MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
An ISO 9001:2015 Certified Institution

DEPARTMENT OF MATHEMATICS

SUBJECT NAME: ANALYTICAL GEOMETRY

SUBJECT CODE: SM23A

SEMESTER: III

PREPARED BY: PROF.S.KAVITHA

UNIVERSITY OF MADRAS

B.Sc. DEGREE COURSE IN MATHEMATICS
SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC05

CORE-V: ANALYTICAL GEOMETRY
(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 5 **YEAR: II**
Credits : 4 **SEMESTER: III**

Learning outcomes:

Students will acquire Knowledge

- To analyze characteristics and properties of two and three dimensional geometric shapes.
- To develop mathematical arguments about geometric relationships.
- In Geometry and its applications in real world.

UNIT I
Chord of contact – polar and pole,- conjugate points and conjugate lines – chord with $(x_1 y_1)$ as its midpoint – diameters – conjugate diameters of an ellipse.- semi diameters- conjugate diameters of hyperbola
Chapter 7: Sections 7.1 to 7.3 , Chapter – 8 Section 8.1 to 8.5.

UNIT II
Polar coordinates: General polar equation of straight line – Polar equation of a circle on A_1A_2 as diameter, Equation of a straight line, circle, conic – Equation of chord, tangent, normal. Equations of the asymptotes of a hyperbola.
Chapter 10 : Sec 10.1 to 10.8.

UNIT III
Introduction.– System of Planes - Length of the perpendicular – Orthogonal projection.
Chapter 2 Sec 2.1 to 2.10.

UNIT IV
Representation of line – angle between a line and a plane- co-planar lines- shortest distance 2 skew lines- Length of the perpendicular- intersection of three planes
Chapter 3 :Sec 3.1 to 3.8.

UNIT V
Equation of a sphere - general equation - section of a sphere by a plane - equation of the circle - tangent plane - angle of intersection of two spheres- condition for the orthogonality - radical plane.
Chapter 6 : Sec 6.1 to 6.8.

Contents and treatment as in

1. Analytical Geometry of 2D by P.Durai Pandian- Muhil publishers for Unit – 1 and 2
2. Analytical Solid Geometry of 3D by Shanthi Narayan and Dr.P.K. Mittal-S.Chand& Co. Pvt.Ltd.- for Unit – 3 to 5

Reference :

1. Analytical Geometry of Two Dimension by T. K. Manikavachakam Pillai and S. Narayanan.S.Viswanathan (Printers and Publishers) Pvt. Ltd.
2. Analytical Geometry of Three Dimension by T. K. Manikavachakam Pillai and S. Narayanan.S.Viswanathan (Printers and Publishers) Pvt. Ltd.

e-Resources:

1. http://mathworld.wolfram.com.
2. http://www.univie.ac.at/future.media/moe/galerie.html

UNIT- 1

CHORD OF CONTACT

POLE AND POLAR OF A CONIC

13. CHORD

13.1 Chord of Contact

It is defined as the line joining the point of intersection of tangents drawn from any point. The equation to the chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

13.2 Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ ($T = S_1$), where T and S_1 have their usual meanings.

13.3 Chord of Hyperbola (Parametric Form)

Note: Chord of ellipse $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

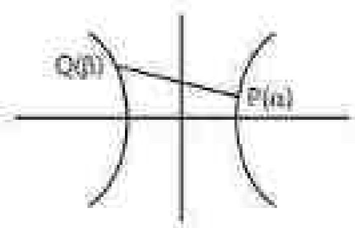


Figure 12.27

For a hyperbola it is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$$

Passing through (d, 0) $\frac{d}{a}\cos\left(\frac{\alpha-\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$

$$\frac{d}{a}=\frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)}$$

$$\frac{d+a}{d-a}=\frac{-2\cos\alpha/2\ \cos\beta/2}{2\cos\alpha/2\ \sin\beta/2}$$

$$\frac{a-d}{a+d}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

if $\quad d = ae \Rightarrow \frac{1-e}{1+e}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

PLANCESS CONCEPTS

Point of intersection of tangents at P(α) and Q(β) can be obtained by comparing COC with the chord at P(α) & Q(β)

Equation of PQ

COC ⇒ $\frac{xh}{a^2}-\frac{yk}{b^2}=1$

PQ ⇒ $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$

$\therefore h=a\frac{\cos((\alpha-\beta)/2)}{\cos((\alpha+\beta)/2)}$, $k=b\frac{\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)}$

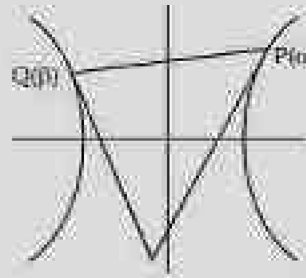


Figure 12.28

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 23: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of intersection of tangents at A and B. **(JEE MAIN)**

Sol: The point of intersection of the tangents at A and B is nothing but the point for which AB is the chord of contact. Use this information to find the locus.

Let P ≡ (h, k) be the point of intersection of tangent at A and B

$\therefore$ Equation of the chord of contact AB is $\frac{xh}{a^2}-\frac{yk}{b^2}=1$ ....(i)

Which touches the parabola. Equation of the tangent to the parabola $y^2 = 4ax$

$y = mx - a/m \Rightarrow mx - y = -a/m$ ....(ii)

equation (i) and (ii) must be same

$\therefore \frac{m}{(h/a^2)}=\frac{-1}{(-(k/b^2))}=\frac{-a/m}{1} \Rightarrow m=\frac{h}{k}\frac{b^2}{a^2}$ and $m=-\frac{ak}{b^2}$

$\therefore \frac{hb^2}{ka^2}=-\frac{ak}{b^2} \Rightarrow$ locus of P is $y^2=-\frac{b^4}{a^3}x$.

Illustration 24: A point P moves such that the chord of contact of a pair of tangents from P to $y^2 = 4x$ touches the rectangular hyperbola $x^2 - y^2 = 9$. If locus of 'P' is an ellipse, find e. **(JEE MAIN)**

Sol: Write the equation of the chord of contact to the parabola w.r.t a point (h , k). Then solve this equation with the equation of the hyperbola.

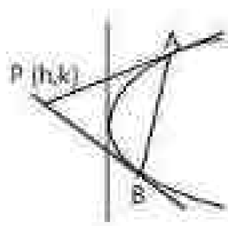


Figure 12.29

$yy_1 = 2a(x + x_1)$; $yk = 2(x + h)$ ⇒ $y=\frac{2x}{k}+\frac{2h}{k}$; $\frac{4h^2}{k^2}=9.\frac{4}{k^2}-9$

$4h^2 = 36 - 9k^2$ $\quad \frac{x^2}{9} + \frac{y^2}{4} = 1$ $\quad e^2 = 1 - \frac{4}{9}$ $\quad e = \frac{\sqrt{5}}{3}$

Illustration 25: Find the locus of the mid-point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **(JEE MAIN)**

Sol: Use the formula $T = S_1$ to get the equation of the chord and substitute the co-ordinates of the focus.

Let P = (h, k) be the mid-point

$\therefore$ Equation of the chord whose mid-point (h, k) is given $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$ since it is a focal chord.

$\therefore$ It passes through the focus, either (ae, 0) or (–ae, 0)

$\therefore$ Locus is $\pm\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Illustration 26: Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line x + y = b. **(JEE ADVANCED)**

Sol: Consider a point on the line x + y = b and then find a chord with this point as the mid-point. Then substitute the point in the equation of the chord to get the condition between 'a' and 'b'.

Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ Equation of the chord whose mid-point is $P(\alpha, b - \alpha)$ is:

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b) $\therefore \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0 \Rightarrow a = b$$

Illustration 27: Locus of the mid points of the focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is another hyperbola whose eccentricity is e. **(JEE ADVANCED)**

Sol: Use the formula $T = S_1$ and proceed further.

$T = S_1$; $\frac{xh}{a^2} - \frac{y0}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

It passes through focus $\Rightarrow \frac{eh}{a} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

$\Rightarrow \frac{x^2}{a^2} - \frac{ex}{a} = \frac{y^2}{b^2} \Rightarrow \frac{1}{a^2}[x^2 - eax] = \frac{y^2}{b^2}$

$\Rightarrow \frac{1}{a^2}\left[\left(x - \frac{ea}{2}\right)^2 - \frac{e^2a^2}{4}\right] = \frac{y^2}{b^2} \Rightarrow \frac{(x-(ea/2))^2}{a^2} - \frac{y^2}{b^2} = \frac{e^2}{4}$

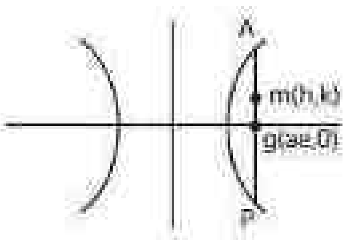


Figure 12.30

Hence the locus is a hyperbola of eccentricity e.

Illustration 28: Find the locus of the midpoint of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtends a right angle at the origin. **(JEE ADVANCED)**

Sol: Use the formula $T = S_1$ and then homogenise the equation of the hyperbola using the equation of the chord to find the locus.

Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2} \quad \text{... (i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (i) is obtained by making a homogeneous hyperbola with the help of (i)

$$\therefore \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{\left((hx/a^2)-(ky/b^2)\right)^2}{\left((h^2/a^2)-(k^2/b^2)\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 x^2-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 y^2=\frac{h^2}{a^4}x^2+\frac{k^2}{b^4}y^2-\frac{2hk}{a^2b^2}xy \quad \text{... (ii)}$$

The lines represented by (ii) will be at right angles if the coefficient of x^2 + the coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{h^2}{a^4}-\frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2-\frac{k^2}{b^4}=0 \quad \Rightarrow \quad \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{h^2}{a^4}+\frac{k^2}{b^4}$$

hence, the locus of (h, k) is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}$

14. DIAMETER

The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter. The point where a diameter intersects the hyperbola is known as the vertex of the diameter.

14.1 Equation of Diameter

The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}x$

14.2 Conjugate Diameters

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chords parallel to the other.

Let $y = m_1x$ and $y = m_2x$ be conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Then, $y = m_1x$ bisects the system of chords parallel to $y = m_2x$. So, its equation is

$$y=\frac{b^2}{a^2m}x \quad \text{... (i)}$$

Clearly, (i) and $y = m_2x$ represent the same line. Therefore, $m_2=\frac{b^2}{a^2m_1} \Rightarrow m_1m_2=\frac{b^2}{a^2}$

Thus, $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $m_1m_2=\frac{b^2}{a^2}$

PLANCESS CONCEPTS

- In a pair of conjugate diameters of a hyperbola, only one meets the hyperbola on a real point.
- Let P(a sec θ, b tan θ) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that CP and CD are conjugate diameters of the hyperbola. Then, the coordinates of D are (a tan θ, b sec θ)
- If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D respectively then $CP^2 - CD^2 = a^2 - b^2$.

Shivam Agarwal (JEE 2009, AIR 27)

15. POLE AND POLAR

Let $P(x_1, y_1)$ be any point inside the hyperbola. A chord through P intersects the hyperbola at A and B respectively. If tangents to the hyperbola at A and B meet at Q(h, k) then the locus of Q is called the polar of P with respect to the hyperbola and the point P is called the pole.

If $P(x_1, y_1)$ is any point outside the hyperbola and tangents are drawn, then the line passing through the contact points is polar of P and P is called the pole of the polar.

Note: If the pole lies outside the hyperbola then the polar passes through the hyperbola. If the pole lies inside the hyperbola then the polar lies completely outside the hyperbola. If pole the lies on the hyperbola then the polar becomes the same as the tangent.

Equation of polar: Equation of the polar of the point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$, i.e., T = 0

Coordinates of Pole: The pole of the line $lx + my + n = 0$ with respect to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$.

Properties of pole and polar:

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points. Condition for conjugate points is $\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}=1$.
2. If the pole of line $l_1x + m_1y + n_1 = 0$ lies on another line $l_2x + m_2y + n_2 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.
3. Pole of a given line is the same as the point of intersection of the tangents at its extremities.
4. Polar of focus is its directrix.

16. ASYMPTOTES

An asymptote to a curve is a straight line, such that distance between the line and curve approaches zero as they tend to infinity.

In other words, the asymptote to a curve touches the curves at infinity i.e. asymptote to a curve is its tangent at infinity.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y = \pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$

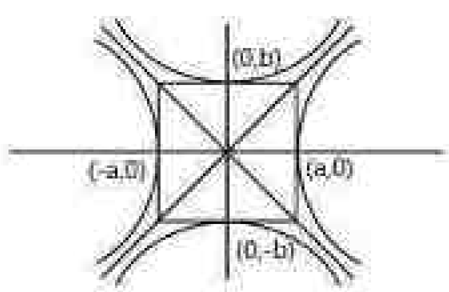


Figure 12.31

Combined equation of asymptote $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Note: If the angle between the asymptotes of the hyperbola is θ, then its eccentricity is sec θ.

PLANCESS CONCEPTS

- The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.
- When b = a, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The equation of the pair of asymptotes differ from the hyperbola and the conjugate hyperbola by the same constant. i.e. Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola
- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angles between the asymptotes are the coordinates axes.
- The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.
- Asymptotes are the tangents to the hyperbola from the centre.
- The tangent at any point P on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the centre C meets asymptotes at Q, R and cut off ΔCQR of constant area = ab.
- The parts of the tangent intercepted between the asymptote is bisected at the point of contact.
- If f(x, y) = 0 is an equation of the hyperbola then the centre of the hyperbola is the point of intersection of $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Ravi Vooda (JEE 2009, AIR 71)

Illustration 29: Find the asymptotes of $xy - 3y - 2x = 0$. **(JEE MAIN)**

Sol: Proceed according to the definition of asymptotes.

Since the equation of a hyperbola and its asymptotes differ in constant terms only

∴ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$ ...(i)

where λ is any constant such that represents two straight lines

∴ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

⇒ $0 + 2 \times -3/2 \times -1 \times 1/2 - 0 - 0 - \lambda (1/2)^2 = 0$

∴ $\lambda = 6$

From (i) the asymptotes of given hyperbola are given by $xy - 3y - 2x + 6 = 0$ or $(y - 2)(x - 3) = 0$

∴ Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

Illustration 30: Find the equation of that diameter which bisects the chord $7x + y - 20 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$. **(JEE ADVANCED)**

Sol: Consider a diameter $y = mx$ and solve it with the equation of the hyperbola to form a quadratic in x. Find the midpoint of the intersection of the chord and hyperbola. Use this point to find the slope of the diameter.

The centre of the hyperbola is (0, 0). Let the diameter be $y = mx$ ... (i)

The ends of the chord are found by solving

$$7x + y - 20 = 0 \quad \text{... (ii)}$$

and $$\frac{x^2}{3} - \frac{y^2}{7} = 1 \quad \text{... (iii)}$$

Solving (ii), (iii) we get $\frac{x^2}{3} - \frac{1}{7}(20 - 7x)^2 = 1$

or $7x^2 - 3(400 - 280x + 49x^2) = 21$ or $140x^2 - 840x + 1221 = 0$

Let the roots be x_1, x_2

Then $x_1 + x_2 = \frac{840}{140} = 6$ ... (iv)

If (x_1, y_1), (x_2, y_2) be ends then $7x_1 + y_1 - 20 = 0$, $7x_2 + y_2 - 20 = 0$

Adding, $7(x_1 + x_2) + (y_1 + y_2) - 40 = 0$

or $42 + y_1 + y_2 - 40 = 0$ using (iv) ; $\therefore$ $y_1 + y_2 = -2$

$\therefore$ The middle point of the chord $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6}{2}, \frac{-2}{2}\right) = (3, -1)$

This lies on (i). So $-1 = 3m$; $\therefore$ $m = -\frac{1}{3}$; the equation of the diameter is $y = -\frac{1}{3}x$.

Illustration 31: The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3) show that its equation is $(2x + 3y - 8)(3x + 2y + 7) = 154$. **(JEE ADVANCED)**

Sol: With the information given, find out the equation of the asymptotes and then use the fact that the point (5, 3) lies on the hyperbola to find the equation of the hyperbola.

Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since the asymptote passes through (1, 2) then $\lambda = -8$ and $\mu = -7$

Thus the equation of the asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Let the equation of the hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + \nu = 0$ ... (i)

It passes through (5, 3), then $(10 + 9 - 8)(15 + 6 - 7) + \nu = 0$

$\Rightarrow$ $11 \times 14 + \nu = 0$

$\therefore$ $\nu = -154$

putting the value of ν in (i) we obtain $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$

which is the equation of the required hyperbola.

17. RECTANGULAR HYPERBOLA

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = \pm \frac{b}{a}x$.

The θ angle between these two asymptotes is given by

$$\tan\theta\left(\frac{(b/a)-(-(b/a))}{1+(b/a)(-(b/a))}\right)=\frac{2b/a}{1-b^2/a^2}=\frac{2ab}{a^2-b^2}$$

If the asymptotes are at right angles, then $\theta=\pi/2 \Rightarrow \tan\theta=\tan\pi/2 \Rightarrow \frac{2ab}{a^2-b^2}=\tan\frac{\pi}{2} \Rightarrow a=b$.

Thus, the transverse and conjugate axes of a rectangular hyperbola are equal and the equation of the hyperbola is $x^2-y^2=a^2$.

Remarks: Since the transverse and conjugate axis of a rectangular hyperbola are equal. So, its eccentricity e is given by

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{2}$$

17.1 With Asymptotes as Coordinate Axes

Equation of the hyperbola referred to the transverse and conjugate axes along the axes of co-ordinates, the equation of the rectangular hyperbola is $x^2-y^2=a^2$ ....(i)

The asymptotes of (i) are $y=x$ and $y=-x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of $-\pi/4$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and, we have

$$x=X\cos(-\pi/4)-Y\sin(-\pi/4)=\frac{X+Y}{\sqrt{2}} \text{ and } y=X\sin(-\pi/4)+Y\cos(-\pi/4)=\frac{Y-X}{\sqrt{2}}$$

Substituting the values of x and y in (i), we obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2-\left(\frac{Y-X}{\sqrt{2}}\right)^2=a^2$

$$\Rightarrow XY=\frac{a^2}{2} \Rightarrow XY=c^2, \text{ where } c^2=\frac{a^2}{2}$$

Thus, the equation of the hyperbola referred to its asymptotes as the coordinates axes is

$$xy=c^2, \text{ where } c^2=\frac{a^2}{2}$$

Remark: The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy=c^2$.

If the asymptotes of a rectangular hyperbola are $x=\alpha$, $y=\beta$, then its equation is

$(x-\alpha)(y-\beta)=c^2$ or $xy-ay-bx+\lambda=0$; $(\lambda\le\alpha\beta)$

17.2 Tangent

Point Form

The equation of the tangent at (x_1, y_1) to the hyperbola $xy=c^2$ is $xy_1+yx_1=2c^2$ or $\frac{x}{x_1}+\frac{y}{y_1}=2$.

Parametric Form

The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy=c^2$ is $\frac{x}{t}+yt=2c$.

Note: Tangent at P $\left(ct_1, \frac{c}{t_1}\right)$ and Q $\left(ct_2, \frac{c}{t_2}\right)$ to the rectangular hyperbola $xy=c^2$ intersect at $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$

17.3 Normal

Point Form

The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$

Parametric Form

The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt - \frac{y}{t} = ct^2 - \frac{c}{t^2}$

Note:

(i) The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree equation in t. So, in general, at most four normals can be drawn from a point to the hyperbola $xy = c^2$.

(ii) The equation of the polar of any point $P(x_1, y_1)$ with respect to $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

(iii) The equation of the chord of the hyperbola $xy = c^2$ whose midpoint (x, y) is $xy_1 + yx_1 = 2x_1y_1$ or $T = S'$, where T and S' have their usual meanings.

(iv) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the rectangular $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

Illustration 32: A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC

(ii) The area of the triangle formed by the tangents A, B and C **(JEE ADVANCED)**

Sol: Use parametric co-ordinates and the formula for the area to get the desired result.

Let co-ordinates of A, B and C on the hyperbola $xy = c^2$ be $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

(i) Area of triangle ABC $= \frac{1}{2}\left(\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right) = \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$

$$= \frac{c^2}{2t_1t_2t_3}\,|\,t_1^2t_3 - t_2^2t_3 + t_1t_2^2 - t_1t_3^2 + t_2t_3^2 - t_1^2t_2\,| = \frac{c^2}{2t_1t_2t_3}\,|\,(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\,|$$

(ii) Equation of tangents at A, B, C are $x + yt_1^2 - 2ct_1 = 0$, $x + yt_2^2 - 2ct_2 = 0$ and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area $= \frac{1}{2\,|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2$ ... (i)

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_3 = t_2^2 - t_1^2$ and $C_2 = t_1^2 - t_3^2$

From (i) $= \frac{1}{2\,|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|}\,4c^2(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2 = 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

$\therefore$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

PROBLEM SOLVING TACTICS

(a) In general convert the given hyperbola equation into the standard form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and compare it with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then solve using the properties of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. So, it is advised to remember the standard results.

(b) Most of the standard results of a hyperbola can be obtained from the results of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ just by changing the sign of b^2.

FORMULAE SHEET

HYPERBOLA

(a) Standard Hyperbola:

Imp. Terms \ Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	(±ae, 0)	(0, ±be)
Equation of directrices	x = ± a/e	y = ± b/e
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of L.R.	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	(a sec φ, b tan φ) $0 \le \phi < 2\pi$	(a tan φ, b sec φ) $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ $S'P = ex_1 + a$	$SP = ey_1 - b$ $S'P = ey_1 + b$
(S′P − SP)	2a	2b
Tangents at the vertices	x = −a, x = a	y = −b, y = b
Equation of the transverse axis	y = 0	x = 0
Equation of the conjugate axis	x = 0	y = 0

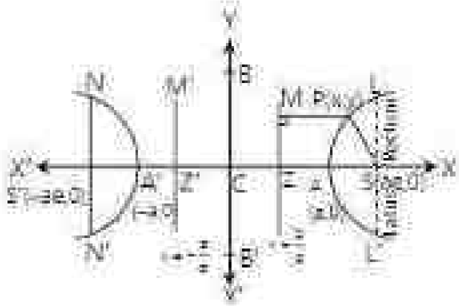


Figure 12.32: Hyperbola

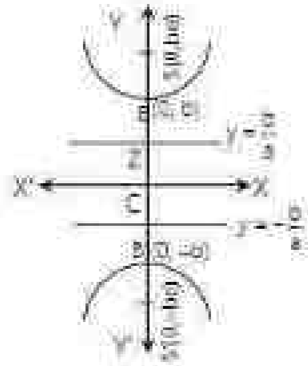


Figure 12.33: Conjugate Hyperbola

(b) Special form of hyperbola: If (h , k) is the centre of a hyperbola and its axes are parallel to the co-ordinate axes, then the equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(c) Parametric equations of a hyperbola: The equation $x = a\sec\phi$ and $y = b\tan\phi$ are known as the parametric equation of the standard hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}$, then $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(d) Position of a point and a line w.r.t. a hyperbola: n The point (x_1, y_1) lies inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ being >, = or < zero.

The line $y = mx + c$ intersects at 2 distinct points, 1 point or does not intersect with the hyperbola according as c^2 >, = or < $a^2m^2 - b^2$.

(e) Tangent:

(i) Point form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(ii) Parametric form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at parametric coordinates $(a\sec\phi, b\tan\phi)$ is $\frac{x}{a}\sec\phi-\frac{y}{b}\phi=1.$

(iii) Slope form: The equation of the tangents having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$y = mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

(f) Equation of a pair of tangents from an external point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1 = T^2$.

(g) Normal:

(i) Point form: The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2.$$

(ii) Parametric form: The equation of the normal at parametric coordinates (a sec θ, b tanθ) to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\quad ax \cos\theta + by \cot\theta = a^2 + b^2.$

(iii) Slope form: The equation of the normal having slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$$

(iv) Condition for normality: y = mx + c is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

$$c^2 = \frac{m^2(a^2 + b^2)^2}{(a^2 - m^2b^2)}$$

(v) Points of contact: Co-ordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2 - b^2m^2}}\right)$.

(h) The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $x^2 + y^2 = a^2 - b^2$.

(i) Equation of the chord of contact of the tangents drawn from the external point (x_1, y_1) to the hyperbola is

given by $\qquad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(j) The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) is $T = S_1$

(k) Equation of a chord joining points P(a sec f_1, b tan f_1) and Q (a sec f_2, b tan f_2) is

$$\frac{x}{a}\cos\left(\frac{\phi_1 - \phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1 + \phi_2}{2}\right) = \cos\left(\frac{\phi_1 + \phi_2}{2}\right)$$

(l) Equation of the polar of the point (x_1, y_1) w.r.t. the hyperbola is given by T = 0.

The pole of the line lx + my + n = 0 w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\left(-\frac{a^2l}{n}, \frac{b^2m}{n}\right)$

(m) The equation of a diameter of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ corresponding to the chords of slope m is $y = \frac{b^2}{a^2m}x$

(n) The diameters $y = m_1x$ and $y = m_2x$ are conjugate if $m_1m_2 = \frac{b^2}{a^2}$

(o) Asymptotes:

- Asymptote to a curve touches the curve at infinity.
- The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$.

- The asymptote of a hyperbola passes through the centre of the hyperbola.
- * The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$
- * The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{a^2}{b^2}$ or $2\sec^{-1}e$.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The bisector of the angles between the asymptotes are the coordinate axes.
- Equation of the hyperbola – Equation of the asymptotes = constant.

(p) Rectangular or Equilateral Hyperbola:

- A hyperbola for which a = b is said to be a rectangular hyperbola, its equation is $x^2-y^2=a^2$.
- $xy=c^2$ represents a rectangular hyperbola with asymptotes x = 0, y = 0.
- Eccentricity of a rectangular hyperbola is $\sqrt{2}$ and the angle between the asymptotes of a rectangular hyperbola is 90°.
- Parametric equation of the hyperbola $xy=c^2$ are $x=ct$, $y=\frac{c}{t}$, where t is a parameter.
- Equation of a chord joining t_1, t_2 on $xy=c^2$ is $x+y\,t_1t_2=c(t_1+t_2)$
- Equation of a tangent at (x_1, y_1) to $xy=c^2$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$.
- Equation of a tangent at t is $x+yt^2=2ct$
- Equation of the normal at (x_1, y_1) to $xy=c^2$ is $xx_1-yy_1=x_1^2-y_1^2$.
- Equation of the normal at t on $xy=c^2$ is $xt^3-yt-ct^4+c=0$.

  (i.e. Four normals can be drawn from a point to the hyperbola $xy=c^2$)
- If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.
- Equation of chord of the hyperbola $xy=c^2$ whose middle point is given is $T=S_1$.
- Point of intersection of tangents at t_1 and t_2 to the hyperbola $xy=c^2$ is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$

Solved Examples

JEE Main/Boards

Example 1: Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol: Calculate the value of 'a', by using the distance between the two foci and eccentricity. Then calculate the value of 'b'. Using these two values find the equation of the hyperbola.

Let S, S' be the foci and C be the centre of the hyperbola S, S' and C lie on the line y = 4. The co-ordinates of the centre are (1, 4).

The equation of the hyperbola is

$$\frac{(x-1)^2}{a^2}-\frac{(y-4)^2}{b^2}=1$$

The distance between the foci is 2ae = 10; $\therefore$ $a=\frac{5}{2}$

$$b^2=a^2(e^2-1)=\frac{25}{4}(4-1)=\frac{75}{4}$$

Hence the equation of the hyperbola is

$$\frac{(x-1)^2}{\frac{25}{4}}-\frac{(y-4)^2}{\frac{75}{4}}=1$$

Example 2: Obtain the equation of hyperbola whose asymptotes are the straight lines x + 2y + 3 = 0 & 3x + 4y + 5 = 0 and which passes through the point (1, –1)

Sol: Use the following formula

Equation of hyperbola – Equation of asymptotes = constant.

The equation of the hyperbola is

$(x + 2y + 3)(3x + 4y + 5) = k$, k being a constant.

This passes through the point (1, –1)

$\therefore (1 + 2(-1) + 3)(3(1) + 4(-1) + 5) = k$

$\Rightarrow k = 2 \times 4 = 8$

$\therefore$ The equation of the hyperbola is

$(x + 2y + 3)(3x + 4y + 5) = 8$

Example 3: If e and e' are the eccentricities of two hyperbolas conjugate to each other,

show that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol: Start with the standard equation of two hyperbolas and eliminate 'a' and 'b'.

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

be the two hyperbolas with eccentricities e and e' respectively

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{1}{e^2} = \frac{a^2}{a^2 + b^2}$

$a^2 = b^2(ee'^2 - 1) \Rightarrow \frac{1}{e'^2} = \frac{b^2}{a^2 + b^2}$

$\therefore \frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{(a^2 + b^2)} + \frac{b^2}{(a^2 + b^2)} = 1$

Example 4: If any point P on the rectangular hyperbola $x^2 - y^2 = a^2$ is joined to its foci S, S' show that SPS¢P = CP^2, where C is the centre of the hyperbola.

