r}{\partial z}\right)$$

$$= 3n\,r^{n-2} + n(n-2)r^{n-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3n\,r^{n-2} + n(n-2)r^{n-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$$\left[\because r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}\right]$$

$$= 3nr^{n-2} + n\,(n-2)r^{n-4}\,[x^2+y^2+z^2]$$

$$= 3nr^{n-2} + n\,(n-2)\,r^{n-4}.r^2 \qquad (\because\ r^2 = x^2+y^2+z^2)$$

$$= r^{n-2}\,[3n + n^2 - 2n] = r^{n-2}\,(n^2+n) = n(n+1)\,r^{n-2}$$

If we put $n = -1$

div grad $(r^{-1}) = -1\,(-1+1)\,r^{-1-2}$

$\Rightarrow$ $\nabla^2\left(\dfrac{1}{r}\right) = 0$

Ques. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$ and $r = |r|$ find div $\left(\dfrac{\boldsymbol{r}}{r^2}\right)\cdot$ (*U.P. I Sem., Dec. 2006*) **Ans.** $\dfrac{1}{r^2}$

EXERCISE 5.8

1. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\dfrac{\vec{r}}{|\bar{r}|^3}\right) = 0$,

(*ii*) div (grad r^n) = $n\ (n + 1)\ r^{n-2}$ (*AMIETE, June 2010*) (*iii*) div ($r\ \phi$) = $3\phi + r$ grad ϕ.

2. Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

(*R.G.P.V., Bhopal, Dec. 2003*)

3. Show that $\nabla.(\phi\ A) = \nabla\phi.A + \phi(\nabla.A)$

4. If ρ, ϕ, z are cylindrical coordinates, show that grad (log ρ) and grad ϕ are solenoidal vectors.

5. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their corresponding expression in orthogonal curvilinear coordinates.

Prove the following:

6. $\vec{\nabla}.(\phi\vec{F}) = (\vec{\nabla}\phi).\vec{F} + \phi(\vec{\nabla}.\vec{F})$

7. (*a*) $\nabla.(\nabla\phi) = \nabla^2\phi$ (*b*) $\vec{\nabla} \times \dfrac{(\vec{A} \times \vec{R})}{r^n} = \dfrac{(2-n)\vec{A}}{r^n} + \dfrac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$

8. div ($f\ \nabla\ g$) – div ($g\ \nabla\ f$) = $f\ \nabla^2 g - g\ \nabla^2 f$

5.31 CURL (*U.P., I semester, Dec. 2006)*

The curl of a vector point function F is defined as below

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} \qquad (\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Curl $\vec{F}$ is a vector quantity.

5.32 PHYSICAL MEANING OF CURL

(*M.D.U., Dec. 2009, U.P. I Semester, Winter 2009, 2000*)

We know that $\vec{V} = \vec{\omega} \times \vec{r}$, where ω is the angular velocity, $\vec{V}$ *is the linear velocity and* $\vec{r}$ *is the position vector of a point on the rotating body.*

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V} \qquad \begin{bmatrix} \vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} \\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \end{bmatrix}$$

$$= \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = \vec{\nabla} \times [(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})]$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \vec{\nabla} \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_2)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k} \quad = 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) = 2\omega$$

Curl $\vec{V} = 2\omega$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name *rotation* used for curl.

If Curl $\overline{F} = 0$, the field F is termed as *irrotational*.

Example 41. *Find the divergence and curl of* $\vec{v} = (x\,y\,z)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$ *at* (*2, –1, 1*) (*Nagpur University, Summer 2003*)

Solution. Here, we have

$$\vec{v} = (x\,y\,z)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$$

$$\text{Div. } \vec{v} = \nabla\phi$$

$$\text{Div } \vec{v} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 3x^2 + 2x\,z - y^2 \qquad = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{Curl } \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & 3x^2 y & xz^2 - y^2 z \end{vmatrix} \qquad = -2yz\,\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

Curl at (2, –1, 1)

$$= -2(-1)(1)\,\hat{i} + \{(2)\,(-1) - 1\}\,\hat{j} + \{6(2)(-1) - 2(1)\}\,\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k}$$ **Ans.**

Example 42. *If* $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of curl* $\vec{V}$.

(*U.P., I Semester, Winter 2000*)

Solution. $$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2 + y^2 + z^2)^{1/2}} & \dfrac{y}{(x^2 + y^2 + z^2)^{1/2}} & \dfrac{z}{(x^2 + y^2 + z^2)^{1/2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2+z^2)^{1/2}}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)\right.$$

$$\left. - \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2+z^2)^{1/2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right)\right]$$

$$= \hat{i}\left[\frac{-yz}{(x^2+y^2+z^2)^{3/2}} + \frac{y.z}{(x^2+y^2+z^2)^{3/2}}\right] - \hat{j}\left[\frac{-zx}{(x^2+y^2+z^2)^{3/2}} + \frac{zx}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \hat{k}\left[\frac{-xy}{(x^2+y^2+z^2)^{3/2}} + \frac{xy}{(x^2+y^2+z^2)^{3/2}}\right] = 0 \quad \textbf{Ans.}$$

Example 43. *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.* (*U.P., I Sem, Dec. 2008*)

Solution. Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0$.

Now, $\vec{\nabla}.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$

$= -2 + 2x - 2x + 2 = 0$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\overline{F} = 0$.

Now, Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$

$= (3z + 2y - 2y + 3z)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)\hat{k}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$

Thus, $\vec{F}$ is irrotational.

Hence, $\vec{F}$ is both solenoidal and irrotational. **Proved.**

Example 44. *Determine the constants a and b such that the curl of vector* $\overline{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ *is zero.* (*U.P. I Semester, Dec 2008*)

Solution. Curl $A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & -3xy - byz \end{vmatrix}$$

$$= [-3x - bz - ax + 8z]\hat{i} - [-3y - 3y]\hat{j} + [2x + az - 2x - 3z]\hat{k}$$

$$= [-x(3+a) + z(8-b)]\hat{i} + 6y\hat{j} + z(-3+a)\hat{k}$$

$$= 0 \qquad \text{(given)}$$

i.e., $3 + a = 0$ and $8 - b = 0$, $\quad -3 + a = 0$

$a = -3, 3 \quad b = 8 \quad \Rightarrow \quad a = 3$ **Ans.**

Example 45. *If a vector field is given by*

$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. *Is this field irrotational ? If so, find its scalar potential.*

(*U.P. I Semester, Dec 2009*)

Solution. Here, we have

$$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

$$\text{Curl } F = \nabla \times \vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + x & -2xy - y & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y + 2y) = 0$$

Hence, vector field $\vec{F}$ is irrotational.

To find the scalar potential function ϕ

$$\vec{F} = \nabla \phi$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \left|\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right| \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \cdot (\vec{d}\,\vec{r}) = \nabla\phi \cdot \vec{d}\,\vec{r} = \vec{F} \cdot \vec{d}\,\vec{r}$$

$$= [(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (x^2 - y^2 + x)dx - (2xy + y)dy.$$

$$\phi = \int [(x^2 - y^2 + x)dx - (2xy + y)dy] + c$$

$$= \int \left[x^2\,dx + x\,dx - y\,dy - y^2\,dx - 2\,xy\,dy\right] + c = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$$

Hence, the scalar potential is $\frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$ **Ans.**

Example 46. *Find the scalar potential function f for* $\vec{A} = y^2\,\hat{i} + 2xy\,\hat{j} - z^2\,\hat{k}$.

(*Gujarat, I Semester, Jan. 2009*)

Solution. We have, $\quad \vec{A} = y^2\,\hat{i} + 2xy\,\hat{j} - z^2\,\hat{k}$

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (y^2\,\hat{i} + 2xy\,\hat{j} - z^2\,\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(2y - 2y) = 0$$

Hence, $\vec{A}$ is irrotational. To find the scalar potential function f.

$$\vec{A} = \nabla f$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad = \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f.dr = \nabla f.d\vec{r}$$

$$= \vec{A}.dr \qquad (A = \nabla f)$$

$$= (y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= y^2\,dx + 2xy\,dy - z^2\,dz \quad = d\,(xy^2) - z^2\,dz$$

$$f = \int d\,(xy^2) - \int z^2\,dz \quad = xy^2 - \frac{z^3}{3} + C \qquad \textbf{Ans.}$$

Example 47. *A vector field is given by* $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. *Show that the field is irrotational and find the scalar potential.*(*Nagpur Univeristy, Summer 2003, Winter 2002*)

Solution. $\vec{A}$ is irrotational if curl $\vec{A} = 0$

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy) = 0$$

Hence, $\vec{A}$ is irrotational. If ϕ is the scalar potential, then

$$\vec{A} = \text{grad } \phi$$

$$d\,\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \text{grad } \phi \,.\, dr$$

$$= \vec{A}.dr = [(x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}].(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (x^2 + xy^2)\,dx + (y^2 + x^2y)\,dy = x^2\,dx + y^2\,dy + (x\,dx)y^2 + (x^2)\,(y\,dy)$$

$$\phi = \int x^2 dx + \int y^2 dy + \int [(x\,dx)\,y^2 + (x^2)\,(y\,dy)] = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c \qquad \textbf{Ans.}$$

Example 48. *Show that* $\vec{V}(x, y, z) = 2x\,y\,z\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$ *is irrotational and find a scalar function u(x, y, z) such that* $\vec{V}$ *= grad (u).*

Solution. $\vec{V}\,(x, y, z) = 2x\,y\,z\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$

$$\text{Curl } \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [2xyz\,\hat{i} + (x^2 z + 2y)\,\hat{j} + x^2 y\,\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2 z + 2y & x^2 y \end{vmatrix}$$

$$= (x^2 - x^2)\,\hat{i} - (2xy - 2xy)\,\hat{j} + (2xz - 2xz)\,\hat{k} = 0$$

Hence, $\vec{V}\,(x, y, z)$ is irrotational.

To find corresponding scalar function u, consider the following relations given

$$\vec{V} = \text{grad } (u)$$

or $$\vec{V} = \vec{\nabla}(u) \qquad ...(1)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad \text{(Total differential coefficient)}$$

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla} u . d\vec{r} = \vec{V} . d\vec{r} \qquad \text{[From (1)]}$$

$$= [2xyz\,\hat{i} + (x^2 z + 2y)\,\hat{j} + x^2 y\,\hat{k}].(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= 2\,x\,y\,z\,dx + (x^2 z + 2y)\,dy + x^2 y\,dz$$

$$= y(2x\,z\,dx + x^2\,dz) + (x^2 z)\,dy + 2y\,dy$$

$$= [yd\,(x^2 z) + (x^2 z)\,dy] + 2y\,dy = d(x^2 yz) + 2y\,dy$$

Integrating, we get $u = x^2 yz + y^2$ **Ans.**

Example 49. *A fluid motion is given by* $\vec{v} = (y + z)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}$. *Show that the motion is irrotational and hence find the velocity potential.*

(*Uttarakhand, I Semester 2006; U.P., I Semester, Winter 2003*)

Solution. $$\text{Curl } \vec{v} = \nabla \times \vec{v}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(y + z)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix} = (1 - 1)\,\hat{i} - (1 - 1)\,\hat{j} + (1 - 1)\,\hat{k} = 0$$

Hence, $\vec{v}$ is irrotational.

To find the corresponding velocity potential ϕ, consider the following relation.

$$\bar{v} = \nabla\phi$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \qquad \text{[Total Differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi.d\vec{r} = \nabla\phi.d\vec{r} = \vec{v}.d\vec{r}$$

$$= [(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}].(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$$

$$= (y+z)\,dx+(z+x)\,dy+(x+y)\,dz$$

$$= y\,dx+z\,dx+z\,dy+x\,dy+x\,dz+y\,dz$$

$$\phi = \int(y\,dx+x\,dy)+\int(z\,dy+y\,dz)+\int(z\,dx+x\,dz)$$

$$\phi = xy+yz+zx+c$$

Velocity potential $= xy+yz+zx+c$ **Ans.**

Example 50. *A fluid motion is given by*

$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

is the motion irrotational? If so, find the velocity potential.

Solution. Curl $\vec{v} = \vec{\nabla}\times\vec{v}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(y\sin z-\sin x)\hat{i}+(x\sin z+2yz)\hat{j}+(xy\cos z+y^2)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z-\sin x & x\sin z+2yz & xy\cos z+y^2 \end{vmatrix}$$

$$= (x\cos z+2y-x\cos z-2y)\hat{i} - [y\cos z - y\cos z]\hat{j} + (\sin z-\sin z)\hat{k} = 0$$

Hence, the motion is irrotational.

So, $\bar{v} = \bar{\nabla}\phi$ where ϕ is called velocity potential.

$$d\phi = \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz) = \vec{\nabla}\phi.d\vec{r} = \vec{v}.d\vec{r}$$

$$= [(y\sin z-\sin x)\hat{i}+(x\sin z+2yz)\hat{j}+(xy\cos z+y^2)\hat{k}].[\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz]$$

$$= (y\sin z-\sin x)\,dx+(x\sin z+2yz)\,dy+(xy\cos z+y^2)\,dz$$

$$= (y\sin z\,dx+x\,dy\sin z+xy\cos z\,dz)-\sin x\,dx+(2yz\,dy+y^2\,dz)$$

$$= d(xy\sin z)+d(\cos x)+d(y^2z)$$

$$\phi = \int d(xy\sin z)+\int d(\cos x)+\int d(y^2z)$$

$$\phi = xy\sin z+\cos x+y^2z+c$$

Hence, Velocity potential $= xy\sin z+\cos x+y^2z+c$. **Ans.**

Example 51. *Prove that* $\vec{F} = r^2\vec{r}$ *is conservative and find the scalar potential* ϕ *such that*

$$\vec{F} = \nabla\phi. \qquad \text{(\textit{Nagpur University, Summer 2004})}$$

Solution. Given $\vec{F} = r^2\vec{r} = r^2(x\hat{i}+y\hat{j}+z\hat{k}) = r^2x\hat{i}+r^2y\hat{j}+r^2z\hat{k}$

Consider $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^2x & r^2y & r^2z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}r^2 z - \frac{\partial}{\partial z}r^2 y\right] - \hat{j}\left[\frac{\partial}{\partial x}r^2 z - \frac{\partial}{\partial z}r^2 x\right] + \hat{k}\left[\frac{\partial}{\partial x}r^2 y - \frac{\partial}{\partial y}r^2 x\right]$$

$$= \hat{i}\left[2rz\frac{\partial r}{\partial y} - 2ry\frac{\partial r}{\partial z}\right] - \hat{j}\left[2rz\frac{\partial r}{\partial x} - 2rx\frac{\partial r}{\partial z}\right] + \hat{k}\left[2ry\frac{\partial r}{\partial x} - 2rx\frac{\partial r}{\partial y}\right]$$

$$\left[\text{But } r^2 = x^2 + y^2 + z^2, \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= \hat{i}\left[2rz\frac{y}{r} - 2ry\frac{z}{r}\right] - \hat{j}\left[2rz\frac{x}{r} - 2rx\frac{z}{r}\right] + \hat{k}\left[2ry\frac{x}{r} - 2rx\frac{y}{r}\right]$$

$$= \hat{i}(2yz - 2yz) - \hat{j}(2zx - 2zx) + \hat{k}(2xy - 2xy) = 0\hat{i} - 0\hat{j} + 0\hat{k} = 0$$

$\therefore \qquad \nabla \times \vec{F} = 0$

$\therefore \qquad \vec{F}$ is irrotational $\qquad \therefore \qquad F$ is conservative.

Consider scalar potential ϕ such that $\vec{F} = \nabla\phi$.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \nabla\phi.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{F}.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \qquad = r^2\vec{r}.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \qquad (\nabla\phi = \vec{F})$$

$$= (x^2 + y^2 + z^2)(\hat{i}x + \hat{j}y + \hat{k}z).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (x^2 + y^2 + z^2)(x\,dx + y\,dy + z\,dz)$$

$$= x^3\,dx + y^3\,dy + z^3\,dz + (x\,dx)\,y^2 + (x^2)\,(y\,dy) + (x\,dx)z^2 + z^2\,(y\,dy) + x^2\,(z\,dz) + y^2\,(z\,dz)$$

$$\phi = \int x^3\,dx + \int y^3\,dy + \int z^3\,dz + \int[(x\,dx)y^2 + (y\,dy)x^2] + \int[(x\,dx)z^2 + (z\,dz)x^2] + \int[(y\,dy)z^2 + (z\,dz)y^2]$$

$$= \frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{2}y^2z^2 + c$$

$$= \frac{1}{4}(x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2) + c \qquad \textbf{Ans.}$$

Example 52. *Show that the vector field* $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ *is irrotational as well as solenoidal. Find the scalar potential.*

(*Nagpur University, Summer 2008, 2001, U.P. I Semester Dec. 2005, 2001*)

Solution. $\qquad F = \dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{-3}{2}\frac{2yz}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{2}\frac{2yz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$-\hat{j}\left[\frac{-3}{2}\frac{2xz}{(x^2+y^2+z^2)^{5/2}} - \left(-\frac{3}{2}\right)\frac{2xz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$+\hat{k}\left[-\frac{3}{2}\frac{2xy}{(x^2+y^2+z^2)^{5/2}} - \left(-\frac{3}{2}\right)\frac{2xy}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= 0$$

Hence, $\vec{F}$ is irrotational.

$\Rightarrow$ $\vec{F} = \vec{\nabla}\phi$, where ϕ is called scalar potential

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \quad = \vec{\nabla}\phi.d\vec{r} = \vec{F}.d\vec{r}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \quad = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\phi = \frac{1}{2}\int\frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{1}{2}\left(-\frac{2}{1}\right)(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = -\frac{1}{|\vec{r}\,|} \qquad \textbf{Ans.}$$

Now, $\quad \text{Div}\,\vec{F} = \vec{\nabla}.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{(x^2+y^2+z^2)^{3/2}\,(1) - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}\,(2x)}{(x^2+y^2+z^2)^3}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2}\,(1) - y\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}\,(2y)}{(x^2+y^2+z^2)^3}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2}\,(1) - z\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}\,(2z)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\ [x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2]$$

$$= 0$$

Hence, $\vec{F}$ is solenoidal. **Proved.**

Example 53. *Given the vector field* $\vec{V} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$ *find curl V. Show that the vectors given by curl V at P_0 (1, 2, –3) and P_1 (2, 3, 12) are orthogonal.*

Solution. $\overline{\text{Curl}\,\vec{V}} = \vec{\nabla} \times \vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}]$$

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= -(x+y)\hat{i} - (2x - 2x)\hat{j} + (z - y + 2y)\hat{k} = -(x+y)\hat{i} + (y+z)\hat{k}$$

$$\text{curl } \vec{V} \text{ at } P_0\ (1, 2, -3) = -(1+2)\hat{i} + (2-3)\hat{k} = -3\hat{i} - \hat{k}$$

$$\text{curl } \vec{V} \text{ at } P_1\ (2, 3, 12) = -(2+3)\hat{i} + (3+12)\hat{k} = -5\hat{i} + 15\hat{k}$$

The curl $\vec{V}$ at (1, 2, –3) and (2, 3, 12) are perpendicular since

$$(-3\hat{i} - \hat{k}).(-5\hat{i} + 15\hat{k}) = +15 - 15 = 0$$ **Proved.**

Example 54. *Find the constants a, b, c, so that*

$$\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k} \qquad ...(1)$$

is irrotational and hence find function ϕ *such that* $\vec{F} = \nabla\phi$.