Sol: The eccentricity of a rectangular hyperbola is $\sqrt{2}$. Consider a parametric point on the hyperbola and simplify the LHS.

Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ is P(a secθ, a tanθ) ; eccentricity of a rectangular hyperbola is $\sqrt{2}$.

S is (ae, 0), S' is (–ae, 0) and C is (0, 0)

$(SP)^2.(S'P)^2 = [(a\sec\theta - ae)^2 + a^2\tan^2 q] \times$

$[(a\sec\theta + ae)^2 + a^2\tan^2 q]$

$= a^4[(\sec^2\theta + \tan^2\theta + e^2)^2 - 4e^2\sec^2 q]$

$= a^4[(2\sec^2\theta - 1 + 2)^2 - 4.2\sec^2 q]$

$= a^4[(2\sec^2\theta + 1)^2 - 8\sec^2 q]$

$= a^4[(2\sec^2\theta - 1)^2]$

$\therefore SPS'P = a^2(2\sec^2\theta - 1)$

$= a^2(\sec^2\theta + \tan^2\theta)$

$= CP^2$.

Example 5: Find the equation of the hyperbola conjugate to the hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y + 2 = 0$

Sol: Use the formula.

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

Let asymptotes be

$2x^2 + 3xy - 2y^2 - 5x + 5y + \lambda = 0$

The equation above represents a pair of lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore \quad \lambda = -5$

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

$\therefore$ Conjugate Hyperbola

= 2(Asymptotes) – Hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$

Example 6: If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin then the eccentricity of the hyperbola is

Sol: Use the definition of the hyperbola S¢P – SP = 2a.

Let S(5, 12) and S'(24, 7) be the two foci and P(0, 0) be a point on the conic then

$SP = \sqrt{25 + 144} = \sqrt{169} = 13$;

$S'P = \sqrt{(24)^2 + 7^2} = \sqrt{625} = 25$

and $SS' = \sqrt{(24 - 5)^2 + (7 - 12)^2} = \sqrt{19^2 + 5^2} = \sqrt{386}$

since the conic is a hyperbola, S¢P – SP = 2a, the length of transverse axis and SS' = 2ae, e being the eccentricity.

$\Rightarrow \quad e = \frac{SS'}{S'P - SP} = \frac{\sqrt{386}}{12}$

Example 7: An equation of a tangent to the hyperbola $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is

Sol: Write the equation of the hyperbola in the standard form and compare to get the equation of the tangent.

Equation of the hyperbola can be written as

$X^2/5^2 - Y^2/4^2 = 1$ ....(i)

where $X = x - 3$ and $Y = y - 2$.

Equation of a tangent which makes an angle $\pi/4$, with the transverse axis $X = 0$ of (i) is

$$Y = \tan\frac{\pi}{4}X \pm \sqrt{25\tan^2\frac{\pi}{4} - 16}$$

$\Rightarrow \quad y - 2 = x - 3 \pm \sqrt{25 - 16}$

$\Rightarrow \quad y - 2 = x - 3 \pm 3$

$\Rightarrow \quad y = x + 2$ or $y = x - 4$.

Example 8: If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola, then prove that Gg=2PC.

Sol: In the equation of a normal, find the point of intersection with the axes and find the coordinates of G and g.

Let $P(x_1, y_1)$ be any point on the hyperbola $x^2 - y^2 = 4$ then equation of the normal at P is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$\Rightarrow \quad x_1y + y_1x = 2x_1y_1$

Then coordinates of G are $(2x_1, 0)$ and of g are $(0, 2y_1)$ so that

$$PG = \sqrt{(2x_1 - x_1)^2 + y_1^2} = \sqrt{x_1^2 + y_1^2} = PC$$

$$Pg = \sqrt{x_1^2 + (2y_1 - y_1)^2} = \sqrt{x_1^2 + y_1^2} = PC$$

and

$$Gg = \sqrt{(2x_1)^2 + (2y_1)^2} = 2\sqrt{x_1^2 + y_1^2} = 2PC$$

Hence proved.

Example 9: The normal to the curve at P(x, y) meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P then the curve is-

Sol: Similar to the previous question.

Equation of the normal at (x, y) is

$Y - y = -\frac{dx}{dy}(X - x)$ which meets the x-axis at G

$\left(0, x + y\frac{dy}{dx}\right)$, then $x + y\frac{dy}{dx} = \pm 2x$

$\Rightarrow \quad x + y\frac{dy}{dx} = 2x \Rightarrow \quad y\,dy = x\,dx$

$\Rightarrow \quad x^2 - y^2 = c$

or $\quad y\,dy = -3x\,dx$

$\Rightarrow \quad 3x^2 + y^2 = c$

Thus the curve is either a hyperbola or an ellipse.

Example 10: Find the centre, eccentricity, foci and directrices of the hyperbola

$16x^2 - 9y^2 + 32x + 36y - 164 = 0$

Sol: Represent the equation of the hyperbola in the standard form and compare.

Here,

$16x^2 + 32x + 16 - (9y^2 - 36y + 36) - 144 = 0$

or $\quad 16(x + 1)^2 - 9(y - 2)^2 = 144$

$\therefore \quad \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Putting $x + 1 = X$ and $y - 2 = Y$, the equation becomes

$$\frac{X^2}{9} - \frac{Y^2}{16} = 1$$

which is in the standard form.

$\because \quad b^2 = a^2(e^2 - 1)$, here $a^2 = 9$ & $b^2 = 16$

$\therefore \quad e^2 - 1 = \frac{16}{9} \Rightarrow \quad e^2 = \frac{25}{9}$, i.e. $e = \frac{5}{3}$

Now, centre $= (0, 0)_{X,Y} = (-1, 2)$

foci $= (\pm ae, 0)_{X,Y} = \left(\pm 3\cdot\frac{5}{3}, 0\right)_{X,Y} = (\pm 5, 0)_{X,Y}$

$= (-1 \pm 5, 2) = (4, 2), (-6, 2)$

Directrices in X, Y coordinates have the equations

$$X \pm \frac{a}{e} = 0 \quad \text{or} \quad x + 1 \pm \frac{3}{5/3} = 0$$

i.e. $\quad x + 1 \pm \frac{9}{5} = 0$

$\therefore \quad x = -\frac{14}{5}$ and $x = \frac{4}{5}$

UNIT-2

CONJUGATE HYPERBOLA

1. INTRODUCTION

A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.

The fixed point is called the focus, the fixed line is called the directrix. The constant ratio is generally denoted by e and is known as the eccentricity of the hyperbola. A hyperbola can also be defined as the locus of a point such that the absolute value of the difference of the distances from the two fixed points (foci) is constant. If S is the focus, ZZ' is the directrix and P is any point on the hyperbola as show in figure.

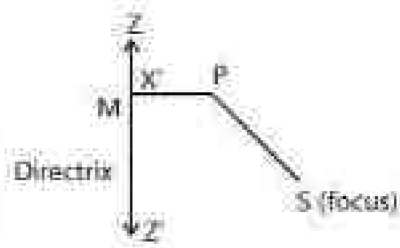


Figure 12.1

Then by definition, we have $\frac{SP}{PM} = e\ (e > 1)$.

Note: The general equation of a conic can be taken as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

This equation represents a hyperbola if it is non-degenerate (i.e. eq. cannot be written into two linear factors)

$\Delta \neq 0,\ h^2 > ab$. Where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

PLANCESS CONCEPTS

1. The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0$ and $\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

   Degeneracy condition depends on the determinant of the 3x3 matrix and the type of conic depends on the determinant of the 2x2 matrix.

2. Also the equation can be taken as the intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

2. STANDARD EQUATION OF HYPERBOLA

Let the center O of the hyperbola be at the origin O and the foci F_1 and F_2 be on the x-axis.

The coordinates of foci F_1 and F_2 are $(-c, 0)$ and $(c, 0)$.

By the definition of hyperbola,

Distance between a point P and focus F_1 − Distance between P and focus F_2 = constant (say 2a)

$PF_1 - PF_2 = 2a; \sqrt{(x+c)^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = 2a$

$\Rightarrow \quad \sqrt{(x+c)^2+(y)^2} = 2a + \sqrt{(x-c)^2+(y)^2}$

Squaring both the sides, we get $\quad (x+c)^2 + y^2 = 4a^2 + 2(2a)\sqrt{(x-c)^2+(y)^2} + (x-c)^2 + y^2$

$\Rightarrow \quad x^2 + 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} + x^2 - 2cx + c^2 + y^2$

$\Rightarrow \quad 4cx = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx = a^2 + a\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx - a^2 = a\sqrt{(x-c)^2+(y)^2}$

Squaring again, we get

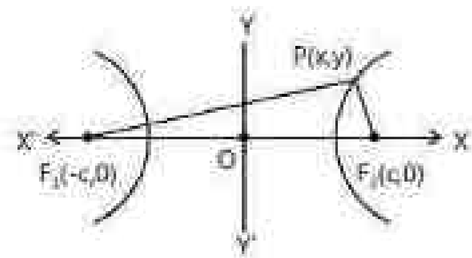


Figure 12.2

$c^2x^2 - 2a^2cx + a^4 = a^2[(x-c)^2 + y^2]$

$c^2x^2 - 2a^2cx + a^4 = a^2[x^2 - 2cx + c^2 + y^2] = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$\Rightarrow \quad (c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2) \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad$ (taking $b^2 = c^2 - a^2$)

Hence, any point P(x, y) on the hyperbola satisfies the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. TERMS ASSOCIATED WITH HYPERBOLA

(a) Focus: The two fixed points are called the foci of the hyperbola and are denoted by F_1 and F_2. The distance between the two foci F_1 and F_2 is denoted by 2c.

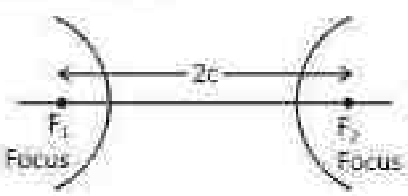


Figure 12.3

(b) Centre: The midpoint of the line joining the foci is called the center of the hyperbola.

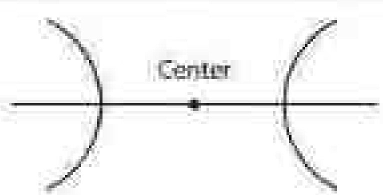


Figure 12.4

(c) Transverse-Axis: The line through the foci is called the transverse axis. Length of the transverse axis is 2a.

Figure 12.5

(d) Conjugate-Axis: The line segment through the center and perpendicular to the transverse axis is called the conjugate axis. Length of the conjugate axis is 2b.

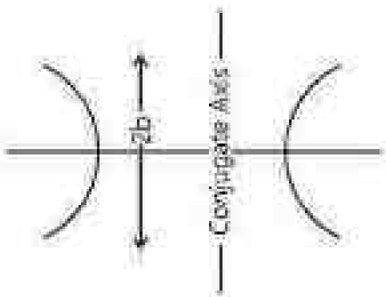


Figure 12.6

(e) Vertices: The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola. The distance between the two vertices is denoted by 2a.

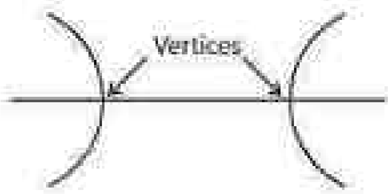
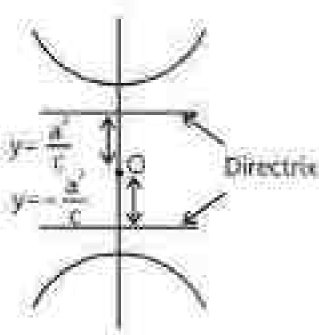


Figure 12.7

(f) Eccentricity: Eccentricity of the hyperbola is defined as $\frac{c}{a}$ and it is denoted by e. And e is always greater than 1 since c is greater than 1.

(g) Directrix: Directrix is a line perpendicular to the transverse axis and cuts it at a distance of $\frac{a^2}{c}$ from the centre.

i.e. $x = \pm\frac{a^2}{c}$ or $y = \pm\frac{a^2}{c}$



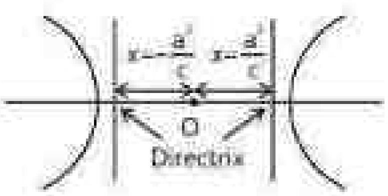


Figure 12.8

(h) Length of The Latus Rectum: The latus rectum of a hyperbola is a line segment perpendicular to the transverse axis and passing through any of the foci and whose end points lie on the hyperbola. Let the length of LF be ℓ. Then, the coordinates of L are (c, ℓ)

Since, L lies on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Therefore, we have $\frac{c^2}{a^2} - \frac{\ell^2}{b^2} = 1$

$$\Rightarrow \quad \frac{\ell^2}{b^2} = \frac{c^2}{a^2} - 1 = \frac{c^2 - a^2}{a^2} \quad \Rightarrow \quad \ell^2 = b^2\left(\frac{b^2}{a^2}\right) = \frac{b^4}{a^2} \Rightarrow \frac{b^2}{a}$$

Latus rectum $\quad LL' = LF + L'F = \frac{b^2}{a} + \frac{b^2}{a} = \frac{2b^2}{a}$

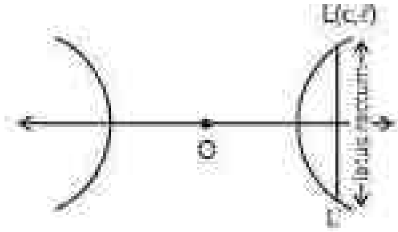


Figure 12.9

(i) Focal Distance of a Point: Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as shown in figure. Then by definition,

We have $SP = e.PM$ and $S'P = e.PM'$

$\Rightarrow SP = e.NK = e(CN - CK) = e\left(x - \frac{a}{e}\right) = ex - a$ and $S'P = e(NK')$

$= e(CN + CK') = e\left(x + \frac{a}{e}\right) = ex + a$

$\Rightarrow S'P - SP = (ex + a) - (ex - a) = 2a$ = length of transverse axis

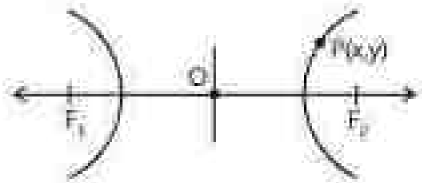


Figure 12.10

Illustration 1: Find the equation of the hyperbola, where the foci are $(\pm 3, 0)$ and the vertices are $(\pm 2, 0)$. **(JEE MAIN)**

Sol: Use the relation $c^2 = a^2 + b^2$, to find the value of b and hence the equation of the hyperbola.

We have, foci $= (\pm c, 0) = (\pm 3, 0) \Rightarrow \quad c = 3$

and vertices $\quad (\pm a, 0) = (\pm 2, 0)$

$a = 2$

But $c^2 = a^2 + b^2 \Rightarrow \quad 9 = 4 + b^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow \quad b^2 = 5$

Here, the foci and vertices lie on the x-axis, therefore the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \Rightarrow \qquad \frac{x^2}{4} - \frac{y^2}{5} = 1$$

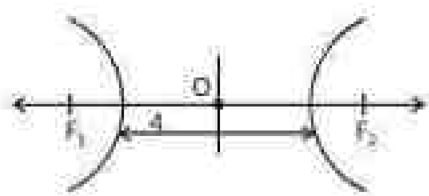


Figure 12.11

Illustration 2: Find the equation of the hyperbola, where the vertices are $(0, \pm 5)$ and the foci are $(0, \pm 8)$. **(JEE MAIN)**

Sol: Similar to the previous question.

We have, vertices $(0, \pm a) = (0, \pm 5) \Rightarrow \quad a = 5$

foci $(0, \pm c) = (0, \pm 8) \quad \Rightarrow \quad c = 8$

But, we know that $\quad c^2 = a^2 + b^2 \quad \Rightarrow \quad 64 = 25 + b^2$

$\Rightarrow \quad b^2 = 64 - 25 = 39$

Here, the foci and vertices lie on the y-axis, therefore the equation of

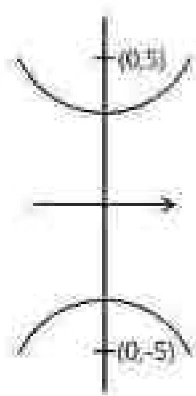


Figure 12.12

hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ i.e. $\frac{y^2}{25} - \frac{x^2}{39} = 1$

which is the required equation of the hyperbola.

Illustration 3: If circle c is a tangent circle to two fixed circles c_1 and c_2, then show that the locus of c is a hyperbola with c_1 and c_2 as the foci. **(JEE MAIN)**

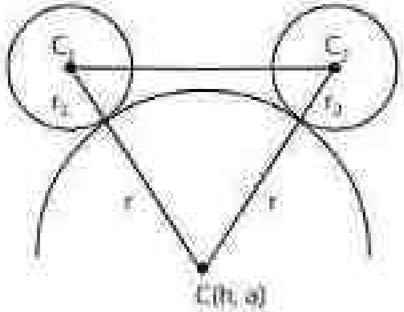


Figure 12.13

Sol: Refer to the definition of a hyperbola.

$cc_1 = r + r_1$; $cc_2 = r + r_2$

$cc_1 - cc_2 = r_1 - r_2 =$ constant

Illustration 4: Find the equation of the hyperbola whose directrix is 2x + y = 1 and focus (1, 2) and eccentricity $\sqrt{3}$. **(JEE MAIN)**

Sol: Use the definition of the hyperbola to derive the equation.

Let P(x, y) be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

They by definition SP = e PM

$\Rightarrow (SP)^2 = e^2(PM)^2 \Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2 \Rightarrow 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 5: Find the equation of the hyperbola when the foci are at $(\pm 3\sqrt{5}, 0)$, and the latus rectum is of length 8. **(JEE ADVANCED)**

Sol: Use the formula for the length of the latus rectum to get a relation between a and b. Then use the foci and the relation between a and b to get the equation of the hyperbola.

Here foci are at $(\pm 3\sqrt{5}, 0)$ $\Rightarrow$ $c = 3\sqrt{5}$

Length of the latus rectum $= \frac{2b^2}{a} = 8$

$\Rightarrow$ $b^2 = 4a$ ... (i)

We know that

$c^2 = a^2 + b^2$

$(3\sqrt{5})^2 = a^2 + 4a$

$45 = a^2 + 4a$

$a^2 + 4a - 45 = 0$

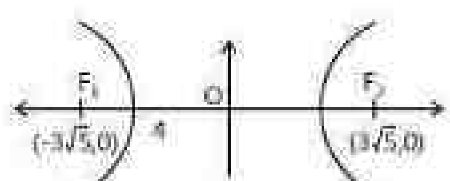


Figure 12.14

$(a + 9)(a - 5) = 0$

$a = -9,\ a = 5$ (a cannot be –ve)

Putting $a = 5$ in (i), we get

$b^2 = 5 \times 4 = 20 \Rightarrow b^2 = 20$

Since, foci lie on the x-axis, therefore the equation of the hyperbola is of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ i.e., $\frac{x^2}{25} - \frac{y^2}{20} = 1$

$\Rightarrow 20x^2 - 25y^2 = 500 \Rightarrow 4x^2 - 5y^2 = 100$

Which is the required equation of hyperbola.

Illustration 6: Find the equation of the hyperbola when the foci are at $(0, \pm\sqrt{10})$, and passing through (2, 3) **(JEE ADVANCED)**

Sol: Start with the standard equation of a hyperbola and use the foci and the point (2 , 3) to find the equation.

Here, foci are at $(0, \pm\sqrt{10})$

$\Rightarrow c = \sqrt{10}$ Here the foci lie at the y-axis.

So the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{... (i)}$$

Point (ii, iii) lies on (i).

$$\text{So: } \frac{9}{a^2} - \frac{4}{b^2} = 1 \Rightarrow \frac{9}{a^2} = 1 + \frac{4}{b^2} \Rightarrow \frac{9}{a^2} = \frac{b^2 + 4}{b^2} a^2 = \frac{9b^2}{b^2 + 4} \quad \text{... (ii)}$$

We know that

$c^2 = a^2 + b^2$

$\Rightarrow 10 = \frac{9b^2}{b^2 + 4} + b^2$

$\Rightarrow \frac{9b^2 + b^4 + 4b^2}{b^2 + 4} = 10$

$\Rightarrow 10b^2 + 40 = b^4 + 13b^2$

$\Rightarrow b^4 + 3b^2 - 40 = 0$

$\Rightarrow (b^2 + 8)(b^2 - 5) = 0$

$\Rightarrow b^2 + 8 = 0,\ b^2 - 5 = 0$

$\Rightarrow b^2 = -8$ & $b^2 = 5$ ($b^2 = -8$ not possible)

$\Rightarrow b^2 = 5$ in (ii), we get

$a^2 = \frac{9 \times 5}{5 + 4} = \frac{45}{9} = 5$

Again putting $a^2 = 5$ and $b^2 = 5$ in (i), we get

$\frac{y^2}{5} - \frac{x^2}{5} = 1 \Rightarrow y^2 - x^2 = 5$

Which is the required equation of the hyperbola.

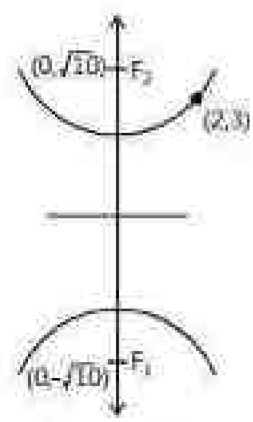


Figure 12.15

Illustration 7: An ellipse and hyperbola are confocal i.e., having same focus and conjugate axis of hyperbola & minor axis of ellipse. If e_1 and e_2 are the eccentricities of the hyperbola and ellipse then find $\frac{1}{e_1^2}+\frac{1}{e_2^2}$.

(JEE ADVANCED)

Sol: Consider the standard equation of an ellipse and hyperbola by taking the eccentricity as e_1 and e_2 respectively. Find the relation between the eccentricities by using the condition that they have the same focus.

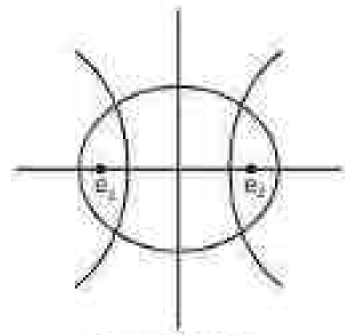


Figure 12.16

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1 \Rightarrow$ $ae_1 = Ae_2$ and $B = b$

$\Rightarrow$ $B^2 = b^2$ $\Rightarrow$ $A^2(e_2^2-1) = a^2(1-e_2)$

$\therefore$ $\frac{a^2e_1^2}{e_2^2}(e_2^2-1) = a^2(1-e_1^2)$ $\therefore$ $\frac{1}{e_1^2}+\frac{1}{e_2^2}=2$

Illustration 8: Find the equation of a hyperbola if the distance of one of its vertices from the foci are 3 and 1. Find all the possible equations. **(JEE ADVANCED)**

Sol: Consider two cases when the major axis is parallel to the X – axis and the minor axis is parallel to the Y-axis and vice versa.

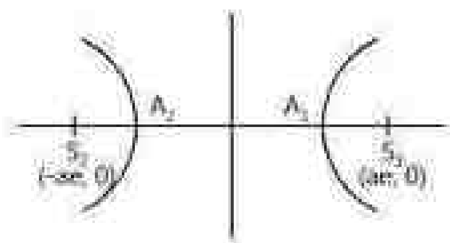


Figure 12.17

Case I:

$ae - a = 1$

$ae + a = 3$

$\Rightarrow$ $e = 2$

$\Rightarrow$ $a = 1$

$\Rightarrow$ $b^2 = 3$

Equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3}=1$

Case II

$b(e - 1) = 1$

$b(e + 1) = 3$

$\Rightarrow$ $e = 2, b = 1, a^2 = 3$

Equation of hyperbola is $\frac{y^2}{3}-\frac{x^2}{1}=1$

4. CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The hyperbola conjugate to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

The eccentricity of the conjugate hyperbola is given by a2 = b2(e2 – 1)

and the length of the latus rectum is $\frac{2a^2}{b}$

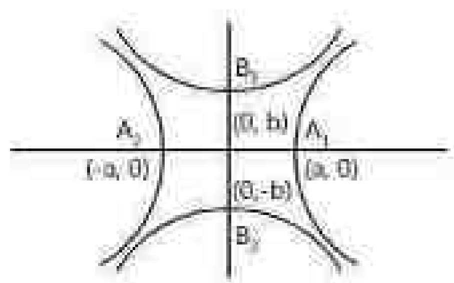
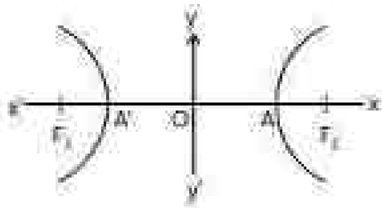
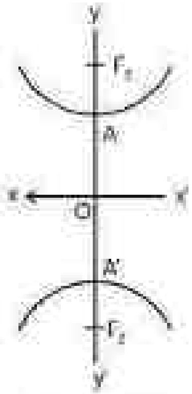


Figure 12.18

Condition of similarity: Two hyperbolas are said to be similar if they have the same value of eccentricity.

Equilateral hyperbola: If a = b or L(T.A.) = L(C.A) then it is an equilateral or rectangular hyperbola.

5. PROPERTIES OF HYPERBOLA/CONJUGATE HYPERBOLA

Equation of the Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Figure	 Figure 12.19	 Figure 12.20
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Transverse axis	2a	2b
Conjugate axis	2b	2a
Relation between a, b, c	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Foci	(±c, 0)	(0, ±c)
Eccentricity	$e = \frac{c}{a}$	$e' = \frac{c}{b}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

PLANCESS CONCEPTS

- If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate hyperbola then

$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=1}$$

$$e_1^2 = 1 + \frac{b^2}{a^2} \qquad e_2^2 = 1 + \frac{a^2}{b^2}$$

$$\therefore \quad \frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

- The foci of a hyperbola and its conjugate hyperbola are CONCYCLIC and form vertices of square.

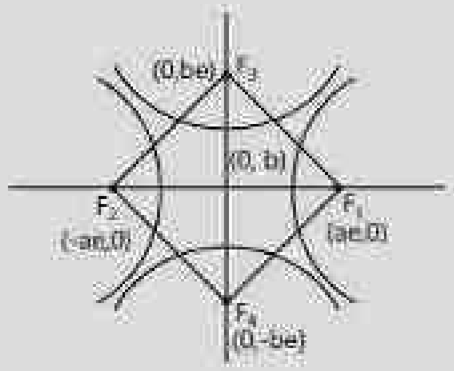


Figure 12.21

Anvit Tawar (JEE 2009, AIR 9)

6. AUXILIARY CIRCLE

A circle described on the transverse axis as diameter is an auxiliary circle and its equation is $x^2 + y^2 = a^2$

Any point of the hyperbola is P ≡ (a sec θ, b tan θ)

P, Q are called corresponding point and θ is eccentric angle of P.

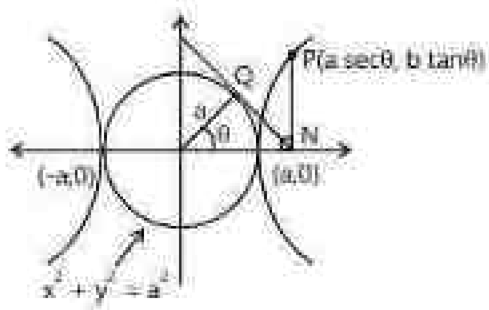


Figure 12.22

PLANCESS CONCEPTS

1. If θ ∈ (0, π/2), P lies on upper right branch.
2. If θ ∈ (π/2, π), P lies on upper left branch.
3. If θ ∈ (π, 3π/2), P lies on lower left branch.
4. If θ ∈ (3π/2, 2π), P lies on lower right branch.

Vaibhav Krishnan (JEE 2009, AIR 22)

7. PARAMETRIC COORDINATES

Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Draw PL perpendicular from P on OX and then a tangent LM from L to the circle described on A'A as diameter.

Then, x = CL = CM sec θ = a sec θ

Putting x = a sec θ in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we obtain y = b tan θ

Thus, the coordinates of any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are (a sec θ, b tan θ), where θ is the parameter such that $0 \le \theta \le 3\pi$. These coordinates are known as the parametric coordinates. The parameter θ is also called the eccentric angle of point P on the hyperbola.

The equation x = a sec θ and y = b tan θ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Note: (i) The circle $x^2 + y^2 = a^2$ is known as the auxiliary circle of the hyperbola.

Let P (a sec θ_1, b tan θ_1) and Q (a sec θ_2, b tan θ_2) be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Then the equation of the chord PQ is

$$y - b\tan q_1 = \frac{b\tan\theta_2 - b\tan\theta_1}{a\sec\theta_2 - a\sec\theta_1}(x - a\sec q_1) \Rightarrow \frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

Illustration 9: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis. **(JEE MAIN)**

Sol: Establish the relation between a and b and then use the eccentricity formula.

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then transverse axis = 2a and latus rectum = $\frac{2b^2}{a}$

According to the question $\frac{2b^2}{a}=\frac{1}{2}(2a)$

$\Rightarrow \quad 2b^2 = a^2 \Rightarrow 2a^2(e^2-1) = a^2 \Rightarrow 2e^2 - 2 = 1 \Rightarrow e^2 = 3/2 \quad \therefore e = \sqrt{3/2}$

Illustration 10: If the chord joining two points (a sec θ_1, b tan θ_1) and (a sec θ_2, b tan θ_2) passes through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{1-e}{1+e}$. **(JEE ADVANCED)**

Sol: Obtain a relation between the two given eccentric angles by substituting the point in the equation of chord.

The equation of the chord joining (a sec θ_1, b tan θ_1) and (a sec θ_2, b tan θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If it passes through the focus (ae, 0) then $e\cos\left(\frac{\theta_1-\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$

$$\Rightarrow \quad \frac{\cos((\theta_1-\theta_2)/2)}{\cos((\theta_1+\theta_2)/2)}=1/e$$

using componendo dividendo rule we get $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{1-e}{1+e}$.

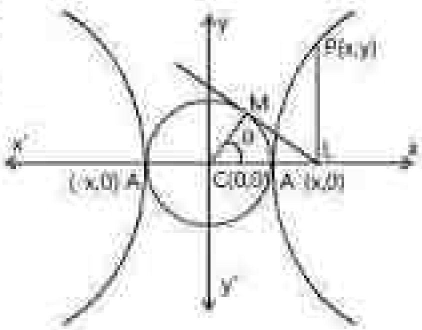


Figure 12.23

8. POINT AND HYPERBOLA

The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ '<' or '=' or '>' 0

Proof: Draw PL perpendicular to x-axis. Suppose it cuts the hyperbola at Q(x_1, y_2).

Clearly, PL > QL

$\Rightarrow \quad y_1 > y_2 \quad \Rightarrow \quad \frac{y_1^2}{b^2} > \frac{y_2^2}{b^2} \quad \Rightarrow \quad -\frac{y_1^2}{b^2} < -\frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$

$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 \qquad \left[\because Q(x_1, y_2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} = 1 \right]$

Thus the point (x_1, y_1) lies outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

Similarly, we can prove that the point (x_1, y_1) will lie inside or on the hyperbola according to

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$ or, $= 0$.

P lies outside/on/inside $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 / = 0 / > 0$

Illustration 11: Find the position of the points (7, −3) and (2, 7) relative to the hyperbola $9x^2 - 4y^2 = 36$.

(JEE MAIN)

Sol: Use the concept of position of a point w.r.t. the hyperbola.

The equation of the given hyperbola is $9x^2 - 4y^2 = 36$ or, $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Now,

$\frac{7^2}{4} - \frac{(-3)^2}{9} - 1 = \frac{41}{4} > 0$ and, $\frac{2^2}{4} - \frac{7^2}{9} \Rightarrow 1 - \frac{49}{9} \Rightarrow 1 = \frac{-49}{9} < 0$.

Hence, the point (7, −3) lies inside the parabola whereas the point (2, 7) lies outside the hyperbola.

Illustration 12: Find the position of the point (5, −4) relative to the hyperbola $9x^2 - y^2 = 1$. **(JEE MAIN)**

Sol: Use the concept of position of a point

Since $9(5)^2 - (4)^2 - 1 = 225 - 16 - 1 = 208 > 0$. So the point (5, −4) inside the hyperbola $9x^2 - y^2 = 1$.

9. LINE AND HYPERBOLA

Consider a line $y = mx + c$ and hyperbola $\frac{a^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving $y = mx + c$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \quad b^2x^2 - a^2(mx + c)^2 = a^2b^2 \quad \Rightarrow \quad (b^2 - a^2m^2)\, x^2 - 2a^2cmx - a^2(b^2 + c^2) = 0$

- $D > 0$: Secant
- $D = 0$: Tangent — $c^2 = a^2m^2 - b^2$ (condition of tangency)
- $D < 0$: Neither secant nor tangent

$\Rightarrow \quad y = mx + \sqrt{a^2m^2 - b^2}$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

PLANCESS CONCEPTS

No. of tangents drawn to a hyperbola passing through a given point (h, k)

Let y = mx+c be tangent to the hyperbola

$\Rightarrow \quad c^2 = a^2m^2 - b^2$

Since line passes through (h, k)

$\Rightarrow \quad (k - mh)^2 = a^2m^2 - b^2$

$\Rightarrow \quad (h^2 - a^2)m^2 - 2hkm + k^2 + b^2 = 0$

Hence a maximum of 2 tangents can be drawn to the hyperbola passing through (h, k)

$m_1 + m_2 = \frac{2hk}{h^2 - a^2} \qquad m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2}$

if $m_1m_2 = -1$ $\quad \boxed{x^2 + y^2 = a^2 - b^2}$

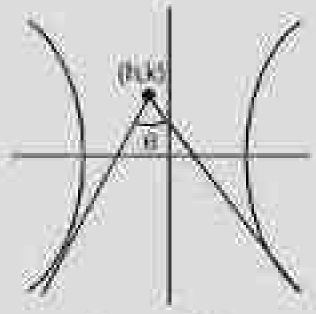


Figure 12.24

Illustration 13: Common tangent to $y^2 = 8x$ and $3x^2 - y^2 = 3$. **(JEE MAIN)**

Sol: Start with the standard equation of a tangent to a parabola and apply the condition for it be a tangent to $3x^2 - y^2 = 3$.

Tangent to the parabola is of the form $y = mx + \frac{2}{m}$. For this line to be tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1 \qquad c^2 = a^2m^2 - b^2$

$\Rightarrow \quad \frac{4}{m^2} = m^2 - 1 \quad \Rightarrow \quad m^2 = 4 \quad \therefore \pm y = 2x + 1$ are the common tangents.

10. TANGENT

Point Form: The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Slope Form: The equation of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$y = mx \pm \sqrt{a^2m^2 - b^2}$

The coordinates of the points of contact are $\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Parametric Form: The equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

Note:

(i) The tangents at the point P $(a \sec \theta_1, b \tan \theta_1)$ and Q $(a \sec \theta_2, b \tan \theta_2)$ intersect at the point R $\left(\frac{a\cos((\theta_1 - \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}, \frac{b\sin((\theta_1 + \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}\right)$.

(ii) If $|\theta_1 + \theta_2| = \pi$, then the tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touch the hyperbola at the extremities of a diameter.

(iv) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2=a^2$ and the product of these perpendiculars is b^2.

(v) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(vi) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as the diameter of the circle.

Illustration 14: Prove that the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2l^2 - b^2m^2 = n^2$. **(JEE MAIN)**

Sol: Apply the condition of tangency and prove the above result.

The given line is $lx + my + n = 0$ or $y = -l/m\ x - n/m$

Comparing this line with $y = Mx + c$ $\quad \therefore M = -l/m$ and $c = -n/m$ ....(i)

This line (i) will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2 = a^2M^2 - b^2$

$\Rightarrow \quad \frac{n^2}{m^2}=\frac{a^2l^2}{m^2}-b^2 \quad$ or $\quad a^2l^2 - b^2m^2 = n^2$. **Hence proved.**

Illustration 15: Find the equations of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$. **(JEE MAIN)**

Sol: Get the slope of the perpendicular line and use it to get the equation of the tangent.

Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$m \times 1 = -1$

$\Rightarrow \quad m = -1$

Since $\quad x^2 - 4y^2 = 36 \quad$ or $\quad \frac{x^2}{36}-\frac{y^2}{9}$

Comparing this with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \therefore a^2 = 36$ and $b^2 = 9$

So the equation of the tangents are $y = (-1)x \pm \sqrt{36\times(-1)^2-9}$

$\Rightarrow \quad y = -x \pm \sqrt{27} \quad \Rightarrow \quad x + y \pm 3\sqrt{3} = 0$

Illustration 16: If two tangents drawn from any point on hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles θ_1 and θ_2 with the axis then $\tan\theta_1.\tan\theta_2$.

Sol: Establish a quadratic in m, where m is the slope of the two tangents. Then use the sum and product of the roots to find $\tan\theta_1.\tan\theta_2$.

Let $c^2 = a^2 - b^2$

Any tangent to the ellipse $y = mx \pm \sqrt{a^2m^2+b^2}$

$c\tan\theta = mc\sec\theta \pm \sqrt{a^2m^2+b^2}$)

$c^2(\tan\theta - m\sec\theta)^2 = a^2m^2 - b^2$

$(c^2\sec^2\theta - a^2)m^2 + (\ldots\ldots)m + c^2\tan^2\theta - b^2 = 0 \Rightarrow \tan\theta_1\tan\theta_2$ = product of the roots

$$= \frac{c^2\tan^2\theta - b^2}{c^2\sec^2\theta - a^2} = 1$$

11. NORMAL

Point Form: The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

Parametric Form: The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a\,x\cos\theta + b\,y\cot\theta = a^2 + b^2$.

Slope Form: The equation of a normal of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}} \quad \text{at the points} \left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp\frac{b^2m}{\sqrt{a^2 - b^2m^2}}\right)$$

Note:

(i) At most four normals can be drawn from any point to a hyperbola.

(ii) Points on the hyperbola through which, normal through a given point pass are called co-normal points.

(iii) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This illustrates the reflection property of the hyperbola as "**An incoming light ray**" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common points.

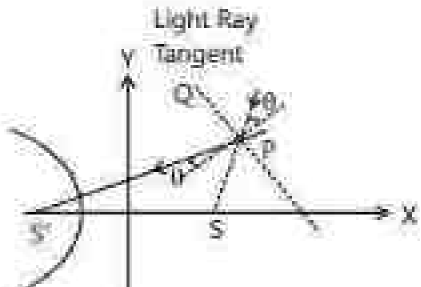


Figure 12.25

(iv) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(v) The sum of the eccentric angles of co-normal points is an odd multiple of π.

(vi) If θ_1, θ_2 and θ_3 are eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The normals at which are concurrent, then $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$

(vii) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then $(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$.

Illustration 17: How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$. Find the equation of these tangents and the angle between them. **(JEE MAIN)**

Sol: Use the concept of Position of a Point w.r.t. the hyperbola to find the number of real tangents.

Given point P = (4, 3)

Hyperbola $S \equiv \frac{x^2}{16}-\frac{y^2}{9}-1=0$

$\because S_1 \equiv \frac{16}{16}-\frac{9}{9}-1=-1<0$

$\Rightarrow$ Point P ≡ (4, 3) lies outside the hyperbola

$\therefore$ Two tangents can be drawn from the point P(4, 3). Equation of a pair of tangents is $SS_1 = T^2$.

$\Rightarrow \left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)(-1)=\left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$

$\Rightarrow -\frac{x^2}{16}+\frac{y^2}{9}+1=\frac{x^2}{16}-\frac{y^2}{9}+1-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3} \Rightarrow 3y^2-4xy-12x+16y=0$ and $\theta=\tan^{-1}\left(\frac{4}{3}\right)$

Illustration 18: Find the equation of common tangents to hyperbolas

$$H_1: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1;\ H_2: \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$ **(JEE MAIN)**

Sol: Compare the equation of the common tangents to H_1 and H_2 and compare the two equations to find the value of m.

Tangent to H_1

$y=mx\pm\sqrt{a^2m^2-b^2}$

$H_2: \frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1$

$a^2m^2-b^2=(-b^2)m^2-(-a^2)$

$\therefore \qquad a^2(m^2-1)=b^2(1-m^2)$

$\qquad m=\pm 1$

Equation of common tangents are $\pm y=x\pm\sqrt{a^2-b^2}$

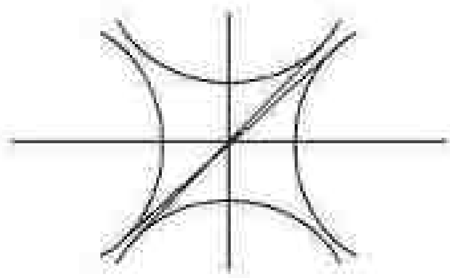


Figure 12.26

Illustration 19: If the normals at (x_r, y_r); r = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q(h, k), prove that the sum of the ordinates of the four points is k. Also prove that the product of the ordinates is $-c^4$. **(JEE ADVANCED)**

Sol: Write the equation of the normal in the parametric form and then use the theory of equations.

Any point on the curve $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$

The equation of the normal to the hyperbola at the point $\left(ct, \frac{c}{t}\right)$ is

$y-\frac{c}{t}=\frac{-1}{\left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}}}(x-ct)$.

Here, $xy=c^2$; or $y=\frac{c^2}{x}$ $\therefore$ $\frac{dy}{dx}=\frac{-c^2}{x^2}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{t=\frac{c}{t}} = \frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

$\therefore$ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$y - \frac{c}{t} = t^2(x - ct)$ or $ty - c = t^3(x - ct)$ or $ct^4 - t^3x + ty - c = 0$

The normal passes through (h, k). So

$ct^4 - t^3h + tk - c = 0$ ... (i)

Let the roots of (i) be t_1, t_2, t_3, t_4. Then $x_i = ct_i\ y_i = \frac{c}{t_i}$

$\therefore$ sum of ordinates $= y_1 + y_2 + y_3 + y_4$

$= \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = c\frac{t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3}{t_1t_2t_3t_4}$

$= c \cdot \frac{-k/c}{-c/c} = k$, (from roots of the equation (i)) and, product of the ordinates

$= y_1y_2y_3y_4 = \frac{c}{t_1}\cdot\frac{c}{t_2}\cdot\frac{c}{t_3}\cdot\frac{c}{t_4} = \frac{c^4}{t_1t_2t_3t_4} = \frac{c^4}{-c/c} = -c^4$

Hence proved.

Illustration 20: The perpendicular from the centre on the normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet at R. Find the locus of R. **(JEE ADVANCED)**

Sol: Solve the equation of the normal and the equation of line perpendicular to it passing through the origin.

Let (x_1, y_1) be any point on the hyperbola.

So, $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ... (i)

The equation of the normal at (x_1, y_1) is $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}$ or $\frac{x_1}{a^2}(y - y_1) + \frac{y_1}{b^2}(x - x_1) = 0$ ... (ii)

'm' of the normal $= -\frac{a^2y_1}{b^2x_1}$

$\therefore$ The equation of the perpendicular from the centre (0, 0) on (ii) is

$y = \frac{b^2x_1}{a^2y_1}.x$ ... (iii)

The intersection of (ii) and (iii) is R and the required locus is obtained by eliminating x_1, y_1 from (i), (ii) and (iii).

From (iii), $\frac{x_1}{a^2y} = \frac{y_1}{b^2x} = t$ (say)

Putting in (ii), $yt(y - b^2xt) + xt(x - a^2yt) = 0$

or $(x^2 + y^2)t - (a^2 + b^2)xyt^2 = 0$.

But $t \neq 0$ for then $(x_1, y_1) = (0, 0)$ which is not true.

$\therefore \quad t = \frac{x^2 + y^2}{xy(a^2 + b^2)}$; $\quad \therefore x_1 = \frac{x^2 + y^2}{xy(a^2 + b^2)}a^2y = \frac{a^2(x^2 + y^2)}{x(a^2 + b^2)}$

and $y_1 = \frac{x^2 - y^2}{xy(a^2 - b^2)} b^2 x = \frac{b^2(x^2 - y^2)}{y(a^2 + b^2)}$

$\therefore$ from (i), $\frac{1}{a^2} \cdot \frac{a^2(x^2 - y^2)^2}{x^2(a^2 - b^2)} - \frac{1}{b^2} \cdot \frac{b^2(x^2 - y^2)^2}{y^2(a^2 + b^2)^2} = 1$

or $(x^2 + y^2)^2 \left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = (a^2 + b^2)^2$.

Illustration 21: A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

(JEE ADVANCED)

Sol: Find the co-ordinates of the point M and N and then eliminate the parameter between the ordinate and abscissae.

The equation of normal at the point Q(a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos\phi + by \cot\phi = a^2 + b^2$ ... (i)

The normal (i) meets the x-axis in M $\left(\frac{a^2 + b^2}{a} \sec\phi, 0\right)$ and y-axis in N $\left(0, \frac{a^2 + b^2}{b} \tan\phi\right)$

$\therefore$ Equation of MP, the line through M and perpendicular to axis, is

$x = \left(\frac{a^2 + b^2}{a}\right) \sec\phi$ or $\sec\phi = \frac{ax}{(a^2 + b^2)}$ ... (ii)

and the equation of NP, the line through N and perpendicular to the y-axis is

$y = \left(\frac{a^2 + b^2}{b}\right) \tan\phi$ or $\tan\phi = \frac{by}{(a^2 + b^2)}$ ... (iii)

The locus of the point is the intersection of MP and NP and will be obtained by eliminating ϕ from (ii) and (iii), so we have $\sec^2\phi - \tan^2\phi = 1$

$\Rightarrow \frac{a^2x^2}{(a^2 + b^2)^2} - \frac{b^2y^2}{(a^2 + b^2)^2} = 1$ or $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

Illustration 22: Prove that the length of the tangent at any point of hyperbola intercepted between the point of contact and the transverse axis is the harmonic mean between the lengths of perpendiculars drawn from the foci on the normal at the same point. **(JEE ADVANCED)**

Sol: Proceed according to the question to prove the above statement.

$\frac{P_1}{P} = \frac{S_1G}{TG} = \frac{e^2x_1 - ae}{e^2x_1 - a\cos\theta} = \frac{ae^2 - ae\cos\theta}{ae^2 - a\cos^2\theta}$

$\therefore \frac{P_1}{P} = \frac{e(e - \cos\theta)}{(e - \cos\theta)(e + \cos\theta)} \Rightarrow \frac{P}{P_1} = \frac{e + \cos\theta}{e} = 1 + \frac{\cos\theta}{e}$

Similarly we get $\frac{P}{P_2} = 1 - \frac{\cos\theta}{e}$ $\quad \therefore \frac{P}{P_1} + \frac{P}{P_2} = 2 \Rightarrow \frac{1}{P_1} + \frac{1}{P_2} = \frac{2}{P}$

Hence Proved.

UNIT-3

PLANES

2·1. General equation of first degree. *Every equation of the first degree in x, y, z represents a plane.*

The most *general* equation of the first degree in x, y, z is

$$ax+by+cz+d=0$$

where a, b, c are not all zero.

The locus of this equation will be a plane if *every* point of the line joining *any* two points on the locus *also* lies on the locus.

To show this, we take any two points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

on the locus, so that we have

$$ax_1+by_1+cz_1+d=0, \quad \text{...(i)}$$

$$ax_2+by_2+cz_2+d=0. \quad \text{...(ii)}$$

Multiplying (*ii*) by k and adding to (*i*), we get

$$a\frac{x_1+kx_2}{1+k}+b\frac{y_1+ky_2}{1+k}+c\frac{z_1+kz_2}{1+k}+d=0. \quad \text{...(iii)}$$

The relation (*iii*) shows that the point

$$\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$$

is *also* on the locus. But, for different values of k, these are the general co-ordinates of any point on the line PQ. Thus every point on the straight line joining *any two arbitrary points* on the locus also lies on the locus.

The given equation, therefore, represents a plane.

Hence every equation of the first degree in x, y, z represents a plane.

Ex. *Find the co-ordinates of the points where the plane*

$$ax+by+cz+d=0$$

meets the three co-ordinate axes.

2·2. Normal form of the equation of a plane. *To find the equation of a plane in terms of p, the length of the normal from the origin to it and l, m, n the direction cosines of that normal; (p is to be always regarded positive).*

Let OK be the normal from O to the given plane; K being the foot of the normal.

Then $OK=p$ and l, m, n are its direction cosines.

Take any point $P(x, y, z)$ on the plane.

Now, $PK \perp OK$, for it lies in the plane which is $\perp OK$.

Therefore the projection of OP on $OK = OK = p$.

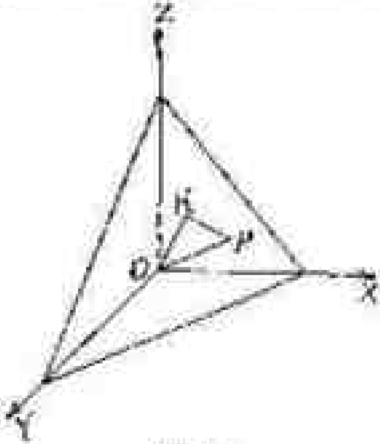


Fig. 12

Also the projection of the line OP joining

$$O(0, 0, 0) \text{ and } P(x, y, z),$$

on the line OK whose direction cosines are

$$l, m, n,$$

is

$$l(x-0)+m(y-0)+n(z-0)=lx+my+nz. \quad (\S 1{\cdot}84, \text{ p. } 14)$$

Hence $$\mathbf{lx+my+nz=p.}$$

This equation, being satisfied by the co-ordinates of any point $P(x, y, z)$ on the given plane, represents the plane and is known as the *normal form* of the equation of a plane.

Cor. *The equation of any plane is of the first degree in* x, y, z.

This is the converse of the theorem proved in § 2·1.

Ex. *Find the equation of the plane containing the lines through the origin with direction cosines proportional to* $(1, -2, 2)$ *and* $(2, 3, -1)$.

[*Ans.* $4x-5y-7z=0$.

✓**2·3. Transformation to the Normal form.** *To transform the equation*

$$ax+by+cz+d=0$$

to the normal form

$$lx+my+nz=p.$$

As these two equations represent the same plane, we have

$$-\frac{d}{p}=\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=\pm\frac{\sqrt{(a^2+b^2+c^2)}}{\sqrt{(l^2+m^2+n^2)}}=\pm\sqrt{(a^2+b^2+c^2)}.$$

Thus, $-d/p=\pm\sqrt{(a^2+b^2+c^2)}$ and as p, according to our convention, is to be always positive, we shall take positive or negative sign with the radical according as, d, is negative or positive.

Thus, if d be positive,

$$l=-\frac{a}{\sqrt{\Sigma a^2}};\ m=-\frac{b}{\sqrt{\Sigma a^2}};\ n=-\frac{c}{\sqrt{\Sigma a^2}};\ p=+\frac{d}{\sqrt{\Sigma a^2}}.$$

If d be negative, we have only to change the signs of all these.

Thus the normal form of the equation $ax+by+cz+d=0$ is

$$-\frac{a}{\sqrt{\Sigma a^2}}x-\frac{b}{\sqrt{\Sigma a^2}}y-\frac{c}{\sqrt{\Sigma a^2}}z=\frac{d}{\sqrt{\Sigma a^2}},\ \text{if } d \text{ be positive ;}$$

$$\frac{a}{\sqrt{\Sigma a^2}}x+\frac{b}{\sqrt{\Sigma a^2}}y+\frac{c}{\sqrt{\Sigma a^2}}z=-\frac{d}{\sqrt{\Sigma a^2}},\ \text{if } d \text{ be negative.}$$

2·31. Direction cosines of normal to a plane. From above we deduce a very important fact that the direction cosines of normal to any plane are proportional to the co-efficients of x, y, z in its equation or, that the direction ratios of the normal to a plane are the co-efficients of x, y, z in its equation.

Thus,

$$a,\ b,\ c$$

are the direction ratios of the normal to the plane

$$ax+by+cz+d=0.$$

Ex. 1. Find the direction cosines of the normals to the planes

(i) $2x-3y+6z=7$, (ii) $x+2y+2z-1=0$.

[*Ans.* (i) $2/7, -3/7, 6/7$, (ii) $1/3, 2/3, 2/3$.

Ex. 2. Show that the normals to the planes

$$x-y+z=1,\ 3x+2y-z+2=0$$

are inclined to each other at an angle 90°.

2·32. Angle between two planes. Angle between two planes is equal to the angle between the normals to them from any point. Thus the angle between the two planes

$$ax+by+cz+d=0, \text{ and } a_1x+b_1y+c_1z+d_1=0$$

is equal to the angle between the lines with direction ratios

$$a,\ b,\ c,$$
$$a_1,\ b_1,\ c_1,$$

and is, therefore,

$$=\cos^{-1}\left(\frac{aa_1+bb_1+cc_1}{\sqrt{(\Sigma a^2)}\ \sqrt{(\Sigma a_1^2)}}\right).$$

2·33. Parallelism and perpendicularity of two planes. Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular. *Thus the two planes*

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0$$

will be parallel, if

$$\mathbf{a/a_1=b/b_1=c/c_1}\ ;$$

and will be perpendicular, if

$$\mathbf{aa_1+bb_1+cc_1=0.}$$

Exercises

1. Find the angles between the planes

(i) $2x-y+2z=3$, $3x+6y+2z=4$. [*Ans.* $\cos^{-1}(4/21)$.

(ii) $2x-y+z=6$, $x+y+2z=7$. [*Ans.* $\pi/3$.

(iii) $3x-4y+5z=0$, $2x-y-2z=5$. [*Ans.* $\pi/2$.

2. Show that the equations

$$ax+by+r=0,\ by+cz+p=0,\ cz+ax+q=0$$

represent planes respectively perpendicular to XY, YZ, ZX planes.

3. Show that $ax+by+cz+d=0$ represents planes, perpendicular respectively to YZ, ZX, XY planes, if a, b, c separately vanish. (Similar to Ex. 2).

4. Show that the plane

$$x+2y-3z+4=0$$

is perpendicular to each of the planes

$$2x+5y+4z+1=0,\ 4x+7y+6z+2=0.$$

2·4. Determination of a plane under given conditions. *The general* equation $ax+by+cz+d=0$ of a plane contains *three* arbitrary constants (ratios of the co-efficients a, b, c, d) and, therefore, a plane can be found to satisfy three conditions each giving rise to only one relation between the constants. The three constants can then be determined from the three resulting relations.

We give below a few sets of conditions which determine a plane:—

(i) passing through *three* non-collinear points ;

(ii) passing through *two* given points and perpendicular to a given plane ;

(iii) passing through a *given* point and perpendicular to *two* given planes.

2·41. Intercept form of the equation of a plane. *To find the equation of a plane in terms of the intercepts a, b, c which it makes on the axes.*

Let the equation of the plane be

$$Ax+By+Cz+D=0. \qquad \ldots(1)$$

The co-ordinates of the point in which this plane meets the X-axis are given to be $(a, 0, 0)$. Substituting these in equation (1), we obtain

$$aA+D=0,$$

or

$$-\frac{A}{D}=\frac{1}{a}.$$

Similarly

$$-\frac{B}{D}=\frac{1}{b}; \quad -\frac{C}{D}=\frac{1}{c}.$$

The equation (1) can be re-written as

$$-\frac{A}{D}x-\frac{B}{D}y-\frac{C}{D}z=1,$$

so that, after substitution, we obtain

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}+\frac{\mathbf{z}}{\mathbf{c}}=\mathbf{1},$$

as the required equation of the plane.

Note. The fact that a plane makes intercepts a, b, c, on the three axes is equivalent to the statement that it passes through the three points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, so that what we have really done here is to determine the three ratios of the co-efficients in (1) in order that the same may pass through these points.

Ex. 1. Find the intercepts of the plane $2x-3y+4z=12$ on the co-ordinate axes. [*Ans.* 6, -4, 3.

Ex. 2. A plane meets the co-ordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c); show that the equation of the plane is $x/a+y/b+z/c=3$.

Ex. 3. Prove that a variable plane which moves so that the sum of the reciprocals of its intercepts on the three co-ordinate axes is constant, passes through a fixed point.

2·42. Plane through three points. *To find the equation of the plane passing through the three non-collinear points*

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).$$

Let the required equation of the plane be

$$ax+by+cz+d=0. \qquad ...(i)$$

As the given points lie on the plane, we have

$$ax_1+by_1+cz_1+d=0, \qquad ...(ii)$$

$$ax_2+by_2+cz_2+d=0, \qquad ...(iii)$$

$$ax_3+by_3+cz_3+d=0. \qquad ...(iv)$$

Eliminating a, b, c, d from (i)—(iv), we have

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}=0$$

which is the required equation of the plane.

Note. In actual numerical exercises, the student would find it more convenient to follow the method of the first exercise below.

Exercises

✓1. *Find the equation of the plane through*

$$P(2, 2, -1), Q(3, 4, 2), R(7, 0, 6).$$

The general equation of a plane through $P(2, 2, -1)$ is

$$a(x-2)+b(y-2)+c(z+1)=0 \quad \text{(Refer 4, § 2·6, p. 25)} \qquad ...(i)$$

It will pass through Q and R, if

$$a+2b+3c=0$$

$$5a-2b+7c=0.$$

These give

$$\frac{a}{20}=\frac{b}{8}=\frac{c}{-12} \text{ or } \frac{a}{5}=\frac{b}{2}=\frac{c}{-3}.$$

Substituting these values in (i), we have

$$5(x-2)+2(y-2)-3(z+1)=0$$

i.e., $$5x+2y-3z-17=0$$

as the required equation.