(*Nagpur University, Summer 2005, Winter 2000; R.G.P.V., Bhopal 2009*)

Solution. We have,

$$\therefore \qquad \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix}$$

$$= (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k}$$

As $\vec{F}$ is irrotational, $\nabla \times \vec{F} = \vec{0}$

i.e., $(c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore \qquad c + 1 = 0, \quad 4 - a = 0 \quad$ and $\quad b - 2 = 0$

i.e., $\qquad a = 4, \quad b = 2, \quad c = -1$

Putting the values of *a, b, c* in (1), we get

$$\vec{F} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k}$$

Now we have to find ϕ such that $\vec{F} = \nabla\phi$

We know that

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{[Total differential coefficient]}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \nabla\phi.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \bar{F}.(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= [(x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k})].(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (x + 2y + 4z)\,dx + (2x - 3y - z)\,dy + (4x - y + 2z)\,dz$$

$$= x\,dx - 3y\,dy + 2z\,dz + (2y\,dx + 2x\,dy) + (4z\,dx + 4x\,dz) + (-z\,dy - y\,dz)$$

$$\phi = \int x\,dx - 3\int y\,dy + 2\int z\,dz + \int(2y\,dx + 2x\,dy) + \int(4z\,dx + 4x\,dz) - \int(z\,dy + y\,dz)$$

$$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c \qquad \textbf{Ans.}$$

Example 55. *Let $\vec{V}(x, y, z)$ be a differentiable vector function and $\phi(x, y, z)$ be a scalar function. Derive an expression for div $(\phi\vec{V})$ in terms of $\phi\,.\,\vec{V}$, div $\vec{V}$ and $\nabla\phi$.*

(*U.P. I Semester, Winter 2003*)

Solution. Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{div }(\phi\vec{V}) = \vec{\nabla}.(\phi\vec{F})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[\phi V_1\hat{i} + \phi V_2\hat{j} + \phi V_3\hat{k}] = \frac{\partial}{\partial x}(\phi V_1) + \frac{\partial}{\partial y}(\phi V_2) + \frac{\partial}{\partial z}(\phi V_3)$$

$$= \left(\phi\frac{\partial V_1}{\partial x} + \frac{\partial\phi}{\partial x}V_1\right) + \left(\phi\frac{\partial V_2}{\partial y} + \frac{\partial\phi}{\partial y}V_2\right) + \left(\phi\frac{\partial V_3}{\partial z} + \frac{\partial\phi}{\partial z}V_3\right)$$

$$= \phi\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + \left(\frac{\partial\phi}{\partial x}V_1 + \frac{\partial\phi}{\partial y}V_2 + \frac{\partial\phi}{\partial z}V_3\right)$$

$$= \phi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}) + \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \phi(\nabla.\vec{V}) + (\vec{\nabla}\phi).\vec{V} = \phi\,(\text{div }\vec{V}) + (\text{grad }\phi).\,\vec{V} \qquad \textbf{Ans.}$$

Example 56. *If $\vec{A}$ is a constant vector and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that*

$$Curl\left[\left(\vec{A}\,.\,\vec{R}\right)\vec{A}\right] = \vec{A} \times \vec{R}$$

(*K. University, Dec. 2009*)

Solution. Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\qquad \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{A}\,.\,\vec{R} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\,.\,(x\hat{i} + y\hat{j} + z\hat{k}) = A_1x + A_2y + A_3z$$

$$[\vec{A}\,.\,\vec{R}]\,\vec{R} = (A_1x + A_2y + A_3z)\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (A_1x^2 + A_2xy + A_3zx)\,\hat{i} + (A_1xy + A_2y^2 + A_3yz)\,\hat{j} + (A_1xz + A_2yz + A_3z^2)\,\hat{k}$$

$$\text{Curl}\left[(\vec{A}\,.\,\vec{R})\,\vec{R}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1x^2 + A_2xy + A_3zx & A_2xy + A_2y^2 + A_3yz & A_1xz + A_2yz + A_3z^2 \end{vmatrix}$$

$$= (A_2z - A_3y)\ \hat{i}\ - [A_1z - A_3x]\ \hat{j}\ \ [A_1y - A_2x]\ \hat{k} \qquad \ldots (1)$$

$$\text{L.H.S.} = \vec{A} \times \vec{R}$$

$$= (A_1\hat{i}\ + A_2\hat{j}\ + A_3k) \times (x\hat{i}\ + y\hat{j}\ + z\ \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (A_2z - A_3y)\ \hat{i}\ - (A_1z - A_3x)\ \hat{j}\ + (A_1y - A_2x)\ \hat{k}$$

$= \text{R.H.S.}$ [From (1)]

Example 57. *Suppose that $\vec{U}, \vec{V}$ and f are continuously differentiable fields then Prove that, div $(\vec{U} \times \vec{V}) = \vec{V}.curl\,\vec{U} - \vec{U}.curl\,\vec{V}$.* (*M.U. 2003, 2005*)

Solution. Let

$$\vec{U} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k},\ \vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\vec{U} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$$\text{div}\ (\vec{U} \times \vec{V}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[(u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}]$$

$$= \frac{\partial}{\partial x}(u_2v_3 - u_3v_2) + \frac{\partial}{\partial y}(-u_1v_3 + u_3v_1) + \frac{\partial}{\partial z}(u_1v_2 - u_2v_1)$$

$$= \left[u_2\frac{\partial v_3}{\partial x} + v_3\frac{\partial u_2}{\partial x} - u_3\frac{\partial v_2}{\partial x} - v_2\frac{\partial u_3}{\partial x}\right] + \left[-u_1\frac{\partial v_3}{\partial y} - v_3\frac{\partial u_1}{\partial y} + u_3\frac{\partial v_1}{\partial y} + v_1\frac{\partial u_3}{\partial y}\right] + \left[u_1\frac{\partial v_2}{\partial z} + v_2\frac{\partial u_1}{\partial z} - u_2\frac{\partial v_1}{\partial z} - v_1\frac{\partial u_2}{\partial z}\right]$$

$$= v_1\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + v_2\left(-\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\right) + v_3\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) + u_1\left(-\frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial z}\right) + u_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + u_3\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)$$

$$= (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}).\left[\hat{i}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + \hat{j}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + \hat{k}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\right]$$
$$-(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}).\left[\hat{i}\left(-\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y}\right) + \hat{j}\left(-\frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(-\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right)\right]$$

$$= V.(\vec{\nabla} \times \vec{U}) - \vec{U}.(\vec{\nabla} \times V) = \vec{V}.\text{curl}\,\vec{U} - \vec{U}.\text{curl}\,\vec{V}$$

Proved.

Example 58. *Prove that*

$$\vec{\nabla} \times (\vec{F} \times \vec{G}) = \vec{F}(\vec{\nabla}.\vec{G}) - \vec{G}(\vec{\nabla}.\vec{F}) + (\vec{G}.\vec{\nabla})\vec{F} - (\vec{F}.\vec{\nabla})\vec{G} \qquad \text{(M.U. 2004, 2005)}$$

Solution.

$$\vec{\nabla} \times (\vec{F} \times \vec{G}) = \Sigma\,\hat{i} \times \frac{\partial}{\partial x}(\vec{F} \times \vec{G})$$

$$= \Sigma\,\hat{i} \times \left(\frac{\partial F}{\partial x} \times \vec{G} + \vec{F} \times \frac{\partial G}{\partial x}\right) = \Sigma\,\hat{i} \times \left(\frac{\partial F}{\partial x} \times \vec{G}\right) + \Sigma\,\hat{i} \times \left(\vec{F} \times \frac{\partial G}{\partial x}\right)$$

$$= \Sigma\left[(\hat{i}.\vec{G})\frac{\partial F}{\partial x} - \left(\hat{i}\frac{\partial F}{\partial x}\right)\vec{G}\right] + \Sigma\left[\left(\hat{i}\frac{\partial G}{\partial x}\right)\vec{F} - (\hat{i}.\vec{F})\frac{\partial G}{\partial x}\right]$$

$$= \Sigma\,(\vec{G}.\hat{i})\frac{\partial F}{\partial x} - \vec{G}\Sigma\left(\hat{i}.\frac{\partial F}{\partial x}\right) + \vec{F}\Sigma\left(\hat{i}.\frac{\partial G}{\partial x}\right) - \Sigma(\vec{F}.\hat{i})\frac{\partial G}{\partial x}$$

$$= \vec{F}\left(\Sigma\,\hat{i}\frac{\partial G}{\partial x}\right) - \vec{G}\,\Sigma\left(\hat{i}.\frac{\partial F}{\partial x}\right) + \Sigma(\vec{G}.\hat{i})\frac{\partial F}{\partial x} - \Sigma(\vec{F}.\hat{i})\frac{\partial G}{\partial x}$$

$$= \vec{F}\,(\vec{\nabla}\,\vec{G}) - \vec{G}\,(\vec{\nabla}.\vec{F}) + (\vec{G}.\vec{\nabla})\vec{F} - (\vec{F}.\vec{\nabla})\vec{G}$$

Proved.

Questions for practice:

Prove that

$$\vec{\nabla}\,(\vec{F}.\vec{G}) = (\vec{G}.\vec{\nabla})\vec{F} + (\vec{F}.\vec{\nabla})\vec{G} + \vec{G} \times (\vec{\nabla} \times \vec{F}) + \vec{F} \times (\vec{\nabla} \times \vec{G})$$

Example 59. *Prove that, for every field $\vec{V}$; div curl $\vec{V}$ = 0.*
(*Nagpur University, Summer 2004; AMIETE, Sem II, June 2010*)

Solution. Let

$$V = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}$$

$$\text{div }(\overline{\text{curl}}\,\vec{V}) = \vec{\nabla}.(\vec{\nabla} \times \vec{V})$$

$$= \vec{\nabla}.\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x\,\partial y} - \frac{\partial^2 V_2}{\partial x\,\partial z} - \frac{\partial^2 V_3}{\partial y\,\partial x} + \frac{\partial^2 V_1}{\partial y\,\partial z} + \frac{\partial^2 V_2}{\partial z\,\partial x} - \frac{\partial^2 V_1}{\partial z\,\partial y}$$

$$= \left(\frac{\partial^2 V_1}{\partial y\,\partial z} - \frac{\partial^2 V_1}{\partial z\,\partial y}\right) + \left(\frac{\partial^2 V_2}{\partial z\,\partial x} - \frac{\partial^2 V_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 V_3}{\partial x\,\partial y} - \frac{\partial^2 V_3}{\partial y\,\partial x}\right)$$

$$= 0$$

Ans.

Example 60. *If $\vec{a}$ is a constant vector, show that*

$$\vec{a} \times (\vec{\nabla} \times \vec{r}) = \vec{\nabla}(\vec{a}.\vec{r}) - (\vec{a}.\vec{\nabla})\,\vec{r}. \qquad \text{(U.P., Ist Semester, Dec. 2007)}$$

Solution.

$$\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}, \quad \vec{r} = r_1\,\hat{i} + r_2\,\hat{j} + r_3\,\hat{k}$$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r_1 & r_2 & r_3 \end{vmatrix} = \left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right)\hat{i} - \left(\frac{\partial r_3}{\partial x} - \frac{\partial r_1}{\partial z}\right)\hat{j} + \left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right)\hat{k}$$

$$\vec{a} \times (\vec{\nabla} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ \dfrac{\partial r_3}{\partial y} - \dfrac{\partial r_2}{\partial z} & -\dfrac{\partial r_3}{\partial x} + \dfrac{\partial r_1}{\partial z} & \dfrac{\partial r_2}{\partial x} - \dfrac{\partial r_1}{\partial y} \end{vmatrix}$$

$$= \left[\left(a_2 \frac{\partial r_2}{\partial x} - a_2 \frac{\partial r_1}{\partial y}\right) - \left(-a_3 \frac{\partial r_3}{\partial x} + a_3 \frac{\partial r_1}{\partial z}\right)\right]\hat{i} - \left[a_1 \frac{\partial r_2}{\partial x} - a_1 \frac{\partial r_1}{\partial y} - a_3 \frac{\partial r_3}{\partial y} + a_3 \frac{\partial r_2}{\partial z}\right]\hat{j}$$
$$+ \left[- a_1 \frac{\partial r_3}{\partial x} + a_1 \frac{\partial r_1}{\partial z} - a_2 \frac{\partial r_3}{\partial y} + a_2 \frac{\partial r_2}{\partial z}\right]\hat{k}$$

$$= \left[\left(a_1 \hat{i} \frac{\partial r_1}{\partial x} + a_2 \hat{i} \frac{\partial r_2}{\partial x} + a_3 \hat{i} \frac{\partial r_3}{\partial x}\right) + \left(a_1 \hat{j} \frac{\partial r_1}{\partial y} + a_2 \hat{j} \frac{\partial r_2}{\partial y} + a_3 \hat{j} \frac{\partial r_3}{\partial y}\right)\right.$$
$$\left. + \left(a_1 \hat{k} \frac{\partial r_1}{\partial z} + a_2 \hat{k} \frac{\partial r_2}{\partial z} + a_3 \hat{k} \frac{\partial r_3}{\partial z}\right)\right] - \left[\left(a_1 \hat{i} \frac{\partial r_1}{\partial x} + a_1 \hat{j} \frac{\partial r_2}{\partial x} + a_1 \hat{k} \frac{\partial r_3}{\partial x}\right)\right.$$
$$\left. + \left(a_2 \hat{i} \frac{\partial r_1}{\partial y} + a_2 \hat{j} \frac{\partial r_2}{\partial y} + a_2 \hat{k} \frac{\partial r_3}{\partial y}\right) + \left(a_3 \hat{i} \frac{\partial r_1}{\partial z} + a_3 \hat{j} \frac{\partial r_2}{\partial z} + a_3 \hat{k} \frac{\partial r_3}{\partial z}\right)\right]$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)(a_1 r_1 + a_2 r_2 + a_3 r_3) - \left[a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right](r_1 \hat{i} + r_2 \hat{j} + r_3 \hat{k})$$

$$= \nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \vec{\nabla})\vec{r}$$ **Proved.**

Example 61. *If r is the distance of a point (x, y, z) from the origin, prove that* $Curl\left(k \times grad\frac{1}{r}\right) + grad\left(k.grad\frac{1}{r}\right) = 0$*, where k is the unit vector in the direction OZ.*
(*U.P., I Semester, Winter 2000*)

Solution.

$$r^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2$$

$$\Rightarrow \quad \frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$$

$$\text{grad}\ \frac{1}{r} = \vec{\nabla}\frac{1}{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\,(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= -(x^2 + y^2 + z^2)^{-3/2}\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$k \times \text{grad}\ \frac{1}{r} = k \times [-(x^2 + y^2 + z^2)^{-3/2}\,(x\hat{i} + y\hat{j} + z\hat{k})]$$

$$= -(x^2 + y^2 + z^2)^{-3/2}\,(x\hat{j} - y\hat{i})$$

$$\text{curl}\left(k \times \text{grad}\ \frac{1}{r}\right) = \vec{\nabla} \times \left(k \times \text{grad}\ \frac{1}{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [-(x^2 + y^2 + z^2)^{-3/2}\ (x\hat{j} - y\hat{i})\,]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{-x}{(x^2+y^2+z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= -\left(-\frac{3}{2}\right)\frac{(-x)\,(2z)}{(x^2+y^2+z^2)^{5/2}}\,\hat{i} + -\frac{3}{2}\frac{y\,(2z)}{(x^2+y^2+z^2)^{5/2}}\,\hat{j} + \left[-\frac{3}{2}\frac{(-x)(2x)}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{(-3/2)\,(y)\,(2y)}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]\hat{k}$$

$$= \frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\,\hat{i} - \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\,\hat{j} + \frac{(3x^2 - x^2 - y^2 - z^2 + 3y^2 - x^2 - y^2 - z^2)}{(x^2+y^2+z^2)^{5/2}}\,\hat{k}$$

$$= \frac{-3xz\,\hat{i} - 3yz\,\hat{j} + (x^2 + y^2 - 2z^2)\,\hat{k}}{(x^2+y^2+z^2)^{5/2}} \qquad \text{...(1)}$$

$$k\,.\,\text{grad}\,\frac{1}{r} = k.[-(x^2+y^2+z^2)^{-3/2}\ (x\hat{i} + y\hat{j} + z\hat{k})] = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{grad}\left(k.\text{grad}\,\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{3}{2}\frac{\hat{i}(-z)(2x)}{(x^2+y^2+z^2)^{5/2}} + -\frac{3}{2}\frac{\hat{j}(-z)(2y)}{(x^2+y^2+z^2)^{5/2}} + \left[-\frac{3}{2}\frac{(-z)(2z)}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]\hat{k}$$

$$= \frac{3xz\,\hat{i} + 3yz\,\hat{j} + (3z^2 - x^2 - y^2 - z^2)\,\hat{k}}{(x^2+y^2+z^2)^{5/2}} = \frac{3xz\,\hat{i} + 3yz\,\hat{j} - (x^2 + y^2 - 2z^2)\,\hat{k}}{(x^2+y^2+z^2)^{5/2}} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$\text{Curl}\left(k \times \text{grad}\,\frac{1}{r}\right) + \text{grad}\left(k.\text{grad}\,\frac{1}{r}\right) = 0$$ **Proved.**

Example 62. *Prove that* $\nabla \times \left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right) = \dfrac{(2-n)\,\vec{a}}{r^n} + \dfrac{n\,(\vec{a}\,.\,\vec{r})\,\vec{r}}{r^{n+2}}$.

(*M.U. 2009, 2005, 2003, 2002; AMIETE, II Sem. June 2010*)

Solution. We have,

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{1}{r^n}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{1}{r^n}\,(a_2 z - a_3 y)\,\hat{i} + \frac{1}{r^n}(a_3 x - a_1 z)\,\hat{j} + \frac{1}{r^n}(a_1 y - a_2 x)\,\hat{k}$$

$$\nabla \times \frac{(\vec{a} \times \vec{r})}{r^n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{a_2 z - a_3 y}{r^n} & \frac{a_3 x - a_1 z}{r^n} & \frac{a_1 y - a_2 x}{r^n} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^n}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{a_3 x - a_1 z}{r^n}\right) - \frac{\partial}{\partial y}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right]$$

Now, $\quad r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\quad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore\ \nabla \times \left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \hat{i}\left[\left\{-nr^{-n-1}.\left(\frac{y}{r}\right)(a_1 y - a_2 x) + \frac{1}{r^n}a_1\right\}\right.$$
$$\left.-\left\{-nr^{-n-1}\left(\frac{z}{r}\right)(a_3 x - a_1 z) + \frac{1}{r^n}(-a_1)\right\}\right] + \text{two similar terms}$$

$$= \hat{i}\left[-\frac{n}{r^{n+2}}(a_1 y^2 - a_2 xy) + \frac{a_1}{r^n} + \frac{n}{r^{n+2}}(a_3 xz - a_1 z^2) + \frac{a_1}{r^n}\right] + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{n}{r^{n+2}}a_1(y^2 + z^2) + \frac{n}{r^{n+2}}(a_2 xy + a_3 xz)\right] + \text{two similar terms}$$

Adding and subtracting $\frac{n}{r^{n+2}}a_1 x^2$ to third and from second term, we get

$$\vec{\nabla} \times \left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}(x^2 + y^2 + z^2) + \frac{n}{r^{n+2}}(a_1 x^2 + a_2 xy + a_3 xz)\right] + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}x(a_1 x + a_2 y + a_3 z)\right] + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^n} + \frac{n}{r^{n+2}}x(a_1 x + a_2 y + a_3 z)\right] + \hat{j}\left[\frac{2a_2}{r^n} - \frac{na_2}{r^n} + \frac{n}{r^{n+2}}y(a_2 y + a_3 z + a_1 x)\right]$$
$$+ \hat{k}\left[\frac{2a_3}{r^n} - \frac{na_3}{r^n} + \frac{n}{r^{n+2}}z(a_3 z + a_1 x + a_2 y)\right]$$

$$= \frac{2}{r^n}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - \frac{n}{r^n}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \frac{n}{r^{n+2}}(a_1 x + a_2 y + a_3 z)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{2-n}{r^n}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \frac{n}{r^{n+2}}(a_1 x + a_2 y + a_3 z)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{2-n}{r^n}\vec{a} + \frac{n}{r^{n+2}}(\vec{a}.\vec{r})\vec{r}$$ **Proved.**

Example 63. *If f and g are two scalar point functions, prove that*

div $(f\nabla g) = f\nabla^2 g + \nabla f\,\nabla g.$ (*U.P., I Semester, compartment, Winter 2001*)

Solution. We have, $\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$$\Rightarrow \quad f\,\nabla g = f\frac{\partial g}{\partial x}\hat{i} + f\frac{\partial g}{\partial y}\hat{j} + f\frac{\partial g}{\partial z}\hat{k}$$

$$\Rightarrow \quad \text{div}\,(f\,\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)g + \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right).\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right)$$

$= f\,\nabla^2 g + \nabla f.\nabla g$ **Proved.**

Example 64. *For a solenoidol vector $\vec{F}$, show that curl curl curl curl $\vec{F} = \nabla^4 \vec{F}$.*

(*M.D.U., Dec. 2009*)

Solution. Since vector $\vec{F}$ is solenoidal, so div $\vec{F} = 0$ ... (1)

We know that curl curl $\vec{F}$ = grad div ($\vec{F} - \nabla^2 \vec{F}$) ... (2)

Using (1) in (2), grad div $\vec{F}$ = grad (0) = 0 ... (3)

On putting the value of grad div F in (2), we get

curl curl $\vec{F} = -\nabla^2 \vec{F}$ ... (4)

Now, curl curl curl curl $\vec{F}$ = curl curl ($-\nabla^2 \vec{F}$) [Using (4)]

$= -$ curl curl $(\nabla^2 \vec{F}) = -$ [grad div $(\nabla^2 \vec{F}) - \nabla^2 (\nabla^2 \vec{F})$] [Using (2)]

$= -$ grad $(\nabla \,.\, \nabla^2 \vec{F}) + \nabla^2 (\nabla^2 \vec{F}) = -$ grad $(\nabla^2 \nabla \,.\, \vec{F}) + \nabla^4 \vec{F}$ $[\nabla \cdot \vec{F} = 0]$

$= 0 + \nabla^4 \vec{F} = \nabla^4 \vec{F}$ [Using (1)] **Proved.**

EXERCISE 5.9

1. Find the divergence and curl of the vector field $V = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$.

Ans. Divergence $= 4x$, Curl $= (2y - x)\hat{i} + y\hat{j} + 4y\hat{k}$

2. If a is constant vector and r is the radius vector, prove that

(*i*) $\nabla(\vec{a}\,.\,\vec{r}) = \vec{a}$ (*ii*) div $(\vec{r} \times \vec{a}) = 0$ (*iii*) curl $(\vec{r} \times \vec{a}) = -2\vec{a}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

3. Prove that:

(*i*) $\nabla.(\phi A) = \nabla\phi.A + \phi(\nabla.A)$

(*ii*) $\nabla(A.B) = (A.\nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$ (*R.G.P.V. Bhopal, June 2004*)

(*iii*) $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$

4. If $F = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that F.curl $F = 0$.

(*R.G.P.V. Bhopal, Feb. 2006, June 2004*)

Prove that

5. $\vec{\nabla} \times (\phi\vec{F}) = (\vec{\nabla}\phi) \times \vec{F} + \phi(\vec{\nabla} \times \vec{F})$

6. $\nabla.(\vec{F} \times \vec{G}) = \vec{G}.(\nabla \times \vec{F}) - \vec{F}.(\nabla \times \vec{G})$

7. Evaluate div $(\vec{A} \times \vec{r})$ if curl $\vec{A} = 0$.

8. Prove that curl $(\vec{a} \times \vec{r}) = 2a$

9. Find div $\vec{F}$ and curl F where F = grad $(x^3 + y^3 + z^3 - 3xyz)$. (*R.G.P.V. Bhopal Dec. 2003*)

Ans. div $\vec{F} = 6(x + y + z)$, curl $\vec{F} = 0$

10. Find out values of *a, b, c* for which $\vec{v} = (x + y + az)\hat{i} + (bx + 3y - z)\hat{j} + (3x + cy + z)\hat{k}$ is irrotational.

Ans. $a = 3,\ b = 1,\ c = -1$

11. Determine the constants *a, b, c*, so that $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. Hence find the scalar potential ϕ such that $\vec{F}$ = grad ϕ.

(*R.G.P.V. Bhopal, Feb. 2005*) **Ans.** $a = 4,\ b = 2,\ c = 1$

Potential $\phi = \left(\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx\right)$

Choose the correct alternative:

12. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is

(*i*) $\frac{2}{3}$ (*ii*) $\frac{3}{2}$ (*iii*) 3 (*iv*) 6 (*A.M.I.E.T.E., Summer 2000*) **Ans.** (*iv*)

13. If $u = x^2 - y^2 + z^2$ and $\bar{V} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\nabla\ (u\bar{V})$ is equal to

(*i*) $5u$ (*ii*) $5|\vec{V}|$ (*iii*) $5(u - |\vec{V}|)$ (*iv*) $5(u - |\vec{V}|)$ (*A.M.I.E.T.E., June 2007*)

14. A unit normal to $x^2 + y^2 + z^2 = 5$ at $(0, 1, 2)$ is equal to

(*i*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} + \hat{k})$ (*ii*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} - \hat{k})$ (*iii*) $\frac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$ (*iv*) $\frac{1}{\sqrt{5}}(\hat{i} - \hat{j} + \hat{k})$

(*A.M.I.E.T.E., Dec. 2008*)

15. The directional derivative of $\phi = x\,y\,z$ at the point $(1, 1, 1)$ in the direction $\hat{i}$ is:

(*i*) -1 (*ii*) $-\frac{1}{3}$ (*iii*) 1 (*iv*) $\frac{1}{3}$ **Ans.** (*iii*)

(*R.G.P.V. Bhopal, II Sem., June 2007*)

16. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ then $\nabla\phi\ (r)$ is:

(*i*) $\phi'(r)\vec{r}$ (*ii*) $\frac{\phi(r)\vec{r}}{r}$ (*iii*) $\frac{\phi'(r)\vec{r}}{r}$ (*iv*) None of these **Ans.** (*iii*)

(*R.G.P.V. Bhopal, II Semester, Feb. 2006*)

17. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then value of $\nabla(\log r)$ is (*U.P., I Sem, Dec 2008*)

(*i*) $\frac{\vec{r}}{r}$ (*ii*) $\frac{\vec{r}}{r^2}$ (*iii*) $-\frac{\vec{r}}{r^3}$ (*iv*) none of the above. **Ans.** (*ii*)

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = r$, then div $\vec{r}$ is:

(*i*) 2 (*ii*) 3 (*iii*) -3 (*iv*) -2 **Ans.** (*ii*)

(*R.G.P.V. Bhopal, II Semester, Feb. 2006*)

19. If $\vec{V} = xy^2\hat{i} + 2yx^2z\hat{j} - 3yz^2\hat{k}$ then curl $\vec{V}$ at point $(1, -1, 1)$ is

(*i*) $-(\hat{j} + 2\hat{k})$ (*ii*) $(\hat{i} + 3\hat{k})$ (*iii*) $-(\hat{i} + 2\hat{k})$ (*iv*) $(\hat{i} + 2\hat{j} + \hat{k})$

(*R.G.P.V. Bhopal, II Semester, Feb 2006*)
Ans. (*iii*)

20. If $\vec{A}$ is such that $\nabla \times \vec{A} = 0$ then $\vec{A}$ is called

(*i*) Irrotational (*ii*) Solenoidal (*iii*) Rotational (*iv*) None of these

(*A.M.I.E.T.E., Dec. 2008*)

21. If $\vec{F}$ is a conservative force field, then the value of curl $\vec{F}$ is

(*i*) 0 (*ii*) 1 (*iii*) $\overline{\nabla F}$ (*iv*) -1 (*A.M.I.E.T.E., June 2007*)

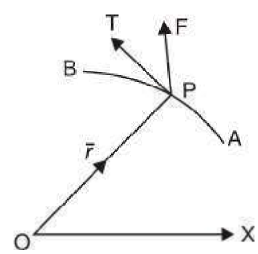
UNIT-5

VECTOR INTEGRATION

LINE INTEGRAL

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.



Component of $\vec{F}$ along a tangent PT at P

$$= \text{Dot product of } \vec{F} \text{ and unit vector along PT}$$

$$= \vec{F} \cdot \frac{d\vec{r}}{ds} \left(\frac{d\vec{r}}{ds} \text{ is a unit vector along tangent PT} \right)$$

$$\text{Line integral} = \sum \vec{F} \cdot \frac{d\vec{r}}{ds} \text{ from A to B along the curve}$$

$$\therefore \text{ Line integral} = \int_c \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_c \vec{F} \cdot d\vec{r}$$

Note (1) Work. If $\overline{F}$ represents the variable force acting on a particle along arc AB, then the total work done = $\int_A^B \vec{F} \cdot d\vec{r}$

(2) Circulation. If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V} \cdot d\vec{r}$ is called the circulation of V round the closed curve c.

If the circulation of V round every closed curve is zero then V is said to be irrotational there.

(3) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

Example 65. *If a force* $\vec{F} = 2x^2 y\hat{i} + 3\,xy\hat{j}$ *displaces a particle in the xy-plane from* $(0, 0)$ *to* $(1, 4)$ *along a curve* $y = 4\,x^2$. *Find the work done.*

Solution. Work done $= \int_c \vec{F} . d\vec{r}$

$$\left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ d\vec{r} = dx\hat{i} + dy\,\hat{j} \end{array} \right]$$

$$= \int_c (2\,x^2 y\,\hat{i} + 3\,xy\,\hat{j}) . (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_c (2\,x^2 y\,dx + 3\,xy\,dy)$$

Putting the values of y and dy, we get $\left(\begin{array}{l} y = 4x^2 \\ dy = 8x\,dx \end{array}\right)$

$$= \int_0^1 [2x^2\,(4x^2)\,dx + 3x\,(4x^2)\,8x\,dx]$$

$$= 104\int_0^1 x^4\,dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5} \qquad \textbf{Ans.}$$

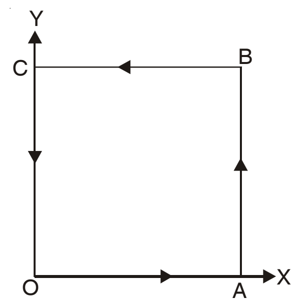
Example 66. *Evaluate* $\int_C \vec{F}\,.\,\vec{dr}$ *where* $\vec{F} = x^2\hat{i} + xy\hat{j}$ *and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

Solution. $\int_C \vec{F}\,.\,\vec{d\,r} = \int_{OA} \vec{F}\,.\,\vec{dr} + \int_{AB} \vec{F}\,.\,\vec{dr} + \int_{BC} \vec{F}\,.\,\vec{dr} + \int_{CO} \vec{F}\,.\,\vec{dr}$

Here $\quad \vec{r} = x\hat{i} + y\hat{j}, \quad \vec{d\,r} = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$

$$\vec{F}\,.\,\vec{dr} = x^2dx + xydy \qquad \text{...(1)}$$



On $\quad OA, y = 0 \qquad \therefore\ \vec{F}\,.\,\vec{dr} = x^2dx$

$$\int_{OA} \vec{F}\,.\,\vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad \text{...(2)}$$

On $\quad AB, x = a \qquad \therefore\ dx = 0$

(1) becomes

$$\therefore\ \vec{F}\,.\,\vec{dr} = aydy$$

$$\int_{Ab} \vec{F}\,.\,\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \qquad \text{...(3)}$$

On $\quad BC, y = a \qquad \therefore\ dy = 0$

$\Rightarrow \quad$ (1) becomes $\qquad \vec{F}\,.\,\vec{dr} = x^2dx$

$$\int_{BC} \vec{F}\,.\,\vec{dr} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \qquad \text{...(4)}$$

On $\quad CO, x = 0, \qquad \therefore\ \vec{F}\,.\,\vec{dr} = 0$

(1) becomes

$$\int_{CO} \vec{F}\,.\,\vec{dr} = 0 \qquad \text{...(5)}$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F}\,.\,\vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ **Ans.**

Example 67. *A vector field is given by*

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}\,.\,\vec{dr}$ *along the path c is x = 2t, y = t, z = t³ from t = 0 to t = 1.*

(*Nagpur University, Winter 2003*)

Solution. $\quad \int_C \vec{F}\,.\,\vec{dr} = \int_C (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$

$$\left[\begin{array}{lll} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore\ \frac{dx}{dt} = 2 & \frac{dy}{dt} = 1 & \frac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t+3)\,(2\,dt) + (2t)\,(t^3)\,dt + (t^4 - 2t)\,(3t^2\,dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**

Example 68. *The acceleration of a particle at time t is given by*

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at t = 0, find* $\vec{v}$ *and* $\vec{r}$ *at any point t.*

Solution. Here, $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$

On integrating, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$$

$$\Rightarrow \quad \vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \vec{c} \qquad \text{...(1)}$$

At $t = 0,\ \vec{v} = \vec{0}$

Putting $t = 0$ and $\vec{v} = 0$ in (1), we get

$$\vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$$

$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

Again integrating, we have

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$$

$$\Rightarrow \quad \vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + t^3\,\hat{k} + \vec{c_1} \qquad \text{...(2)}$$

At, $t = 0,\ \vec{r} = 0$

Putting $t = 0$ and $\vec{r} = 0$ in (2), we get

$$\therefore \quad \vec{0} = -2\hat{i} + \vec{C_1} \quad \Rightarrow \quad \vec{C_1} = 2\hat{i}$$

Hence, $\vec{r} = 2(1 - \cos 3t)\,\hat{i} + 2(\sin 2t - 2t)\,\hat{j} + t^3\hat{k}$ **Ans.**

Example 69. *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\vec{dr}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*

$x = t,\ y = t^2,\ z = t^3.$ (*Uttarakhand, I Semester, Dec. 2006*)

Solution. We have,

$$\int_C \vec{A}.\vec{dr} = \int_C [(3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= \int_C [(3x^2 + 6y)\,dx - 14yzdy + 20xz^2dz]$$

If $x = t,\ y = t^2,\ z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

Now, $\displaystyle\int_C \vec{A}.\vec{dr} = \int_{t=0}^{t=1} [(3t^2 + 6t^2)\,d(t) - 14t^2\,t^3\,d(t^2) + 20t\,(t^3)^2\,d(t^3)]$

$$= \int_{t=0}^{t=1} [9t^2\,dt - 14t^5\,.\,2t\,dt + 20\,t^7\,.\,3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 \qquad = 3 - 4 + 6 = 5 \qquad \textbf{Ans.}$$

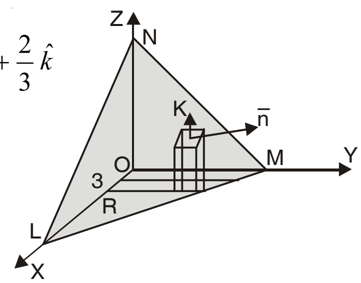
Example 70. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\,ds$ *where* $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (*Nagpur University, Summer 2000*)

Solution. A vector normal to the surface "S" is given by

$$\nabla\,(2x + y + 2z) = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

And $\hat{n}$ = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}$$



$$\hat{k}\,.\,\hat{n} = \hat{k}\,.\left(\frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}\right) = \frac{2}{3}$$

$$\therefore \qquad \iint_S \overline{A}\,.\,\hat{n}\,ds = \iint_R \overline{A}\,.\,\hat{n}\,\frac{dx\,dy}{\hat{k}\,.\,\overline{n}}$$

Where R is the projection of S.

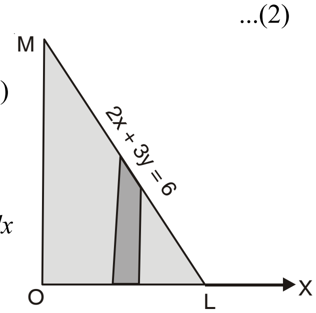
Now, $\quad \vec{A}\,.\,\hat{n} = [\,(x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}].\left(\frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}\right)$

$$= \frac{2}{3}\,(x + y^2) - \frac{2}{3}\,x + \frac{4}{3}\,yz = \frac{2}{3}\,y^2 + \frac{4}{3}\,yz \qquad ...(1)$$

Putting the value of z in (1), we get

$$\vec{A}\,.\,\hat{n} = \frac{2}{3}\,y^2 + \frac{4}{3}\,y\left(\frac{6 - 2\,x - y}{2}\right) \qquad \left(\begin{array}{l}\because \text{on the plane } 2x + y + 2z = 6, \\ z = \dfrac{(6 - 2x - y)}{2}\end{array}\right)$$

$$\vec{A}\,.\,\hat{n} = \frac{2}{3}\,y\,(y + 6 - 2x - y) = \frac{4}{3}\,y\,(3 - x) \qquad ...(2)$$



Hence, $$\iint_S \vec{A}\,.\,\hat{n}\,ds = \iint_R \overline{A}\,.\,\overline{n}\,\frac{dx\,dy}{|\,\hat{k}\,.\,\overline{n}\,|} \qquad ...(3)$$

Putting the value of $\vec{A}\,.\,\hat{n}$ from (2) in (3), we get

$$\iint_S \vec{A}\,.\,\hat{n}\,ds = \iint_R \frac{4}{3}\,y\,(3 - x)\,.\,\frac{3}{2}\,dx\,dy = \int_0^3 \int_0^{6-2x} 2y\,(3 - x)\,dydx$$

$$= \int_0^3 2\,(3 - x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3 - x)\,(6 - 2x)^2\,dx = 4\int_0^3 (3 - x)^3\,dx$$

$$= 4.\left[\frac{(3 - x)^4}{4\,(-1)}\right]_0^3 = -\,(0 - 81) = 81 \qquad \textbf{Ans.}$$

Example 71. *Compute* $\int_c \vec{F}\,.\,\vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*

Solution. $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z,\ d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\int_c \vec{F}\,.\,d\vec{r} = \int_c \frac{\hat{i}y - \hat{j}x}{x^2 + y^2} \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \frac{ydx - xdy}{x^2 + y^2} = \int_c (ydx - xdy) \qquad \ldots(1)\ [\because\ x^2 + y^2 = 1]$$

Parametric equation of the circle are $x = \cos\theta$, $y = \sin\theta$.
Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$ in (1), we get

$$\int_C \vec{F}\,d\vec{r} = \int_0^{2\pi} \sin\theta\,(-\sin\theta\, d\theta) - \cos\theta\,(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta = -(\theta)_0^{2\pi} = -2\pi \qquad \textbf{Ans.}$$

Example 72. *Show that the vector field* $\vec{F} = 2x(y^2 + z^3)\hat{i} + 2x^2 y\hat{j} + 3x^2 z^2\hat{k}$ *is conservative. Find its scalar potential and the work done in moving a particle from* (–1, 2, 1) to (2, 3, 4).
(*A.M.I.E.T.E. June 2010, 2009*)

Solution. Here, we have

$$\vec{F} = 2x(y^2 + z^3)\,\hat{i} + 2x^2 y\,\hat{j} + 3x^2 z^2\hat{k}$$

$$\text{Curl}\ \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x(y^2 + z^3) & 2x^2 y & 3x^2 z^2 \end{vmatrix} = (0-0)i - (6xz^2 - 6xz^2)\hat{j} + (4xy - 4xy)\hat{k} = 0$$

Hence, vector field $\vec{F}$ is irrotational.
To find the scalar potential function ϕ

$$\vec{F} = \vec{\nabla}\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.\left(d\vec{r}\right) = \nabla\phi.d\vec{r} = \vec{F}\,.\,d\vec{r}$$

$$= \left[2x(y^2 + z^3)\hat{i} + 2x^2 y\hat{j} + 3x^2 z^2\hat{k}\right](\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2x\,(y^2 + z^3)\,dx + 2x^2 y\,dy + 3x^2 z^2 dz$$

$$\phi = \int\left[2x(y^2 + z^3)dx + 2x^2 ydy + 3x^2 z^2 dz\right] + C$$

$$\int(2xy^2 dx + 2x^2 ydy) + (2xz^3 dx + 3x^2 z^2 dz) + C = x^2y^2 + x^2z^3 + C$$

Hence, the scalar potential is $x^2y^2 + x^2z^3 + C$
Now, for conservative field

$$\text{Work done} = \int_{(-1,2,1)}^{(2,3,4)} \vec{F}\,.\,d\vec{r} = \int_{(-1,2,1)}^{(2,3,4)} d\phi = [\phi]_{(-1,2,1)}^{(2,3,4)} = \left[x^2 y^2 + x^2 z^3 + c\right]_{(-1,2,1)}^{(2,3,4)}$$

$$= (36 + 256) - (2 - 1) = 291 \qquad \textbf{Ans.}$$

Example 73. *A vector field is given by* $\vec{F} = (\sin y)\,\hat{i} + x\,(1 + \cos y)\,\hat{j}$. *Evaluate the line integral over a circular path* $x^2 + y^2 = a^2$, $z = 0$. `(*Nagpur University, Winter 2001*)

Solution. We have,

$$\text{Work done} = \int_C \vec{F}\,.\,\vec{dr}$$

$$= \int_C [\,(\sin y)\,\hat{i} + x\,(1 + \cos y)\,\hat{j}\,]\,.\,[dx\hat{i} + dy\hat{j}] \quad (\because\ z = 0 \text{ hence } dz = 0)$$

$$\Rightarrow \quad \int_C \vec{F}\,.\,\vec{d}\,r = \int_C \sin y\, dx + x\,(1 + \cos y)\, dy = \int_C (\sin y\, dx + x\cos y\, dy + x\,dy)$$

$$= \int_C d\,(x \sin y) + \int_C x\, dy$$

(where d is differential operator).

The parametric equations of given path

$x^2 + y^2 = a^2$ are $x = a\cos\theta$, $y = a\sin\theta$,

Where θ varies form 0 to 2π

$$\therefore \quad \int_C \vec{F}\,.\,\vec{d}\,r = \int_0^{2\pi} d\,[\,a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a\cos\theta\,.\,a\cos\theta\, d\,\theta$$

$$= \int_0^{2\pi} d\,[a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a^2\cos^2\theta\,.\,d\,\theta$$

$$= [a\cos\theta\sin(a\sin\theta)\,]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta\, d\,\theta$$

$$= 0 + a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\,.\,2\pi = \pi a^2$$ **Ans.**

Example 74. *Determine whether the line integral* $\int (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$ *is independent of the path of integration ? If so, then evaluate it from (1, 0, 1) to* $\left(0, \frac{\pi}{2}, 1\right)$.

Solution. $\int_c (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

$$= \int_c [(2xyz^2\hat{i}) + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \vec{F}\cdot\vec{dr}$$

This integral is independent of path of integration if

$$\vec{F} = \nabla\phi \Rightarrow \nabla\times\vec{F} = 0$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz) = \hat{i} - (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$
$= 0$

Hence, the line integral is independent of path.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{(Total differentiation)}$$

$$= \left(\hat{i}\,\frac{\partial \phi}{\partial x} + \hat{j}\,\frac{\partial \phi}{\partial y} + \hat{k}\,\frac{\partial \phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz) \quad = \nabla\phi\cdot dr = \vec{F}\cdot\vec{d}\,r$$

$$= [(2xyz^2)\,\hat{i} + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$

$$= [(2x\,dx)\,yz^2 + x^2\,(dy)\,z^2 + x^2y\,(2z\,dz)] + [(\cos yz\,dy)\,z + (\cos yz\,dz)\,y]$$

$$= d\,(x^2yz^2) + d\,(\sin yz)$$

$$\phi = \int d\,(x^2yz^2) + \int d\,(\sin yz) = x^2yz^2 + \sin yz$$

$$[\phi]_A^B = \phi\,(B) - \phi\,(A)$$

$$= [x^2yz^2 + \sin yz]_{(0,\frac{\pi}{2},1)} - [x^2yz^2 + \sin yz]_{(1,0,1)} = \left[0 + \sin\left(\frac{\pi}{2}\times 1\right)\right] - [0+0]$$

$$= 1$$ **Ans.**

Example 75. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\;d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

Solution. Here, $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $f\,(x, y, z) = 2x + 3y + 6z - 12$

Normal vector $= \nabla f = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$

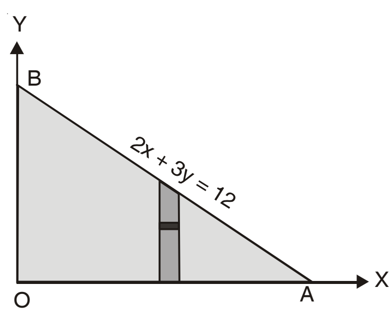
$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{1}{7}\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$dS = \frac{dx\,dy}{\hat{n}\,.\,\hat{k}} = \frac{dx\,dy}{\frac{1}{7}\,(2\hat{i} + 3\hat{j} + 6\hat{k})\,.\,\hat{k}} = \frac{dxdy}{\frac{6}{7}} = \frac{7}{6}\,dx\,dy$$

Now, $\iint \vec{A}\,.\,\hat{n}\;dS = \iint (18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k})\,.\,\frac{1}{7}\,(2\hat{i} + 3\hat{j} + 6\hat{k})\,\frac{7}{6}\,dx\,dy$

$$= \iint (36z - 36 + 18y)\,\frac{dx\,dy}{6} \qquad = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get



$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\,dx\,dy$$

$$= \int_0^6 (6 - 2x)\,dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\,dx\,(y)_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6 - 2x)\,\frac{1}{3}\,(12 - 2x)\,dx = \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx$$

$$= \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6 = \frac{1}{3}[4\times 36\times 2 - 18\times 36 + 72\times 6] = \frac{72}{3}\,[4 - 9 + 6] = 24$$ **Ans.**

EXERCISE 5.10

1. Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $(\hat{i} + \hat{j})$. **Ans.** 1

2. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$ moves a particle from origin to (1, 1) along a parabola $y^2 = x$. **Ans.** $\frac{2}{3}$

3. Show that $\vec{V} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find its scalar potential ϕ such that $\vec{V}$ = grad ϕ. Find the work done by the force $\vec{V}$ in moving a particle from (1, – 2, 1) to (3, 1, 4). **Ans.** $x^2y + xz^3$, 202

4. Show that the line integral $\int_c (2xy + 3)\,dx + (x^2 - 4z)\,dy - 4y\,dz$

where c is any path joining (0, 0, 0) to (1, – 1, 3) does not depend on the path c and evaluate the line integral. **Ans.** 14

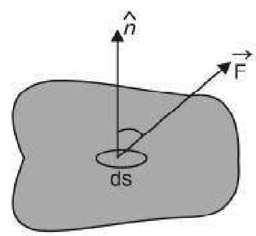
5. Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$, under the field of force given by $F = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$. Is the field of force conservative? (*A.M.I.E.T.E., Winter 2000*) **Ans.** 40π

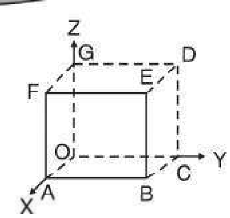
6. If $\vec{\nabla}\phi = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\,\hat{j} + (z^3 - 3x^2yz^2)\,\hat{k}$, find ϕ. **Ans.** $3y + \frac{z^4}{4} + xy^2 - x^2yz^3$

7. $\int_C \vec{R} \cdot d\vec{R}$ is independent of the path joining any two point if it is. (*A.M.I.E.T.E., June 2010*)

(*i*) irrotational field (*ii*) solenoidal field (*iii*) rotational field (*iv*) vector field. **Ans.** (*i*)

5.34 SURFACE INTEGRAL





A surface $r = f(u, v)$ is called smooth if $f\ (u, v)$ posses continous first order partial derivative.