✓2. Find the equation of the plane through the three points $(1, 1, 1)$, $(1, -1, 1)$, $(-7, -3, -5)$ and show that it is perpendicular to the XZ plane.

[*Ans.* $3x-4z+1=0$.

3. Obtain the equation of the plane passing through the point $(-2, -2, 2)$ and containing the line joining the points $(1, 1, 1)$ and $(1, -1, 2)$.
[*Ans.* $x-3y-6z+8=0$.

4. If, from the point $P(a, b, c)$, perpendiculars PL, PM be drawn to YZ and ZX planes, find the equation of the plane OLM. [*Ans.* $bcx+acy-abz=0$.

5. Show that the four points $(-6, 3, 2)$, $(3, -2, 4)$, $(5, 7, 3)$ and $(-13, 17, -1)$ are coplanar.

6. Show that the join of points $(6, -4, 4)$, $(0, 0, -4)$ intersects the join of $(-1, -2, -3)$, $(1, 2, -5)$.

7. Show that $(-1, 4, -3)$ is the circumcentre of the triangle formed by the points $(3, 2, -5)$, $(-3, 8, -5)$, $(-3, 2, 1)$.

8. *Find the equation of the plane through the points*

$$(2, 2, 1) \text{ and } (9, 3, 6)$$

and perpendicular to the plane

$$2x+6y+6z=9.$$

Any plane through $(2, 2, 1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0. \qquad \ldots(i)$$

It will pass through $(9, 3, 6)$ if

$$a(9-2)+b(3-2)+c(6-1)=0$$

i.e.,

$$7a+b+5c=0 \qquad \ldots(ii)$$

The plane (*i*) will be perpendicular to the given plane if

$$2a+6b+6c=0. \qquad \ldots(iii)$$

From (*ii*) and (*iii*), we have

$$\frac{a}{-24}=\frac{b}{-32}=\frac{c}{40} \quad \text{or} \quad \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}.$$

Substituting in (*i*), we see that the equation of the required plane is

$$3(x-2)+4(y-2)-5(z-1)=0$$

or

$$3x+4y-5z=9.$$

9. Show that the equations of the three planes passing through the points $(1, -2, 4)$, $(3, -4, 5)$ and perpendicular to XY, YZ, ZX planes are $x+y+1=0$; $x-2z+7=0$; $y+2z=6$ respectively.

10. Obtain the equation of the plane through the point $(-1, 3, 2)$ and perpendicular to the two planes $x+2y+2z=5$; $3x+3y+2z=8$.
[*Ans.* $2x-4y+3z+8=0$.

11. Find the equation of the plane through $A(-1, 1, 1)$ and $B(1, -1, 1)$ and perpendicular to the plane $x+2y+2z=5$. [*Ans.* $2x+2y-3z+3=0$.

12. Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin, which cut off from the axes intercepts whose sum is zero. (*M.T.*)
[*Ans.* $2x-3y-6z=6$; $6x+3y-2z=18$.

13. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

2·5. Systems of planes. The equation of a plane satisfying two conditions will involve one arbitrary constant which can be chosen in an infinite number of ways, thus giving rise to an infinite number of planes, called a *system of planes*.

The arbitrary constant which is different for different members of the system is called a *parameter*.

Similarly the equation of a plane satisfying one condition will involve two parameters.

The following are the equations of a few systems of planes involving one or two arbitrary constants.

EXERCISES

1. The equation

$$ax+by+cz+k=0$$

represents a system of planes parallel to the plane

$$ax+by+cz+d=0,$$

k being the parameter. (§ 2·23, *p.* 21).

2. The equation

$$ax+by+cz+k=0$$

represents a system of planes perpendicular to the line with direction ratios a, b, c ; k being the parameter. (§2·31, *p.* 21).

3. The equation

$$(ax+by+cz+d)+k(a_1x+b_1y+c_1z+d_1)=0 \qquad ...(1)$$

represents a system of planes passing through the line of intersection of the planes

$$ax+by+cz+d=0, \qquad ...(2)$$

$$a_1x+b_1y+c_1z+d_1=0 \; ; \qquad ...(3)$$

k being the parameter, for

(*i*) the equation, being of the first degree in x, y, z, represents a plane ;

(*ii*) it is evidently satisfied by the co-ordinates of the points which satisfy (2) and (3), whatever value k may have.

4. The system of planes passing through the point (x_1, y_1, z_1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

where the required *two* parameters are the two ratios of the co-efficients A, B, C ; for, the equation is of the first degree and is clearly satisfied by the point (x_1, y_1, z_1), whatever be the values of the ratios of the co-efficients.

Exercises

1. *Find the equation of the plane passing through the intersection of the planes*

$$x+y+z=6 \text{ and } 2x+3y+4z+5=0$$

and the point (1, 1, 1).

The plane

$$x+y+z-6+k(2x+3y+4z+5)=0, \qquad ...(i)$$

passes through the intersection of the given planes for all values of k.

It will pass through (1, 1, 1) if

$$-3+14k=0 \text{ or } k=3/14.$$

Putting $k=3/14$ in (*i*), we obtain

$$20x+23y+26z-69=0,$$

which is the required equation of the plane.

2. Obtain the equation of the plane through the intersection of the planes

$$x+2y+3z+4=0 \text{ and } 4x+3y+2z+1=0$$

and the origin. [*Ans.* $3x+2y+z=0$.

3. *Find the equation of the plane passing through the line of intersection of the planes*

$$2x-y=0 \text{ and } 3z-y=0$$

and perpendicular to the plane

$$4x+5y-3z=8.$$

The plane

$$2x-y+k(3z-y)=0, \text{ i.e., } 2x-(1+k)y+3kz=0,$$

passes through the line of intersection of the given planes whatever k may be. It will be perpendicular to

$$4x+5y-3z=8,$$

if
$$2 \cdot 4-(1+k) \cdot 5+3k(-3)=0, \text{ i.e. } 14k=3.$$

$\therefore$
$$k=\frac{3}{14}.$$

Thus the required equation is

$$2x-y+\left(\frac{3}{14}\right)(3z-y)=0,$$

i.e.,
$$28x-17y+9z=0.$$

4. Find the equation of the plane which is perpendicular to the plane

$$5x+3y+6z+8=0$$

and which contains the line of intersection of the planes

$$x+2y+3z-4=0, \; 2x+y-z+5=0. \qquad (I.U. \; 1938)$$

[Ans. $51x+15y-50z+173=0$.

5. The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$, find the equation of the plane in its new position. [Ans. $22x+5y-4z-35=0$.

6. Find the equation of the plane through the intersection of the planes

$$ax+by+cz+d=0, \; a_1x+b_1y+c_1z+d_1=0$$

and perpendicular to the XY plane.

[Ans. $x(ac_1-a_1c)+y(bc_1-b_1c)+(dc_1-d_1c)=0$.

7. *Obtain the equation of the plane through the point (x_1, y_1, z_1) and parallel to the plane $ax+by+cz+d=0$.*

The plane

$$ax+by+cz+k=0$$

is parallel to the given plane for all values of k.

It will pass through (x_1, y_1, z_1), if

$$ax_1+by_1+cz_1+k=0.$$

Subtracting, we get

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0,$$

which is the required equation.

8. Find the equation of the plane through the point $(2, 3, 4)$ and parallel to the plane $5x-6y+7z=3$. [Ans. $5x-6y+7z=20$.

9. Find the equation of the plane that passes through $(3, -3, 1)$ and is normal to the line joining the points $(3, 4, -1)$ and $(2, -1, 5)$.

[Ans. $x+5y-6z+18=0$.

10. Obtain the equation of the plane that bisects the line joining $(1, 2, 3)$, $(3, 4, 5)$, at right angles.

11.
$$x+2y-z-3=0, \; 3x-y+2z-1=0,$$
$$2x-2y+3z-2=0, \; x-y+z+1=0$$

are four planes. Show that the line of intersection of the first two planes is coplanar with the line of intersection of the latter two and find the equation of the plane containing the two lines.

The planes

$$x+2y-z-3+k(3x-y+2z-1)=0$$

and
$$2x-2y+3z-2+k'(x-y+z+1)=0$$

i.e.,
$$(1+3k)x+(2-k)y+(-1+2k)z+(-3-k)=0$$

and
$$(2+k')x+(-2-k')y+(3+k')z+(-2+k')=0,$$

separately contain the two lines. The two lines will be coplanar if, for some values of k and k', they become identical. This requires

$$\frac{1+2k}{2+k'}=\frac{2-k}{-2-k'}=\frac{1+2k}{3+k'}=\frac{-3-k}{-2+k'},$$

or
$$6+4k+3k'+2kk'=0, \qquad ...(i)$$
$$4+k+kk'=0, \qquad ...(ii)$$
$$11-k+2k'+3kk'=0. \qquad ...(iii)$$

(i) and (ii) give
$$k=-3/2,\ k'=5,$$
and
$$k=2,\qquad k'=-2.$$

Of these two sets of values, $k=-3/2$ and $k'=5$ satisfy (iii) also. Thus the two planes become identical for $k=-3/2$ and $k'=5$. Hence the two lines are coplanar and the equation of the plane containing them is
$$7x-7y+8z+3=0.$$

12. Show that the line of intersection of the planes
$$7x-4y+7z+16=0,\ 4x+3y-2z+3=0$$
is coplanar with the line of intersection of
$$x-3y+4z+6=0,\ x-y+z+1=0.$$

Obtain the equation of the plane through both.

Ans. $8x-7y+6z+13=0.$

13. A variable plane passes through a fixed point (a, b, c) and meets the co-ordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the co-ordinate planes is
$$a/x+b/y+c/z=1.$$

2·6 Two sides of a plane.

Two points
$$A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$$
lie on the same or different sides of the plane
$$ax+by+cz+d=0,$$
according as the expressions
$$ax_1+by_1+cz_1+d,\qquad ax_2+by_2+cz_2+d$$
are of the same or different signs.

Let the line AB meet the given plane in a point P and let P divide AB in the ratio $r : 1$ so that r is positive or negative according as P divides AB internally or externally, *i.e.*, according as A and B lie on the opposite or the same side of the plane.

Since the point P whose co-ordinates are
$$\left(\frac{rx_2+x_1}{r+1},\ \frac{ry_2+y_1}{r+1},\ \frac{rz_2+z_1}{r+1}\right)$$
lies on the given plane, therefore
$$a\frac{rx_2+x_1}{r+1}+b\frac{ry_2+y_1}{r+1}+c\frac{rz_2+z_1}{r+1}+d=0,$$
or
$$r(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0,$$
or
$$r=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

This shows that r is negative or positive according as
$$ax_1+by_1+cz_1+d,\ ax_2+by_2+cz_2+d$$
are of the same or different signs.

Thus the theorem is proved.

Ex. Show that the origin and the point $(2, -4, 3)$ lie on different sides of the plane $x+3y-5z+7=0$.

2·7. Length of the perpendicular from a point to a plane. *To find the perpendicular distance of the point*

$$P(x_1, y_1, z_1)$$

from the plane

$$lx + my + nz = p.$$

The equation of the plane through $P(x_1, y_1, z_1)$ parallel to the given plane is

$$lx + my + nz = p_1,$$

where $\qquad lx_1 + my_1 + nz_1 = p_1.$

Let OKK' be the perpendicular from the origin O to the two parallel planes meeting them in K and K' so that

$$OK = p \text{ and } OK' = p_1.$$

Draw $PL \perp$ given plane.

Then $\qquad LP = OK' - OK$

$$= p_1 - p = lx_1 + my_1 + nz_1 - p.$$

Cor. *To find the length of the perpendicular from* (x_1, y_1, z_1) *to the plane* $ax + by + cz + d = 0$.

The normal form of the given equation of the plane being

$$\pm \frac{a}{\sqrt{\Sigma a^2}} x \pm \frac{b}{\sqrt{\Sigma a^2}} y \pm \frac{c}{\sqrt{\Sigma a^2}} z \pm \frac{d}{\sqrt{\Sigma a^2}} = 0,$$

the required length of the perpendicular is

$$\pm \frac{\mathbf{a x_1 + b y_1 + c z_1 + d}}{\sqrt{(\mathbf{a^2 + b^2 + c^2})}}.$$

Thus the length of the perpendicular from (x_1, y_1, z_1) *to the plane*

$$ax + by + cz + d = 0$$

is obtained by substituting

$$x_1, y_1, z_1, \textit{ for } x, y, z,$$

respectively in the expression,

$$ax + by + cz + d,$$

and dividing the same by

$$\sqrt{(a^2 + b^2 + c^2)}.$$

Exercises

1. Find the distances of the points (2, 3, 4) and (1, 1, 4) from the plane
$$3x - 6y + 2z + 11 = 0. \qquad [\textit{Ans. } 1 ; 16/7.$$

2. Show that the distance between the parallel planes
$$2x - 2y + z + 3 = 0 \text{ and } 4x - 4y + 2z + 5 = 0$$
is 1/6.

(The distance between two parallel planes is the distance of any point on one from the other).

3. Find the locus of the point whose distance from the origin is three times its distance from the plane $2x - y + 2z = 3$.

[*Ans.* $3x^2 + 3z^2 - 4xy + 8xz - 4yz - 12x + 6y - 12z + 9 = 0.$

4. Show that $(1/8, 1/8, 1/8)$ is in the centre of the tetrahedron formed by the four planes $x=0$, $y=0$, $z=0$, $x+2y+2z=1$.

5. Sum of the distances of any number of fixed points from a variable plane is zero; show that the plane passes through a fixed point.

6. A variable plane which remains at a constant distance, $3p$, from the origin cuts the co-ordinate axes at A, B, C. Show that the locus of centroid of the triangle ABC is

$$x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

2·71. Bisectors of angles between two planes. *To find the equations of the bisectors of the angles between the planes*

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0.$$

If (x, y, z) be any point on any one of the planes bisecting the angles between the planes, then the perpendiculars from this point to the two planes must be equal (in magnitude).

Hence

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

are the equations of the two bisecting planes.

Of these two bisecting planes, one bisects the acute and the other the obtuse angle between the given planes.

The bisector of the acute angle makes with either of the planes an angle which is less than 45° and the bisector of the obtuse angle makes with either of them an angle which is greater than 45°. This gives a test for determining which angle, acute or obtuse, each bisecting plane bisects.

Ex. *Find the equations of the planes bisecting the angles between the planes*

$$x+2y+2z-3=0, \qquad \ldots(i)$$

$$3x+4y+12z+1=0, \qquad \ldots(ii)$$

and specify the one which bisects the acute angle.

The equations of the two bisecting planes are

$$\frac{x+2y+2z-3}{3}=\pm\frac{3x+4y+12z+1}{13}$$

or
$$2x+7y-5z-21=0, \qquad \ldots(iii)$$

and
$$11x+19y+31z-18=0. \qquad \ldots(iv)$$

If θ be the angle between the planes (i) and (iii), we have

$$\cos\theta=2/\sqrt{78}$$

so that $\tan\theta=\sqrt{(74/2)}$, which being greater than 1, we see that θ is greater than 45°. Hence (iii) bisects the obtuse angle, and consequently, (iv) bisects the acute angle.

Note. Sometimes we distinguish between the two bisecting planes by finding that plane which bisects the angle between the given planes containing the origin. To do this, we express the equations of the given planes so that d and d_1 are positive. Consider the equation

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} \qquad \ldots(A)$$

Since, by virtue of the equality (A), the expressions $ax+by+cz+d$ and $a_1x+b_1y+c_1z+d_1$ must have the same sign (denominators being both positive), the points (x, y, z) on the locus lie on the origin or the non-origin side of both the planes, i.e., the points on the locus lie in the angle between the planes containing the origin. Thus the equation (A) represents the plane bisecting that angle between the planes which contains the origin.

Similarly,

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=-\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

represents the plane bisecting the other angle between the given planes.

Exercises

1. Find the bisector of the acute angle between the planes

$$2x-y+2z+3=0,\quad 3x-2y+6z+8=0.$$

[*Ans.* $23x-13y+32z+45=0$.

2. Show that the plane

$$14x-8y+13=0$$

bisects the obtuse angle between the planes

$$3x+4y-5z+1=0,\ 5x+12y-13z=0.$$

3. Find the bisector of that angle between the planes

$$3x-6y+2z+5=0,\ 4x-12y+3z-3=0$$

which contains the origin. [*Ans.* $67x-162y+47z+44=0$.

2·8. Joint equation of two planes. *To find the condition so that the homogeneous second degree equation*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad ...(1)$$

may represent two planes.

Let the two planes represented by (1) be

$$lx+my+nz=0, \text{ and } l'x+m'y+n'z=0.$$

There cannot appear constant terms in the equations of the planes, for, otherwise, their joint equation will not be homogeneous.

We have

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy\equiv(lx+my+nz)(l'x+m'y+n'z)$$

so that comparing co-efficients, we obtain

$$a=ll',\ b=mm',\ c=nn'$$

and $$2f=m'n+mn',\ 2g=ln'+l'n,\ 2h=lm'+l'm.$$

In order to find the required condition, we have to eliminate $l, m, n;\ l', m', n'$ from the above six relations and this can be easily effected as follows. We have

$$0=\begin{vmatrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{vmatrix}\times\begin{vmatrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{vmatrix}$$

$$=\begin{vmatrix} ll'+l'l, & l'm+lm', & l'n+ln' \\ lm'+l'm, & mm'+m'm, & m'n+mn' \\ n'l+nl', & n'm+nm', & n'n+nn' \end{vmatrix}$$

$$=8\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=8(abc+2fgh-af^2-bg^2-ch^2)$$

Hence

$$abc+2fgh-af^2-bg^2-ch^2=0$$

is the required condition.

Cor. Angle between planes. If θ be the angle between the planes represented by (1), we have

$$\tan\theta=\frac{\sqrt{[(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2]}}{ll'+mm'+nn'}$$

$$=\frac{2\sqrt{(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}.$$

The planes will be at right angles if $a+b+c=0$, for then θ is 90°.

Ex. Show that the following equations represent pairs of planes and also find the angles between each pair:

(i) $12x^2-2y^2-6z^2-2xy+7yz+6zx=0$. [*Ans.* $\cos^{-1}(4/21)$.

(ii) $2x^2-2y^2+4z^2+6xz+2yz+3xy=0$. [*Ans.* $\cos^{-1}(4/9)$.

2·9. Orthogonal projection on a plane. Determination of Plane Areas. Def. *The foot of the perpendicular drawn from any point P to a given plane, π, is called the orthogonal projection of the point P on the plane π.*

This plane, π, is called the plane of the projection.

Thus (Fig. 1, p. 1) L, M, N are respectively the orthogonal projections of the point P on the YZ, ZX and XY planes.

The projection of a curve on the plane of projection is the locus of the projection on the plane of any point on the curve.

The projection of the area enclosed by a plane curve is the area enclosed by the projection of the curve on the plane of projection.

In particular, the projection of a straight line is the locus of the foot of the perpendicular drawn from any point on it to the plane of the projection.

2·91. The following simple results of *Pure solid geometry* are assumed without proof :—

(1) The projection of a straight line is a straight line.

(2) If a line AB in a plane, be perpendicular to the line of intersection of this plane with the plane of projection, then the length of its projection is $AB\cos\theta$; θ being the angle between the two planes.

In case AB is parallel to the plane of projection, then the length of the projection is the same as that of AB.

(3) The projection of the area, A, enclosed by any curve in a plane is $A\cos\theta$; θ being the angle between the plane of the area and the plane of projection.

Theorem. *If A_x, A_y, A_z be the areas of the projections of an area, A, on the three co-ordinate planes, then*

$$A^2=A_x^2+A_y^2+A_z^2$$

Let l, m, n be the direction cosines of the normal to the plane of the area A.

Since l is the cosine of the angle between the YZ plane and the plane of the area A, therefore

$$A_x = l.A.$$

Similarly, $$A_y = m.A,$$

and $$A_z = n.A.$$

Hence $$A_x^2 + A_y^2 + A_z^2 = A^2(l^2 + m^2 + n^2) = A^2.$$

Exercises

1. *Find the area of the triangle whose vertices are the points*

$$(1, 2, 3), (-2, 1, -4), (3, 4, -2).$$ *(D.U. Hons., 1947)*

To find the area A of this triangle, we find the areas A_x, A_y, A_z, of the projection of the same on the co-ordinate planes.

The vertices of the projection of the triangle on the XY plane are

$$(1, 2, 0), (-2, 1, 0), (3, 4, 0),$$

so that

$$A_z = \frac{1}{2}\begin{vmatrix} 1, & 2, & 1 \\ -2, & 1, & 1 \\ 3, & 4, & 1 \end{vmatrix} = -2.$$

Similarly, $$A_y = \frac{1}{2}\begin{vmatrix} 1, & 3, & 1 \\ -2, & -4, & 1 \\ 3, & -2, & 1 \end{vmatrix} = \frac{29}{2},$$

and $$A_x = \frac{1}{2}\begin{vmatrix} 2, & 3, & 1 \\ 1, & -4, & 1 \\ 4, & -2, & 1 \end{vmatrix} = \frac{19}{2}.$$

Therefore, the area of the triangle

$$= \sqrt{(A_x^2 + A_y^2 + A_z^2)} = \sqrt{\left(4 + \frac{(29)^2}{4} + \frac{(19)^2}{4}\right)} = \frac{\sqrt{1218}}{2}.$$

2. Find the areas of the triangles whose vertices are the points

(i) $(a, 0, 0), (0, b, 0), (0, 0, c)$.

(ii) $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$.

3. From a point $P(x', y', z')$, a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B, C; prove that the area of the triangle ABC is $r^5/2x'y'z'$, where r is the measure of OP.

2·10. Volume of a tetrahedron. *To find the volume of a tetrahedron in terms of the co-ordinates*

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$$

of its vertices A, B, C, D.

Let V be the volume of the tetrahedron $ABCD$.

Fig. 13

Then

$$V = \tfrac{1}{3}p.\triangle \qquad ...(i)$$

where p is the length of the perpendicular AL from any vertex A to the opposite face BCD; and $\triangle$ is the area of the triangle BCD.

The equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0$$

$$\text{or } x\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix} - y\begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix} + z\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix} = 0.$$

$\therefore$ the length of the perpendicular, $p =$

$$\frac{x_1\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix}}{\left\{\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}} \quad \ldots\text{(ii)}$$

$$\text{The numerator of } p = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

If $\triangle_x$, $\triangle_y$, $\triangle_z$ be the areas of the projections of $\triangle$ on the YZ, ZX, XY planes respectively, we obtain

$$2\triangle_x = \begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}, \quad 2\triangle_y = \begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix},$$

$$2\triangle_z = \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix},$$

Therefore, denominator of $p=[4(\triangle_x^2+\triangle_y^2+\triangle_z^2)]^{\frac{1}{2}}=2\triangle$.

From (i) and (ii), we see that the required volume

$$=\tfrac{1}{3}\triangle\, p=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

Exercises

1. The vertices of a tetrahedron are (0, 1, 2), (3, 0, 1), (4, 3, 6), (2, 3, 2); show that its volume is 6.

2. *A, B, C* are three fixed points and a variable point *P* moves so that the volume of the tetrahedron *PABC* is constant; show that the locus of the point *P* is a plane parallel to the plane *ABC*.

3. A variable plane makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Find

   (i) the locus of the centroid of the tetrahedron. [*Ans.* $xyz=6k^3$.

   (ii) the locus of the foot of the perpendicular from the origin to the plane. [*Ans.* $(x^2+y^2+z^2)^3=384k^3xyz$.

4. *Find the volume of the tetrahedron in terms of three edges which meet in a point and of the angles which they make with each other.* (P.U. 1939)

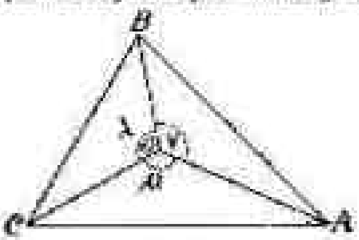


Fig. 14

Let *OABC* be a tetrahedron.

Let

$$OA=a,\ OB=b,\ OC=c.$$

Let

$$\angle BOC=\lambda,\ \angle COA=\mu,$$

and $$\angle AOB=\nu.$$

We take *O* as origin and any system of three mutually perpendicular lines through *O* as co-ordinate axes. Let the direction cosines of *OA, OB, OC* be

$$l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3.$$

Therefore, the co-ordinates of *A, B, C* are

$$(l_1a, m_1a, n_1a);\ (l_2b, m_2b, n_2b);\ (l_3c, m_3c, n_3c) \qquad (\S\ 1\cdot61)$$

Therefore, the volume of the tetrahedron *OABC*

$$=\frac{1}{6}\begin{vmatrix} 0, & 0, & 0, & 1 \\ l_1a, & m_1a, & n_1a, & 1 \\ l_2b, & m_2b, & n_2b, & 1 \\ l_3c, & m_3c, & n_3c, & 1 \end{vmatrix}=-\frac{1}{6}\begin{vmatrix} l_1a, & m_1a, & n_1a \\ l_2b, & m_2b, & n_2b \\ l_3c, & m_3c, & n_3c \end{vmatrix}=-\frac{abc}{6}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

Now

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2=\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}\times\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

$$\begin{vmatrix} \Sigma l_1^2, & \Sigma l_1 l_2, & \Sigma l_1 l_3 \\ \Sigma l_1 l_2, & \Sigma l_2^2, & \Sigma l_2 l_3 \\ \Sigma l_3 l_1, & \Sigma l_3 l_2, & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}$$

Therefore, the volume of the tetrahedron $OABC$

$$= \frac{abc}{6} \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}$$

5. *Show that the volume of the tetrahedron, the equations of whose faces are*

$$a_r x + b_r y + c_r z + d_r = 0, \; r = (1, 2, 3, 4)$$

is

$$\frac{\Delta^3}{6 D_1 D_2 D_3 D_4}$$

where Δ is the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$$

and D_1, D_2, D_3, D_4 are the co-factors of d_1, d_2, d_3, d_4 respectively in the determinant Δ.

Let (x_1, y_1, z_1) be the point of intersection of the three planes

$$a_r x + b_r y + c_r z + d_r = 0, \; r = (2, 3, 4),$$

so that (x_1, y_1, z_1) is one of the vertices of the tetrahedron.

Let (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the other vertices, similarly obtained.

We write

$$a_1 x_1 + b_1 y_1 + c_1 z_1 + d_1 = k_1$$

i.e.,

$$a_1 x_1 + b_1 y_1 + c_1 z_1 + (d_1 - k_1) = 0. \quad \dots(1)$$

Also, we have

$$a_2 x_1 + b_2 y_1 + c_2 z_1 + d_2 = 0, \quad \dots(2)$$
$$a_3 x_1 + b_3 y_1 + c_3 z_1 + d_3 = 0, \quad \dots(3)$$
$$a_4 x_1 + b_4 y_1 + c_4 z_1 + d_4 = 0. \quad \dots(4)$$

Eliminating x_1, y_1, z_1 from (1), (2), (3), (4), we have

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 - k_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = 0,$$

or

$$\Delta + k_1 \begin{vmatrix} a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \\ a_4, & b_4, & c_4 \end{vmatrix} = 0,$$

or $$\Delta - k_1 D_1 = 0$$

∴ $$k_1 = \frac{\Delta}{D_1}.$$

Similarly

$$a_2 x_2 + b_2 y_2 + c_2 z_2 + d_2 = k_2 = \frac{\Delta}{D_2},$$

$$a_3 x_3 + b_3 y_3 + c_3 z_3 + d_3 = k_3 = \frac{\Delta}{D_3},$$

$$a_4 x_4 + b_4 y_4 + c_4 z_4 + d_4 = k_4 = \frac{\Delta}{D_4}.$$

We, now, have

$$\begin{vmatrix} a_1, b_1, c_1, d_1 \\ a_2, b_2, c_2, d_2 \\ a_3, b_3, c_3, d_3 \\ a_4, b_4, c_4, d_4 \end{vmatrix} \times \begin{vmatrix} x_1, y_1, z_1, 1 \\ x_2, y_2, z_2, 1 \\ x_3, y_3, z_3, 1 \\ x_4, y_4, z_4, 1 \end{vmatrix} = \begin{vmatrix} k_1, 0, 0, 0 \\ 0, k_2, 0, 0 \\ 0, 0, k_3, 0 \\ 0, 0, 0, k_4 \end{vmatrix}$$

$$= k_1 k_2 k_3 k_4$$

$$= \frac{\Delta^4}{D_1 D_2 D_3 D_4}.$$

Therefore the required volume $= \dfrac{\Delta^3}{6 D_1 D_2 D_3 D_4}$

6. Find the volume of the tetrahedron formed by planes whose equations are

$y+z=0,\ z+x=0,\ x+y=0$ and $x+y+z=1$. *(P. U. 1942)*

[*Ans.* 2/3.

UNIT-4

STRAIGHT LINES

3·1. Equations of a line. A line may be determined as the intersection of any two planes through it.

Now, if

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0$$

be the equations of any two planes through the given line, then these two equations, taken together, give the equations of the line. This follows from the fact that any point on the line lies on both these planes and, therefore, its co-ordinates satisfy both the equations and conversely, any point whose co-ordinates satisfy the two equations lies on both these planes, and, therefore, on the line.

Thus, *a straight line in space is represented by* **two** *equations of the first degree in x, y, z.*

Of course any given line can be represented by *different pairs* of first degree equations, for we may take any pair of planes through the line and the equations of the same will constitute the equations of the line.

In particular, as the X-axis is the intersection of the XZ and XY planes, its equations are $y=0, z=0$ taken together. Similarly the equations of the Y-axis are $x=0, z=0$ and of the Z-axis are $x=0, y=0$.

Ex. Find the intersection of the line

$$x-2y+4z+4=0,\ x+y+z-8=0$$

with the plane

$$x-y+2z+1=0. \qquad [\textit{Ans.}\ (2, 5, 1)$$

3·11. Symmetrical form of the equations of a line. *To find the equations of the line passing through a given point $A(x_1, y_1, z_1)$, and having direction cosines, l, m, n.*

Let $P(x, y, z)$ be any point on the line and let $AP=r$.

Projecting AP on the co-ordinate axes, we obtain

$$x-x_1=lr,\ y-y_1=mr,\ z-z_1=nr \qquad ...(i)$$

so that for all points (x, y, z) on the given line,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.$$

Thus

$$\mathbf{\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}} \qquad ...(ii)$$

are the two required equations of the line.

Clearly, the equations (*ii*) of the line are not altered if we replace the direction cosines l, m, n by three numbers proportional to them.

so that it suffices to use direction ratios in place of direction cosines while writing down the equations of a line.

Cor. From the relations (i), we have

$$\mathbf{x}=\mathbf{x}_1+\mathbf{l}\mathbf{r},\ \mathbf{y}=\mathbf{y}_1+\mathbf{m}\mathbf{r},\ \mathbf{z}=\mathbf{z}_1+\mathbf{n}\mathbf{r},$$

which are the general co-ordinates of any point on the line in terms of the *parameter* r.

Any value of r will give *some* point on the line and *any* point on the line arises from *some* value of r.

It should be noted, that it is only when l, m, n are the actual direction cosines that r gives the distance between the points (x_1, y_1, z_1) and (x, y, z).

Note 1. The symmetrical form (ii) of the equations of a straight line proves useful when we are concerned with the direction cosines of the line or when we wish to obtain the general co-ordinates of any point on the line in terms of a parameter r.

Note 2. The equation

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}$$

of first degree, being free of z, represents a plane through the line drawn perpendicular of the XOY plane. Similar statements may be made about the equations

$$\frac{y-y_1}{m}=\frac{z-z_1}{n},\quad \frac{z-z_1}{n}=\frac{x-x_1}{l}.$$

The equations

$$(x-x_1)/l=(y-y_1)/m,\ (y-y_1)/m=(z-z_1)/n$$

represent a pair of planes through the given line.

3·12. Line through two points. *To find the equations of the line through the two points*

$$(x_1, y_1, z_1)\ \text{and}\ (x_2, y_2, z_2).$$

Since

$$x_2-x_1,\ y_2-y_1,\ z_2-z_1$$

are proportional to the direction cosines of the line, the required equations are

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{x}_2-\mathbf{x}_1}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{y}_2-\mathbf{y}_1}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{z}_2-\mathbf{z}_1}.$$

Note. Results obtained in Cor. 2 page 6 may be regarded as the parametric equations of the line through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2); λ being the parameter.

Exercises

1. Find k so that the lines

$$\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$$

$$\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$$

may be perpendicular to each other. [*Ans.* $-10/7$.

2. Find two points on the line

$$\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z+5}{2}$$

on either side of $(2, -3, -5)$ and at a distance 3 from it.

[*Ans.* $(3, -5, -3)$; $(1, -1, -7)$.

EXERCISES

3. *Find the co-ordinates of the point of intersection of the line*

$$\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}$$

with the plane

$$3x+4y+5z=5.$$

Let

$$\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}=r,$$

so that the point

$$r-1,\ 3r-3,\ -2r+2$$

lies on the given line for all values of r.

If it also lies on the given plane, we have

$$3r-3+12r-12-10r+10=5 \quad \text{or} \quad r=2.$$

Hence the required point of intersection is $(1, 3, -2)$.

Its distance from the point $(-1, -3, 2)$ is $\sqrt{56}$ which is different from the value 2 of r. (*Why?*)

4. Find the point where the line joining $(2, -3, 1)$, $(3, -4, -5)$ cuts the plane $2x+y+z=7$. [*Ans.* $(1, -2, 7)$.

5. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{1}{2}(x-2)=\frac{1}{3}(y+1)=\frac{1}{12}(z-2)$ and the plane

$$x-y+z=5.$$

(*P.U.* 1934) [*Ans.* 13.

6. Find the distance of the point $(3, -4, 5)$ from the plane

$$2x+5y-6z=16$$

measured along a line with direction cosines proportional to $(2, 1, -2)$.

[*Ans.* 60/7.

7. *Find the image of the point P $(1, 3, 4)$ in the plane*

$$2x-y+z+3=0.$$

If two points P, Q be such that the line is bisected perpendicularly by a plane, then either of the points is the image of the other in the plane.

The line through P perpendicular to the given plane is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1},$$

so that the co-ordinates of Q are of the form

$$(2r+1,\ -r+3,\ r+4)$$

Making use of the fact that the mid point

$$(r+1,\ -\tfrac{1}{2}r+3,\ \tfrac{1}{2}r+4),$$

of PQ lies on the given plane, we see that

$$r=-2$$

so that the image of P is $(-3, 5, 2)$.

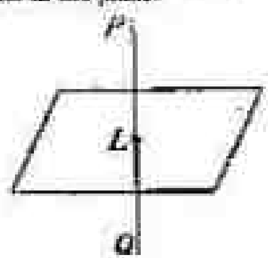


Fig. 18

8. Find the equations to the line through $(-1, 3, 2)$ and perpendicular to the plane $x+2y+2z=3$, the length of the perpendicular and the co-ordinates of its foot. [*Ans.* 2 ; $(-5/3, 5/3, 2/3)$.

9. Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x+3y-4z+1=0$; also find the co-ordinates of the point which is the image of the origin in the plane. (*P.U. Supp.*)

[*Ans.* $(-2/29, -3/29, 4/29)$; $(-4/29, -6/29, 8/29)$.

10. Find the equations to the line through (x_1, y_1, z_1) perpendicular to the plane $ax+by+cz+d=0$ and the co-ordinates of its foot. Deduce the expression for the perpendicular distance of the given point from the given plane.

[*Ans.* $(ar+x_1,\ br+y_1,\ cr+z_1)$ where $r=-(ax_1+by_1+cz_1+d)/(a^2+b^2+c^2)$.

11. Show that the line

$$\tfrac{1}{2}(x-7)=-(y+3)=(z-4)$$

intersects the planes

$$6x+4y-5z=4 \text{ and } x-5y+2z=12$$

in the same point and deduce that the line is co-planar with the line of intersection of the planes.

12. Show that the line

$$(x-3)/3=(2-y)/4=(z+1)/1$$

intersects the line

$$x+2y+3z=0,\ 2x+4y+3z+3=0.$$

Find their point of intersection. [*Ans.* $(9, -6, 1)$.

13. Show that the equations to the straight line through (a, b, c) parallel to the X-axis are $(x-a)/1=(y-b)/0=(z-c)/0$.

14. Show that

$$(x-a)/l=(y-b)/m=(z-c)/0$$

is a straight line perpendicular to the Z-axis.

15. Show that the straight line

$$(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$$

meets the locus of the equation

$$ax^2+by^2+cz^2=1,$$

in two points.

Deduce the conditions for the two points to coincide at (α, β, γ).

[*Ans.* $al\alpha+bm\beta+cn\gamma=0$; $a\alpha^2+b\beta^2+c\gamma^2=1$.

16. P is any point on the plane $lx+my+nz=p$ and a point Q is taken on the line OP such that $OP.OQ=p^2$; show that the locus of Q is

$$p(lx+my+nz)=x^2+y^2+z^2.$$

17. A variable plane makes intercepts on the co-ordinate axes the sum of whose squares is constant and equal to k^2. Find the locus of the foot of the perpendicular from the origin to the plane.

[*Ans.* $(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)^2=k^2$.

18. Show that the equations of the lines bisecting the angles between the lines

$$\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-7}{-2},\quad \frac{x-3}{4}=\frac{y+4}{-12}=\frac{z-5}{3}$$

are

$$\frac{x-3}{38}=\frac{y+4}{-49}=\frac{z-5}{-17},\quad \frac{x-3}{14}=\frac{y+4}{23}=\frac{z-5}{-35}.$$

3·13. It has been seen in §§ 3·11, 3·12, that the equations of a straight line which we generally employ are of two forms.

One is the *symmetrical* form deduced from the consideration that a straight line is completely determined when we know its direction and the co-ordinates of any one point on it, or when any two points on the line are given.

The second form is *unsymmetrical* and is deduced from the consideration that a straight line is the locus of points common to any two planes through it.

In the next section it will be seen how one form of equations can be transformed into the other.

3·14. Transformation from unsymmetrical to the symmetrical form. *To transform the equations*

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0$$

of a line to the symmetrical form.

To transform these to the symmetrical form, we require

(i) the direction ratios of the line, and

(ii) the co-ordinates of any one point on it.

Let l, m, n be the direction ratios of the line. Since the line lies in both the planes

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0,$$

it is perpendicular to the normals to both of them. As the direction ratios of the normals to the two planes are

$$a, b, c \,;\, a_1, b_1, c_1,$$

we have

$$al+bm+cn=0,$$
$$a_1l+b_1m+c_1n=0.$$

$$\therefore \qquad \frac{l}{bc_1-b_1c}=\frac{m}{ca_1-c_1a}=\frac{n}{ab_1-a_1b}.$$

Now, we require the co-ordinates of *any one* point on the line and there is an infinite number of points from which to choose. We, for the sake of convenience, find the point of intersection of the line with the plane $z=0$. This point which is given by the equations

$$ax+by+d=0 \text{ and } a_1x+b_1y+d_1=0,$$

is

$$\left(\frac{bd_1-b_1d}{ab_1-a_1b}, \frac{a_1d-ad_1}{ab_1-a_1b}, 0\right).$$

Thus, in the symmetrical form, the equations of the given line are

$$\frac{x-(bd_1-b_1d)/(ab_1-a_1b)}{bc_1-b_1c}=\frac{y-(a_1d-ad_1)/(ab_1-a_1b)}{ca_1-c_1a}=\frac{z-0}{ab_1-a_1b}.$$

Exercises

1. Find, in a symmetrical form, the equations of the line

$$x+y+z+1=0,\ 4x+y-2z+2=0$$

and find its direction cosines. (P.U. 1937)

$$\left[\text{Ans. } \frac{x+1/3}{1}=\frac{y+2/3}{-2}=\frac{z}{1}; \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}.\right.$$

2. Obtain the symmetrical form of the equations of the line

$$x-2y+3z=4,\ 2x-3y+4z=5.$$

[Ans. $(x+2)=\frac{1}{2}(y+3)=z$.

3. Find out the points of intersection of the line

$$x+y-z+1=0=14x+9y-7z-1$$

with the XY and YZ planes, and hence put down the symmetrical form of its equations. [Ans. $-(x)/2=(y-4)/7=(z-5)/5$.

4. Find the equation of the plane through the point (1, 1, 1) and perpendicular to the line

$$x-2y+z=2,\ 4x+3y-z+1=0.$$

[Ans. $x-5y-11z+15=0$.

5. Find the equations of the line through the point (1, 2, 4) parallel to the line

$$3x+2y-z=4,\ x-2y-2z=5.$$

[Ans. $(x-1)/6=(2-y)/5=(z-4)/8$.

6. Find the angle between the lines in which the planes

$$3x-7y-5z=1,\ 5x-13y+3z+2=0$$

cut the plane

$$8x-11y+2z=0 \qquad [Ans.\ 90^\circ$$

7. *Find the angle between the lines*

$$3x+2y+z-5=0=x+y-2z-3,$$

$$2x-y-z=0=7x+10y-8z. \qquad (L.U.)\ [Ans.\ 90^\circ$$

8. Show that the condition for the lines

$$x=az+b,\ y=cz+d\ ;\ x=a_1z+b_1,\ y=c_1z+d_1,$$

to be perpendicular is

$$aa_1+cc_1+1=0.$$

9. What are the symmetrical forms of the equations of the lines

(i) $y=b,\ z=c$ [*Ans.* $x/1=(y-b)/0=(z-c)/0$.

(ii) $x=a,\ by+cz=d$ [*Ans.* $(x-a)/0=(y-d/b)/c=z/-b$.

3·2. *To find the angle between*

the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

and the plane

$$ax+by+cz+d=0.$$

Angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

Since the direction cosines of the normal to the given plane and of the given line are proportional to a, b, c and l, m, n respectively, we have

$$\sin\theta=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\ \sqrt{(l^2+m^2+n^2)}},$$

where θ is the required angle.

The straight line is parallel to the plane, if $\theta=0$

i.e.,

$$al+bm+cn=0,$$

which is also evident from the fact *that if a line be parallel to a plane, it is perpendicular to the normal to it.*

Exercises

1. Show that the line $\frac{1}{2}(x-2)=\frac{1}{3}(y-3)=\frac{1}{4}(z-4)$ is parallel to the plane $2x+y-2z=3$.

2. Find the equations of the line through the point $(-2, 3, 4)$, and parallel to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$.

[*Ans.* $(x+2)=-\frac{1}{2}(y-3)=(z-4)$.

[*Hint.* The direction ratios, l, m, n, of the line are given by the relations $2l+3m+4n=0=3l+4m+5n$.]

3. Find the equation of the plane through the points

$$(1, 0, -1),\ (3, 2, 2)$$

and parallel to the line

$$(x-1)=(1-y)/2=(z-2)/3. \qquad [Ans.\ 4x-y-2z=6.$$

4. Show that the equation of the plane parallel to the join of

$$(3, 2, -5) \text{ and } (0, -4, -11)$$

and passing through the points

$$(-2, 1, -3) \text{ and } (4, 3, 3)$$

is

$$4x+3y-5z=10.$$

5. Find the equation of the plane containing the line

$$2x-5y+2z=6,\ 2x+3y-z=5$$

and parallel to the line $x=-y/6=z/7$. [*Ans.* $6x+y-16=0$.

6. Prove that the equation of the plane through the line

$$u_1 \equiv a_1x+b_1y+c_1z+d_1=0,\ u_2 \equiv a_2x+b_2y+c_2z+d_2=0$$

and parallel to the line

$$x/l=y/m=z/n$$

is

$$u_1(a_2l+b_2m+c_2n)=u_2(a_1l+b_1m+c_1n). \qquad \textit{(D.U. Hons. 1937)}$$

7. Find the equation of the plane through the point (f, g, h) and parallel to the lines $x/l_r=y/m_r=z/n_r$; $r=1, 2$. [*Ans.* $\Sigma(x-f)(m_1n_2-m_2n_1)=0$.

8. Find the equations of the two planes through the origin which are parallel to the line

$$(x-1)/2=-(y+3)=-(z+1)/2$$

and distant $5/3$ from it ; show that the two planes are perpendicular. [*Ans.* $2x+2y+z=0,\ x-2y+2z=0$.

3·3. Conditions for a line to lie in a plane. *To find the conditions that the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

may lie in the plane

$$ax+by+cz+d=0.$$

The line would lie in the given plane, if, and only if, the co-ordinates

$$lr+x_1,\ mr+y_1,\ nr+z_1$$

of any point on the line satisfy the equation of the plane for all values of r so that

$$r(al+bm+cn)+(ax_1+by_1+cz_1+d)=0,$$

is an identity.

This gives

$$\mathbf{al+bm+cn=0,}$$

$$\mathbf{ax_1+by_1+cz_1+d=0}\ ;$$

which are the required *two* conditions.

These conditions lead to the geometrical facts that a line will lie in a given plane, if

(*i*) *the normal to the plane is perpendicular to the line,*

and (*ii*) *any one point on the line lies in the plane.*

Cor. *General equation of the plane containing the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

where

$$Al+Bm+Cn=0. \qquad \dots(1)$$

Here, $A : B : C$ are the parameters subjected to the condition (1).

Exercises

1. Show that the line $x+10=(8-y)/2=z$ lies in the plane

$$x+2y+3z=6$$

and the line $\frac{1}{2}(x-2)=-(y+2)=\frac{1}{3}(z-3)$ in the plane

$$2x+3y-z+3=0.$$

2. *Find the equation to the plane through the point* (x_1, y_1, z_1) *and through the line*

$$(x-a)/l=(y-b)/m=(z-c)/n. \qquad (P.U.\ 1939)$$

The general equation of the plane containing the given line is

$$A(x-a)+B(y-b)+C(z-c)=0, \qquad ...(i)$$

where A, B, C are any numbers subjected to the condition

$$Al+Bm+Cn=0. \qquad ...(ii)$$

The plane (i) will pass through (x_1, y_1, z_1), if

$$A(x_1-a)+B(y_1-b)+C(z_1-c)=0. \qquad ...(iii)$$

Eliminating A, B, C from (i), (ii) and (iii), we have

$$\begin{vmatrix} x-a, & y-b, & z-c \\ l, & m, & n \\ x_1-a, & y_1-b, & z_1-c \end{vmatrix}=0,$$

as the required equation.

3. Find the equation of the plane containing the line

$$\frac{1}{2}(x+2)=\frac{1}{3}(y+3)=-\frac{1}{4}(z-4)$$

and the point (3, 0, 0). [Ans. [illegible] $- 12 = 0$.

4. $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

are two straight lines. Find the equation of the plane containing the first line and parallel to the second. [Ans. $\Sigma(x-x_1)(m_1n_2-m_2n_1)=0$.

5. Show that the equation of the plane through the line

$$\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2} \text{ and parallel to } \frac{x-2}{2}=\frac{y-1}{-3}=\frac{z+4}{5}$$

is $26x-11y-17z-109=0$ and show that the point $(2, 1, -4)$ lies on it. What is the geometrical relation between the two lines and the plane ?

6. Find the equation of the plane containing the line

$$-\frac{1}{3}(x+1)=\frac{1}{2}(y-3)=(z+2)$$

and the point $(0, 7, -7)$ and show that the line $x=\frac{1}{3}(7-y)=\frac{1}{2}(z+7)$ lies in the same plane. [Ans. $x+y+z=0$.

7. Find the equation of the plane which contains the line

$$(x-1)/2=-y-1=(z-3)/4$$

and is perpendicular to the plane

$$x+2y+z=12.$$

Deduce the direction cosines of the projection of the given line on the given plane. (L.U.)

[Ans. $9x-2y-5z+4=0$; $4k, -7k, 10k$, where $k=1/\sqrt{(165)}$.

8. Find the equations, in the symmetrical form, of the projection of the line

$$\frac{1}{2}(x+1)=-\frac{1}{3}(y+7)=\frac{1}{4}(z+3)$$

on the plane

$$x-2y+3z-4=0.$$

[Ans. $(x-\frac{3}{4})/10=(y+13/8)/29=(z-0)/16$.

3·4. Coplanar Lines. Condition for the coplanarity of lines. *To find the condition that the two given straight lines should intersect, i.e., be coplanar.*

Let the given straight lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \quad ...(1)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(2)$$

If the lines intersect, they must lie in a plane. Equation of *any* plane containing the line (1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \quad ...(i)$$

with the condition

$$Al_1+Bm_1+Cn_1=0. \quad ...(ii)$$

The plane (*i*) will contain the line (2), if the point (x_2, y_2, z_2) lies upon it and the line is perpendicular to the normal to it. (§ 3·3). This requires

$$A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0, \quad ...(iii)$$

$$Al_2+Bm_2+Cn_2=0. \quad ...(iv)$$

Eliminating A, B, C from (*ii*), (*iii*), (*iv*), we get

$$\begin{vmatrix} x_2-x_1, & y_2-y_1, & z_2-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0, \quad ...(A)$$

which is the required condition for the lines to intersect. Again eliminating A, B, C from (*i*), (*ii*), (*iv*) we get

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0,$$

which is the equation of the plane containing the two lines, in case they intersect.

Second Method. The condition for intersection may also be obtained as follows :—

$(l_1r_1+x_1, m_1r_1+y_1, n_1r_1+z_1)$ and $(l_2r_2+x_2, m_2r_2+y_2, n_2r_2+z_2)$ are the general co-ordinates of the points on the lines (1) and (2) respectively.

In case the lines intersect, these points should coincide for some values of r_1 and r_2. This requires

$$(x_1-x_2)+l_1r_1-l_2r_2=0,$$
$$(y_1-y_2)+m_1r_1-m_2r_2=0,$$
$$(z_1-z_2)+n_1r_1-n_2r_2=0.$$

Eliminating r_1, r_2, we have

$$\begin{vmatrix} x_1-x_2, & l_1, & l_2 \\ y_1-y_2, & m_1, & m_2 \\ z_1-z_2, & n_1, & n_2 \end{vmatrix}=0$$

which is the same condition as (A).

Note 1. In general, the equation

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0$$

represents the plane through (1) and parallel to (2), and the equation

$$\begin{vmatrix} x-x_2, & y-y_2, & z-z_2 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0$$

represents the plane through (2) and parallel to (1).

In case the lines are coplanar, the condition (A) shows that the point (x_2, y_2, z_2) lies on the first plane and the point (x_1, y_1, z_1) on the second. These two planes are then identical and contain both the intersecting lines.

Thus the equation of a plane containing two intersecting lines is obtained by finding the plane through one line and parallel to the other or, through one line and any point on the other.

Note 2. Two lines will intersect if, and only if, there exists a point whose co-ordinates satisfy the *four* equations, two of each line. But we know that three unknowns can be determined so as to satisfy three equations. Thus for intersection, we require that the four equations should be consistent among themselves, *i.e.*, the values of the unknowns x, y, z, as obtained from any three equations, should satisfy the fourth also. The condition of consistency of four equations containing three unknowns is obtained by eliminating the unknowns. It is sometimes comparatively more convenient to follow this method to obtain the condition of intersection or to prove the fact of intersection of two lines.

Note 3. The condition for the lines, whose equations, given in the unsymmetrical form, are

$$a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0;$$
$$a_3x+b_3y+c_3z+d_3=0,\ a_4x+b_4y+c_4z+d_4=0;$$

to be coplanar, *i.e.*, to intersect, as obtained by eliminating x, y, z from these equations, is

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}=0.$$

In case, this condition is satisfied, the co-ordinates of the point of intersection are obtained by solving any three of the four equations simultaneously.

Examples

1. *Prove that the lines*

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7},\quad \frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$$

intersect and find the co-ordinates of their point of intersection.

Now,

$$(r+4,\ -4r-3,\ 7r-1) \text{ and } (2r'+1,\ -3r'-1,\ 8r'-10)$$

are the general co-ordinates of points on the two lines respectively.

They will intersect if the three equations

$$r-2r'+3=0, \qquad \dots (i)$$
$$4r-3r'+2=0, \qquad \dots (ii)$$
$$7r-8r'+9=0, \qquad \dots (iii)$$

are simultaneously true.

(i) and (ii) give $r=1$, $r'=2$ which also, clearly, satisfy (iii). Hence the lines intersect and their point of intersection obtained by putting $r=1$, or $r'=2$ is $(5, -7, 6)$.

Note. This equation can also be solved by first finding the point satisfying three equations

$$\frac{x-4}{1}=\frac{y+3}{-4};\ \frac{y+3}{-4}=\frac{z+1}{7};\ \frac{x-1}{2}=\frac{y+1}{-3},$$

and then showing that the same point also satisfies the equation

$$\frac{y+1}{-3}=\frac{z+10}{8}.$$

2. *Show that the lines*

$$\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3};\ \frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$$

are coplanar and find the equation of the plane containing them.

The equation of the plane containing the first line and parallel to the second is

$$\begin{vmatrix} x+3, & y+5, & z-7 \\ 2, & 3, & -3 \\ 4, & 5, & -1 \end{vmatrix}=0$$

or $$6x-5y-z=0.$$

which is clearly satisfied by the point $(-1, -1, -1)$, a point on the second line. Hence this plane contains also the second line. Thus the two lines are coplanar and the equation of the plane containing them is

$$6x-5y-z=0.$$

3. *Show that the lines*

$$\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2},$$
$$3x+2y+z-2=0=x-3y+2z-13$$

are coplanar and find the equation to the plane in which they lie.

The *general* equation of the plane through the second line is

$$3x+2y+z-2+k(x-3y+2z-13)=0$$

or

$$x(3+k)+y(2-3k)+z(1+2k)-2-13k=0.$$

This will be parallel to the first line

if $$3(3+k)+(2-3k)-2(1+2k)=0,\ i.e.,\ k=\tfrac{9}{4}.$$

Hence the equation of the plane containing the second line and parallel to the first is

$$21x-19y+22z-125=0$$

which clearly passes through the point $(-5, -4, 7)$ and so contains also the first line.

Thus the two lines are coplanar and lie in the plane

$$21x-19y+22z-125=0.$$

Exercises

1. Show that the lines

$$\tfrac{1}{3}(x+4)=\tfrac{1}{5}(y+6)=-\tfrac{1}{2}(z-1)$$
$$3x-2y+z+5=0=2x+3y+4z-4$$

are coplanar. Find also the co-ordinates of their point of intersection and the equation of the plane in which they lie.
[*Ans.* $(2, 4, -3)$; $45x-17y+25z+53=0$.

2. Prove that the lines

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8};\ \frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$$

intersect. Find also their point of intersection and the plane through them.
[*Ans.* $(5, -7, 6)$; $11x=6y+5z+67$.

3. Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7};\ \frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$$

intersect. Find their point of intersection and the plane in which they lie.
[*Ans.* $(1/2, -1/2, -3/2)$; $x-2y+z=0$.

4. Show that the lines

$$x+2y-5z+9=0=3x-y+2z-5;$$
$$2x+3y-z-3=0=4x-5y+z+3$$

are coplanar.

5. Prove that the lines

$$x-3y+2z+4=0=2x+y+4z+1;$$
$$3x+2y+5z-1=0=2y+z$$

intersect and find the co-ordinates of their point of intersection.
[*Ans.* $(3, 1, -2)$.

6. Prove that the lines

$$\frac{x-a}{a'}=\frac{y-b}{b'}=\frac{z-c}{c'}\ \text{and}\ \frac{x-a'}{a}=\frac{y-b'}{b}=\frac{z-c'}{c}$$

intersect and find the co-ordinates of the point of intersection and the equation of the plane in which they lie. [*Ans.* $(a+a', b+b', c+c')$; $\Sigma x(bc'-b'c)=0$.

7. Show that the condition that the two straight lines

$$x=mz+a,\ y=nz+b,\ \text{and}\ x=m'z+a',\ y=n'z+b'$$

should intersect is

$$(a-a')(n-n')=(b-b')(m-m').$$

8. Show that the plane which contains the two parallel lines

$$x-4=-\tfrac{1}{4}(y-3)=\tfrac{1}{5}(z-2),\ x-3=-\tfrac{1}{4}(y+2)=\tfrac{1}{5}z$$

is given by

$$11x-y-3z=35.$$

9. Find the equation of the plane passing through $x/l=y/m=z/n$, and perpendicular to the plane containing

$$x/m=y/n=z/l\ \text{and}\ x/n=y/l=z/m.$$ (*D. U. Hons. 1949*)
[*Ans.* $\Sigma(m-n)x=0$.

10. Show that the line $x+a=y+b=z+c$ intersects the four lines

(i) $x=0,\ y+z=2a$; (ii) $y=0,\ z+x=2b$; (iii) $z=0;\ x+y=2c$

(iv) $x+y+z=2k,\ a(a-k)^{-1}x+b(b-k)^{-1}y+c(c-k)^{-1}z=0$

at right angles if $a+b+c=0$.

11. Obtain the condition for the line

$$(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$$

to intersect the locus of the equations $ax^2+by^2=1,\ z=0$.

[*Ans.* $a(\alpha n-l\gamma)^2+b(\beta n-m\gamma)^2=n^2$.