Let $\vec{F}$ be a vector function and S be the given surface.

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal

$= \vec{F} \cdot \hat{n}$, where n is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|} \qquad ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

Surface integral of F over S

$$= \Sigma \vec{F}\cdot\hat{n} \qquad = \iint_S (\vec{F}\cdot\hat{n})\,ds$$

Note. (1) Flux $= \iint_S (\vec{F}\cdot\hat{n})\,ds$ where, $\overline{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F}\cdot\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a *solenoidal* vector point function.

Example 76. *Evaluate* $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\vec{ds}$ *where S is the surface of the sphere*

$x^2 + y^2 + z^2 = a^2$ *in the first octant.* (*U.P., I Semester, Dec. 2004*)

Solution. Here, $\phi = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\phi = \hat{i}\,\dfrac{\partial\phi}{\partial x} + \hat{j}\,\dfrac{\partial\phi}{\partial y} + \hat{k}\,\dfrac{\partial\phi}{\partial z}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because\ x^2 + y^2 + z^2 = a^2]$$

Here, $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

$$\vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $\displaystyle\iint_S F\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\,\frac{dx\,dy}{|\hat{k}\,.\,\hat{n}|} = \int_0^a\int_0^{\sqrt{a^2 - x^2}} \frac{3xyz\,dx\,dy}{a\left(\dfrac{z}{a}\right)}$

$$= 3\int_0^a\int_0^{\sqrt{a^2 - x^2}} xy\,dy\,dx = 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2 - x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2 - x^2)\,dx = \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}. \qquad \textbf{Ans.}$$

Example 77. *Show that* $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$, *where* $\vec{F} = 4\,xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes,*

x= 0, x = 1, y = 0, y = 1, z = 0, z = 1.

S.No.	Surface	Outward normal	ds	
1	OABC	$-k$	$dx\,dy$	$z = 0$
2	DEFG	k	$dx\,dy$	$z = 1$
3	OAGF	$-j$	$dx\,dz$	$y = 0$
4	BCED	j	$dx\,dz$	$y = 1$
5	ABDG	i	$dy\,dz$	$x = 1$
6	OCEF	$-i$	$dy\,dz$	$x = 0$

Solution. $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC}\vec{F}\cdot\hat{n}\,ds$

$$+\iint_{DEFG}\vec{F}\cdot\hat{n}\,ds + \iint_{OAGF}\vec{F}\cdot\hat{n}\,ds$$

$$+\iint_{BCED}\vec{F}\cdot\hat{n}\,ds + \iint_{ABDG}\vec{F}\cdot\hat{n}\,ds$$

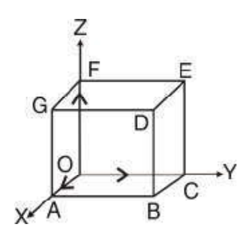
$$+\iint_{OCEF}\vec{F}\cdot\hat{n}\,ds \qquad ...(1)$$

Now, $\displaystyle\iint_{OABC}\vec{F}\cdot n\,ds = \iint_{OABC}(4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\,(-k)\,dx\,dy = \int_0^1\int_0^1 - yz\,dx\,dy = 0$ (as $z = 0$)

$$\iint_{DEFG}(4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot\hat{k}\,dx\,dy$$

$$= \iint_{DEFG} yz\,dx\,dy = \int_0^1\int_0^1 y\,(1)\,dx\,dy$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = [x]_0^1\,\frac{1}{2} = \frac{1}{2}$$



$\displaystyle\iint_{OAGF}(4\,xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-j)\,dx\,dz = \iint_{OAGF} y^2\,dx\,dz = 0$ (as $y = 0$)

$$\iint_{BCED}(4xz\,\hat{i}-y^2\,\hat{j}+yz\,\hat{k})\cdot\hat{j}\;dx\,dz \;=\; \iint_{BCED}(-\,y^2)\;dx\,dz$$

$$= -\int_0^1 dx\int_0^1 dz = -\,(x)_0^1\;(z)_0^1 = -\,1 \qquad\qquad (\text{as } y=1)$$

$$\iint_{ABDG}(4xz\hat{i}-y^2\,\hat{j}+yz\hat{k})\cdot\hat{i}\;dy\,dz \;=\; \iint 4xz\;dy\,dz = \int_0^1\int_0^1 4\,(1)\;z\;dy\,dz$$

$$= 4\,(y)_0^1\left(\frac{z^2}{2}\right)_0^1 = 4\,(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCEF}(4xz\,\hat{i}-y^2\,\hat{j}+yz\,\hat{k})\,(-\,\hat{i})\;dy\,dz \;=\; \int_0^1\int_0^1 -\,4xz\;dy\,dz = 0 \qquad\qquad (\text{as } x=0)$$

On putting these values in (1), we get

$$\iint_S \bar{F}\cdot\hat{n}\,ds \;=\; 0+\frac{1}{2}+0-1+2+0 \;=\; \frac{3}{2}$$ **Proved.**

EXERCISE 5.11

1. Evaluate $\iint_S \vec{A}\,.\,\hat{n}\;ds$, where $\vec{A} \;=\; (x+y^2)\,\hat{i}-2x\hat{j}+2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant. **Ans.** 81

2. Evaluate $\iint_S \vec{A}\,.\,\hat{n}\;ds$, where $\vec{A} \;=\; z\hat{i}+x\hat{j}-3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$. **Ans.** 90

3. If $\vec{r} \;=\; t\hat{i}-t^2\hat{j}+(t-1)\,\hat{k}$ and $\vec{S} \;=\; 2t^2\hat{i}+6t\hat{k}$, evaluate $\int_0^2 \vec{r}\cdot\vec{S}\;dt$. **Ans.** 12

4. Evaluate $\iint_S \vec{F}\cdot\hat{n}\;dS$, where, $\vec{F} = 18z\,\hat{i}-12\,\hat{j}+3y\hat{k}$ and S is the surface of the plane $2x+3y+6z=12$ in the first octant. **Ans.** 24

5. Evaluate $\iint_S \vec{F}\cdot\hat{n}\;ds$, where, $F = \;2yx\,\hat{i}-yz\hat{j}+x^2\hat{k}$ over the surface S of the cube bounded by the coordinate planes and planes $x=a$, $y=a$ and $z=a$. **Ans.** $\frac{1}{2}a^4$

6. If $\vec{F} = 2y\,\hat{i}-3\,\hat{j}+x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4$, and $z=6$, then evaluate $\iint_S \vec{F}\cdot\hat{n}\;dS$. **Ans.** 132

5.35 VOLUME INTEGRAL

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \vec{F}\;dv$

Example 78. *If* $\vec{F} \;=2\,z\,\hat{i}\;-x\,\hat{j}\;+y\,\hat{k}$, *evaluate* $\iiint_V \vec{F}\;dv$ *where, v is the region bounded by the surfaces*

$x=0,\;\; y=0,\;\; x=2,\;\; y=4,\;\; z=x^2,\;\; z=2.$

Solution. $\iiint_V \vec{F}\;dv \;=\; \iiint(2\,z\,\hat{i}-x\hat{j}+y\hat{k})\;dx\,dy\,dz$

$$= \int_0^2 dx\int_0^4 dy\int_{x^2}^2 (2\,z\hat{i}-x\hat{j}+y\hat{k})\;dz \quad = \int_0^2 dx\int_0^4 dy\;[z^2\,\hat{i}-xz\hat{j}+yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx\int_0^4 dy\;[4\,\hat{i}-2\,x\hat{j}+2\,y\hat{k}-x^4\,\hat{i}+x^3\,\hat{j}-x^2\,y\hat{k}]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k} \right]_0^4$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\, dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k} \right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$ **Ans.**

EXERCISE 5.12

1. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \vec{F}\, dV$, where V is bounded by the plane $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}$

2. Evaluate $\iiint_V \phi\, dV$, where $\phi = 45\, x^2y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$ **Ans.** 128

3. If $\vec{F} = (2x^2 - 3z)\,\hat{i} - 2xy\,\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \times \vec{F}\, dV$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}(\hat{j} - \hat{k})$

4. Evaluate $\iiint_V (2x + y)\, dV$, where V is closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$. **Ans.** $\frac{80}{3}$

5. If $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, evaluate $\iiint \vec{F}\, dV$ over the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$ and $z = x^2$, $z = 4$. **Ans.** $(16\hat{i} - 3\hat{j} + 48\hat{k})$

5.36 GREEN'S THEOREM (For a plane)

Statement. *If* $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ *be continuous functions over a region R bounded by simple closed curve C in x – y plane, then*

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y} \right) dx\, dy$$ (*AMIETE, June 2010, U.P., I Semester, Dec. 2007*)

Proof. Let the curve C be divided into two curves C_1 (ABC) and C_1 (CDA).

Let the equation of the curve C_1 (ABC) be $y = y_1(x)$ and equation of the curve C_2 (CDA) be $y = y_2(x)$.

Let us see the value of

$$\iint_R \frac{\partial\phi}{\partial y}\, dx\, dy = \int_{x=a}^{x=c} \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial\phi}{\partial y}\, dy \right] dx = \int_a^c [\phi(x, y)]_{y=y_1(x)}^{y=y_2(x)}\, dx$$

$$= \int_a^c [\phi(x, y_2) - \phi(x, y_1)]\, dx = -\int_c^a \phi(x, y_2)\, dx - \int_a^c \phi(x, y_1)\, dx$$

$$= -\left[\int_c^a \phi(x, y_2)\, dx + \int_a^c \phi(x, y_1)\, dx \right]$$

$$= -\left[\int_{c_2} \phi(x, y)\, dx + \int_{c_1} \phi(x, y)\, dx \right] = -\oint_c \phi(x, y)\, dx$$

Thus, $\oint_c \phi\, dx = -\iint_R \frac{\partial \phi}{\partial y}\, dx\, dy$ ...(1)

Similarly, it can be shown that

$$\oint_c \psi\, dy = \iint \frac{\partial \psi}{\partial x}\, dx\, dy \qquad ...(2)$$

On adding (1) and (2), we get

$$\oint (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy \quad \textbf{Proved.}$$

Note. Green's Theorem in vector form

$$\int_c \vec{F}\cdot d\vec{r} = \iint_R (\nabla \times \vec{F})\cdot \hat{k}\, dR$$

where, $\vec{F} = \phi\,\hat{i} + \psi\,\hat{j}, \bar{r} = x\hat{i} + y\hat{j}, \hat{k}$ is a unit vector along z-axis and $dR = dx\, dy$.

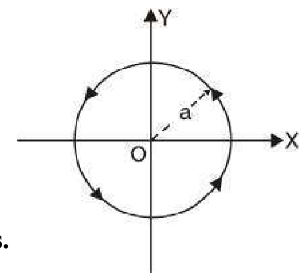
Example 79. *A vector field* $\vec{F}$ *is given by* $\vec{F} = \sin y\hat{i} + x(1+\cos y)\,\hat{j}$.

Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ *where C is the circular path given by* $x^2 + y^2 = a^2$.

Solution. $\vec{F} = \sin y\hat{i} + x(1+\cos y)\,\hat{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [\sin y\hat{i} + x(1+\cos y)\,\hat{j}]\cdot(\hat{i}dx + \hat{j}dy) = \int_C \sin y\, dx + x(1+\cos y)\, dy$$

On applying Green's Theorem, we have



$$\oint_c (\phi\, dx + \psi\, dy) = \iint_s \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \iint_s [(1+\cos y) - \cos y]\, dx\, dy$$

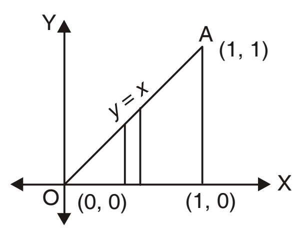
where s is the circular plane surface of radius a.

$$= \iint_s dx\, dy = \text{Area of circle} = \pi a^2. \quad \textbf{Ans.}$$

Example 80. *Using Green's Theorem, evaluate* $\int_c (x^2\, y dx + x^2\, dy)$, *where c is the boundary described counter clockwise of the triangle with vertices* (*0, 0*), (*1, 0*), (*1, 1*).

(*U.P., I Semester, Winter 2003*)

Solution. By Green's Theorem, we have



$$\int_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$\int_c (x^2 y\, dx + x^2\, dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy = \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$

$$= \int_0^1 (2x - x^2)(x)\, dx = \int_0^1 (2x^2 - x^3)\, dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}$$ **Ans.**

Example 81. *State and verify Green's Theorem in the plane for* $\oint (3x^2 - 8y^2)\, dx + (4y - 6xy)$ *dy where C is the boundary of the region bounded by* $x \ge 0$, $y \le 0$ and $2x - 3y = 6$.

(*Uttarakhand, I Semester, Dec. 2006*)

Solution. Statement: See Article 24.4 on page 576.

Here the closed curve C consists of straight lines OB, BA and AO, where coordinates of A and B are (3, 0) and (0, – 2) respectively. Let R be the region bounded by C.

Then by Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\,dy \qquad ...(1)$$

$$= \iint_R (-6y + 16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

$$= 10\int_0^3 dx \int_{\frac{1}{3}(2x-6)}^0 y\,dy = 10\int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^0 = -\frac{5}{9}\int_0^3 dx\,(2x-6)^2$$

$$= -\frac{5}{9}\left[\frac{(2x-6)^3}{3\times 2}\right]_0^3 = -\frac{5}{54}(0+6)^3 = -\frac{5}{54}(216) = -20 \qquad ...(2)$$

Now we evaluate L.H.S. of (1) along OB, BA and AO.

Along OB, $x = 0$, $dx = 0$ and y varies form 0 to –2.

Along BA, $x = \frac{1}{2}(6+3y)$, $dx = \frac{3}{2}\,dy$ and y varies from –2 to 0.

and along AO, $y = 0$, $dy = 0$ and x varies from 3 to 0.

L.H.S. of (1) $= \oint [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$

$$= \int_{OB} [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy] + \int_{BA} [(3x^2 - 8y^2)\,dx + (4x - 6xy)\,dy]$$
$$+ \int_{AO} [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \int_0^{-2} 4y\,dy + \int_{-2}^0 \left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}\,dy\right) + [4y - 3(6+3y)\,y]\,dy + \int_3^0 3x^2\,dx$$

$$= [2y^2]_0^{-2} + \int_{-2}^0 \left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right] dy + (x^3)_3^0$$

$$= 2[4] + \int_{-2}^0 \left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right] dy + (0 - 27)$$

$$= 8 + \left[\frac{9}{8}\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^0 - 27 = -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2\right]$$

$$= -19 + 27 - 56 + 28 = -20 \qquad ...(3)$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

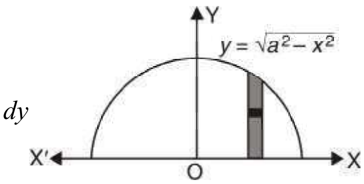
Example 82. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.

(*M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004*)

Solution. $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$

By Green's Theorem, we've $\int_C (\phi\,dx + \psi\,dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}(2x^2-y^2)\right]dx\,dy$$



$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(2x+2y)\,dx\,dy = 2\int_{-a}^{a}dx\int_{0}^{\sqrt{a^2-x^2}}(x+y)\,dy$$

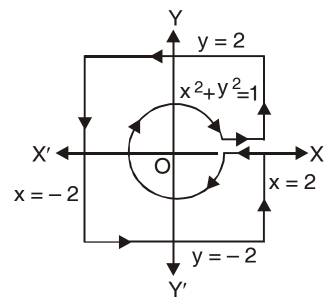
$$= 2\int_{-a}^{a}dx\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} = 2\int_{-a}^{a}\left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx$$

$$= 2\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx+\int_{-a}^{a}(a^2-x^2)\,dx \qquad \left[\begin{array}{ll}\int_{-a}^{a}f(x)\,dx = 2\int_0^a f(x)\,dx, & f \text{ is even}\\ \quad = 0, & f \text{ is odd}\end{array}\right]$$

$$= 0+2\int_0^a(a^2-x^2)\,dx = 2\left(a^2x-\frac{x^3}{3}\right)_0^a = 2\left(a^3-\frac{a^3}{3}\right) = \frac{4a^3}{3}$$ **Ans.**

Example 83. *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2 = 1$ *and* $C_2 : x = \pm 2, y = \pm 2$. (*Gujarat, I Semester, Jan 2009*)

Solution. $\oint_C -\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$



$$= \iint\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}+\frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)dx\,dy$$

$$= \iint\left[\frac{(x^2+y^2)1-2x\,(x)}{(x^2+y^2)^2}+\frac{(x^2+y^2)1-2y\,(y)}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}+\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint\left[\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}\right]dx\,dy = \iint\frac{0}{(x^2+y^2)^2}\,dx\,dy = 0$$ **Ans.**

5.37 AREA OF THE PLANE REGION BY GREEN'S THEOREM

Proof. We know that

$$\int_C M\,dx+N\,dy = \iint_A\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \qquad \text{...(1)}$$

On putting $N = x \left(\frac{\partial N}{\partial x} = 1\right)$ and $M = -y \left(\frac{\partial M}{\partial y} = 1\right)$ in (1), we get

$$\int_C -y\,dx+x\,dy = \iint_A[1-(-1)]\,dx\,dy = 2\iint dx\,dy = 2A$$

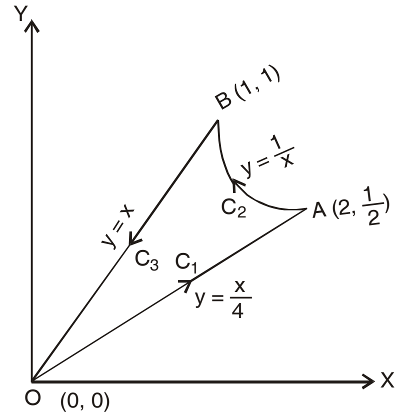
$$\text{Area} = \frac{1}{2}\int_C(x\,dy-y\,dx)$$

Example 84. *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$ (*U.P. I, Semester, Dec. 2008*)

Solution. By Green's Theorem Area A of the region bounded by a closed curve C is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$



Here, C consists of the curves $C_1 : y = \frac{x}{4},\ C_2 : y = \frac{1}{x}$ and $C_3 : y = x$ So

$$\left[A = \frac{1}{2}\oint_C = \frac{1}{2}\left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] = \frac{1}{2}(I_1 + I_2 + I_3)\right]$$

Along $\quad C_1 : y = \frac{x}{4},\ dy = \frac{1}{4}dx,\ x : 0 \text{ to } 2$

$$I_1 = \int_{C_1}(xdy - ydx) = \int_{C_1}\left(x\frac{1}{4}dx - \frac{x}{4}dx\right) = 0$$

Along $\quad C_2 : y = \frac{1}{x},\ dy = -\frac{1}{x^2}dx,\ x : 2 \text{ to } 1$

$$I_2 = \int_{C_2}(xdy - ydx) = \int_2^1\left[x\left(-\frac{1}{x^2}\right)dx - \frac{1}{2}dx\right] = [-2\log x]_2^1 = 2\log 2$$

Along $\quad C_3 : y = x,\ dy = dx\ ;\ x : 1 \text{ to } 0\ ;$

$$I_3 = \int_{C_3}(xdy - ydx) = \int(xdx - xdx) = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{2}(0 + 2\log 2 + 0) = \log 2$$ **Ans.**

EXERCISE 5.13

1. Evaluate $\int_c [(3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz]$ from (0, 0, 0) to (1, 1, 1) along the path c given by the straight line from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (1, 1, 1).

2. Verify Green's Theorem in plane for $\int_C (x^2 + 2xy)\,dx + (y^2 + x^3y)\,dy$, where c is a square with the vertices P (0, 0), Q (1, 0), R (1, 1) and S (0, 1). **Ans.** $-\frac{1}{2}$

3. Verify Green's Theorem for $\int_c (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4. Use Green's Theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where c is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy-plane. **Ans.** $\frac{4}{3}$

5. Apply Green's Theoem to evaluate $\int_c [(y - \sin x)\,dy + \cos x\,dy]$, where c is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$. **Ans.** $-\frac{\pi^2 + 8}{4\pi}$

6. Either directly or by Green's Therorem, evaluate the line integral $\int_c e^{-x}(\cos y\,dx - \sin y\,dy)$, where c is the rectangle with vertices (0, 0), (π, 0,), $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. **Ans.** $2\,(1 - e^{-\pi})$

(*AMIETE II Sem June 2010*)

7. Verify the Green's Theorem to evaluate the line integral $\int_c (2y^2\,dx + 3x\,dy)$, where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

(*U.P., I Semester, Dec. 20005, AMIETE Summer 2004, Winter 2001*) **Ans.** $\frac{27}{4}$

8. Evaluate $\iint_s \bar{F}\,.\,\hat{n}.ds$, where $\vec{F} = x\,y\,\hat{i} - x^2\hat{j} + (x+z)\,\hat{k}$ and s is the region of the plane $2x + 2y + z = 6$ in the first octant. (*A.M.I.E.T.E., Summer 2004, Winter 2001*) **Ans.** $\frac{27}{4}$

9. Verify Green's Theorem for $\int_C \left[(xy + y^2)\,dx + x^2 dy\right]$ where C is the boundary by $y = x$ and $y = x^2$. (*AMIETE, June 2010*)

5.38 STOKE'S THEOREM (Relation between Line Integral and Surface Integral)

(*Uttarakhand, I Sem. 2008*, *U.P., Ist Semester, Dec. 2006*)

Statement. Surface integral of the component of curl $\bar{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically

$$\oint \vec{F}\,.\,d\,\bar{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds,

Proof. Let

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$F = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

On putting the values of $\vec{F}, d\,\vec{r}$ in the statement of the theorem

$$\oint_c (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \iint_S \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (F_1\,\hat{i} + F_2\,\hat{j} + F3\,\hat{k}).\ (\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\,ds$$

$$\oint (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right].$$

$$(\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma)\,ds$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]ds \qquad ...(1)$$

Let us first prove

$$\oint_c F_1\,dx = \iint_S \left[\left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\right]ds \qquad ...(2)$$

Let the equation of the surface S be $z = g\,(x, y)$. The projection of the surface on $x - y$ plane is region R.

$$\oint_c F_1\,(x, y, z)\,dx = \oint_c F_1\,[x, y, g(x, y)]\,dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1\,(x, y, g)\,dx\,dy \qquad \text{[By Green's Theorem]}$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right)dx\,dy \qquad ...(3)$$

The direction consines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{\frac{-\partial g}{\partial x}} = \frac{\cos\beta}{\frac{-\partial g}{\partial y}} = \frac{\cos\gamma}{1}$$

And $dx\, dy$ = projection of ds on the xy-plane = $ds \cos\gamma$

Putting the values of ds in R.H.S. of (2)

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds = \iint_R \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\frac{dx\, dy}{\cos\gamma}$$

$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial F_1}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right) dx\, dy \qquad ...(4)$$

From (3) and (4), we get

$$\oint_c F_1\, dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds \qquad ...(5)$$

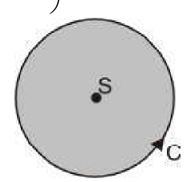
Similarly,
$$\oint_c F_2\, dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) ds \qquad ...(6)$$

and
$$\oint_c F_3\, dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds \qquad ...(7)$$

On adding (5), (6) and (7), we get

$$\oint_c (F_1\, dx + F_2\, dy + F_3\, dz) = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma + \frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right.$$
$$\left. + \frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds \quad \textbf{Proved.}$$

5.39 ANOTHER METHOD OF PROVING STOKE'S THEOREM



The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F}\cdot d\,\overline{r} = \iint_S curl\ \vec{F}\cdot\hat{n}\ d\,\vec{s} = \iint_S curl\ \vec{F}\cdot d\,\vec{S}$$

Proof : The projection of any curved surface over xy-plane can be treated as kernal of the surface integral over actual surface

Now, $$\iint_S (\nabla\times\vec{F})\cdot\hat{k}\ d\,\vec{S} = \iint_S (\nabla\times\vec{F})\cdot(\vec{i}\times\vec{j})\, dx\, dy \qquad [\hat{k} = \hat{i}\times\hat{j}]$$

$$= \iint_S [(\nabla\cdot\hat{i})\,(\vec{F}\cdot\hat{j}) - (\nabla\cdot\hat{j})\,(\vec{F}\cdot\hat{i})]\, dx\, dy = \iint_S \left[\frac{\partial}{\partial x}(F_y) - \frac{\partial}{\partial y}(F_x)\right] dx\, dy$$

$$= \iint_S [F_x\, dx + F_y\, dy] \text{ [By Green's theorem]}$$

$$= \iint_S [\hat{i}\, F_x + \hat{j}\, F_y]\cdot(\hat{i}\, dx + \hat{j}\, dy) = \oint_c \vec{F}\,.\,\vec{dr}$$

$$\iint_S curl\ \vec{F}\cdot\hat{n}\, dS = \oint_c \vec{F}\,.\,\vec{d}\,r.$$

where, $\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$ and $\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\, dx + zx\, dy + xy\, dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (*M.D.U., Dec 2009*)

Solution. Here we have $\oint yz\, dx + zx\, dy + xy\, dz$

$$= \int (yz\hat{i} + zx\hat{j} + xy\hat{k}).\,(\hat{i}dx + \hat{j}dy + kdz)$$

$$= \oint F.dx \qquad\qquad \text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \int curl\ F.\eta\ ds \qquad\qquad = (x-x)\ \hat{i}\ + (y-y)\ \hat{j}\ + (z-z)\ \hat{k}$$

$$= 0 \quad = 0$$ **Ans.**

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\, dx - yz^2 dy - y^2 z\, dz]$$

where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

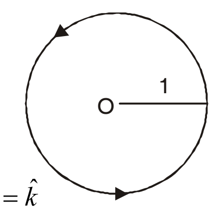
(*U.P., I Semester, Winter 2001*)

Solution. $\int_c [(2x - y)\, dx - yz^2 dy - y^2 z\, dz]$

$$= \int_c [(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}] \cdot (\hat{i}\ dx + \hat{j}\ dy + \hat{k}\ dz)$$

By Stoke's theorem $\oint \vec{F} \cdot d\,\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\ ds$ ...(1)

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$



$$= (-2\,yz + 2\,yz)\,\hat{i} - (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$

Putting the value of curl $\vec{F}$ in (1), we get

$$= \iint \hat{k} \cdot \hat{n}\ ds = \iint \hat{k} \cdot \hat{n}\, \frac{dx\ dy}{\hat{n} \cdot \hat{k}} = \iint dx\ dy = \text{Area of the circle} = \pi \qquad \left[\because ds = \frac{dx\ dy}{(\hat{n} \cdot \hat{k})}\right]$$

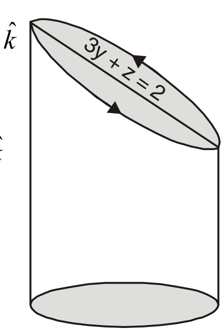
Example 87. *Evaluate* $\int_C \vec{F} \,.\, d\,\vec{r}$, *where* $F\,(x, y, z) = -\,y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.* (*Gujarat, I sem. Jan. 2009*)

Solution. $\oint_C \vec{F} \,.\, \vec{dr} = \iint_S \text{curl } \vec{F} \,.\, \hat{n}\, ds = \iint_S \text{curl } (-\,y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k})\,\hat{n}\, ds$ ...(1)

$$F\,(x, y, z) = -\,y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k}$$ (*By Stoke's Theorem*)

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial\, x} & \dfrac{\partial}{\partial\, y} & \dfrac{\partial}{\partial\, z} \\ -\,y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}\ (0 - 0) - \hat{j}\ (0 - 0) + \hat{k}\ (1 + 2y) = (1 + 2y)\ \hat{k}$$



$x^2 + y^2 = 1$

Normal vector $= \nabla\ \vec{F}$

$$= \left(\hat{i}\,\frac{\partial}{\partial\, x} + \hat{j}\,\frac{\partial}{\partial\, y} + \hat{k}\,\frac{\partial}{\partial\, z}\right)(y + z - 2) = \hat{j} + \hat{k}$$

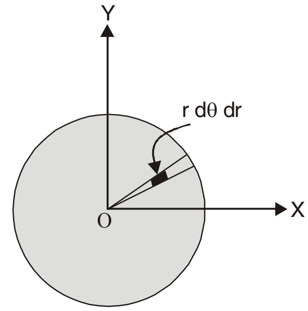
Unit normal vector $\hat{n} = \dfrac{\hat{j} + \hat{k}}{\sqrt{2}}$

$$ds = \frac{dx\ dy}{\hat{\eta}\ .\ \hat{k}}$$

On putting the values of curl $\vec{F}, \hat{n}$ and ds in (1), we get

$$\int_C \vec{F}\,.\,\vec{dr} = \iint_S (1+2y)\,\hat{k}\,.\,\frac{\hat{j}+\hat{k}}{\sqrt{2}}\,\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right).\,\hat{k}}$$

$$= \iint \frac{1+2y}{\sqrt{2}}\,\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$



$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi \qquad \textbf{Ans.}$$

Example 88. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

where c is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$. (*Nagpur, Summer 2001*)

Solution. $\int_c (y\,dx + z\,dy + x\,dz)$

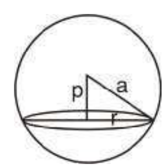
$$= \int_c (y\hat{i}+z\hat{j}+x\hat{k})\cdot(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz) = \int_C (y\hat{i}+z\hat{j}+x\hat{k})\cdot\overline{dr}$$

$$= \iint_S \text{curl}\,(y\hat{i}+z\hat{j}+x\hat{k})\cdot\hat{n}\,ds \qquad \text{(By Stoke's Theorem)}$$

$$= \iint_S\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(y\hat{i}+z\hat{j}+x\hat{k})\cdot\hat{n}\,ds = \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\hat{n}\,ds \quad ...(1)$$

where S is the circle formed by the intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+z-a)}{|\nabla\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{1+1}}$$



$$\therefore \qquad \hat{n} = \frac{\hat{i}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}$$

Putting the vlaue of $\hat{n}$ in (1), we have

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}\right)ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds \qquad \left[\text{Use } r^2 = R^2-p^2 = a^2-\frac{a^2}{2} = \frac{a^2}{2}\right]$$

$$= \frac{-2}{\sqrt{2}}\iint_S ds = \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}} \qquad \textbf{Ans.}$$

Example 89. *Directly or by Stoke's Theorem, evaluate $\iint_s \text{curl}\,\vec{v}\cdot\hat{n}\,ds$, $\vec{v} = \hat{i}y+\hat{j}z+\hat{k}x$, s is the surface of the paraboloid $z = 1-x^2-y^2$, $z^3 \geq 0$ and $\hat{n}$ is the unit vector normal to s.*

Solution. $$\overline{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Obviously $\hat{n} = \hat{k}.$

Therefore $(\nabla \times \vec{v}) \cdot \hat{n} = (-\hat{i} - \hat{j} - \hat{k}).\, \hat{k} = -1$

Hence $$\iint_S (\nabla \times \overline{v}) \cdot \hat{n}\, ds = \iint_S (-1)\, dx\, dy = -\iint_S dx\, dy$$
$$= -\pi\,(1)^2 = -\pi.$$ (Area of circle = πr^2) **Ans.**

Example 90. *Use Stoke's Theorem to evaluate* $\int_c \vec{v} \cdot \vec{dr}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\,\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9,\ z > 0$, *oriented in the positive direction.*

Solution. By Stoke's theorem

$$\int_c \vec{v} \cdot \vec{dr} = \iint_S (\text{curl}\ \vec{v}) \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, ds$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = (0-0)\,\hat{i} - (z-0)\,\hat{j} + (y - 2\,y)\,\hat{k}$$
$$= -z\hat{j} - y\hat{k}$$

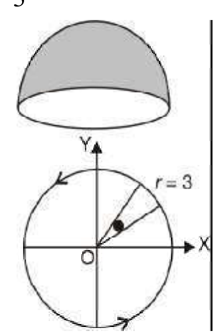
$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)}{|\nabla \phi|}$$

$$= \frac{2\,x\hat{i} + 2\,y\hat{j} + 2\,z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4\,z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}$$

$$(\nabla \times \vec{v}) \cdot \hat{n} = (-z\hat{j} - y\hat{k}) \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3} = \frac{-yz - yz}{3} = \frac{-2\,yz}{3}$$

$$\hat{n} \cdot \hat{k}\, ds = dx\, dy \Rightarrow \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}.\,\hat{k}\ dx = dx\, dy \Rightarrow \frac{z}{3}\, ds = dx\, dy$$

$\therefore$ $$ds = \frac{3}{z}\, dx\, dy$$



$$\iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, ds = \iint \left(\frac{-2\,yz}{3}\right)\left(\frac{3}{z}\, dx\, dy\right) = -\iint 2\,y\, dx\, dy$$
$$= -\iint 2r \sin\theta\, r\, d\,\theta\, dr = -2\int_0^{2\pi} \sin\theta\, d\,\theta \int_0^3 r^2\, dr$$
$$= -2\,(-\cos\theta)_0^{2\pi} \cdot \left[\frac{r^3}{3}\right]_0^3 = -2\,(-1+1)\,9 = 0$$ **Ans.**

Example 91. *Evaluate the surface integral* $\iint_S \text{curl}\ \vec{F}\,.\,\hat{n}\, d\,S$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$ and $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}\,\cdot$ (*K. University, Dec. 2008*)

Solution. $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Obviously $\hat{n} = \hat{k}.$

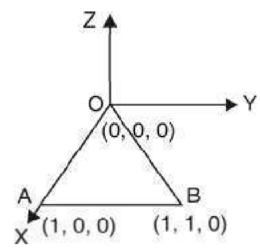
Therefore $(\nabla \times \vec{F}) \cdot \hat{n} = (-\hat{i} - \hat{j} - \hat{k}).\, \hat{k} = -1$

Hence $$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (-1)\, dx\, dy = -\iint_S dx\, dy$$

$$= -\pi\,(1)^2 = -\pi.$$ (Area of circle $= \pi r^2$) **Ans.**

Example 92. *Evaluate* $\oint_C \vec{F} \cdot \vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$ *and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*
(*U.P., I Semester, Winter 2000*)

Solution. We have, curl $\vec{F} = \nabla \times \vec{F}$



$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0.\,\hat{i} + \hat{j} + 2\,(x-y)\,\hat{k}.$$

We observe that z co-ordinate of each vertex of the triangle is zero. Therefore, the triangle lies in the xy-plane.

$\therefore \quad \hat{n} = \hat{k}$

$\therefore \quad$ curl $\vec{F} \cdot \hat{n} = [\hat{j} + 2\,(x-y)\hat{k}]\,.\,\hat{k} = 2\,(x-y).$

In the figure, only xy-plane is considered.
The equation of the line OB is $y = x$
By Stoke's theorem, we have

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\text{curl}\, \vec{F} \cdot \hat{n})\, ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2\,(x-y)\, dx\, dy = 2\int_0^1 \left[xy - \frac{y^2}{2} \right]_0^x dx$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx = 2\int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2\, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$ **Ans.**

Example 93. *Evaluate* $\oint_C \vec{F} \cdot \vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = (x^2 + y^2)\,\hat{i} - 2\,xy\,\hat{j}$ *and C is the boundary of the rectangle* $x = \pm\, a$, $y = 0$ *and* $y = b$. (*U.P., I Semester, Winter 2002*)

Solution. Since the z co-ordinate of each vertex of the given rectangle is zero, hence the given rectangle must lie in the xy-plane.

Here, the co-ordinates of A, B, C and D are $(a, 0)$, (a, b), $(-a, b)$ and $(-a, 0)$ respectively.

$$\therefore \quad \text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = -4\,y\,k$$

Here, $\hat{n} = \hat{k}$, so by Stoke's theorem, we've

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\, \overline{F} \cdot \hat{n}\, d\, s$$

$$= \iint_S (-4\, y\, \hat{k}) \cdot (\hat{k})\, d\, x\, d\, y \quad = -4 \int_{x=-a}^{a} \int_{y=0}^{b} y\, dx\, dy$$

$$= -4 \int_{-a}^{a} \left[\frac{y^2}{2} \right]_0^b dx = -2\, b^2 \int_{-a}^{a} dx = -4\, a\, b^2 \qquad \textbf{Ans.}$$

Example 94. *Apply Stoke's Theorem to calculate* $\int_c 4\, y\, dx + 2\, z\, dy + 6\, y\, dz$ *where c is the curve of intersection of* $x^2 + y^2 + z^2 = 6\, z$ *and* $z = x + 3$.

Solution.
$$\int_c \vec{F} \cdot \vec{dr} = \int_c 4\, y\, dx + 2\, z\, dy + 6\, y\, dz$$

$$= \int_c (4\, y\hat{i} + 2\, z\hat{j} + 6\, y\hat{k}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\vec{F} = 4\, y\hat{i} + 2\, z\hat{j} + 6\, y\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4\, y & 2\, z & 6\, y \end{vmatrix} \begin{aligned} &= (6-2)\, \hat{i} - (0-0)\, \hat{j} + (0-4)\, \hat{k} \\ &= 4\, \hat{i} - 4\, \hat{k} \end{aligned}$$

S is the surface of the circle $x^2 + y^2 + z^2 = 6z$, $z = x + 3$, $\hat{n}$ is normal to the plane $x - z + 3 = 0$

$$\hat{n} = \frac{\nabla\, \phi}{|\nabla\, \phi|} = \frac{\left(\hat{i}\, \dfrac{\partial}{\partial x} + \hat{j}\, \dfrac{\partial}{\partial y} + \hat{k}\, \dfrac{\partial}{\partial z} \right)(x - z + 3)}{|\nabla\, \phi|} = \frac{\hat{i} - \hat{k}}{\sqrt{1+1}} = \frac{\hat{i} - \hat{k}}{\sqrt{2}}$$

$$(\nabla \times F) \cdot \hat{n} = (4\, \hat{i} - 4\, \hat{k}) \cdot \frac{\hat{i} - \hat{k}}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = 4\sqrt{2}$$

$$\int_c \vec{F} \cdot \vec{dr} = \iint_S (\text{curl}\, F) \cdot \hat{n}\, ds = \iint_S 4\sqrt{2}\, (dx\, dz) = 4\sqrt{2} \text{ (area of circle)}$$

Centre of the sphere $x^2 + y^2 + (z - 3)^2 = 9$, $(0, 0, 3)$ lies on the plane $z = x + 3$. It means that the given circle is a great circle of sphere, where radius of the circle is equal to the radius of the sphere.

Radius of circle = 3, Area = $\pi\, (3)^2 = 9\, \pi$

$$\iint_S (\nabla \times F) \cdot \hat{n}\, ds = 4\sqrt{2}(9\, \pi) = 36\sqrt{2}\, \pi \qquad \textbf{Ans.}$$

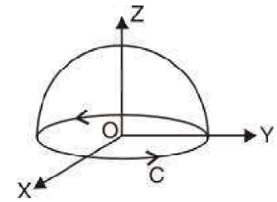
Example 95. *Verify Stoke's Theorem for the function* $\overline{F} = z\, \hat{i} + x\, \hat{j} + y\, \hat{k}$, *where C is the unit circle in xy-plane bounding the hemisphere* $z = \sqrt{(1 - x^2 - y^2)}$. (*U.P., I Semester Comp. 2002*)

Solution. Here
$$\overline{F} = z\, \hat{i} + x\, \hat{j} + y\, \hat{k}. \qquad \text{...(1)}$$

Also,
$$\overline{r} = x\hat{i} + y\, \hat{j} + z\, \hat{k} \quad \Rightarrow \quad \overline{dr} = d\, x\, \hat{i} + dy\, \hat{j} + dz\, \hat{k}.$$

$\therefore$
$$\overline{F} \cdot \overline{dr} = z\, dx + x\, dy + y\, dz.$$

$\therefore$
$$\oint_C \overline{F} \cdot \overline{dr} = \oint_C (z\, dx + x\, dy + y\, dz). \qquad \text{...(2)}$$



On the circle C, $x^2 + y^2 = 1$, $z = 0$ on the xy-plane. Hence on C, we have $z = 0$ so that $dz = 0$. Hence (2) reduces to

$$\oint_C \overline{F} \cdot \overline{dr} = \oint_C x\, dy. \qquad ...(3)$$

Now the parametric equations of C, *i.e.*, $x^2 + y^2 = 1$ are

$$x = \cos\phi,\ y = \sin\phi. \qquad ...(4)$$

Using (4), (3) reduces to $\oint_C \vec{F} \,.\, \vec{dr} = \int_{\phi=0}^{2\pi} \cos\phi \cos\phi\, d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2}\, d\phi$

$$= \frac{1}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi} = \pi \qquad ...(5)$$

Let $P(x, y, z)$ be any point on the surface of the hemisphere $x^2 + y^2 + z^2 = 1$, O origin is the centre of the sphere.

Radius $= \text{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ $\qquad$ Normal $= x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = x\hat{i} + y\hat{j} + z\hat{k}$$

(Radius is $\perp$ to tangent *i.e*. Radius is normal)

$$x = \sin\theta\cos\phi,\quad y = \sin\theta\sin\phi,\quad z = \cos\theta \qquad ...(6)$$

$$\hat{n} = \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k}$$

Also, $\qquad \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k} \qquad ...(7)$

$$\text{Curl } \vec{F}\cdot\hat{n} = (\hat{i} + \hat{j} + \hat{k})\,.\,(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \sin\theta\,\hat{k})$$
$$= \sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta$$

$\therefore$ $$\iint_S \text{Curl } \vec{\boldsymbol{F}}\cdot\hat{\boldsymbol{n}}\, dS = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (\hat{i} + \hat{j} + \hat{k})$$
$$.\ (\sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k})\sin\theta\, d\theta\, d\phi$$
$$= \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\, d\phi$$

$[\because\ dS$ = Elementary area on hemisphere $= \sin\theta\, d\theta\, d\phi]$

$$= \int_0^{\pi/2} \sin\theta\, d\theta\, [\sin\theta\sin\phi + \sin\theta(-\cos\phi) + \phi\cos\theta]_0^{2\pi} = \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \int_0^{\pi/2} (0 + 0 + 2\pi\sin\theta\cos\theta)\, d\theta = \pi\int_0^{\pi/2} \sin 2\theta\, d\theta = \pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -(\pi/2)\,[-1 - 1] = \pi.$$

From (5) and (8), $\oint_C \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{dr}} = \iint_S \text{curl } \overline{F}\cdot\hat{n}\, d\,S,$ which verifies Stokes's theorem.

Example 96. *Verify Stoke's theorem for the vector field* $\overline{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$ *over the upper half of the surface* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on xy- plane.*

(*Nagpur University, Summer 2001*)

Solution. Let S be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary C or S is a circle in the xy plane of radius unity and centre O. The equation of C are $x^2 + y^2 = 1$, $z = 0$ whose parametric form is

$$x = \cos t,\ y = \sin t,\ z = 0,\ 0 < t < 2\pi$$

$$\int_C \vec{F}\,.\,\vec{dr} = \int_C [(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}].[\hat{i}\ dx + \hat{j}\,dy + \hat{k}\,dz]$$

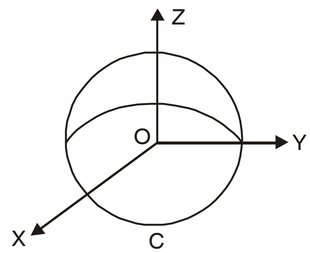
$= \int_C [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz] = \int_C (2x - y)\,dx,$ since on C, $z = 0$ and $2z = 0$

$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt$

$= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\,dt = \int_0^{2\pi}\left(-\sin 2t + \frac{1 - \cos 2t}{2}\right)dt$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi \qquad \text{...(1)}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\,\hat{i} + (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$



$\text{Curl } \vec{F}\,.\,\hat{n} = \hat{k}\,.\,\hat{n} = \hat{n}\,.\,\hat{k}$

$\iint_S \text{Curl } \vec{F}\,.\,\hat{n}\,ds = \iint_S \hat{n}\,.\,\hat{k}\,ds = \iint_R \hat{n}\,.\,\hat{k}\,.\,\frac{dx}{\hat{n}}\,.\,\frac{dy}{\hat{k}}$

Where R is the projection of S on xy-plane.