3·5. Number of arbitrary constants in the equations of a straight line. *To show that there are four arbitrary constants in the equations of a straight line.*

A line PQ can be regarded as the intersection of *any* two planes through it. In particular, we may take the two planes perpendicular to two of the co-ordinate planes, say, YZ and ZX planes.

The equations of the planes through PQ perpendicular to the YZ and ZX planes are respectively of the forms

$$x=cy+d, \text{ and } z=ax+b$$

which are, therefore, the equations of the line PQ and contain four arbitrary constants a, b, c, d.

Hence the *equations of a straight line involve four arbitrary constants* as it is always possible to express them in the above form.

Note. The symmetrical form of the equations of a line apparently involves six constants x_1, y_1, z_1; l, m, n, but they are really equivalent to four arbitrary constants only as is shown below :

l, m, n, which are connected by the relation $l^2+m^2+n^2=1$ are equivalent to two independent constants only.

Also, of the three apparently independent numbers x_1, y_1, z_1, only two are independent as one of them can always be arbitrarily chosen as described below :—

A line cannot be parallel to all the co-ordinate planes. Let the given line, in particular, be not parallel to the YZ plane. If, now, x_1 be assigned any value, we may take the point where the line meets the plane $x=x_1$ as the point (x_1, y_1, z_1).

Hence we may give to x_1 any value we please. The three numbers x_1, y_1, z_1 are, therefore, equivalent to two independent constants only.

The fact that the general equations of a straight line contain four arbitrary constants may also be seen *directly* as follows :—

We see that

$$\frac{x-x_1}{l}=\frac{y-y_1}{m},\quad \frac{y-y_1}{m}=\frac{z-z_1}{n}$$

are equivalent to

$$x=\frac{l}{m}y+\frac{(mx_1-ly_1)}{m},\quad y=\frac{m}{n}z+\frac{(ny_1-mz_1)}{n}$$

respectively, so that

$$\frac{l}{m},\quad \frac{m}{n},\quad \frac{mx_1-ly_1}{m},\quad \frac{ny_1-mz_1}{n}$$

are the four arbitrary constants or parameters.

3·51. Determination of lines satisfying given conditions.

We now consider the various *sets of conditions* which determine a line.

We know that the equations of a straight line involve four arbitrary constants and hence any four geometrical conditions, each giving rise to one relation between the constants, fix a straight line.

It may be noted that the conditions for a line to intersect a given line or be perpendicular to it separately involve one relation between the constants and hence three more relations are required to fix the line.

A given condition may sometimes give rise to two relations between the constants as, for instance, the condition of the line

(*i*) to pass through a given point,

or (*ii*) to have a given direction.

In such cases only two more relations will be required to fix the straight line.

Equations of lines have already been discussed under the following sets of conditions :

(*i*) passing through a given point and having a given direction ;
(*ii*) passing through two given points ;
(*iii*) passing through a point and parallel to two given planes ;
(*iv*) passing through a point and perpendicular to two given lines.

Some further sets of conditions which determine a line are given below :—

(*v*) passing through a given point and intersecting two given lines ;
(*vi*) intersecting two given lines and having a given direction ;
(*vii*) intersecting a given line at right angles and passing through a given point ;
(*viii*) intersecting two given lines at right angles ;
(*ix*) intersecting a given line parallel to a given line and passing through a given point ;
(*x*) passing through a given point and perpendicular to two given lines ;

and so on.

An Important Note : *If*

$$u_1=0=v_1 \text{ and } u_2=0=v_2,$$

be two straight lines, then the general equations of a straight line intersecting them both are

$$u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2,$$

where λ_1, λ_2 are any two constant numbers.

The line $u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2$ lies in the plane $u_1+\lambda_1 v_1=0$ which again contains the line $u_1=0=v_1$.

The two lines

$$u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2 \,;\; u_1=0=v_1$$

are, therefore, coplanar and hence they intersect.

Similarly, the same line intersects the line $u_3=0=v_3$.

This conclusion will be found very helpful in what follows.

For the sake of illustration, we give below a few examples.

Examples

1. *Find the equations of the line that intersects the lines*

$$2x+y-4=0=y+2z\ ;\ x+3z=4,\ 2x+5z=8$$

and passes through the point $(2, -1, 1)$.

The line

$$2x+y-4+\lambda_1(y+2z)=0,\ x+3z-4+\lambda_2(2x+5z-8)=0$$

intersects the two given lines for all values of λ_1, λ_2.

This line will pass through the point $(2, -1, 1)$, if

$$-1+\lambda_1=0 \text{ and } 1+\lambda_2=0,$$

i.e., if $$\lambda_1=1,\ \lambda_2=-1.$$

The required equations, therefore, are

$$x+y+z=2 \text{ and } x+2z=4.$$

2. *Find the equations, to the line that intersects the lines*

$$2x+y-1=0=x-2y+3z\ ;$$
$$3x-y+z+2=0=4x+5y-2z-3$$

and is parallel to the line

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}.$$

The general equations of the lines intersecting the two given lines are

$$2x+y-1+\lambda_1(x-2y+3z)=0$$
$$3x-y+z+2+\lambda_2(4x+5y-2z-3)=0$$

which will be parallel to the given line if λ_1, λ_2 be so chosen that the two planes representing it are separately parallel to the given line.

This requires

$$(2+\lambda_1)+2(1-2\lambda_1)+3(3\lambda_1)=0,\ i.e.,\ \lambda_1=-\tfrac{2}{3}.$$

and $$(3+4\lambda_2)+2(-1+5\lambda_2)+3(1-2\lambda_2)=0,\ i.e.,\ \lambda_2=-\tfrac{1}{2}.$$

The required equations of the line, therefore, are

$$4x+7y-6z-3=0,\ 2x-7y+4z+7=0.$$

3. *A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equations*

$$x=y+a=z\ ;\ x+a=2y=2z\ ;$$

find out the co-ordinates of each of the points of intersection.

$P(r, r-a, r)$ and $P'(2r'-a, r', r')$ are the general co-ordinates of points on the two given lines

$$\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1},\ \frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}.$$

The direction cosines of PP' are proportional to

$$r-2r'+a,\ r-r'-a,\ r-r'.$$

Now, we choose r and r' such that the line PP' has direction cosines proportional to (2, 1, 2).

$$\therefore \qquad \frac{r-2r'+a}{2}=\frac{r-r'-a}{1}=\frac{r-r'}{2},$$

which give

$$r=3a,\ r'=a.$$

Putting $r=3a$ and $r'=a$ in the co-ordinates of P and P', we get

$$(3a,\ 2a,\ 3a)\ \text{and}\ (a,\ a,\ a)$$

which are the required points of intersection.

4. *Find the equations of the perpendicular from the point* $(3, -1, 11)$ *to the line*

$$\tfrac{1}{2}x=\tfrac{1}{3}(y-2)=\tfrac{1}{4}(z-3).$$

Obtain also the foot of the perpendicular.

The co-ordinates of any point on the given line are

$$2r,\ 3r+2,\ 4r+3.$$

This will be the required foot of the perpendicular if the line joining it to the point (3, −1, 11) be perpendicular to the given line. This requires

$$2(2r-3)+3(3r+2+1)+4(4r+3-11)=0 \text{ or } r=1.$$

Therefore the required foot is (2, 5, 7) and the required equations of the perpendiculars are

$$\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}.$$

Exercises

1. Find the equations of the perpendicular from

(i) $(2, 4, -1)$ to $(x+5)=\tfrac{1}{4}(y+3)=-\tfrac{1}{9}(z-6)$,

(ii) $(-2, 2, -3)$ to $(x-3)=\tfrac{1}{2}(y+1)=-\tfrac{1}{4}(z-2)$,

(iii) $(0, 0, 0)$ to $x+2y+3z+4=0=2x+3y+4z+5$,

(iv) $(-2, 2, -3)$ to $2x+y+z-7=0=4x+z-14$.

Obtain also the foot of the perpendiculars.

[*Ans.* (i) $\tfrac{1}{6}(x-2)=\tfrac{1}{3}(y-4)=\tfrac{1}{2}(z+1)$, $(-4, 1, -3)$.
(ii) $\tfrac{1}{6}(x+2)=-(y-2)=(z+3)$, $(4, 1, -2)$.
(iii) $-x/2=y=z/4$, $(2/3, -1/3, -4/3)$.
(iv) $\tfrac{1}{6}(x+2)=-(y-2)=(z+3)$, $(4, 1, -2)$.

2. A line with direction cosines proportional to $(7, 4, -1)$ is drawn to intersect the lines

$$\frac{x-1}{3}=\frac{y-7}{-1}=\frac{z+2}{1};\quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}.$$

Find the co-ordinates of the points of intersection and the length intercepted on it. [*Ans.* $(7, 5, 0)$, $(0, 1, 1)$, $\sqrt{(66)}$.

3. Find the equations to the line that intersects the lines

$$x+y+z=1,\ 2x-y-z=2;\ x-y-z=3,\ 2x+4y-z=4$$

and passes through the point (1, 1, 1). Find also the points of intersection.

(P.U. 1939)

[*Ans.* $(x-1)/0=(y-1)/1=(z-1)/3$; $(1, \tfrac{1}{2}, -\tfrac{1}{2})$; $(1, 0, -2)$.

4. Find the equations to the straight lines drawn from the origin to intersect the lines

$$3x+2y+4z-5=0=2x-3y+4z+1\ ;\ 2x-4y+z+6=0=3x-4y+z-3.$$

(*P.U. 1942*)

[*Ans.* $13x-15y+24z=0=8x-13y+3z$.

5. Obtain the equations of the line drawn through the point $(1, 0, -1)$, and intersecting the lines

$$x=2y=2z\ ;\ 3x+4y=1\ ;\ 4x+5z=2.$$

[*Ans.* $-(x-1)/6=y=-(z+1)/8$.

6. Find the equations to the line drawn parallel to $x/2=y/3=z/4$ so as to intersect the lines

$$9x+y+z+4=0=5x+y+3z\ ;\ x+2y-3z-3=0=2x-5y+3z+3.$$

[*Ans.* $(x+1)/2=y/3=z/4$.

7. Find the equations of the line drawn through the point $(-4, 3, 1)$, parallel to the plane $x+2y-z=5$ so as to intersect the line

$$-(x+1)/3=(y-3)/2=-(z-2).$$

Find also the point of intersection.

[*Ans.* $(x+4)/3=-(y-3)=(z-1)$; $(2, 1, 3)$.

8. Find the distance of the point $(-2, 3, -4)$ from the line

$$(x+2)/3=(2y+3)/4=(3z+4)/5$$

measured parallel to the plane

$$4x+12y-3z+1=0.$$ [*Ans.* $17/2$.

9. Find the equations of the straight line through the point $(2, 3, 4)$ perpendicular to the X-axis and intersecting the line $x=y=z$.

[*Ans.* $x=2,\ 2y-z=2$.

10. Find the equations of the straight line through the origin which will intersect the lines

$$(x-1)/2=(y+3)/4=(z-5)/3,\quad (x-4)/2=(y+3)/3=(z-14)/4$$

and prove that the secant is divided at the origin in the ratio $1:2$.

11. Find the equations of the two lines through the origin which intersect the line $(x-3)/2=y-3=z$ at angles of $60°$.

[*Ans.* $x=y/2=-z$; $x=-y=z/2$.

12. The straight line which passes through the points $(11, 11, 18)$, $(2, -1, 3)$ is intersected by a straight line drawn through $(15, 20, 8)$ at right angles to Z-axis ; show that the two lines intersect at the point $(5, 3, 8)$.

13. A straight line is drawn through the origin meeting perpendicularly the straight line through (a, b, c) with direction cosines l, m, n ; prove that the direction cosines of the line are proportional to

$$a-lk,\ b-mk,\ c-nk \text{ where } k=al+bm+cn.$$

14. From the point $P(a, b, c)$ perpendiculars PA, PB are drawn to the lines $y=2x$, $z=1$ and $y=-2x$, $z=-1$; find the co-ordinates of A and B.

Prove that, if P moves so that the angle APB is always a right angle, P always lies on the surface $12x^2-3y^2+25z^2=25$.

[*Ans.* $A[(2b+a)/5,\ (4b+2a)/5,\ 1]$; $B[(a-2b)/5,\ (4b-2a)/5,\ -1]$.

3·6. The shortest distance between two lines. *To show that the shortest distance between two lines lies along the line meeting them both at right angles.*

Let AB, CD be two given lines.

A line is completely determined if it intersects two lines at right angles. [See § 3·51. *Case viii*).

Thus, there is one and only one line which intersects the two given lines at right angles, say, at G and H.

GH is, then, the shortest distance between the two lines for, if, A, C be *any* two points, one on each of the two given lines, then GH is clearly the projection of AC on itself and, therefore,

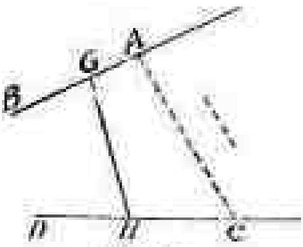


Fig. 19.

$$GH = AC \cos \theta,$$

where θ is the angle between GH and AC.

Hence $\qquad GH < AC.$

Thus GH is the shortest distance between the two lines AB and CD.

3·61. *To find the magnitude and the equations of the line of shortest distance between two straight lines.*

If AB, CD be two given lines and GH the line which meets them both at right angles at G and H, then GH is the line of shortest distance between the given lines and the length GH is the magnitude.

Let the equations of the given lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad ...(i)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}, \qquad ...(ii)$$

and let the shortest distance GH lie along the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad ...(iii)$$

Line (*iii*) is perpendicular to both the lines (*i*) and (*ii*). Therefore, we have

$$ll_1+mm_1+nn_1=0,$$
$$ll_2+mm_2+nn_2=0,$$

or

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$
$$=\frac{1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}=\frac{1}{\sin\theta}$$

where θ is the angle between the given lines.

$$\therefore \qquad l=\frac{m_1n_2-m_2n_1}{\sin\theta},\ m=\frac{n_1l_2-n_2l_1}{\sin\theta},\ n=\frac{l_1m_2-l_2m_1}{\sin\theta} \qquad ...(iv)$$

The line of shortest distance is perpendicular to both the lines. Therefore the magnitude of the shortest distance is the projection on the line of shortest distance of the line joining *any* two points, one on each of the given lines (*i*) and (*ii*).

Taking the projection of the join of (x_1, y_1, z_1), (x_2, y_2, z_2) on the line with direction cosines l, m, n, we see that the shortest distance

$$=(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n,$$

where l, m, n have the values as given in (*iv*).

To find the equations of the line of shortest distance, we observe that it is coplanar with both the given lines.

The equation of the plane containing the coplanar lines (*i*) and (*iii*) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix}=0, \qquad \ldots(v)$$

and that of the plane containing the coplanar lines (*ii*) and (*iii*) is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}=0, \qquad \ldots(vi)$$

Thus (*v*) and (*vi*) are the two equations of the line of shortest distance, where l, m, n are given in (*iv*).

Note. Other methods of determining the shortest distance are given below where an example has been solved by three different methods.

Examples

1. *Find the magnitude and the equations of the line of shortest distance between the lines :*

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}, \qquad \ldots(i)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}. \qquad \ldots(ii)$$

First Method

Let l, m, n be the direction cosines of the line of shortest distance.

As it is perpendicular to the two lines, we have

$$3l-16m+7n=0,$$

and

$$3l+8m-5n=0.$$

$\therefore$

$$\frac{l}{24}=\frac{m}{36}=\frac{n}{72},$$

or

$$\frac{l}{2}=\frac{m}{3}=\frac{n}{6}.$$

Hence

$$l=\tfrac{2}{7},\ m=\tfrac{3}{7},\ n=\tfrac{6}{7}.$$

The magnitude of the shortest distance is the projection of the join of the points (8, −9, 10), (15, 29, 5), on the line of the shortest distance and is, therefore,

$$=7\cdot\tfrac{2}{7}+38\cdot\tfrac{3}{7}-5\cdot\tfrac{6}{7}=14.$$

Again, the equation of the plane containing the first of the two given lines and the line of shortest distance is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 2, & 3, & 6 \end{vmatrix}=0,$$

or

$$117x+4y-41z-490=0.$$

Also the equation of the plane containing the second line and the shortest distance line is

$$\begin{vmatrix} x-15, & y-29, & z-5 \\ 3, & 8, & -5 \\ 2, & 3, & 6 \end{vmatrix}=0,$$

or

$$9x-4y-z=14.$$

Hence the equations of the shortest distance line are

$$117x+4y-41z-490=0=9x-4y-z-14.$$

Second Method

$$P(3r+8, -16r-9, 7r+10), P'(3r'+15, 8r'+29, -5r'+5)$$

are the general co-ordinates of the points on the two lines respectively. The direction cosines of PP' are proportional to

$$3r-3r'-7, -16r-8r'-38, 7r+5r'+5.$$

Now PP' will be the required line of shortest distance, if it is perpendicular to both the given lines, which requires

$$3(3r-3r'-7)-16(-16r-8r'-38)+7(7r+5r'+5)=0,$$

and $$3(3r-3r'-7)+8(-16r-8r'-38)-5(7r+5r'+5)=0,$$

or $$157r+77r'+311=0 \text{ and } 11r+7r'+25=0$$

which give $$r=-1, r'=-2.$$

Therefore co-ordinates of P and P' are

$$(5, 7, 3) \text{ and } (9, 13, 15).$$

Hence, the shortest distance $PP'=14$ and its equations are

$$\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}.$$

This method is sometimes very convenient and is specially useful when we require also the points where the line of shortest distance meets the two lines.

Third Method. This method depends upon the following considerations :—

Let AB, CD be the given lines and GH, the line of shortest distance between them.

Let 'α' denote the plane through AB and parallel to CD and let 'β' be the plane through CD and parallel to AB.

The line of shortest distance GH, being perpendicular to both AB, CD is normal to the two planes so that the two planes are parallel. The length GH of the shortest distance is, therefore, the distance between the parallel planes α and β. This distance between parallel planes being the distance of *any* point on one from the other, we see that it is enough to determine only one plane say 'α' and then the magnitude of the shortest distance is the distance of any point on the second line from the plane 'α'.

Again, we easily see that the plane through the lines AB, GH is perpendicular to the plane 'α' and the plane through CD, GH is perpendicular to the plane 'β' and, therefore, also to 'α'. Thus GH, *the line of shortest distance, is the line of intersection of the planes separately drawn through* AB, CD *perpendicular to the plane* 'α'.

We now solve the equation.

The equation of the plane containing the line (*i*) and parallel to the line (*ii*) is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 3, & 8, & -5 \end{vmatrix}=0$$

or $$2x+3y+6z-49=0 \qquad \text{...(iii)}$$

Perpendicular distance of the point (15, 29, 5), lying on the second line, from this plane

$$=\frac{30+87+30-49}{7}$$

$$=14,$$

which is the required magnitude of the shortest distance.

The equation of the plane through (*i*) perpendicular to the plane (*iii*) is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 2, & 3, & 6 \end{vmatrix}=0$$

or $$117x+4y-41z-490=0. \qquad \text{...(iv)}$$

The equation of the plane through (*ii*) and perpendicular to the plane (*iii*) is

$$\begin{vmatrix} x-15, & y-29, & z-5 \\ 3, & 8, & -5 \\ 2, & 3, & 6 \end{vmatrix}=0$$

or $$9x-4y-z=14. \qquad \text{...(v)}$$

Hence (*iv*), (*v*) are the equations of the line of shortest distance.

2. ***Find the shortest distance between the axis of z and the line***

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0.$$

(D.U. Hons. 1948, B.U. 1955)

The *third method* given on page 56 will prove very convenient in this case.

Now, any plane through the second given line is

$$ax+by+cz+d+k(a'x+b'y+c'z+d')=0,$$

i.e., $$(a+ka')x+(b+kb')y+(c+kc')z+(d+kd')=0. \quad ...(i)$$

It will be parallel to z-axis whose direction cosines are 0, 0, 1, if the normal to the plane is $\perp$ z-axis, *i.e.,* if,

$$0.(a+ka')+0.(b+kb')+1.(c+kc')=0,$$

i.e., $$k=-c/c'.$$

Substituting this value of k in (i), we see that the equation of the plane through the second line parallel to the first is

$$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0 \quad ...(ii)$$

The required S.D. is the distance of *any* point on z-axis from the plane (ii).

$\therefore$ S.D. = perpendicular from (0, 0, 0), (a point on z-axis)

$$=\pm\frac{dc'-d'c}{\sqrt{[(ac'-a'c)^2+(bc'-b'c)^2]}}.$$

Exercises

1. Find the magnitude and the equations of the line of shortest distance between the two lines :

(i) $$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5};\ \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}.$$

(ii) $$\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1};\ \frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}.$$

[*Ans.* (i) $x=y=z$; $4\sqrt{3}$.

(ii) $(x-4)=(y-2)/3=-(z+3)/5$; $\sqrt{35}$.

2. Find the length and the equations of the shortest distance line between

$$5x-y-z=0,\quad x-2y+z+3=0\ ;$$
$$7x-4y-2z=0,\quad x-y+z-3=0.$$

[*Hint.* Transform the equations to the symmetrical form.]

[*Ans.* $17x+20y-19z-39=0=8x+5y-31z+67$; $13/\sqrt{75}$.

3. Find the magnitude and the position of the shortest distance between the lines

(i) $2x+y-z=0,\ x-y+2z=0\ ;\ x+2y-3z=4,\ 2x-3y+4z=5.$

(ii) $\dfrac{x}{4}=\dfrac{y+1}{3}=\dfrac{z-2}{2}$; $5x-2y-3z+6=0,\ x-3y+2z-3=0.$

[*Ans.* (i) $3x+z=0=22x-5y+4z-67$, $2\sqrt{14}/7$.

(ii) $7x-2y-11z+20=0=13x-12z+21$; $17\sqrt{6/29}$.

4. Obtain the co-ordinates of the points where the shortest distance between the lines

$$\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3},\ \frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$$

meets them. [*Ans.* (11, 11, 31) and (3, 5, 7).

5. Find the co-ordinates of the point on the join of $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes

$$3x-y-3z+32=0 \text{ and } 3x+2y-15z-8=0.$$

[*Ans.* $(-7, -1, -9)$.

6. Show that the shortest distance between the lines

$$x+a=2y=-12z \text{ and } x=y+2a=6z-6a$$

is $2a$.

7. Find the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}; \quad \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5};$$

show also that the lines are co-planar. *(P.U. 1926)*

8. Find the length and equations of the line of shortest distance between the lines

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2}, \quad \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}. \qquad \textit{(B.U. 1956)}$$

[*Ans.* 9; $22x+34y+12z-108=0$, $12x+33y+15z-81=0$.

9. Show that the length of the shortest distance between the line $z=x\tan\alpha$, $y=0$ and any tangent to the ellipse $x^2\sin^2\alpha+y^2=a^2$; $z=0$ is constant.

10. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes

$$y+z=0, \; z+x=0, \; x+y=0, \; x+y+z=a$$

is $2a/\sqrt{6}$ and that the three lines of shortest distance intersect at the point $x=y=z=-a$. *(D.U. Hons. 1960)*

3·7. Length of the perpendicular from a point to a line. *To find the length of the perpendicular from a given point $P(x_1, y_1, z_1)$ to a given line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

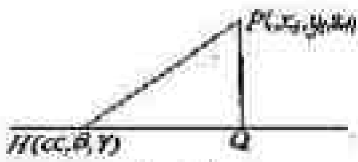


Fig. 17.

If H be the point (α, β, γ) on the given line and Q the foot of the perpendicular from P on it, we have,

$$PQ^2=HP^2-HQ^2.$$

But $\qquad HP^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2,$

and $\qquad HQ=$ projection of HP on the given line

$$=l(x_1-\alpha)+m(y_1-\beta)+n(z-\gamma),$$

provided l, m, n, are the actual direction cosines.

$$\therefore \quad PQ^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2 - [l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2.$$

The expression for PQ^2 can be put in an elegant form as follows:

We have, by Lagrange's identity,

$$PQ^2=[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2][l^2+m^2+n^2] - [l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2$$

$$=[l(y_1-\beta)-m(x_1-\alpha)]^2+[m(z_1-\gamma)-n(y_1-\beta)]^2 + [n(x_1-\alpha)-l(z_1-\gamma)]^2$$

$$=\begin{vmatrix} x_1-\alpha, & y_1-\beta \\ l, & m \end{vmatrix}^2+\begin{vmatrix} y_1-\beta, & z_1-\gamma \\ m, & n \end{vmatrix}^2+\begin{vmatrix} z_1-\gamma, & x_1-\alpha \\ n, & l \end{vmatrix}^2.$$

Exercises

1. *Find the length of the perpendicular from the point* $(4, -5, 3)$ *to the line*

$$\frac{x-5}{3}=\frac{y+2}{-4}=\frac{z-6}{5}. \qquad \left[\textit{Ans.}\ \frac{\sqrt{(470)}}{5}.\right.$$

2. *Find the locus of the point which moves so that its distance from the line* $x=y=z$ *is twice its distance from the plane* $x+y+z=1$.

$$\left[\textit{Ans.}\ x^2+y^2+z^2+5xy+5yz+5zx-4x-4y-4z+2=0.\right.$$

3. *Find the length of the perpendicular from the point* $P(3, 4, -1)$ *upon the line* $\frac{1}{2}(x-1)=\frac{1}{3}y=\frac{1}{4}z$. $\left[\textit{Ans.}\ \sqrt{(2100/110)}.\right.$

3·8. Intersection of three planes. *To find the conditions that the three planes*

$$a_rx+b_ry+c_rz+d_r=0\ ; \quad (r=1, 2, 3)$$

should have a common line of intersection.

If these three planes have a common line of intersection, then

$$a_3x+b_3y+c_3z+d_3=0 \qquad \ldots(i)$$

must represent the same plane as

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0, \qquad \ldots(ii)$$

for some value of λ.

Comparing (i) and (ii), we get

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=k, \text{ (suppose)}$$

$$\therefore \quad \begin{aligned} a_1+\lambda a_2-ka_3&=0,\\ b_1+\lambda b_2-kb_3&=0,\\ c_1+\lambda c_2-kc_3&=0,\\ d_1+\lambda d_2-kd_3&=0. \end{aligned}$$

Eliminating λ and k from these four equations, taking them three by three, we obtain

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0, \quad \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0,$$

which are the required conditions.

Only *two* of these four conditions are *independent* for, if the planes have two points in common, they have the whole line in common and this fact requires only two conditions.

These four determinants will respectively be denoted by the letters $\triangle, \triangle_1, \triangle_2, \triangle_3$.

Note. The following is the Algebraic proof of the fact that only two of the above *four* conditions are independent, *i.e.*, if two of these four determinants vanish the other two must also vanish.

$$\text{Let} \quad \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0=\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}.$$

$$\therefore \quad \begin{aligned} a_1A_1+a_2A_2+a_3A_3&=0\\ d_1A_1+d_2A_2+d_3A_3&=0 \end{aligned}$$

$$\therefore \quad \frac{A_1}{a_2d_3-a_3d_2}=\frac{A_2}{a_3d_1-a_1d_3}=\frac{A_3}{a_1d_2-a_2d_1}=\frac{1}{k}, \text{ (suppose)}$$

or $\quad a_2d_3-a_3d_2=kA_1,\ a_3d_1-a_1d_3=kA_2,\ a_1d_2-a_2d_1=kA_3.$

Thus we obtain

$$b_1(a_2d_3-a_3d_2)+b_2(a_3d_1-a_1d_3)+b_3(a_1d_2-a_2d_1)$$
$$=k(A_1b_1+A_2b_2+A_3b_3)=0,$$

i.e.,
$$\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0.$$

Similarly it can be proved that

$$\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}=0.$$

Note. The same conditions will also be obtained in § 3·82 in a different manner.

3·81. Triangular prism. Def. *Three planes are said to form a triangular prism if the three lines of intersection of the three planes, taken in pairs, are parallel.*

Clearly, the three planes will form a triangular prism if the line of intersection of two of them be parallel to the third.

3·82. *To find the condition that the three planes*

$$a_rx+b_ry+c_rz+d_r=0\ ;\ (r=1, 2, 3)$$

should form a prism or intersect in a line.

The line of intersection of the first two planes is

$$\frac{x-(b_1d_2-b_2d_1)/(a_1b_2-a_2b_1)}{b_1c_2-b_2c_1}=\frac{y-(a_2d_1-a_1d_2)/(a_1b_2-a_2b_1)}{a_2c_1-a_1c_2}$$
$$=\frac{z}{a_1b_2-a_2b_1} \quad \text{...(i)} \quad \text{(See Page 41)}$$

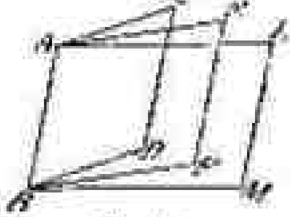


Fig. 18.

Fig. 19.

The three planes will form a triangular prism if this line is parallel to the third plane but does not lie in the same.

Then line (*i*) will be parallel to the third plane, if

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0,$$

or

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0,$$

i.e.,

$$\Delta=0.$$

Again, the planes will intersect in a line if the line (*i*) lies in the plane $a_3x+b_3y+c_3z+d_3=0$. This requires :

(1) this line is parallel to the third plane which gives $\Delta=0$, and

(2) the point $\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}, \dfrac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}, 0\right)$ lies on it which gives

$$a_3(b_1d_2-b_2d_1)+b_3(a_2d_1-a_1d_2)+d_3(a_1b_2-a_2b_1)=0,$$

or

$$\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0,$$

i.e.,

$$\Delta_3=0.$$

Thus the three planes will intersect in a line, if

$$\Delta=\Delta_3=0,$$

and will form a triangular prism, if

$$\Delta=0 \text{ and } \Delta_3\neq 0.$$

Note. Three distinct non-parallel planes behave in relation to each other in any of the following three ways :—

(*i*) They may intersect in a line which requires that two of the four determinants Δ, Δ_1, Δ_2, Δ_3 should vanish.

(*ii*) They may form a prism which requires that *only* Δ should vanish.

(*iii*) They may intersect in a unique finite point which requires that $\Delta\neq 0$.

Exercises

1. Show that the following sets of planes intersect in lines :

   (*i*) $4x+3y+2z+7=0,\ 2x+y-4z+1=0,\ x-7z-2=0.$

   (*ii*) $2x+y+z+4=0,\ y-z+4=0,\ 3x+2y+z+8=0.$

2. Show that the following sets of planes form triangular prisms :

   (*i*) $x+y+z+3=0,\ 3x+y-2z+2=0,\ 2x+4y+7z-7=0.$

   (*ii*) $x-z-1=0,\ x+y-2z-3=0,\ x-2y+z-3=0.$

3. Examine the nature of the intersection of the following sets of planes ;

   (*i*) $4x-5y-2z-2=0,\ 5x-4y+2z+2=0,\ 2x+2y+8z-1=0.$

   (*ii*) $2x+3y-z-2=0,\ 3x+3y+z-4=0,\ x-y+2z-5=0.$

   (*iii*) $5x+3y+7z-4=0,\ 3x+26y+2z-9=0,\ 7x+2y+10z-5=0.$

   (*iv*) $2x+6y+11=0,\ 6x+20y-6z+3=0,\ 6y-18z+1=0.$

   [*Ans.* (*i*) prism, (*ii*) point, (*iii*) line, (*iv*) prism.

EXERCISES

4. Prove that the planes

$$x = cy + bz, \; y = az + cx, \; z = bx + ay,$$

pass through one line if

$$a^2 + b^2 + c^2 + 2abc = 1,$$

and show that the line of intersection, then, is

$$\frac{x}{\sqrt{(1-a^2)}} = \frac{y}{\sqrt{(1-b^2)}} = \frac{z}{\sqrt{(1-c^2)}}. \qquad \textit{(B.U.)}$$

5. Show that the planes

$$bx - ay = n, \; cy - bz = l, \; az - cx = m,$$

will intersect in a line if

$$al + bm + cn = 0,$$

and the direction ratios of the line, then, are, a, b, c.

6. Prove that the three planes

$$bz - cy = b - c, \; cx - az = c - a, \; ay - bx = a - b,$$

pass through one line (say l), and the three planes

$$(c-a)z - (a-b)y = -b + c,$$
$$(a-b)x - (b-c)z = -c + a,$$
$$(b-c)y - (c-a)x = -a + b,$$

pass through another line, say (l'). Show that the lines l and l' are at right angles to each other.

UNIT-5

THE SPHERE

6·11. Def. *A* **sphere** *is the locus of a point which remains at a constant distance from a fixed point.*

The constant distance is called the *radius* and the fixed point the *centre* of the sphere.

6·12. Equation of a sphere. Let (a, b, c) be the centre and r the radius of a given sphere.

Equating the radius r to the distance of any point (x, y, z) on the sphere from its centre (a, b, c), we have

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

or $$x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2-r^2)=0 \qquad \text{...(A)}$$

which is the required equation of the given sphere.

We note the following *characteristics* of the equation (A) of the sphere :

1. It is of the second degree in x, y, z ;
2. The co-efficients of x^2, y^2, z^2 are all equal ;
3. The product terms xy, yz, zx are absent.

Conversely, we shall now show that the general equation

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0,\ a\neq 0 \qquad \text{.. (B)}$$

having the above three characteristics represents a sphere.

The equation (B) can be re-written as

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2},$$

and this manner of re-writing shows that the distance between the variable point (x, y, z) and the fixed point

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$$

is

$$\frac{\sqrt{(u^2+v^2+w^2-ad)}}{a}$$

and is, therefore, constant.

The locus of the equation (B) is thus a sphere.

The radius and, therefore, the sphere is imaginary when

$$u^2+v^2+w^2-ad<0$$

and in this case we call it a *virtual sphere*.

6·13. General equation of a sphere.

The equation (B), when written in the form,

$$x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z+\frac{d}{a}=0,\ a\neq 0$$

or $$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0,$$
is taken as the *general equation of a sphere.*

Ex. 1. Find the centres and radii of the spheres :

(i) $x^2+y^2+z^2-6x+8y-10z+1=0,$

(ii) $x^2+y^2+z^2+2x-4y-6z+5=0,$

(iii) $2x^2+2y^2+2z^2-2x+4y+2z+3=0,$

[*Ans.* (i) $(3, -4, 5)$: 7. (ii) $(-1, 2, 3)$: 3. (iii) $(\frac{1}{2}, -1, -\frac{1}{2})$: 0.

2. Obtain the equation of the sphere described on the join of
$$A(2, -3, 4), B(-5, 6, -7)$$
as diameter. [*Ans.* $x^2+y^2+z^2+3(x-y+z)-56=0.$

3. *A point moves so that the sum of the squares of its distances from the six faces of a cube is constant ; show that its locus is a sphere.*

Take the centre of the cube as the origin and the planes through the centre parallel to its faces as co-ordinate planes.

Let each edge of the cube to be equal to $2a$.

Then the equations of the faces of the cube are
$$x=a \,;\, x=-a \,;\, y=a \,,\, y=-a \,;\, z=a,\, z=-a.$$

If (f, g, h) be any point of the locus, we have
$$(f-a)^2+(f+a)^2+(g-a)^2+(g+a)^2+(h-a)^2+(h+a)^2=k^2 \quad (k, \text{ a constant})$$
or
$$2(f^2+g^2+h^2+3a^2)=k^2$$
so that the locus is
$$2(x^2+y^2+z^2+3a^2)=k^2,$$
which is a sphere.

4. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere
$$x^2+y^2+z^2-ax-by-cz=0.$$

5. Through a point P three mutually perpendicular straight lines are drawn ; one passes through a fixed point C on the z-axis, while the others intersect the x-axis and y-axis, respectively ; show that the locus of P is a sphere of which C is the centre.

6·2. The sphere through four given points. General equation of a sphere contains *four* effective constants and, therefore, a sphere can be uniquely determined so as to satisfy four conditions, each of which is such that it gives rise to one relation between the constants.

In particular, we can find a sphere through four non-coplanar points
$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4).$$

Let
$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \qquad ...(i)$$
be the equation of the sphere through the four given points.

We have then the equation
$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0, \qquad ...(ii)$$
and three more similar equations corresponding to the remaining three points.

Eliminating u, v, w, d, from the equation (i) and from the four equations (ii) just obtained, we have

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1 \\ x_1^2+y_1^2+z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2, & x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0,$$

which is the equation of the sphere through the four given points.

Note. In numerical questions we may first find the values of u, v, w, d from the *four conditions* (ii) and then substitute them in the equation (i).

Exercises

1. Find the equation of the sphere through the four points

$$(4, -1, 2), (0, -2, 3), (1, -5, -1), (2, 0, 1).$$

[*Ans.* $x^2+y^2+z^2-4x+6y-2z+5=0$.

2. Find the equation of the sphere through the four points

$$(0, 0, 0), (-a, b, c), (a, -b, c), (a, b, -c)$$

and determine its radius. (*D.U. Hons. 1947*)

$\left[\textit{Ans.} \quad \dfrac{x^2+y^2+z^2}{a^2+b^2+c^2} - \dfrac{x}{a} - \dfrac{y}{b} - \dfrac{z}{c} = 0 ; \ \tfrac{1}{2}(a^2+b^2+c^2)\sqrt{(a^{-2}+b^{-2}+c^{-2})}. \right.$

3. Obtain the equation of the sphere circumscribing the tetrahedron whose faces are

$$x=0, y=0, z=0, x/a+y/b+z/c=1.$$

[*Ans.* $x^2+y^2+z^2-ax-by-cz=0$.

4. Obtain the equation of the sphere which passes through the points

$$(1, 0, 0), (0, 1, 0), (0, 0, 1),$$

and has its radius as small as possible.

[*Ans.* $3(x^2+y^2+z^2)-2(x+y+z)-1=0$.

5. Show that the equation of the sphere passing through the three points $(3, 0, 2)$, $(-1, 1, 1)$, $(2, -5, 4)$ and having its centre on the plane $2x+3y+4z=6$ is $x^2+y^2+z^2+4y-6z=1$.

6. Obtain the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the two points $(0, -2, -4)$, $(2, -1, -1)$.

[*Ans.* $x^2+y^2+z^2-6x-4y+10z+12=0$.

7. A sphere whose centre lies in the positive octant passes through the origin and cuts the planes $x=0, y=0, z=0$, in circles of radii $\sqrt{2}a, \sqrt{2}b, \sqrt{2}c$, respectively; show that its equation is

$$x^2+y^2+z^2-2\sqrt{(b^2+c^2-a^2)}\,x-2\sqrt{(c^2+a^2-b^2)}\,y-2\sqrt{(a^2+b^2-c^2)}\,z=0.$$

8. *A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is*

$$a/x+b/y+c/z=2. \qquad \textit{(D.U. Hons., 1958, 60)}$$

Let the sphere $OABC$ be

$$x^2+y^2+z^2+2ux+2vy+2wz=0, \qquad ...(i)$$

so that u, v, w are different for different spheres. The points A, B, C where it cuts the three axes are $(-2u, 0, 0)$, $(0, -2v, 0)$, $(0, 0, -2w)$. The equation of the plane ABC is

$$\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1.$$

Since it passes through (a, b, c) we have

$$\frac{a}{-2u}+\frac{b}{-2v}+\frac{c}{-2w}=1. \qquad \ldots(2)$$

If x, y, z be the centre of the sphere (1),

$$x=-u,\ y=-v,\ z=-w, \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$$

as the required locus.

9. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC. (P.U. 1940; B.U. 1955)

[*Ans.* $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

10. If O be the centre of a sphere of radius unity and A, B be two points in a line with O such that

$$OA.OB=1$$

and if P be any variable point on the sphere, show that

$$PA : PB=\text{constant}. \qquad (P.U.\ 1941)$$

11. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is the sphere

$$x^2+y^2+z^2=k^2.$$

6·31. Plane section of a sphere. *A plane section of a sphere, i.e., the locus of points common to a sphere and a plane, is a circle.*

Let O be the centre of the sphere and P, any point on the plane section. Let ON be perpendicular to the given plane; N being the foot of the perpendicular.

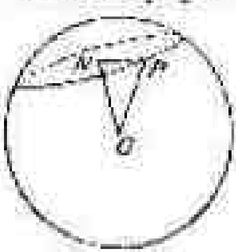


Fig. 23

As ON is perpendicular to the plane which contains the line NP, we have

$$ON \perp NP$$

Hence

$$NP^2=OP^2-ON^2.$$

Now, O and N being fixed points, this relation shows that NP is constant for all positions of P on the section.

Hence the locus of P is a circle whose centre is N, the foot of the perpendicular from the centre of the sphere to the plane.

The section of a sphere by a plane through its centre is known as a *great circle*.

The centre and radius of a great circle are the same as those of the sphere.

Cor. *The circle through three given points lies entirely on any sphere through the same three points.*

Thus the condition of a sphere containing a given circle is equivalent to that of its passing through any three of its points.

6·32. Intersection of two spheres. *The curve of intersection of two spheres is a circle.*

The co-ordinates of points common to any two spheres

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

satisfy both these equations and, therefore, they also satisfy the equation

$$S_1-S_2 \equiv 2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+(d_1-d_2)=0$$

which, being of the first degree, represents a plane.

Thus the points of intersection of the two spheres are the same as those of any one of them and this plane and, therefore, they lie on a circle. [See § 6·31].

6·33. Sphere with a given diameter. *To find the equation of the sphere described on the line joining the points*

$$A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$$

as diameter.

Let P be any point (x, y, z) on the sphere described on AB as diameter.

Since the section of the sphere by the plane through the three points P, A, B is a great circle having AB as diameter, P lies on a semi-circle and, therefore,

$$PA \perp PB.$$

The direction cosines of PA, PB are proportional to

$$x-x_1,\ y-y_1,\ z-z_1 \text{ and } x-x_2,\ y-y_2,\ z-z_2$$

respectively. Therefore they will be perpendicular, if

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

which is the required equation of the sphere.

Ex. Show that the condition for the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

to cut the sphere

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

in a great circle is

$$2uu_1+2vv_1+2ww_1-(d+d_1)=2r_1^2,$$

where r_1 is the radius of the latter sphere.

6·4. Equations of a circle. Any circle is the intersection of its plane with some sphere through it. Therefore a circle can be represented by *two* equations, one being of a sphere and the other of the plane.

Thus the two equations

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,\ lx+my+nz=p$$

taken together represent a circle.

A circle can also be represented by the equations of any two spheres through it.

Note.—The student *may* note that the equations

$$x^2+y^2+2gx+2fy+c=0,\ z=0$$

also represent a circle which is the intersection of the cylinder

$$x^2+y^2+2gx+2fy+c=0,$$

with the plane

$$z=0.$$

Examples

1. *Find the equations of the circle circumscribing the triangle formed by the three points*

$$(a, 0, 0),\ (0, b, 0),\ (0, 0, c).$$

Obtain also the co-ordinates of the centre of this circle.

The equation of the plane passing through these three points is

$$x/a+y/b+z/c=1.$$

The required circle is the curve of intersection of this plane with any sphere through the three points.

To find the equation of this sphere, a fourth point is necessary, which, for the sake of convenience, we take as origin.

If

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

be the sphere through these four points, we have

$$a^2+2ua+d=0\ ;\ b^2+2vb+d=0\ ;\ c^2+2wc+d=0\ ;$$
$$d=0.$$

These give

$$d=0,\ u=-\tfrac{1}{2}a,\ v=-\tfrac{1}{2}b,\ w=-\tfrac{1}{2}c.$$

Thus the equation of the sphere is

$$x^2+y^2+z^2-ax-by-cz=0.$$

Hence the equations of the circle are

$$x^2+y^2+z^2-ax-by-cz=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

To find the centre of this circle, we obtain the foot of the perpendicular from the centre $(\tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c)$ of the sphere to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

The equations of the perpendicular are

$$\frac{x-\frac{1}{2}a}{1/a}=\frac{y-\frac{1}{2}b}{1/b}=\frac{z-\frac{1}{2}c}{1/c}=r, \text{ say}$$

so that

$$\left(\frac{r}{a}+\frac{a}{2},\ \frac{r}{b}+\frac{b}{2},\ \frac{r}{c}+\frac{c}{2}\right),$$

is any point on the line. Its intersection with the plane is given by

$$r\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{2}=0 \text{ or } r=-\frac{1}{(2\Sigma a^{-2})}.$$

Thus the centre is

$$\left[\frac{a(b^{-2}+c^{-2})}{2\Sigma a^{-2}},\ \frac{b(c^{-2}+a^{-2})}{2\Sigma a^{-2}},\ \frac{c(a^{-2}+b^{-2})}{2\Sigma a^{-2}}\right].$$

2. *Show that the centres of all sections of the sphere*

$$x^2+y^2+z^2=r^2$$

by planes through a point (x', y', z') lie on the sphere

$$x(x-x')+y(y-y')+z(z-z')=0.$$

The plane which cuts the sphere in a circle with centre (f, g, h) is

$$f(x-f)+g(y-g)+h(z-h)=0.$$

It will pass through (x', y', z'), if

$$f(x'-f)+g(y'-g)+h(z'-h)=0,$$

and accordingly the locus of (f, g, h) is the sphere

$$x(x'-x)+y(y'-y)+z(z'-z)=0.$$

Exercises

1. Find the centre and the radius of the circle

$$x+2y+2z=15,\ x^2+y^2+z^2-2y-4z=11.$$

[*Ans.* $(1, 3, 4)$, $\sqrt{7}$.

2. Find the equation of that section of the sphere

$$x^2+y^2+z^2=a^2,$$

of which a given internal point (x_1, y_1, z_1) is the centre. (*P.U. 1939 Suppl.*)

(The plane through (x_1, y_1, z_1) drawn perpendicular to the line joining this point to the centre $(0, 0, 0)$ of the sphere determines the required section.)

3. Obtain the equations of the circle lying on the sphere

$$x^2+y^2+z^2-2x+4y-6z+3=0$$

and having its centre at $(2, 3, -4)$.

[*Ans.* $x^2+y^2+z^2-2x+4y-6z+3=0=x+5y-7z-45$.

4. O is the origin and A, B, C, are the points

$$(4a, 4b, 4c),\ (4b, 4c, 4a),\ (4c, 4a, 4b).$$

Show that the sphere

$$x^2+y^2+z^2-2(x+y+z)(a+b+c)+8(bc+ca+ab)=0$$

passes through the nine-point circles of the faces of the tetrahedron $OABC$.

5. Find the equation of the diameter of the sphere $x^2+y^2+z^2=29$ such that a rotation about it will transfer the point $(4, -3, 2)$ to the point $(5, 0, -2)$ along a great circle of the sphere. Find also the angle through which the sphere must be so rotated. (*L.U.*) [*Ans.* $\frac{1}{2}x=\frac{1}{6}y=\frac{1}{5}z$, $\cos^{-1}(16/29)$.

6. Show that the following points are concyclic :—

(*i*) $(5, 0, 2)$, $(2, -6, 0)$, $(7, -3, 8)$, $(4, -9, 6)$;

(*ii*) $(-8, 5, 2)$, $(-5, 2, 2)$, $(-7, 6, 6)$, $(-4, 3, 6)$.

6·41. Spheres through a given circle. The equation

$$S+\lambda U=0,$$

obviously represents a general sphere passing through the circle with equations

$$S=0,\ U=0,$$

where

$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d,$$
$$U\equiv lx+my+nz-p.$$

Also, the equation

$$S+\lambda S'=0$$

represents a general sphere through the circle with equations

$$S=0,\ S'=0,$$

where

$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d,$$
$$S'\equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'.$$

Here λ is an arbitrary constant which may be so chosen that these equations fulfil one more condition.

Note 1. We notice that the equation of the plane of the circle through the two given spheres is

$$S-S' \equiv 2(u-u')x+2(v-v')y+2(w-w')z+d-d'=0.$$

From this we see that the equation of any sphere through the circle

$$S=0,\ S'=0$$

can also be taken of the form

$$S+k(S-S')=0;$$

k, being any arbitrary constant.

This form sometimes proves comparatively more convenient.

Note 2. It is important to remember that the general equation of a sphere through the circle

$$x^2+y^2+2gx+2fy+c=0,\ z=0$$

is

$$x^2+y^2+z^2+2gx+2fy+kz+c=0,$$

where k is different for the different spheres.

Examples

1. *Find the equation of the sphere through the circle*

$$x^2+y^2+z^2=9,\quad 2x+3y+4z=5$$

and the point (1, 2, 3).

The sphere

$$x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0$$

passes through the given circle for all values of λ.

It will pass through (1, 2, 3) if

$$5+15\lambda=0 \text{ or } \lambda=-\tfrac{1}{3}.$$

The required equation of the sphere, therefore, is

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0.$$

2. *Show that the two circles*

$$x^2+y^2+z^2-y+2z=0,\ x-y+z-2=0\ ;$$

$$x^2+y^2+z^2+x-3y+z-5=0,\ 2x-y+4z-1=0\ ;$$

lie on the same sphere and find its equation. *(D.U. Hons. 1947)*

The equation of *any* sphere through the first circle is

$$x^2+y^2+z^2-y+2z+\lambda(x-y+z-2)=0, \qquad \ldots(i)$$

and that of *any* sphere through the second circle is

$$x^2+y^2+z^2+x-3y+z-5+\mu(2x-y+4z-1)=0. \qquad \ldots(ii)$$

The equations (*i*) and (*ii*) will represent the same sphere, if λ, μ can be chosen so as to satisfy the four equations

$$\lambda=2\mu+1,$$

$$-1-\lambda=-\mu-3,$$

$$2+\lambda=4\mu+1,$$

$$-2\lambda=-\mu-5.$$

The first two of these equations give $\lambda=3$, $\mu=1$, and these values clearly satisfy the remaining two equations also. These four equations in λ, μ being consistent, the two circles lie on the same sphere, *viz.*,

$$x^2+y^2+z^2-y+2z+3(x-y+z-2)=0,$$

i.e.,

$$x^2+y^2+z^2+3x-4y+5z-6=0.$$

Exercises

1. Find the equation of the sphere through the circle

$$x^2+y^2+z^2+2x+3y+6=0,\ x-2y+4z-9=0$$

and the centre of the sphere

$$x^2+y^2+z^2-2x+4y-6z+5=0.$$

[*Ans.* $x^2+y^2+z^2+7y-8z+24=0$.

2. Show that the equation of the sphere having its centre on the plane

$$4x-5y-z=3$$

and passing through the circle with equations

$$x^2+y^2+z^2-2x-3y+4z+8=0,\ x^2+y^2+z^2+4x+5y-6z+2=0\ ;$$

is

$$x^2+y^2+z^2+7x+9y-11z-1=0.$$

3. Obtain the equation of the sphere having the circle

$$x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$$

as the great circle.

[The centre of the required sphere lies on the plane $x+y+z=3$.]

[*Ans.* $x^2+y^2+z^2-4x+6y-8z+4=0$.

4. A sphere S has points $(0, 1, 0)$, $(3, -5, 2)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of S with the plane

$$5x-2y+4z+7=0$$

as a great circle.

[*Ans.* $x^2+y^2+z^2+2x+2y+2z+2=0$.

5. Obtain the equation of the sphere which passes through the circle $x^2+y^2=4$, $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

[*Ans.* $x^2+y^2+z^2\pm 6z-4=0$.

6. Show that the two circles

$$2(x^2+y^2+z^2)+8x-13y+17z-17=0,\ 2x+y-3z+1=0\ ;$$
$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0\ ;$$

lie on the same sphere and find its equation.

[*Ans.* $x^2+y^2+z^2+5x-6y+7z-8=0$.

7. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0\ ;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0\ ;$$

lie on the sphere and find its equation. (*D.U. Hons.*, 1945)

[*Ans.* $x^2+y^2+z^2-2x-2y-2z-6=0$.

6·5. Intersection of a sphere and a line.

Let

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{...(1)}$$

and

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \text{...(2)}$$

be the equations of a sphere and a line respectively.

The point $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$ which lies on the given line (2) for all values of r, will also lie on the given sphere (1), if r satisfies the equation

$$r^2(l^2+m^2+n^2)+2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0, \qquad \ldots(A)$$

and this latter being a quadratic equation in r, gives two values say, r_1, r_2 of r. Then

$$(lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma),\ (lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma)$$

are the two points of intersection.

Thus every straight line meets a sphere in two points which may be real, imaginary or coincident.

Ex. Find the co-ordinates of the points where the line

$$\tfrac{1}{4}(x+3)=\tfrac{1}{3}(y+4)=-\tfrac{1}{5}(z-8)$$

intersects the sphere

$$x^2+y^2+z^2+2x-10y=23.$$

[*Ans.* $(1,-1,3)$; $(5,2,-2)$.

6·51. Power of a point. If l, m, n, are the actual direction cosines of the given line (2) in § 6·5, so that $l^2+m^2+n^2=1$, then r_1, r_2, are the distances of the point A (α, β, γ) from the points of intersection P and Q.

$$\therefore \quad AP\times AQ=r_1r_2=\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d$$

which is independent of the direction cosines, l, m, n.

Thus *if from a fixed point A, chords be drawn in any direction to intersect a given sphere in P and Q, then AP.AQ is constant.* This constant is called the *power* of the point A with respect to the sphere.

Example

Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular straight lines through a fixed point is constant.

Take the fixed point O as the origin and any three mutually perpendicular lines through it as the co-ordinate axes. With this choice of axes, let the equation of the given sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

The x-axis, $(y=0=z)$ meets the sphere in points given by

$$x^2+2ux+d=0,$$

so that if x_1, x_2 be its roots, the two points of intersection are

$$(x_1, 0, 0),\ (x_2, 0, 0).$$

Also we have

$$x_1+x_2=-2u,\ x_1x_2=d.$$

$$\therefore \quad (\text{intercept on } x\text{-axis})^2=(x_1-x_2)^2$$
$$=(x_1+x_2)^2-4x_1x_2=4(u^2-d).$$

Similarly

$$(\text{intercept on } y\text{-axis})^2=4(v^2-d),$$
$$(\text{intercept on } z\text{-axis})^2=4(w^2-d).$$

The sum of the squares of the intercepts

$$=4(u^2+v^2+w^2-3d)$$
$$=4(u^2+v^2+w^2-d)-8d$$
$$=4r^2-8p,$$

where r is the radius of the given sphere and p is the power of the given point with respect to the sphere.

Since the sphere and the point are both given, r and p are both constants.

Hence the result.

Note. The co-efficients u, v, w and d in the equation of the sphere will be different for different sets of mutually perpendicular lines through O as axes.

Since, however, the sphere is fixed and the point O is also fixed, the expressions

$$r^2=u^2+v^2+w^2-d$$

for the square of the radius and

$$p=d,$$

for the power of the point, w.r.t. the sphere will be invariants.

Exercises

1. Find the locus of a point whose powers with respect to two given spheres are in a constant ratio.

2. Show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chords.

6·6. Equation of tangent plane. *To find the equation of the tangent plane at any point (α, β, γ) of the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

As (α, β, γ) lies on the sphere, we have

$$\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0. \qquad ...(i)$$

The points of intersection of any line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad ...(ii)$$

through (α, β, γ) with the given sphere are

$$(lr+\alpha,\ mr+\beta,\ nr+\gamma)$$

where the values of r are the roots of the equation

$$r^2(l^2+m^2+n^2)+2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0.$$

By virtue of the condition (*i*), one root of this quadratic equation is zero so that one of the points of intersection coincides with (α, β, γ).

In order that the second point of intersection may also coincide with (α, β, γ), the second value of r must also vanish and this requires

$$l(\alpha+u)+m(\beta+v)+n(\gamma+w)=0. \qquad ...(iii)$$

Thus the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

meets the sphere in two coincident points at (α, β, γ) and so is a *tangent line* to it thereat for any set of values of l, m, n which satisfy the condition (*iii*).

The locus of the tangent lines at (α, β, γ) is, thus, obtained by eliminating l, m, n between (*iii*) and the equations (*ii*) of the line and this gives

$$(x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)=0$$

or

$$\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d$$
$$=\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0, \text{ from } (i)$$

which is a plane known as the *tangent plane* at (α, β, γ).

Hence

$$(\alpha+u)x+(\beta+v)y+(\gamma+w)z+(u\alpha+v\beta+w\gamma+d)=0$$

is the equation of the tangent plane to the given sphere at (α, β, γ).

Cor. 1. *The line joining the centre of a sphere to any point on it is perpendicular to the tangent plane thereat*, for the direction cosines of the line joining

the centre $(-u, -v, -w)$ to the point (α, β, γ)

on the sphere are proportional to

$$(\alpha+u, \beta+v, \gamma+w)$$

which are also the co-efficients of x, y, z in the equation of the tangent plane at (α, β, γ).

Cor. 2. If a plane or a line touch a sphere, then the length of the perpendicular from its centre to the plane or the line is equal to its radius.

Note. Any line in the tangent plane through its point of contact touches the section of the sphere by any plane through that line.

Examples

1. *Show that the plane $lx+my+nz=p$ will touch the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

if

$$(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d).$$

Equating the radius $\sqrt{(u^2+v^2+w^2-d)}$ of the sphere to the length of the perpendicular from the centre $(-u, -v, -w)$ to the plane

$$lx+my+nz=p,$$

we get the required condition.