$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy \quad = \int_{-1}^{1} 2\sqrt{1 - x^2}\,dx = 4\int_0^1 \sqrt{1 - x^2}\,dx$

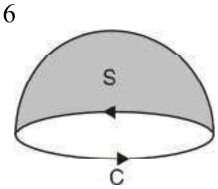
$$= 4\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = 4\left[\frac{1}{2}.\frac{\pi}{2}\right] = \pi \qquad \text{...(2)}$$

From (1) and (2), we have

$\therefore \quad \int_C \vec{F}\,.\,\vec{dr} = \iint \text{Curl } \bar{F}\,.\,\hat{n}\,ds$ which is the Stoke's theorem. **Ans.**

Example 97. *Verify Stoke's Theorem for* $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

Solution. $\int_c \vec{F}\cdot\vec{dr},$ where c is the boundary of the circle $x^2 + y^2 + z^2 = 16$ (bounding the hemispherical surface)



$= \int_c [(x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}]\cdot(\hat{i}dx + \hat{j}dy)$

$= \int_c [(x^2 + y - 4)\,dx + 3\,xy\,dy)]$

Putting $\quad x = 4\cos\theta$, $y = 4\sin\theta$, $dx = -4\sin\theta\,d\theta$, $dy = 4\cos\theta\,d\theta$

$= \int_0^{2\pi} [(16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\,d\theta) + (192\sin\theta\cos^2\theta\,d\theta)]$

$= 16\int_0^{2\pi} [-4\cos^2\theta\sin\theta - \sin^2\theta + \sin\theta + 12\sin\theta\cos^2\theta]\,d\theta$

$= 16\int_0^{2\pi} (8\sin\theta\cos^2\theta - \sin^2\theta + \sin\theta)\,d\theta$

$= -16\int_0^{2\pi} \sin^2\theta\,d\theta \qquad \left\{\begin{array}{l} \int_0^{2\pi} \sin^n\theta\cos\theta\,d\theta = 0 \\ \int_0^{2\pi} \cos^n\theta\sin\theta\,d\theta = 0 \end{array}\right\}$

$= -16 \times 4\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = -64\left(\frac{1}{2}\,\frac{\pi}{2}\right) = -16\,\pi.$

To evaluate surface integral $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$

$$= (0-0)\,\hat{i} - (2z-0)\,\hat{j} + (3y-1)\,\hat{k} = -2z\hat{j} + (3y-1)\,\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-16)}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4}$$

$$(\nabla\times\vec{F})\cdot\hat{n} = [-2z\hat{j} + (3y-1)\,\hat{k}]\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{4} = \frac{-2yz + (3y-1)z}{4}$$

$$\hat{k}\cdot\hat{n}\cdot ds = dx\,dy \Rightarrow \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4}\,.\,k\,ds = dx\,dy \Rightarrow \frac{z}{4}ds = dx\,dy$$

$$\therefore \qquad ds = \frac{4}{z}\,dx\,dy$$

$$\iint(\nabla\times F)\cdot\hat{n}\,ds = \iint\frac{-2yz+(3y-1)z}{4}\left(\frac{4}{z}\,dx\,dy\right) = \iint[-2y+(3y-1)]\,dx\,dy = \iint(y-1)\,dx\,dy$$

On putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$, we get

$$= \iint(r\sin\theta - 1)\,r\,d\theta\,dr = \int d\theta\int(r^2\sin\theta - r)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{r^3}{3}\sin\theta - \frac{r^2}{2}\right)_0^4 = \int_0^{2\pi} d\theta\left(\frac{64}{3}\sin\theta - 8\right)$$

$$= \left(-\frac{64}{3}\cos\theta - 8\theta\right)_0^{2\pi} = \frac{-64}{3} - 16\pi + \frac{64}{3} = -16\pi$$

The line integral is equal to the surface integral, hence Stoke's Theorem is verified. **Proved.**

Example 98. *Verify Stoke's theorem for a vector field defined by* $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ *in the rectangular in xy-plane bounded by lines* $x = 0$, $x = a$, $y = 0$, $y = b$.

(*Nagpur University, Summer 2000*)

Solution. Here we have to verify Stoke's theorem $\int_C \vec{F}\,.\,\vec{dr} = \iint_S(\nabla\times\overline{F})\,.\,\hat{n}\,ds$

Where 'C' be the boundary of rectangle (*ABCD*) and *S* be the surface enclosed by curve *C*.

$$\vec{F} = (x^2-y^2)\,\hat{i} + (2xy)\,\hat{j}$$

$$\vec{F}\,.\,\vec{dr} = [(x^2-y^2)\,\hat{i} + 2xy\,\hat{j}]\,.\,[\hat{i}\,dx + \hat{j}\,dy]$$

$$\Rightarrow \qquad \vec{F}\,.\,\vec{dr} = (x^2+y^2)\,dx + 2xy\,dy \qquad ...(1)$$

$$\text{Now,} \qquad \int_C \vec{F}\,.\,\vec{dr} = \int_{OA} F\,.\,\vec{dr} + \int_{AB}\vec{F}\,.\,\vec{dr} + \int_{BC}\vec{F}\,.\,\vec{dr} + \int_{CO}\vec{F}\,.\,\vec{dr} \qquad ...(2)$$

Along *OA*, put $y = 0$ so that $k\,dy = 0$ in (1) and $\vec{F}\,.\,\vec{d}r = x^2\,dx$,
Where x is from 0 to a.

$$\therefore \qquad \int_{OA}\vec{F}\,.\,\vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ...(3)$$

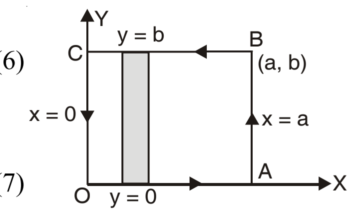
Along *AB*, put $x = a$ so that $dx = 0$ in (1), we get $\vec{F}\,.\,\vec{d}r = 2ay\,dy$
Where y is from 0 to b.

$$\therefore \qquad \int_{AB}\vec{F}\,.\,\vec{d}r = \int_0^b 2ay\,dy = [ay^2]_0^b = ab^2 \qquad ...(4)$$

Along BC, put $y = b$ and $dy = 0$ in (1) we get $\overline{F}\,.\,\overline{dr} = (x^2 - b^2)\,dx$, where x is from a to 0.

$$\therefore \qquad \int_{BC} \vec{F}\,.\,\vec{dr} = \int_a^0 (x^2 - b^2)\,dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = \frac{-a^3}{3} + b^2 a \qquad ...(5)$$

Along CO, put $x = 0$ and $dx = 0$ in (1), we get $\vec{F}\,.\,\vec{dr} = 0$



$$\therefore \qquad \int_{CO} \vec{F}.\,\vec{dr} = 0 \qquad ...(6)$$

Putting the values of integrals (3), (4), (5) and (6) in (2), we get

$$\int_C \vec{F}.\,\vec{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2 \qquad ...(7)$$

Now we have to evaluate R.H.S. of Stoke's Theorem *i.e.* $\iint_S (\nabla \times \vec{F})\,.\,\hat{n}\,ds$

We have,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = (2y + 2y)\,\hat{k} = 4y\,\hat{k}$$

Also the unit vector normal to the surface S in outward direction is $\hat{n} = k$

($\because$ z-axis is normal to surface S)

Also in xy-plane $ds = dx\,dy$

$$\therefore \qquad \iint_S (\nabla \times \vec{F})\,.\,\hat{n}.\,ds = \iint_R 4y\,\hat{k}\,.\,\hat{k}\,dx\,dy = \iint_R 4y\,dx\,dy.$$

Where R be the region of the surface S.

Consider a strip parallel to y-axis. This strip starts on line y = 0 (*i.e.* x-axis) and end on the line $y = b$, We move this strip from $x = 0$ (y-axis) to $x = a$ to cover complete region R.

$$\therefore \qquad \iint_S (\nabla \times \vec{F})\,.\,\hat{n}.\,ds = \int_0^a \left[\int_0^b 4y\,dy\right] dx = \int_0^a [2y^2]_0^b\,dx$$

$$= \int_0^a 2b^2\,dx = 2b^2\,[x]_0^a = 2ab^2 \qquad ...(8)$$

$\therefore$ From (7) and (8), we get

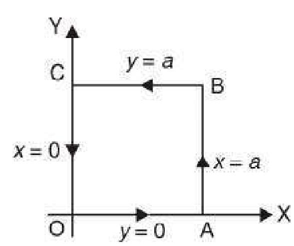
$\int_C \vec{F}\,.\,\vec{dr} = \iint_S (\nabla \times \overline{F})\,.\,\hat{n}\,ds$ and hence the Stoke's theorem is verified.

Example 99. *Verify Stoke's Theorem for the function*

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane z = 0 and bounded by the lines

$x = 0,\ y = 0,\ x = a,\ y = a$.



Solution. We have, $\vec{F} = x^2\hat{i} - xy\hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= (0 - 0)\,\hat{i} - (0 - 0)\,\hat{j} + (-y - 0)\,\hat{k} = -y\hat{k} \qquad (\hat{n} \perp \text{ to } xy \text{ plane } i.e.\ \hat{k})$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (-yk) \cdot k\, dx\, dy$$

$$= \int_0^a dx \int_0^a -y dy = \int_0^a dx \left[-\frac{y^2}{2} \right]_0^a = -\frac{a^2}{2} (x)_0^a = -\frac{a^3}{2} \qquad \text{...(1)}$$

To obtain line integral

$$\int_C \vec{F} \cdot \vec{dr} = \int_C (x^2 \hat{i} - xy\hat{j}) \cdot (\hat{i}\, dx + \hat{j}\, dy) = \int_C (x^2\, dx - xy\, dy)$$

where c is the path $OABCO$ as shown in the figure.

Also, $$\int_C \vec{F} \cdot \vec{dr} = \int_{OABCO} \vec{F} \cdot dr = \int_{OA} \vec{F} \cdot dr + \int_{AB} \vec{F} \cdot dr + \int_{BC} \vec{F} \cdot dr + \int_{CO} \vec{F} \cdot dr \qquad \text{...(2)}$$

Along OA, $y = 0$, $dy = 0$

line	Eq. of line		Lower limit	Upper limit
OA	$y = 0$	$dy = 0$	$x = 0$	$x = a$
AB	$x = a$	$dx = 0$	$y = 0$	$y = a$
BC	$y = a$	$dy = 0$	$x = a$	$x = 0$
CO	$x = 0$	$dx = 0$	$y = a$	$y = 0$

$$\int_{OA} \vec{F} \cdot \vec{dr} = \int_{OA} (x^2 dx - xy dy)$$

$$= \int_0^a x^2 dx = \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3}$$

Along AB, $x = a$, $dx = 0$

$$\int_{AB} \vec{F} \cdot \vec{dr} = \int_{AB} (x^2\, dx - x\, y\, d\, y)$$

$$= \int_0^a -a\, y\, d\, y = -a \left[\frac{y^2}{2} \right]_0^a = -\frac{a^3}{2}$$

Along BC, $y = a$, $dy = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_{BC} (x^2 dx - xy\, dy) = \int_a^0 x^2 dx = \left[\frac{x^3}{3} \right]_a^0 = -\frac{a^3}{3}$$

Along CO, $x = 0$, $dx = 0$

$$\int_{CO} \vec{F} \cdot \vec{dr} = \int_{CO} (x^2 dx - xy\, dy) = 0$$

Putting the values of these integrals in (2), we have

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \qquad \text{...(3)}$$

From (1) and (3), $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \int_C \vec{F} \cdot \vec{dr}$

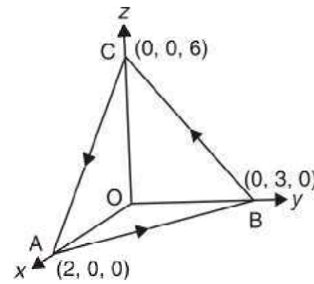
Hence, Stoke's Theorem is verified. **Ans.**

Example 100. *Verify Stoke's Theorem for* $\vec{F} = (x + y)\ \hat{i}\ + (2x - z)\ \hat{j}\ + (y + z)\ \hat{k}$ *for the surface of a triangular lamina with vertices* $(2, 0, 0)$, $(0, 3, 0)$ *and* $(0, 0, 6)$.

(*Nagpur University 2004, K. U. Dec. 2009, 2008, A.M.I.E.T.E., Summer 2000*)

Solution. Here the path of integration c consists of the straight lines AB, BC, CA where the co-ordinates of A, B, C and (2, 0, 0), (0, 3, 0) and (0, 0, 6) respectively. Let S be the plane surface of triangle ABC bounded by C. Let $\hat{n}$ be unit normal vector to surface S. Then by Stoke's Theorem, we must have

$$\oint_c \vec{F} \cdot \vec{dr} = \iint_s \text{curl}\, \vec{F} \cdot \hat{n}\, ds \qquad \text{...(1)}$$



L.H.S. of (1)= $\int_{ABC}^{c} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CA} \vec{F}\cdot\vec{dr}$

Along line AB, $z = 0$, equation of AB is $\frac{x}{2} + \frac{y}{3} = 1$

$\Rightarrow \qquad y = \frac{3}{2}(2 - x),\ dy = -\frac{3}{2}dx$

At A, $x = 2$, At B, $x = 0$, $\bar{r} = x\hat{i} + y\hat{j}$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{AB}[(x+y)\hat{i} + 2x\hat{j} + y\hat{k}]\cdot(\hat{i}dx + \hat{j}dy)$$

$$= \int_{AB}(x+y)\,dx + 2xdy$$

$$= \int_{AB}\left(x + 3 - \frac{3x}{2}\right)dx + 2x\left(-\frac{3}{2}dx\right)$$

$$= \int_{2}^{0}\left(-\frac{7x}{2} + 3\right)dx = \left(-\frac{7x^2}{4} + 3x\right)_{2}^{0}$$

$$= (7 - 6) = +1$$

line	Eq. of line		Lower limit	Upper limit
AB	$\frac{x}{2} + \frac{y}{3} = 1$, $z = 0$	$dy = -\frac{3}{2}dx$	At A $x = 2$	At B $x = 0$
BC	$\frac{y}{3} + \frac{z}{6} = 1$, $x = 0$	$dz = -2dy$	At B $y = 3$	At C $y = 0$
CA	$\frac{x}{2} + \frac{z}{6} = 1$, $y = 0$	$dz = -3dx$	At C $x = 0$	At A $x = 2$

Along line BC, $x = 0$, Equation of BC is $\frac{y}{3} + \frac{z}{6} = 1$ or $z = 6 - 2y,\ dz = -2dy$

At B, $y = 3$, At C, $y = 0$, $\vec{r} = y\hat{j} + z\hat{k}$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{BC}[yi + zj + (y+z)k]\cdot(jdy + kdz) = \int_{BC} -zdy + (y+z)\,dz$$

$$= \int_{3}^{0}(-6 + 2y)\,dy + (y + 6 - 2y)(-2dy)$$

$$= \int_{3}^{0}(4y - 18)\,dy = (2y^2 - 18y)_3^0 = 36$$

Along line CA, $y = 0$, Eq. of CA, $\frac{x}{2} + \frac{z}{6} = 1$ or $z = 6 - 3x$, $dz = -3dx$

At C, $x = 0$, at A, $x = 2$, $\vec{r} = x\hat{i} + z\hat{k}$

$$\int_{CA} \vec{F}\cdot\vec{dr} = \int_{CA}[x\hat{i} + (2x - z)\,\hat{j} + z\hat{k}]\cdot[dx\hat{i} + dz\hat{k}] = \int_{CA}(xdx + zdz)$$

$$= \int_{0}^{2} xdx + (6 - 3x)(-3dx) = \int_{0}^{2}(10x - 18)\,dx = [5x^2 - 18x]_0^2 = -16$$

$$\text{L.H.S. of (1)} = \int_{ABC} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CA} \vec{F}\cdot\vec{dr} = 1 + 36 - 16 = 21 \quad ...(2)$$

$$\text{Curl } \overline{F} = \nabla \times \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(x+y)\,\hat{i} + (2x-z)\,\hat{j} + (y+z)\,k]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = (1+1)\,\hat{i} - (0-0)\,\hat{j} + (2-1)\,\hat{k} = 2\hat{i} + \hat{k}$$

Equation of the plane of ABC is $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right) = \frac{\hat{i}}{2} + \frac{\hat{j}}{3} + \frac{\hat{k}}{6}$$

$$\text{Unit Normal Vector} = \frac{\dfrac{\hat{i}}{2} + \dfrac{\hat{j}}{3} + \dfrac{\hat{k}}{6}}{\sqrt{\dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{36}}}$$

$$\hat{n} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$$

$$\text{R.H.S. of (1)} = \iint_s \text{curl } \vec{F}\cdot\hat{n}\, ds = \iint_s (2\hat{i} + \hat{k})\cdot\frac{1}{\sqrt{4}}(3\hat{i} + 2\hat{j} + \hat{k})\frac{dx\,dy}{\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).\hat{k}}$$

$$= \iint_s \frac{(6+1)}{\sqrt{14}}\frac{dx\,dy}{\frac{1}{\sqrt{14}}} = 7\iint dx\,dy = 7 \text{ Area of } \Delta \text{ OAB}$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right) = 21 \quad ...(3)$$

with the help of (2) and (3) we find (1) is true and so Stoke's *Theorem* is verified.

Example 101. *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2)\ \hat{i} + (yz + 4)\ \hat{j} - (xz)\ \hat{k}$$

over the surface of a cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the XOY plane (*open the bottom*).

Solution. Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_c \vec{F}\cdot\vec{d}\,r = \int[(y - z + 2)\,\hat{i} + (yz + 4)\,\hat{j} - (xz)\,\hat{k}]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c (y - z + 2)\,dx + (yz + 4)\,dy - xzdz$$

$$\int_c \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr} \quad ...(1)$$

(1) Along *OA*, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

	Line	Equ. of line		Lower limit	Upper limit	$\bar{F}.\bar{dr}$
1	OA	$y=0$ $z=0$	$dy=0$ $dz=0$	$x=0$	$x=2$	$2\,dx$
2	AB	$x=2$ $z=0$	$dx=0$ $dz=0$	$y=0$	$y=2$	$4\,dy$
3	BC	$y=2$ $z=0$	$dy=0$ $dz=0$	$x=2$	$x=0$	$4\,dx$
4	CO	$x=0$ $z=0$	$dx=0$ $dz=0$	$y=2$	$y=0$	$4\,dy$

$$\int_{OA} \vec{F}\cdot\vec{dr} = \int_0^2 2\,dx = [2x]_0^2 = 4$$

(2) Along AB, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_0^2 4\,dy = 4\,(y)_0^2 = 8$$

(3) Along BC, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_0^2 (2-0+2)\,dx = (4x)_2^0 = -\,8$$

(4) Along CO, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F}\cdot\vec{dr} = \int (y-0+2)\times 0 + (0+4)\,dy - 0$$

$$= 4\int dy = 4\,(y)_2^0 = -\,8$$

On putting the values of these integrals in (1), we get

$$\int_c \vec{F}\cdot\vec{dr} = 4 + 8 - 8 - 8 = -\,4$$

To obtain surface integral

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z+2 & yz+4 & -\,xz \end{vmatrix}$$

$$= (0-y)\;\hat{i} - (-\,z+1)\;\hat{j} + (0-1)\;\hat{k} = -\,y\,\hat{i} + (z-1)\;\hat{j} - \hat{k}$$

Here we have to integrate over the five surfaces, *ABDE, OCGF, BCGD, OAEF, DEFG.*

Over the surface $ABDE$ ($x = 2$), $\hat{n} = i$, $ds = dy\,dz$

$$\iint (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint [-\,yi + (z-1)\,j - k]\cdot i\,dx\,dz = \iint -\,y\,dy\,dz$$

$$= \iint_R \left[F_3\,(x,y,z)\right]_{z=f_1(x,y)}^{z=f_2(x,y)}\,dx\,dy$$

	Surface	Outward normal	ds	
1	*ABDE*	i	$dy\,dz$	$x = 2$
2	*OCGF*	$-i$	$dy\,dz$	$x = 0$
3	*BCGD*	j	$dx\,dz$	$y = 2$
4	*OAEF*	$-j$	$dx\,dz$	$y = 0$
5	*DEFG*	k	$dx\,dy$	$z = 2$

$$= -\int_0^2 y\,dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface OCGF ($x = 0$), $\hat{n} = -i$, $ds = dy\,dz$

$$\iint (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\,\hat{j} - \hat{k}] \cdot (-\hat{i})\,dy\,dz$$

$$= \iint y\,dy\,dz = \int_0^2 y\,dy \int_0^2 dz = 2\left[\frac{y^2}{2}\right]_0^2 = 4$$

(3) Over the surface *BCGD*, ($y = 2$), $\hat{n} = j$, $ds = dx\,dz$

$$\iint (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\,\hat{j} - \hat{k}] \cdot \hat{j}\,dx\,dz$$

$$= -\iint (z-1)\,dx\,dz = \int_0^2 dx \int_0^2 (z-1)\,dz = -(x)_0^2 \left(\frac{z^2}{2} - z\right)_0^2 = 0$$

(4) Over the surface *OAEF*, ($y = 0$), $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\hat{j} - \hat{k}] \cdot (-\hat{j})\,dx\,dz$$

$$= -\iint (z-1)\,dx\,dz = -\int_0^2 dx \int_0^2 (z-1)\,dz = -(x)_0^2 \left(\frac{z^2}{2} - z\right)_0^2 = 0$$

(5) Over the surface *DEFG,* ($z = 2$), $\hat{n} = k$, $ds = dx\,dy$

$$\iint (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\hat{j} - \hat{k}] \cdot \hat{k}\,dx\,dy = -\iint dx\,dy$$

$$= -\int_0^2 dx \int_0^2 dy = -[x]_0^2\,[y]_0^2 = -4$$

Total surface integral $= -4 + 4 + 0 + 0 - 4 = -4$

Thus $\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds = \int_c \vec{F} \cdot \vec{dr} = -4$

which verifies Stoke's Theorem. **Ans.**

EXERCISE 5.14

1. Use the Stoke's Theorem to evaluate $\int_C y^2 dx + xy\, dy + xz\, dz$,

where C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the positive direction. **Ans.** 0

2. Evaluate $\int_s (\text{curl } F) \cdot \hat{n}\, dA$, using the Stoke's Theorem, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and s is the paraboloid $z = f(x, y) = 1 - x^2 - y^2$, $z \geq 0$. **Ans.** π

3. Evaluate the integral for $\int_C y^2 dx + z^2 dy + x^2 dz$, where C is the triangular closed path joining the points (0, 0, 0), (0, a, 0) and (0, 0, a) by transforming the integral to surface integral using Stoke's Theorem.

Ans. $\frac{a^3}{3}$.

4. Verify Stoke's Theorem for $\vec{A} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and c is its boundary traversed in the clockwise direction. **Ans.** -20π

5. Evaluate $\int_C \vec{F} \cdot \vec{d}R$ where $\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$, C is the circl $x^2 + y^2 = 4$, $z = 1.5$ **Ans.** $\frac{19}{2}\pi$

6. If S is the surface of the sphere $x^2 + y^2 + z^2 = 9$. Prove that $\int_S \text{curl } \vec{F} \cdot ds = 0$.

7. Verify Stoke's Theorem for the vector field

$$\vec{F} = (2y + z)\,\hat{i} + (x - z)\,\hat{j} + (y - x)\,\hat{k}$$

over the portion of the plane $x + y + z = 1$ cut off by the co-ordinate planes.

8. Evaluate $\int_c \vec{F} \cdot dr$ by Stoke's Theorem for $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,k$ and C is the curve of intersection of $x^2 + y^2 = 1$ and $y = z^2$. **Ans.** 0

9. If $\vec{F} = (x - z)\,\hat{i} + (x^3 + yz)\,\hat{j} + 3xy^2\,\hat{k}$ and S is the surface of the cone $z = a - \sqrt{(x^2 + y^2)}$ above the xy-plane, show that $\iint_s \text{curl } \vec{F} \cdot dS = 3\pi a^4 / 4$.

10. If $\vec{F} = 3y\hat{i} - xy\hat{j} + yz^2\hat{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$, show by using Stoke's Theorem that $\iint_s (\nabla \times \vec{F}) \cdot dS = 20\pi$.

11. If $\vec{F} = (y^2 + z^2 - x^2)\,\hat{i} + (z^2 + x^2 - y^2)\,\hat{j} + (x^2 + y^2 - z^2)\,\hat{k}$, evaluate $\int \text{curl } \vec{F} \cdot \hat{n}\, ds$ integrated over the portion of the surface $x^2 + y^2 - 2ax + az = 0$ above the plane $z = 0$ and verify Stoke's Theorem; where $\hat{n}$ is unit vector normal to the surface. (*A.M.I.E.T.E., Winter 20002*) **Ans.** $2\pi a^3$

12. Evaluate by using Stoke's Theorem $\int_C [\sin z\, dx - \cos x\, dy + \sin y\, dz]$ where C is the boundary of rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $z = 3$. (*AMIETE, June 2010*)

5.40 GAUSS'S THEOREM OF DIVERGENCE

(Relation between surface integral and volume integral)

(*U.P., Ist Semester, Jan., 2011, Dec, 2006*)

Statement. The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically

$$\iint_S \vec{F} \,.\, \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Proof. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Putting the values of $\vec{F}, \hat{n}$ in the statement of the divergence theorem, we have

$$\iint_S F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\cdot\hat{n}\,ds = \iiint_V \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\,dx\,dy\,dz.$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \qquad ...(1)$$

We require to prove (1).

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy$$

$$= \iint_R [F_3\,(x, y, f_2) - F_3\,(x, y, f_1)]\,dx\,dy \qquad ...(2)$$

For the upper part of the surface *i.e.* S_2, we have

$$dx\,dy = ds_2 \cos r_2 = \hat{n}_2 . \hat{k}\;\; ds_2$$

Again for the lower part of the surface *i.e.* S_1, we have,

$$dx\,dy = -\cos r_1,\, ds_1 = \hat{n}_1 .\;\hat{k}\,ds_1$$

$$\iint_R F_3\,(x, y, f_2)\,dx\,dy = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,ds_2$$

and $\quad \iint_R F_3\,(x, y, f_1)\,dx\,dy = -\iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,ds_1$

Putting these values in (2), we have

$$\iiint_V \frac{\partial F_3}{\partial z}\,dv = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,ds_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,ds_1 = \iint_S F_3\,\hat{n}\cdot\hat{k}\,ds \qquad ...(3)$$

Similarly, it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\,dv = \iint_S F_2\,\hat{n}\cdot\hat{j}\,ds \qquad ...(4)$$

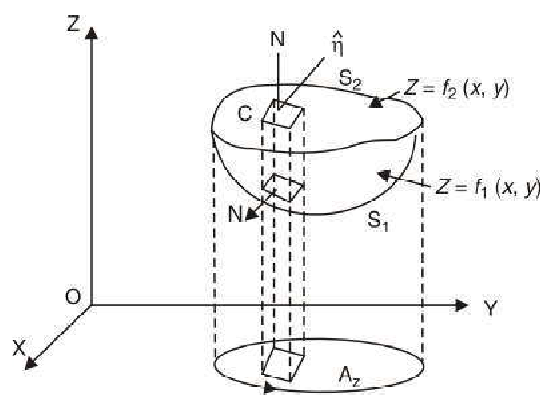
$$\iiint_V \frac{\partial F_1}{\partial x}\,dv = \iint_S F_1\,\hat{n}\cdot\hat{i}\,ds \qquad ...(5)$$

Adding (3), (4) & (5), we have

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$$

$$= \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot\hat{n}\cdot ds$$

$$\Rightarrow \iiint_V (\nabla\cdot\overline{F})\,dv = \iint_S \overline{F}\cdot\hat{n}\cdot ds$$ **Proved.**



Example 102. *State Gauss's Divergence theorem* $\iint_S \vec{F}\,.\hat{n}\,ds = \iiint Div\,\vec{F}\,dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$.

(*Nagpur University, Winter 2004*)

Solution. Statement of Gauss's Divergence theorem is given in Art 24.8 on page 597.

Thus by Gauss's divergence theorem,

$$\iint_S \vec{F}\,.\,\hat{n}\,ds = \iint_v\int \nabla\,.\,\vec{F}\,dv \qquad \text{Here } \vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$$

$$\nabla . \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . (3x\hat{i} + 4y\hat{j} + 5z\hat{k})$$

$$\nabla . \vec{F} = 3 + 4 + 5 = 14$$

Putting the value of ∇ . F, we get

$$\iint_S \vec{F} . \hat{n}\, ds = \iint_v \int 14 . dv \qquad \text{where } v \text{ is volume of a sphere}$$

$$= 14\, v$$

$$= 14 \frac{4}{3} \pi (4)^3 = \frac{3584\pi}{3} \qquad \textbf{Ans.}$$

Example 103. *Evaluate* $\iint_S \vec{F} . \hat{n}\, ds$ *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

(*U.P., Ist Semester, 2009, Nagpur University, Winter 2003*)

Solution. By Divergence theorem,

$$\iint_S \vec{F} . \hat{n}\, ds = \iint_v \int (\nabla . \vec{F})\, dv$$

$$= \iiint_v \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\, dv$$

$$= \iint_v \int \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) \right] dx\, dy\, dz$$

$$= \iint_v \int (4z - 2y + y)\, dx\, dy\, dz$$

$$= \iint_v \int (4z - y)\, dx\, dy\, dz = \int_0^1 \int_0^1 \left(\frac{4z^2}{2} - yz \right)_0^1 dx\, dy$$

$$= \int_0^1 \int_0^1 (2z^2 - yz)_0^1\, dx\, dy = \int_0^1 \int_0^1 (2 - y)\, dx\, dy$$

$$= \int_0^1 \left(2y - \frac{y^2}{2} \right)_0^1 dx = \frac{3}{2} \int_0^1 dx = \frac{3}{2} [x]_0^1 = \frac{3}{2}(1) = \frac{3}{2} \quad \textbf{Ans.}$$

Note: This question is directly solved as on example 14 on Page 574.

Example 104. *Find* $\iint \vec{F} \cdot \hat{n} \cdot ds$, *where* $\vec{F} = (2x + 3z)\,\hat{i} - (xz + y)\,\hat{j} + (y^2 + 2z)\,\hat{k}$ *and S is the surface of the sphere having centre (3, – 1, 2) and radius 3.*

(*AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000*)

Solution. Let V be the volume enclosed by the surface S.
By Divergence theorem, we've

$$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V div\, \vec{F}\, dv.$$

Now, $div\, \vec{F} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}[-(xz + y)] + \frac{\partial}{\partial z}(y^2 + 2z) = 2 - 1 + 2 = 3$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V 3\, dv = 3 \iiint_V dv = 3V.$$

Again V is the volume of a sphere of radius 3. Therefore

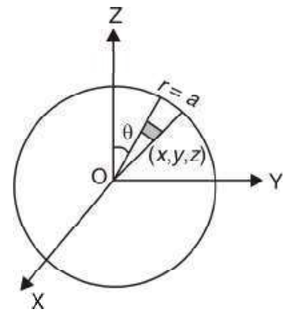
$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (3)^3 = 36\, \pi.$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n} \cdot ds = 3V = 3 \times 36\, \pi = 108\, \pi \qquad \textbf{Ans.}$$

Example 105. *Use Divergence Theorem to evaluate* $\iint_S \vec{A} \cdot \vec{ds}$,

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*AMIETE, Dec. 2009*)



Solution. $\iint_S \vec{A} \cdot \vec{ds} = \iiint_v \text{div } \vec{A}\, dV$

$$= \iiint_v \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\, dV$$

$$= \iiint_v (3x^2 + 3y^2 + 3z^2)\, dV = 3\iiint_v (x^2 + y^2 + z^2)\, dV$$

On putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, we get

$$= 3\iiint_V r^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = 3 \times 8 \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= 24\, (\phi)_0^{\frac{\pi}{2}} (-\cos\theta)_0^{\frac{\pi}{2}} \left(\frac{r^5}{5} \right)_0^a = 24\left(\frac{\pi}{2}\right)(-0+1)\left(\frac{a^5}{5}\right) = \frac{12\pi a^5}{5}$$ **Ans.**

Example 106. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2)\, d\vec{s} = 6\, V$$

where S is any closed surface enclosing volume V. (*U.P., I Semester, Winter 2002*)

Solution. Here $\nabla\ (x^2 + y^2 + z^2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (x^2 + y^2 + z^2)$

$$= 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k} = 2\,(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

$$\therefore \quad \iint_S \nabla (x^2 + y^2 + z^2) \cdot ds = \iint_S \nabla (x^2 + y^2 + z^2) \cdot \hat{n}\, ds$$

$\hat{n}$ being outward drawn unit normal vector to S

$$= \iint_S 2\,(x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{n}\, ds$$

$$= 2\iiint_V div\,(x\hat{i} + y\hat{j} + z\hat{k})\, dv \quad \text{...(1)}$$

(By Divergence Theorem)
(V being volume enclosed by S)

Now, div. $(x\,\hat{i} + y\,\hat{j} + z\hat{k}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \quad \text{...(2)}$$

From (1) & (2), we have

$$\iint \nabla (x^2 + y^2 + z^2) \cdot dS = 2\iiint_V 3\, dv = 6\iiint_V dv = 6\, V$$ **Proved.**

Example 107. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Solution. Let V be the volume enclosed by the surface S. Then by divergence Theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, \hat{n}\, dS = \iiint_V div\,(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2) \right] dV = \iint_V 2z\, y^2\, dV = 2\iint_V zy^2\, dV$$

Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi,\qquad y = r\sin\theta\sin\phi,\qquad z = r\cos\theta,\ dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore\quad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\cdot dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

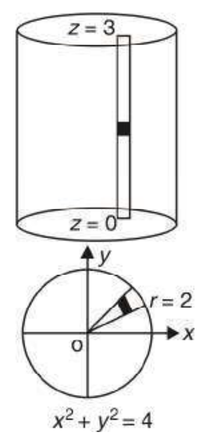
$$= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi}\sin^2\phi\cdot\frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\pi}{12} \qquad \textbf{Ans.}$$

Example 108. *Use Divergence Theorem to evaluate* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.

(*A.M.I.E.T.E., Summer 2003, 2001*)

Solution. By Divergence Theorem,



$$\iint_S \vec{F}\cdot dS = \iiint_V div\,\vec{F}\, dV$$

$$= \iiint_V\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\, dV$$

$$= \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \iint dx\, dy\int_0^3 (4 - 4y + 2z)dz = \iint dx\, dy\,[4z - 4yz + z^2]_0^3$$

$$= \iint (12 - 12y + 9)\, dx\, dy = \iint (21 - 12y)\, dx\, dy$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$

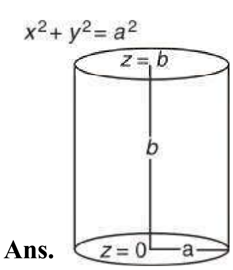
$$= \iint (21 - 12r\sin\theta)\, r\, d\theta\, dr = \int_0^{2\pi} d\theta\int_0^2 (21r - 12r^2\sin\theta)\, dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{21r^2}{2} - 4r^3\sin\theta\right]_0^2 = \int_0^{2\pi} d\theta\,(42 - 32\sin\theta) = (42\theta + 32\cos\theta)_0^{2\pi}$$

$= 84\pi + 32 - 32 = 84\pi$ **Ans.**

Example 109. *Apply the Divergence Theorem to compute* $\iint \vec{u}\cdot\hat{n}\, ds$, *where s is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = \hat{i}x - \hat{j}y + \hat{k}z$.

Solution. By Gauss's Divergence Theorem



$$\iint \vec{u}\cdot\hat{n}ds = \iiint_V (\nabla\cdot\vec{u})\, dv$$

$$= \iiint_V\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x - \hat{j}y + \hat{k}z)dv$$

$$= \iiint_V\left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)dv = \iiint_V (1 - 1 + 1)\, dv$$

$$= \iiint_V dv = \iiint_V dx\, dy\, dz = \text{Volume of the cylinder} = \pi a^2 b \quad \textbf{Ans.}$$

Example 110. *Apply Divergence Theorem to evaluate* $\iiint_V \vec{F}\,.\,\hat{n}\,ds,$ *where* $\vec{F} = 4x^3\hat{i} - x^2 y\,\hat{j} + x^2 z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$ *and* $z = b$. (*U.P. Ist Semester, Dec. 2006*)

Solution. We have,

$$\vec{F} = 4x^3\hat{i} - x^2 y\,\hat{j} + x^2 z\hat{k}$$

$$\therefore \quad div\,\overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot(4x^3\hat{i} - x^2 y\hat{j} + x^2 z\hat{k})$$

$$= \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(-x^2 y) + \frac{\partial}{\partial z}(x^2 z) = 12x^2 - x^2 + x^2 = 12\,x^2$$

Now, $$\iiint_V div\,\vec{F}\,dV = 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\,dz\,dy\,dx$$

$$= 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,(z)_0^b\,dy\,dx = 12\,b\int_{-a}^{a} x^2\,(y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx$$

$$= 12\,b\int_{-a}^{a} x^2 . 2\sqrt{a^2-x^2}\,dx = 24\,b\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx$$

$$= 48\,b\int_{0}^{a} x^2\sqrt{a^2-x^2}\,dx \qquad [\text{Put } x = a\sin\theta,\ dx = a\cos\theta\,d\theta]$$

$$= 48\,b\int_{0}^{\pi/2} a^2\sin^2\theta\,a\cos\theta\,a\cos\theta\,d\theta$$

$$= 48\,ba^4\int_{0}^{\pi/2}\sin^2\theta\cdot\cos^2\theta\,d\theta = 48\,ba^4\,\frac{\left\lceil\frac{3}{2}\right.\left\lceil\frac{3}{2}\right.}{2\left\lceil 3\right.}$$

$$= 48\,ba^4\,\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 2} = 3\,b\,a^4\pi$$ **Ans.**

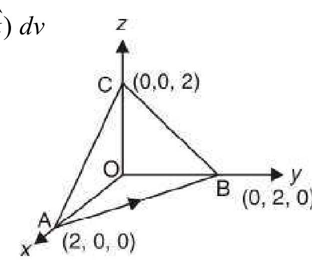
Example 111. *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\,ds,$ *where* $\vec{F} = (x^2 + y^2 + z^2)\ (\hat{i} + \hat{j} + \hat{k}),$ *S is the surface of the tetrahedron* $x = 0,\ y = 0,\ z = 0,\ x + y + z = 2$ *and n is the unit normal in the outward direction to the closed surface S.*

Solution. By Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\cdot dv$$

where S is the surface of tetrahedron $x = 0,\ y = 0,\ z = 0,\ x + y + z = 2$

$$= \iiint_V\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 + z^2)\,(\hat{i} + \hat{j} + \hat{k})\,dv$$



$$= \iiint_V (2x + 2y + 2z)\,dv$$

$$= 2\iiint_V (x + y + z)\,dx\,dy\,dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\int_0^{2-x-y}(x + y + z)\,dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\left(xz + yz + \frac{z^2}{2}\right)_0^{2-x-y}$$

$$= 2\int_0^2 dx \int_0^{2-x} dy \left(2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2}\right]$$

$$= 2\int_0^2 dx \left[2xy - x^2 y - x y^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6}\right]_0^{2-x}$$

$$= 2\int_0^2 dx \left[2x(2-x) - x^2(2-x) - x(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\int_0^2 \left(4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[-\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2 = 2\left[\frac{8}{3} - \frac{16}{12} + \frac{16}{24}\right] = 4$$ **Ans.**

Example 112. *Use the Divergence Theorem to evaluate*

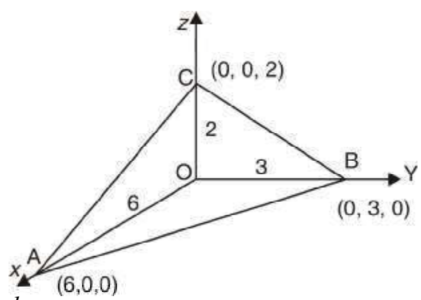
$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where S is the portion of the plane $x + 2y + 3z = 6$ which lies in the first Octant.
(*U.P., I Semester, Winter 2003*)

Solution. $\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy)$

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where S is a closed surface bounding a volume V.



$\therefore$ $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$

$$= \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz = 3\iiint_V dx\,dy\,dz$$

= 3 (Volume of tetrahedron *OABC*)

$$= 3\,[(\frac{1}{3}\text{ Area of the base } \Delta\, OAB) \times \text{height } OC]$$

$$= 3\left[\frac{1}{3}\left(\frac{1}{2} \times 6 \times 3\right) \times 2\right] = 18$$ **Ans.**

Example 113. *Use Divergence Theorem to evaluate :* $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *over the surface of a sphere radius a.* (*K. University, Dec. 2009*)

Solution. Here, we have

$$\iint_S [x\,dy\,dz + y\,dx\,dz + z\,dx\,dy]$$

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz \quad = \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint_V (1 + 1 + 1)\,dx\,dy\,dz = 3 \text{ (volume of the sphere)}$$

$$= 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$ **Ans.**

Example 114. *Using the divergence theorem, evaluate the surface integral* $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ *where* $S : x^2 + y^2 + z^2 = 4$.

(*AMIETE, Dec. 2010, UP, I Sem., Dec 2008*)

Solution. $\iint_S (f_1\, dy\,dz + f_2\, dx\,dz + f_3\, dx\,dy)$

$$= \iiint_v \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where S is closed surface bounding a volume V.

$\therefore \quad \iint_S (yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy)$

$$= \iiint_v \left(\frac{\partial (yz)}{\partial x} + \frac{\partial (zx)}{\partial y} + \frac{\partial (xy)}{\partial z}\right) dx\,dy\,dz = \iiint_v (0 + 0 + 0)\, dx\,dy\,dz$$

$= 0$ **Ans.**

Example 115. *Evaluate* $\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$
where S is the surface of hemispherical region bounded by

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = 0.$$

Solution. $\iint_S (f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy) = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz$

where S is a closed surface bounding a volume V.

$\therefore \quad \iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$

$$= \iiint_V \left[\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2 y - z^3) + \frac{\partial}{\partial z}(2xy + y^2 z)\right] dx\, dy\, dz$$

(Here V is the volume of hemisphere)

$$= \iiint_V (z^2 + x^2 + y^2)\, dx\, dy\, dz$$

Let $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

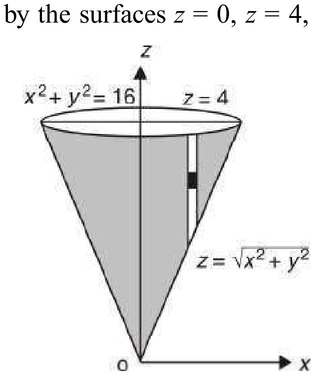
$$= \iiint r^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= (\phi)_0^{2\pi} (-\cos\theta)_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^a = 2\pi(-0 + 1)\frac{a^5}{5} = \frac{2\pi a^5}{5}$$ **Ans.**

Example 116. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $F = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.

Solution. If V is the volume enclosed by S, then V is bounded by the surfaces $z = 0$, $z = 4$, $z^2 = x^2 + y^2$.