2. *Find the two tangent planes to the sphere*

$$x^2+y^2+z^2-4x+2y-6z+5=0$$

which are parallel to the plane

$$2x+2y=z.$$

The general equation of a plane parallel to the given plane

$$2x+2y-z=0,$$

is

$$2x+2y-z+\lambda=0.$$

This will be a tangent plane, if its distance from the centre (2, −1, 3) of the sphere is equal to the radius 3 and this requires

$$\frac{-1+\lambda}{\pm 3}=3.$$

Thus

$$\lambda=10 \text{ or } -8.$$

Hence the required tangent planes are

$$2x+2y-z+10=0, \text{ and } 2x+2y-z-8=0.$$

3. *Find the equation of the sphere which touches the sphere*

$$x^2+y^2+z^2-x+3y+2z-3=0,$$

at (1, 1, −1) *and passes through the origin.*

The tangent plane to the given sphere at (1, 1, −1) is

$$x+5y-6=0.$$

The equation of the required sphere is, therefore, of the form

$$x^2+y^2+z^2-x+3y+2z-3+k(x+5y-6)=0.$$

This will pass through the origin if

$$k=-\tfrac{1}{2}.$$

Thus the required equation is

$$2(x^2+y^2+z^2)-3x+y+4z=0.$$

4. *Find the equations of the sphere through the circle*

$$x^2+y^2+z^2=1,\ 2x+4y+5z=6$$

and touching the plane

$$z=0.$$

The sphere

$$x^2+y^2+z^2-1+\lambda(2x+4y+5z-6)=0$$

passes through the given circle for all values of λ.

Its centre is $(-\lambda, -2\lambda, -\frac{5}{2}\lambda)$, and radius is

$$\sqrt{(\lambda^2+4\lambda^2+\tfrac{25}{4}\lambda^2+1+6\lambda)}.$$

Since it touches $z=0$, we have by Cor. 2,

$$-\tfrac{5}{2}\lambda=\pm\sqrt{(5\lambda^2+\tfrac{25}{4}\lambda^2+1+6\lambda)},$$

or

$$5\lambda^2+6\lambda+1=0.$$

This gives

$$\lambda=-1 \text{ or } -\tfrac{1}{5}.$$

The two corresponding spheres, therefore, are

$$x^2+y^2+z^2-2x-4y-5z+5=0,$$
$$5(x^2+y^2+z^2)-2x-4y-5z+1=0.$$

5. *Find the equations of the two tangent planes to the sphere*

$$x^2+y^2+z^2=9,$$

which passes through the line

$$x+y=6,\ x-2z=3.$$

Any plane

$$x+y-6+\lambda(x-2z-3)=0$$

through the given line will touch the given sphere if

$$\pm\frac{-6-3\lambda}{\sqrt{[(1+\lambda)^2+1+4\lambda^2]}}=3,$$

or

$$2\lambda^2-\lambda-1=0,$$

This gives

$$\lambda=1, -\tfrac{1}{2}.$$

The two corresponding planes, therefore, are

$$2x+y-2z=9, \quad x+2y+2z=9.$$

Exercises

1. Find the equation of the tangent plane to the sphere

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

at the point (1, 2, 3).

[*Ans.* $4x+9y+14z-64=0$.

2. Find the equations of the tangent line to the circle

$$x^2+y^2+z^2+5x-7y+2z-8=0, \quad 3x-2y+4z+3=0$$

at the point (−3, 5, 4).

[*Ans.* $(x+3)/22=(y-5)/34=-(z-4)/7$.

3. Find the value of a for which the plane

$$x+y+z=a\sqrt{3}$$

touches the sphere

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

[*Ans.* $\sqrt{3}\pm 3$.

4. Show that the plane $2x-2y+z+12=0$ touches the sphere

$$x^2+y^2+z^2-2x-4y+2z=3$$

and find the point of contact.

[*Ans.* (−1, 4, −2).

[The point of contact of a tangent plane is the point where the line through the centre perpendicular to the plane meets the sphere.]

5. Find the co-ordinates of the points on the sphere

$$x^2+y^2+z^2-4x+2y=4$$

the tangent planes at which are parallel to the plane

$$2x-y+2z=1.$$

[*Ans.* (4, −2, 2), (0, 0, −2).

6. Show that the equation of the sphere which touches the sphere

$$4(x^2+y^2+z^2)+10x-25y-2z=0,$$

at (1, 2, −2) and passes through the point (−1, 0, 0) is

$$x^2+y^2+z^2+2x-6y+1=0.$$

7. Obtain the equations of the tangent planes to the sphere

$$x^2+y^2+z^2+6x-2z+1=0$$

which pass through the line

$$3(16-x)=3z=2y+30.$$

[*Ans.* $2x+2y-z-2=0$, $x+2y-2z+14=0$.

8. Obtain the equations of the sphere which pass through the circle

$$x^2+y^2+z^2-2x+2y+4z-3=0, \quad 2x+y+z=4$$

and touch the plane $3x+4y=14$.

[*Ans.* $x^2+y^2+z^2+2x+4y+6z-11=0$, $x^2+y^2+z^2-2x+2y-4z-3=0$.

9. Find the equation of the sphere which has its centre at the origin and which touches the line

$$2(x+1)=2-y=z+3.$$

[*Ans.* $9(x^2+y^2+z^2)=5$.

10. Find the equation of a sphere of radius r which touches the three co-ordinate axes. How many spheres can be so drawn.

[*Ans.* $2(x^2+y^2+z^2)+2\sqrt{2}(\pm x\pm y\pm z)r+r^2=0$; eight.

11. Prove that the equation of the sphere which lies in the octant $OXYZ$ and touches the co-ordinate planes is of the form

$$x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0.$$

Show that, in general, two spheres can be drawn through a given point to touch the co-ordinate planes and find for what positions of the point the spheres are

(*a*) real, (*b*) coincident. (*P.U.* *1944*)

[The distances of the centre from the co-ordinate planes are all equal to the radius so that we may suppose that λ is the radius and $(\lambda, \lambda, \lambda)$ is the centre; λ being the parameter.]

12. Show that the spheres

$$x^2+y^2+z^2=25$$
$$x^2+y^2+z^2-24x-40y-18z+225=0$$

touch externally and find the point of the contact.

[*Ans.* $(12/5, 20/5, 9/5)$.

13. Find the centres of the two spheres which touch the plane

$$4x+3y=47$$

at the point $(8, 5, 4)$ and which touch the sphere

$$x^2+y^2+z^2=1.$$

[*Ans.* $(4, 2, 4)$, $(64/21, 27/21, 4)$.

14. Obtain the equations of spheres that pass through the points $(4, 1, 0)$, $(2, -3, 4)$, $(1, 0, 0)$ and touch the plane $2x+2y-z=11$. (*P.U.* *1934*)

[*Ans.* $x^2+y^2+z^2-6x+2y-4z+5=0$; $16(x^2+y^2+z^2)-102x+50y-49z+86=0$.

15. Find the equation of the sphere inscribed in the tetrahedron whose faces are

(*i*) $x=0$, $y=0$, $z=0$, $x+2y+2z=1$.

(*ii*) $x=0$, $y=0$, $z=0$, $2x-6y+3z+6=0$.

[*Ans.* (*i*) $32(x^2+y^2+z^2)-8(x+y+z)+1=0$. (*ii*) $9(x^2+y^2+z^2)+6(x-y+z)+2=0$.

16. Tangent plane at any point of the sphere $x^2+y^2+z^2=r^2$ meets the co-ordinate axes at A, B, C. Show that the locus of the point of intersection of planes drawn parallel to the co-ordinate planes through A, B, C is the surface $x^{-2}+y^{-2}+z^{-2}=r^{-2}$.

6·61. Plane of Contact. *To find the locus of the points of contact of the tangent planes which pass through a given point* (α, β, γ).

Let (x', y', z') be any point on the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

The tangent plane

$$x(x'+u)+y(y'+v)+z(z'+w)+(ux'+vy'+wz'+d)=0,$$

at this point will pass through (α, β, γ), if

$$\alpha(x'+u)+\beta(y'+v)+\gamma(z'+w)+(ux'+vy'+wz'+d)=0,$$

or

$$x'(\alpha+u)+y'(\beta+v)+z'(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0,$$

which is the condition that the point (x', y', z') should lie on the plane

$$x(\alpha+u)+y(\beta+v)+z(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0.$$

It is called the *plane of contact* for the point (α, β, γ).

Thus the locus of points of contact is the circle in which the plane cuts the sphere.

Ex. 1. *Show that the line joining any point P to the centre of a given sphere is perpendicular to the plane of contact of P and if OP meets it in Q, then*

$$OP \cdot OQ=(\text{radius})^2.$$

Ex. 2. *Show that the planes of contact of all points on the line*

$$x/2=(y-a)/3=(z+3a)/4$$

with respect to the sphere $x^2+y^2+z^2=a^2$ pass through the line

$$-(2x+3a)/13=(y-a)/3=z/1.$$

6·62. The polar plane. *If a variable line is drawn through a fixed point A meeting a given sphere in P, Q and point R is taken on this line such that the points A, R divide this line internally and externally in the same ratio, then the locus of R is a plane called polar plane of A w.r. to the sphere.*

It may be easily seen that if the points A, R divide PQ internally and externally in the same ratio, then the points P, Q also divide AR internally and externally in the same ratio.

Consider the sphere

$$x^2+y^2+z^2=a^2, \qquad \ldots(1)$$

and let A be the point (α, β, γ).

Let (x, y, z) be the co-ordinates of the point R on any line through A. The co-ordinates of the point dividing AR in the ratio $\lambda : 1$ are

$$\left[\left(\frac{\lambda x+\alpha}{\lambda+1}\right), \left(\frac{\lambda y+\beta}{\lambda+1}\right), \left(\frac{\lambda z+\gamma}{\lambda+1}\right)\right].$$

This point will be on the sphere (1) for values of λ which are roots of the quadratic equation

$$\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2+\left(\frac{\lambda y+\beta}{\lambda+1}\right)^2+\left(\frac{\lambda z+\gamma}{\lambda+1}\right)^2=a^2,$$

i.e.,

$$\lambda^2(x^2+y^2+z^2-a^2)+2\lambda(\alpha x+\beta y+\gamma z-a^2)+(\alpha^2+\beta^2+\gamma^2-a^2)=0 \quad \ldots(2)$$

Its roots λ_1 and λ_2 are the ratios in which the points P, Q divide AR.

Since P, Q divide AR internally and externally in the same ratio, we have

$$\lambda_1+\lambda_2=0.$$

Thus from (2), we have

$$\alpha x+\beta y+\gamma z-a^2=0, \qquad \ldots(3)$$

which is the relation satisfied by the co-ordinates (x, y, z) of R.

Hence (3) is the locus of R. Clearly it is a plane.

Thus we have seen here that the equation of the polar plane of the point (α, β, γ) with respect to the sphere

$$x^2+y^2+z^2=a^2,$$

is

$$\alpha x+\beta y+\gamma z=a^2.$$

It may similarly be shown that the polar plane of (α, β, γ) with respect to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

is

$$(\alpha+u)x+(\beta+v)y+(\gamma+w)z+(u\alpha+v\beta+w\gamma+d)=0.$$

On comparing the equation of the polar plane with that of the tangent plane (§ 6·6) and the plane of contact (§ 6·61), we see that the polar plane of a point lying on the sphere is the tangent plane at the point and that of a point, lying outside it, is its plane of contact.

If π be the polar plane of a point P, then P is called the pole of the plane π.

6·63. Pole of a plane. *To find the pole of the plane*

$$lx+my+nz=p \qquad \text{...(i)}$$

with respect to the sphere

$$x^2+y^2+z^2=a^2.$$

If (α, β, γ) be the required pole, then we see that the equation (*i*) is identical with

$$\alpha x+\beta y+\gamma z=a^2 \qquad \text{...(ii)}$$

so that, on comparing (*i*) and (*ii*), we obtain

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}=\frac{a^2}{p},$$

or

$$\alpha=a^2l/p,\ \beta=a^2m/p,\ \gamma=a^2n/p.$$

Thus

$$(a^2l/p,\ a^2m/p,\ a^2n/p)$$

is the required pole.

6·64. Some results concerning poles and polars. In the following discussion, we always take the equation of a sphere in the form

$$x^2+y^2+z^2=a^2.$$

1. *The line joining the centre O of a sphere to any point P is perpendicular to the polar plane of P.*

The direction ratios of the line joining the centre O (0, 0, 0) to the point $P(\alpha, \beta, \gamma)$ are α, β, γ and these are also the direction ratios of the normal to the polar plane $\alpha x+\beta y+\gamma z=a^2$ of $P(\alpha, \beta, \gamma)$.

2. *If the line joining the centre O of a sphere to any point P meets the polar plane of P in Q, then*

$$OP.OQ=a^2,$$

where a is the radius of the sphere.

We have,

$$OP=\sqrt{(\alpha^2+\beta^2+\gamma^2)},$$

Also, OQ, which is the length of the perpendicular from the centre $O(0, 0, 0)$ to the polar plane $\alpha x+\beta y+\gamma z=a^2$ of P, is given by

$$OQ=\frac{a^2}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}}.$$

Hence the result.

3. *If the polar plane of a point P passes through another point Q, then the polar plane of Q passes through P.*

The condition that the polar plane

$$\alpha_1 x+\beta_1 y+\gamma_1 z=a^2,$$

of $P(\alpha_1, \beta_1, \gamma_1)$ passes through $Q(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=a^2,$$

which is also, by symmetry, or directly, the condition that the polar plane of Q passes through P.

Conjugate points. *Two points such that the polar plane of either passes through the other are called conjugate points.*

4. *If the pole of a plane π_1 lies on another plane π_2, then the pole of π_2 also lies on π_1.*

The condition that the pole

$$\left(\frac{a^2l_1}{p_1}, \frac{a^2m_1}{p_1}, \frac{a^2n_1}{p_1}\right)$$

of the plane π_1

$$l_1x+m_1y+n_1z=p_1$$

lies on the plane π_2

$$l_2x+m_2y+n_2z=p_2$$

is

$$a^2(l_1l_2+m_1m_2+n_1n_2)=p_1p_2$$

which is also, clearly, the condition that the pole

$$(a^2l_2/p_2, a^2m_2/p_2, a^2n_2/p_2)$$

of π_2 lies on π_1.

Conjugate planes. *Two planes such that the pole of either lies on the other are called conjugate planes.*

5. *The polar planes of all the points on a line l pass through another line l'.*

The polar plane of any point,

$$(lr+\alpha, mr+\beta, nr+\gamma),$$

on the line, l,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is

$$(lr+\alpha)x+(mr+\beta)y+(nr+\gamma)z=a^2,$$

or

$$(\alpha x+\beta y+\gamma z-a^2)+r(lx+my+nz)=0,$$

which clearly passes through the line

$$\alpha x+\beta y+\gamma z-a^2=0,\ lx+my+nz=0,$$

whatever value, r, may have. Hence the result.

Let this line be l'. We shall now prove that the polar plane of every point on l' passes through l.

Now, as the polar plane of any arbitrary point P on l passes through *every* point of l', therefore, the polar plane of every point of l' passes through the point of P on l and as, P is arbitrary, it passes through every point of l *i.e.*, it passes through l.

Thus we see that if l' is the line such that the polar planes, of all the points on a line l, pass through it, then the polar planes of all the points on l' pass through l.

Polar Lines. *Two lines such that the polar plane of every point on either passes through the other are called* **Polar Lines.**

Exercises

1. Show that polar line of

$$(x+1)/2=(y-2)/3=(z+3),$$

with respect to the sphere

$$x^2+y^2+z^2=1,$$

is the line

$$\frac{7x+3}{11}=\frac{2-7y}{5}=\frac{z}{-4}.$$

2. Show that if a line l is coplanar with the polar line of a line l', then, l' is coplanar with the polar line of l.

3. If PA, QB be drawn perpendicular to the polars of Q and P respectively, with respect to a sphere, centre O, then

$$\frac{PA}{QB}=\frac{OP}{OQ}.$$

4. Show that, for a given sphere, there exist an unlimited number of tetrahedra such that each vertex is the pole of the opposite face with respect to the sphere.

(Such a tetrahedron is known as a *self-conjugate* or *self-polar* tetrahedron.)

6·7. Angle of Intersection of two spheres.

Def. *The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point* and is, therefore, also equal to the angle between the radii of the spheres to the common point; the radii being perpendicular to the respective tangent planes at the point.

The angle of intersection at *every* common point of the spheres is the same, for if P, P' be any two common points and C, C' the centres of the spheres, the triangles $CC'P$ and $CC'P'$ are congruent and accordingly

$$\angle CPC'=\angle CP'C'.$$

The spheres are said to be **orthogonal** if the angle of intersection of two spheres is a right angle. In this case

$$CC'^2=CP^2+C'P^2.$$

6·71. Condition for the orthogonality of two spheres.

To find the condition for the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

to be orthogonal.

The spheres will be orthogonal if *the square of the distance between their centres is equal to the sum of the squares of their radii* and this requires

$$(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2$$
$$=(u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2)$$

or

$$\mathbf{2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2.}$$

Exercises

1. Find the equation of the sphere that passes through the circle

$$x^2+y^2+z^2-2x+3y-4z+6=0,\ 3x-4y+5z-15=0$$

and cuts the sphere

$$x^2+y^2+z^2+2x+4y-6z+11=0$$

orthogonally.

(*Ans.* $5(x^2+y^2+z^2)-13x+19y-25z+45=0$.

2. Find the equation of the sphere that passes through the two points

$$(0, 3, 0),\ (-2, -1, -4)$$

and cuts orthogonally the two spheres

$$x^2+y^2+z^2+x-3z-2=0,\ 2(x^2+y^2+z^2)+x+3y+4=0.$$

(*Ans.* $x^2+y^2+z^2+2x-2y+4z-3=0$.

3. Find the equation of the sphere which touches the plane

$$3x+2y-z+2=0$$

at the point $(1, -2, 1)$ and cuts orthogonally the sphere

$$x^2+y^2+z^2-4x+6y+4=0. \qquad (L.U.)$$

(*Ans.* $x^2+y^2+z^2+7x+10y-5z+12=0$.

4. Show that every sphere through the circle

$$x^2+y^2-2ax+r^2=0,\ z=0,$$

cuts orthogonally every sphere through the circle

$$x^2+z^2=r^2,\ y=0.$$

5. Two points P, Q are conjugate with respect to a sphere S; show that the sphere on PQ as diameter cuts S orthogonally.

6. If two spheres S_1 and S_2 are orthogonal, the polar plane of any point on S_1 with respect to S_2 passes through the other end of the diameter of S_1 through P.

Example

Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is

$$r_1r_2/\sqrt{(r_1^2+r_2^2)}.$$

Let the common circle be

$$x^2+y^2=a^2,\ z=0.$$

The general equation of the sphere through this circle being

$$x^2+y^2+z^2+2kz-a^2=0,$$

let the two given spheres through the circle be

$$x^2+y^2+z^2+2k_1z-a^2=0,\ x^2+y^2+z^2+2k_2z-a^2=0.$$

We have

$$r_1^2=k_1^2+a^2,\ r_2^2=k_2^2+a^2. \qquad \ldots(i)$$

Since the spheres cut orthogonally, we have

$$2k_1k_2=-a^2-a^2=-2a^2. \qquad \ldots(ii)$$

From (i) and (ii), eliminating k_1, k_2, we have

$$(r_1^2-a^2)(r_2^2-a^2)=a^4,$$

or

$$a^2=r_1^2r_2^2/(r_1^2+r_2^2).$$

Hence the result.

6·8. Radical plane. *To show that the locus of points whose powers with respect to two spheres are equal is a plane perpendicular to the line joining their centres.*

The powers of the point $P(x, y, z)$ with respect to the spheres

$$S_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2=x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

are

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2$$

respectively.

Equating these, we obtain

$$2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+(d_1-d_2)=0,$$

which is the required locus, and being of the first degree in (x, y, z), it represents a plane which is obviously perpendicular to the line joining the centres of the two spheres and is called the *radical plane* of the two spheres.

Thus *the radical plane of the two spheres*

$$S_1=0,\ S_2=0,$$

in both of which the co-efficients of the second degree terms are equal to unity, is

$$S_1-S_2=0.$$

In case the two spheres intersect, the plane of their common circle is their radical plane. (§ 6·32).

6·81. Radical line. *The three radical planes of three spheres taken two by two intersect in a line.*

If

$$S_1=0,\ S_2=0,\ S_3=0$$

be the three spheres, their radical planes

$$S_1-S_2=0,\ S_2-S_3=0,\ S_3-S_1=0,$$

clearly meet in the line

$$S_1=S_2=S_3.$$

This line is called the *radical line* of the three spheres.

6·82. Radical Centre. *The four radical lines of four spheres taken three by three intersect at a point.*

The point common to the three planes

$$S_1=S_2=S_3=S_4$$

is clearly common to the radical lines, taken three by three, of the four spheres

$$S_1=0,\ S_2=0,\ S_3=0,\ S_4=0.$$

This point is called the *radical centre* of the four spheres.

6·83. Theorem. *If $S_1=0$, $S_2=0$, be two spheres, then the equation*

$$S_1+\lambda S_2=0,$$

λ being the parameter, represents a system of spheres such that any two members of the system have the same radical plane.

Let

$$S_1+\lambda_1 S_2=0 \text{ and } S_1+\lambda_2 S_2=0,$$

be any two members of the system.

Making the co-efficients of second degree terms unity, we write them in the form

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}=0,\ \frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0.$$

The radical plane of these two spheres is

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0,$$

or

$$S_1-S_2=0.$$

Since this equation is independent of λ_1 and λ_2, we see that every two members of the system have the same radical plane.

Co-axal System. Def. *A system of spheres such that any two members thereof have the same radical plane is called a co-axal system of spheres.*

Thus the system of spheres

$$S_1+\lambda S_2=0$$

is co-axal and we say that it is determined by the two spheres

$$S_1=0,\ S_2=0.$$

The common radical plane is

$$S_1-S_2=0$$

This co-axal system is also given by the equation

$$S_1+k_2(S_1-S_2)=0.$$

Refer Note 1, § 6·41, *P.* 92.

Note. It can similarly be proved that the system of spheres

$$S+\lambda U=0$$

is co-axal ; $S=0$ being a sphere and $U=0$ a plane ; the common radical plane is $U=0$.

Cor. *The locus of the centres of spheres of a co-axal system is a line.*

For, if (x, y, z) be the centre of the sphere

$$S_1+\lambda S_2=0,$$

we have

$$x=-\frac{u_1+\lambda u_2}{1+\lambda},\quad y=-\frac{v_1+\lambda v_2}{1+\lambda},\quad z=-\frac{w_1+\lambda w_2}{1+\lambda}.$$

On eliminating λ, we find that it lies on the line

$$\frac{x+u_1}{u_1-u_2}=\frac{y+v_1}{v_1-v_2}=\frac{z+w_1}{w_1-w_2}.$$

This result is also otherwise clear as the line joining the centres of any two spheres is perpendicular to their common radical plane.

6·9. A simplified form of the equation of the two spheres.

By taking the line joining the centres of two given spheres as X-axis, their equations take the form

$$x^2+y^2+z^2+2u_1x+d_1=0,\quad x^2+y^2+z^2+2u_2x+d_2=0.$$

Their radical plane is

$$2x(u_1-u_2)+(d_1-d_2)=0.$$

Further, if we take the radical plane as the YZ plane, *i.e.*, $x=0$, we have $d_1=d_2=d$, (say).

Thus by taking the line joining the centres as X-axis and the radical plane as the YZ plane, the equations of *any* two spheres can be put in the simplified form

$$x^2+y^2+z^2+2u_1x+d=0,\quad x^2+y^2+z^2+2u_2x+d=0,$$

where u_1, u_2 are different.

Cor. 1. The equation

$$x^2+y^2+z^2+2kx+d=0$$

represents a co-axal system of spheres for different values of k; d being constant. The YZ plane is the common radical plane and X-axis is the line of centres.

Cor. 2. Limiting points. The equation

$$x^2+y^2+z^2+2kx+d=0$$

can be written as

$$(x+k)^2+y^2+z^2=k^2-d.$$

For $k=\pm\sqrt{d}$, we get spheres of the system with radius zero and thus the *system includes the two point spheres*

$$(-\sqrt{d},0,0),\ (\sqrt{d},0,0).$$

These two points are called the *limiting points* and are real only when d is positive, *i.e.*, when the spheres do not meet the radical plane in a real circle.

Def. Limiting points *of a co-axal system of spheres are the point spheres of the system.*

Examples

1. *Find the limiting points of the co-axal system defined by the spheres*

$$x^2+y^2+z^2+3x-3y+6=0,\quad x^2+y^2+z^2-6y-6z+6=0.$$

The equation of the plane of the circle through the two given spheres is

$$3x+3y+6z=0, \text{ i.e., } x+y+2z=0.$$

Then the equation of the co-axal system determined by given spheres is

$$x^2+y^2+z^2+3x-3y+6+\lambda(x+y+2z)=0,$$

i.e., $$x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0. \qquad ...(1)$$

The centre of (1) is

$$\left[-\frac{3+\lambda}{2}, -\frac{\lambda-3}{2}, -\lambda\right],$$

and radius is

$$\sqrt{\left[\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6\right]}.$$

Equating this radius to zero, we obtain

$$6\lambda^2-6=0,$$

i.e., $$\lambda=\pm 1.$$

The spheres corresponding to these values of λ become point spheres coinciding with their centres and are the limiting points of the given system of spheres.

The limiting points, therefore, are

$$(-1, 2, 1) \text{ and } (-2, 1, -1).$$

2. *Show that spheres which cut two given spheres along great circles all pass through two fixed points.* (*P.U. 1944 Suppl.*)

With proper choice of axes, the equations of the given spheres take the form

$$x^2+y^2+z^2+2u_1x+d=0, \qquad ...(i)$$

$$x^2+y^2+z^2+2u_2x+d=0. \qquad ...(ii)$$

The equation of any other sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \qquad ...(iii)$$

where u, v, w, c are different for different spheres.

The plane

$$2x(u-u_1)+2vy+2wz+(c-d)=0,$$

of the circle common to (*i*) and (*iii*) will pass through the centre

$$(-u_1, 0, 0)$$

of (*i*), if

$$-2u_1(u-u_1)+(c-d)=0, \qquad ...(iv)$$

which is thus the condition for the sphere (iii) to cut the sphere (*i*) along a great circle.

Similarly

$$-2u_2(u-u_2)+(c-d)=0, \qquad ...(v)$$

is the condition for the sphere (*iii*) to cut the sphere (*ii*) along a great circle.

Solving (*iv*) and (*v*) for u and c, we get

$$u=u_1+u_2 \,;\; c=2u_1u_2+d,$$

so that u, c are constants, being dependent on u_1, u_2, d only.

The sphere (iii) cuts X-axis at points whose x-co-ordinates are the roots of the equation

$$x^2+2ux+c=0.$$

The roots of this equation are constant, depending as they do upon the constants u and c only.

Thus every sphere (iii) meets the X-axis at the same two points and hence the result.

Exercises

1. Show that the sphere

$$x^2+y^2+z^2+2vy+2wz-d=0$$

passes through the limiting points of the co-axal system

$$x^2+y^2+z^2+2ux+d=0$$

and cuts every member of the system orthogonally, whatever be the values of v, w.

Hence deduce that every sphere that passes through the limiting points of a co-axal system cuts every sphere of that system orthogonally.

2. Show that the locus of the point spheres of the system

$$x^2+y^2+z^2+2vy+2wz-d=0$$

is the common circle of the system

$$x^2+y^2+z^2+2ux+d=0\,;$$

u, v, w being the parameters and d a constant.

3. Show that the sphere which cuts two spheres orthogonally will cut every member of the co-axal system determined by them orthogonally.

4. Find the limiting points of the co-axal system of spheres

$$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0.$$

[*Ans.* $(2,-3,4)$; $(-2,3,-4)$.

5. Three spheres of radii r_1, r_2, r_3, have their centres A, B, C at the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and $r_1^2+r_2^2+r_3^2=a^2+b^2+c^2$. A fourth sphere passes through the origin and A, B, C. Show that the radical centre of the four spheres lies on the plane $ax+by+cz=0$. (*D.U.*)

6. Show that the locus of a point from which equal tangents may be drawn to the three spheres

$$x^2+y^2+z^2+2x+3y+3z+2=0,$$
$$x^2+y^2+z^2+4x+4z+4=0,$$
$$x^2+y^2+z^2+x+6y-4z-2=0,$$

is the straight line

$$x/2=(y-1)/5=z/3.$$

7. Show that there are, in general, two spheres of a co-axal system which touch a given plane.

Find the equations to the two spheres of the co-axal system

$$x^2+y^2+z^2-5+\lambda(2x+y+3z-3)=0,$$

which touch the plane

$$3x+4y=15.$$

[*Ans.* $x^2+y^2+z^2+4x+2y+6z-11=0$, $5(x^2+y^2+z^2)-8x-4y-12z-13=0$.

8. P is a variable point on a given line and A, B, C are its projections on the axes. Show that the sphere $OABC$ passes through a fixed circle.

9. Show that the radical planes of the sphere of a co-axal system and of any given sphere pass through a line.