By divergence theorem, we have



$$\iint_S \overline{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F}\, dx\, dy\, dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)\right] dx\, dy\, dz$$

$$= \iiint_V (4z + xz^2 + 3)\, dx\, dy\, dz$$

Limits of z are $\sqrt{x^2 + y^2}$ and 4.

$$\iiint_{\sqrt{x^2+y^2}}^{4} (4z + xz^2 + 3)\, dz\, dy\, dx = \iint \left[2z^2 + \frac{xz^3}{3} + 3z \right]_{\sqrt{x^2+y^2}}^{4} dy\, dx$$

$$= \iint \left[\left(32 + \frac{64x}{3} + 12 \right) - \{ 2(x^2+y^2) + x(x^2+y^2)^{3/2} + 3\sqrt{x^2+y^2} \} \right] dy\, dx$$

$$= \iint \left(44 + \frac{64x}{3} - 2(x^2+y^2) - x(x^2+y^2)^{3/2} - 3\sqrt{x^2+y^2} \right) dy\, dx$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$= \iint \left(44 + \frac{64\, r\cos\theta}{3} - 2r^2 - r\cos\theta\, r^3 - 3r \right) r\, d\theta\, dr$$

Limits of r are 0 to 4.
and limits of θ are 0 to 2π.

$$= \int_0^{2\pi} \int_0^4 \left(44r + \frac{64r^2\cos\theta}{3} - 2r^3 - r^5\cos\theta - 3r^2 \right) d\theta\, dr$$

$$= \int_0^{2\pi} \left[22r^2 + \frac{64 \times r^3\cos\theta}{9} - \frac{r^4}{2} - \frac{r^6}{6}\cos\theta - r^3 \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left[22(4)^2 + \frac{64 \times (4)^3\cos\theta}{9} - \frac{(4)^4}{2} - \frac{(4)^6}{6}\cos\theta - (4)^3 \right] d\theta$$

$$= \int_0^{2\pi} \left[352 + \frac{64 \times 64}{9}\cos\theta - 128 - \frac{(4)^6}{6}\cos\theta - 64 \right] d\theta$$

$$= \int_0^{2\pi} \left[160 + \left(\frac{64 \times 64}{9} - \frac{(4)^6}{6} \right)\cos\theta \right] d\theta$$

$$= \left[160\,\theta + \left(\frac{64 \times 64}{9} - \frac{(4)^6}{6} \right)\sin\theta \right]_0^{2\pi} = 160\,(2\pi) + \left(\frac{64 \times 64}{9} - \frac{(4)^6}{6} \right)\sin 2\pi$$

$= 320\,\pi$ **Ans.**

Example 117. *The vector field* $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ *is defined over the volume of the cuboid given by* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, *enclosing the surface S. Evaluate the surface integral*

$$\iint_S \vec{F} \,.\, \vec{ds}$$ (*U.P., I Semester, Winter 2001*)

Solution. By Divergence Theorem, we have

$$\iint_S (x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,.ds = \iiint_v div\,(x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,dv,$$

where V is the volume of the cuboid enclosing the surface S.

$$= \iiint_v \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} \right) . (x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,dv$$

$$= \iiint_v \left\{ \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) \right\} dx\,dy\,dz$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} (2x + y)\,dx\,dy\,dz = \int_0^a dx \int_0^b dy \int_0^c (2x + y)\,dz$$

$$= \int_0^a dx \int_0^b [2xz + yz]_0^c\,dy = \int_0^a dx \int_0^b (2xc + yc)\,dy$$

$$= c\int_0^a dx\int_0^b (2x+y)\,dy = c\int_0^a\left[2xy+\frac{y^2}{2}\right]_0^b dx = c\int_0^a\left(2bx+\frac{b^2}{2}\right)dx$$

$$= c\left[\frac{2bx^2}{2}+\frac{b^2x}{2}\right]_0^a = c\left[a^2b+\frac{ab^2}{2}\right] = abc\left(a+\frac{b}{2}\right) \qquad \textbf{Ans.}$$

Example 118. *Verify the divergence Theorem for the function* $\overline{F} = 2x^2yi - y^2j + 4xz^2k$ *taken over the region in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$.

Solution. $\iiint_V \nabla\cdot\vec{F}\,dV = \iiint\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(2x^2y\hat{i}-y^2\hat{j}+4xz^2k)\,dV$

$$= \iiint (4xy-2y+8xz)\,dx\,dy\,dz = \int_0^2 dx\int_0^3 dy\int_0^{\sqrt{9-y^2}}(4xy-2y+8xz)\,dz$$

$$= \int_0^2 dx\int_0^3 dy\,(4xyz-2yz+4xz^2)_0^{\sqrt{9-y^2}}$$

$$= \int_0^2 dx\int_0^3 [4xy\sqrt{9-y^2}-2y\sqrt{9-y^2}+4x(9-y^2)]\,dy$$

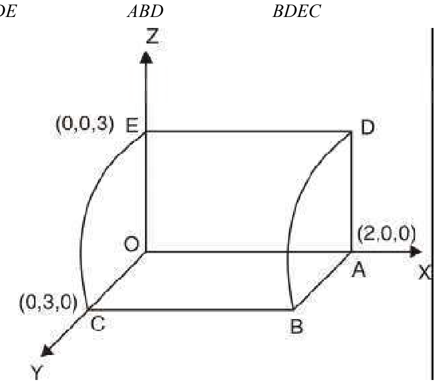
$$= \int_0^2 dx\left[-\frac{4x}{2}\frac{2}{3}(9-y^2)^{3/2}+\frac{2}{3}(9-y^2)^{3/2}+36xy-\frac{4xy^3}{3}\right]_0^3$$

$$= \int_0^2 (0+0+108x-36x+36x-18)\,dx = \int_0^2 (108x-18)\,dx = \left[108\frac{x^2}{2}-18x\right]_0^2$$

$= 216 - 36 = 180$ ...(1)

Here $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC}\vec{F}\cdot\hat{n}\,ds + \iint_{OCE}\vec{F}\cdot\hat{n}\,ds + \iint_{OADE}\vec{F}\cdot\hat{n}\,ds + \iint_{ABD}\vec{F}\cdot\hat{n}\,ds + \iint_{BDEC}\vec{F}\cdot\hat{n}\,ds$

$$\iint_{BDEC}\overline{F}\cdot\hat{n}\,ds = \iint_{BDEC}(2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k})\,.\,\hat{n}\,ds$$



Normal vector

$$= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(y^2+z^2-9)$$

$$= 2y\hat{j}+2z\hat{k}$$

Unit normal vector $= \hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \dfrac{y\hat{j}+z\hat{k}}{\sqrt{y^2+z^2}}$

$$= \frac{y\hat{j}+z\hat{k}}{\sqrt{9}} = \frac{y\hat{j}+z\hat{k}}{3}$$

$$\iint_{BDEC}(2x^2y\hat{i}-y^2\hat{j}+4xz^2k)\cdot\frac{y\hat{j}+zk}{3}\,ds = \frac{1}{3}\iint_{BDEC}(-y^3+4xz^3)\,ds$$

$$\left[dx\,dy = ds\,(\hat{n}\cdot k) = ds\left(\frac{y\hat{j}+z\hat{k}}{3}\cdot\hat{k}\right) = ds\,\frac{z}{3}\text{ or } ds = \frac{dx\,dy}{\frac{z}{3}}\right]$$

$$= \frac{1}{3}\iint_{BDEC}(-y^3+4xz^3)\frac{dxdy}{\frac{z}{3}} = \int_0^2 dx\int_0^3\left(-\frac{y^3}{z}+4xz^2\right)dy \qquad \left(\begin{aligned} y &= 3\sin\theta,\\ z &= 3\cos\theta\end{aligned}\right)$$

$$= \int_0^2 dx\int_0^{\frac{\pi}{2}}\left[\frac{-27\sin^3\theta}{3\cos\theta}+4x\,(9\cos^2\theta)\right]$$

$$= \int_0^2 dx\left(-27\times\frac{2}{3}+108\,x\times\frac{2}{3}\right) = \int_0^2 (-18+72\,x)\,dx$$

$$= \Big[-18x+36x^2\Big]_0^2 = 108 \qquad ...(2)$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{k})\,ds$$

$$= \iint_{OABC} 4xz^2\,ds = 0$$ ...(3) because in $OABC$ xy-plane, $z = 0$

$$\iint_{OADE} \vec{F}\cdot\hat{n}\,ds = \iint_{OADE} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{j})\,ds = \iint_{OADE} y^2\,ds = 0 \qquad ...(4)$$

because in $OADE$ xz-plane, $y = 0$

$$\iint_{OCE} \vec{F}\cdot\hat{n}\,ds = \iint_{OCE} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{i})\,ds = \iint_{OCE} -2x^2y\,ds = 0 \qquad ...(5)$$

because in OCE yz-plane, $x = 0$

$$\iint_{ABD} \vec{F}\cdot\hat{n}\,ds = \iint_{ABD} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(\hat{i})\,ds = \iint_{ABD} 2x^2y\,ds$$

$$= \iint 2x^2y\,dy\,dz = \int_0^3 dz\int_0^{\sqrt{9-z^2}} 2\,(2)^2\,y\,dy$$ because in ABD plane, $x = 2$

$$= 8\int_0^3 dz\left[\frac{y^2}{2}\right]_0^{\sqrt{9-z^2}} = 4\int_0^3 dz\,(9-z^2) = 4\left[9z-\frac{z^3}{3}\right]_0^3 = 4\,[27-9] = 72 \qquad ...(6)$$

On adding (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 108+0+0+0+72 = 180 \qquad ...(7)$$

From (1) and (7), we have $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$

Hence the theorem is verified.

Example 119. *Verify the Gauss divergence Theorem for*

$\vec{F} = (x^2-yz)\,\hat{i} + (y^2-zx)\,\hat{j} + (z^2-xy)\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. (*U.P., I Semester, Compartment 2002*)

Solution. We have

$$\text{div }\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot[(x^2-yz)\,\hat{i}+(y^2-zx)\,\hat{j}+(z^2-xy)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy) = 2x+2y+2z$$

$$\therefore \text{ Volume integral} = \iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 2\,(x+y+z)\,dV$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dx\,dy\,dz = 2\int_0^a dx\int_0^b dy\int_0^c (x+y+z)\,dz$$

$$= 2\int_0^a dx\int_0^b dy\left(xz+yz+\frac{z^2}{2}\right)_0^c = 2\int_0^a dx\int_0^b dy\left(cx+cy+\frac{c^2}{2}\right)$$

$$= 2\int_0^a dx\left(c\,x\,y+c\,\frac{y^2}{2}+\frac{c^2y}{2}\right)_0^b = 2\int_0^a dx\left(bcx+\frac{b^2c}{2}+\frac{bc^2}{2}\right)$$

$$= 2\left[\frac{bc\,x^2}{2} + \frac{b^2\,cx}{2} + \frac{bc^2\,x}{2}\right]_0^a = [a^2bc + ab^2c + abc^2]$$

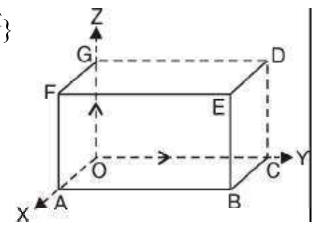
$$= abc\,(a + b + c) \qquad \text{...(A)}$$

To evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where S consists of six plane surfaces.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAFG} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{BCDE} \vec{F}\cdot\hat{n}\,ds + \iint_{ABEF} \vec{F}\cdot\hat{n}\,ds + \iint_{OCDG} \vec{F}\cdot\hat{n}\,ds$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \{(x^2 - yz)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\,\hat{k}\}$$



$$= -\int_0^a\int_0^b (z^2 - xy)\,dx\,dy$$

$$= -\int_0^a\int_0^b (0 - xy)\,dx\,dy = \frac{a^2\,b^2}{4} \qquad \text{...(1)}$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFG} \{(x^2 - yz)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\,(\hat{k})\,dx\,dy$$

$$= \int_0^a\int_0^b (z^2 - xy)\,dx\,dy = \int_0^a\int_0^b (c^2 - xy)\,dx\,dy$$

$$= \int_0^a \left[c^2 y - \frac{xy^2}{2}\right]_0^b dx = \int_0^a \left(c^2 b - \frac{x\,b^2}{2}\right) dx$$

$$= \left[c^2bx - \frac{x^2\,b^2}{4}\right]_0^a = abc^2 - \frac{a^2\,b^2}{4} \qquad \text{...(2)}$$

S.No.	Surface	Outward normal	ds	
1	OABC	$-k$	$dx\,dy$	$z = 0$
2	DEFG	k	$dx\,dy$	$z = c$
3	OAFG	$-j$	$dx\,dz$	$y = 0$
4	BCDE	j	$dx\,dz$	$y = b$
5	ABEF	i	$dy\,dz$	$x = a$
6	OCDG	$-i$	$dy\,dz$	$x = 0$

$$\iint_{OAFG} \vec{F}\cdot\hat{n}\,ds = \iint_{OAFG} \{(x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\,(-\hat{j})\,dx\,dz$$

$$= -\iint_{OAFG} (y^2 - zx)\,dx\,dz$$

$$= -\int_0^a dx \int_0^c (0 - zx)\,dz = \int_0^a dx \left(\frac{x\,z^2}{2}\right)_0^c = \int_0^a \frac{x\,c^2}{2}\,dx = \left[\frac{x^2\,c^2}{4}\right]_0^a = \frac{a^2\,c^2}{4} \qquad \text{...(3)}$$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \iint \{(x^2 - y\,z)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\cdot\hat{j}\,dx\,dz = \iint_{BCDE} (y^2 - xz)\,dx\,dz$$

$$= -\int_0^a dx \int_0^c (b^2 - x\,z)\,dz = \int_0^a \left(b^2\,z - \frac{x\,z^2}{2}\right)_0^c dx = \int_0^a \left(b^2\,c - \frac{x\,c^2}{2}\right) dx$$

$$= \left[b^2\,c\,x - \frac{x^2\,c^2}{4}\right]_0^a = a\,b^2\,c - \frac{a^2c^2}{4} \qquad \text{...(4)}$$

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \iint_{ABEF} \{(x^2 - y\,z)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - x\,y)\,\hat{k}\}\cdot\hat{i}\,dy\,dz$$

$$= \iint_{ABEF} (x^2 - y\,z)\,dy\,dz = \int_0^b dy \int_0^c (a^2 - y\,z)\,d\,z = \int_0^b dy \left(a^2 z - \frac{y\,z^2}{2}\right)_0^c$$

$$= \int_0^b \left(a^2 c - \frac{y c^2}{2}\right) dy = \left[a^2 cy - \frac{y^2 c^2}{4}\right]_0^b = a^2 bc - \frac{b^2 c^2}{4} \qquad ...(5)$$

$$\iint_{OCDG} \vec{F} \cdot \hat{n}\, ds = \iint_{OCDG} \{(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\} \cdot (-\hat{i})\, dy\, dz$$

$$= \int_0^b \int_0^c (x^2 - yz)\, dy\, dz = -\int_0^b dy \int_0^c (-yz)\, dz = -\int_0^b dy \left[\frac{-yz^2}{2}\right]_0^c$$

$$= \int_0^b \frac{yc^2}{2}\, dy = \left[\frac{y^2 c^2}{4}\right]_0^b = \frac{b^2 c^2}{4} \qquad ...(6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint \vec{F} \cdot \hat{n}\, ds = \left(\frac{a^2 b^2}{4}\right) + \left(abc^2 - \frac{a^2 b^2}{4}\right) + \left(\frac{a^2 c^2}{4}\right) + \left(ab^2 c - \frac{a^2 c^2}{4}\right) + \left(\frac{b^2 c^2}{4}\right) + \left(a^2 bc - \frac{b^2 c^2}{4}\right)$$

$= abc^2 + ab^2c + a^2bc$

$= abc\,(a + b + c)$ ...(B)

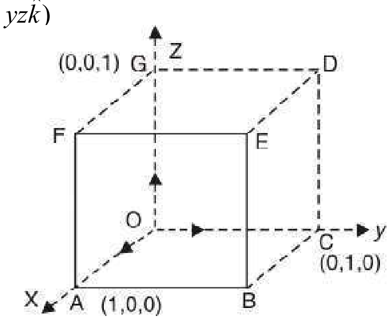
From (A) and (B), Gauss divergence Theorem is verified. **Verified.**

Example 120. *Verify Divergence Theorem, given that* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.*

Solution. $\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (4zx\hat{i} - y^2\hat{j} + yz\hat{k})$

$= 4z - 2y + y$

$= 4z - y$



Volume Integral $= \iiint \nabla \cdot \vec{F}\, dv$

$$= \iiint (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y)\, dz$$

$$= \int_0^1 dx \int_0^1 dy\, (2z^2 - yz)_0^1 = \int_0^1 dx \int_0^1 dy\, (2 - y)$$

$$= \int_0^1 dx \left(2y - \frac{y^2}{2}\right)_0^1 = \int_0^1 dx \left(2 - \frac{1}{2}\right) = \frac{3}{2}\int_0^1 dx = \frac{3}{2}(x)_0^1 = \frac{3}{2} \qquad ...(1)$$

To evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$, where S consists of six plane surfaces.

Over the face $OABC$, $z = 0$, $dz = 0$, $\hat{n} = -\hat{k}$, $ds = dx\, dy$

$$\iint \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 (-y^2\hat{j}) \cdot (-\hat{k})\, dx\, dy = 0$$

Over the face $BCDE$, $y = 1$, $dy = 0$

$$\iint \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4\,xz\hat{i} - \hat{j} + z\hat{k})\cdot(\hat{j})\, dx\, dz$$

$$\hat{n} = \hat{j},\ ds = dx\, dz = \int_0^1\int_0^1 - dxdz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1\,(z)_0^1 = -(1)\,(1) = -1$$

Over the face *DEFG*, $z = 1$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\, dy$

$$\iint \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 [4\,x\,(1) - y^2\,\hat{j} + y\,(1)\,\hat{k}]\cdot(\hat{k})\, dx\, dy$$

$$= \int_0^1\int_0^1 y\, dx\, dy = \int_0^1 dx \int_0^1 y\, dy = (x)_0^1 \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Over the face *OCDG*, $x = 0$, $dx = 0$, $\hat{n} = -\,\hat{i}$, $ds = dy\, dz$

$$\iint \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (0\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\, dy\, dz = 0$$

Over the face *AOGF*, $y = 0$, $dy = 0$, $\hat{n} = -\,\hat{j}$, $ds = dx\, dz$

$$\iint \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4\,xz\hat{i})\cdot(-\hat{j})\, dx\, dz = 0$$

Over the face *ABEF*, $x = 1$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\, dz$

$$\iint \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 [(4\,z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})]\, dy\, dz = \int_0^1\int_0^1 4\,z\, dy\, dz$$

$$= \int_0^1 dy \int_0^1 4\,z\, dz = \int_0^1 dy\,(2\,z^2)_0^1 = 2\int_0^1 dy = 2\,(y)_0^1 = 2$$

On adding we see that over the whole surface

$$\iint \vec{F}\cdot\hat{n}\, ds = \left(0 - 1 + \frac{1}{2} + 0 + 0 + 2\right) = \frac{3}{2} \qquad \text{...(2)}$$

From (1) and (2), we have $\iiint_V \nabla\cdot\vec{F}\, dv = \iint_S \vec{F}\cdot\hat{n}\, ds$ **Verified.**

EXERCISE 5.15

1. Use Divergence Theorem to evaluate $\iint_s (y^2z^2\hat{i} + z^2x^2\hat{j} + x^2y^2\hat{k}).\overline{ds}$,

where S is the upper part of the sphere $x^2 + y^2 + z^2 = 9$ above xy- plane. **Ans.** $\frac{243\pi}{8}$

2. Evaluate $\iint_S (\nabla\times\vec{F})\,.\,ds$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane and $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$. **Ans.** $-4\,\pi$

3. Evaluate $\iint_s [xz^2\, dy\, dz + (x^2y - z^3)\, dz\, dx + (2\,xy + y^2z)\, dx\, dy]$, where S is the surface enclosing a region bounded by hemisphere $x^2 + y^2 + z^2 = 4$ above XY-plane.

4. Verify Divergence Theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$, taken over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

5. Evaluate $\iint_S (2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k})\cdot\vec{ds}$ over the surface of the region bounded by

$x = 0$, $y = 0$, $y = 3$, $z = 0$ and $x + 2\,z = 6$ **Ans.** $\frac{351}{2}$

6. Verify Divergence Theorem for $\vec{F} = (x + y^2)\,\hat{i} - 2\,x\hat{j} + 2\,yz\hat{k}$ and the volume of a tetrahedron bounded by co-ordinate planes and the plane $2\,x + y + 2\,z = 6$.
(*Nagpur, Winter 2000, A.M.I.E.T.E.. Winter 2000*)

7. Verify Divergence Theorem for the function $\vec{F} = y\hat{i} + x\hat{j} + z^2\,\hat{k}$ over the region bounded by
$x^2 + y^2 = 9,\quad z = 0$ and $z = 2$.

8. Use the Divergence Theorem to evaluate $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$,
where S is the surface of the region bounded by the closed cylinder
$x^2 + y^2 = a^2$, $(0 \le z \le b)$ and $z = 0$, $z = b$. **Ans.** $\dfrac{5\pi a^4 b}{4}$

9. Evaluate the integral $\iint_S (z^2 - x)\,dy\,dz - xy\,dx\,dz + 3z\,dx\,dy$, where S is the surface of closed region bounded by $z = 4 - y^2$ and planes $x = 0$, $x = 3$, $z = 0$ by transforming it with the help of Divergence Theorem to a triple integral. **Ans.** 16

10. Evaluate $\iint_S \dfrac{ds}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$ over the closed surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ by applying Divergence Theorem. **Ans.** $\dfrac{4\pi}{\sqrt{(a\,b\,c)}}$

11. Apply Divergence Theorem to evaluate $\iint (l\,x^2 + m\,y^2 + n\,z^2)\,ds$
taken over the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$, l, m, n being the direction cosines of the external normal to the sphere. (*AMIETE June 2010, 2009*) **Ans.** $\dfrac{8\pi}{3}(a + b + c)\,r^3$

12. Show that $\iiint_V (u\,\nabla \cdot \vec{V} + \nabla u \cdot \vec{V})\,dv = \iint_S u\,\vec{V} \cdot ds$.

13. If E = grad ϕ and $\nabla^2\phi = 4\,\pi\,\rho$, prove that $\iint_S \vec{E} \cdot \vec{n}\,ds = -4\,\pi \iiint_V \rho\,dv$
where $\vec{n}$ is the outward unit normal vector, while dS and dV are respectively surface and volume elements.

Pick up the correct option from the following:

14. If $\vec{F}$ is the velocity of a fluid particle then $\int_C \vec{F} . \vec{dr}$ represents.
(*a*) Work done (*b*) Circulation (*c*) Flux (*d*) Conservative field.
(*U.P. Ist Semester, Dec 2009*) **Ans.** (*b*)

15. If $\vec{f} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$, a, b, c, constants, then $\iint f.dS$ where S is the surface of a unit sphere is
(*a*) $\dfrac{\pi}{3}(a + b + c)$ (*b*) $\dfrac{4}{3}\pi\,(a + b + c)$ (*c*) $2\,\pi\,(a + b + c)$ (*d*) $\pi\,(a + b + c)$
(*U.P., Ist Semester, 2009*) **Ans.** (*b*)

16. A force field $\vec{F}$ is said to be conservative if
(*a*) Curl $\vec{F} = 0$ (*b*) grad $\bar{F} = 0$ (*c*) Div $\bar{F} = 0$ (*d*) Curl (grad $\bar{F}$) = 0
(*AMIETE, Dec. 2006*) **Ans.** (*a*)

17. The line integral $\int_c x^2\,dx + y^2\,dy$, where C is the boundary of the region $x^2 + y^2 < a^2$ equals
(*a*) 0, (*b*) a (*c*) $\pi\,a^2$ (*d*) $\dfrac{1}{2}\pi\,a^2$
(*AMIETE, Dec. 2006*) **Ans.** (*b*)