MAR GREGORIOS COLLEGE OF ARTS & SCIENCE

Block No.8, College Road, Mogappair West, Chennai – 37

Affiliated to the University of Madras
Approved by the Government of Tamil Nadu
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DEPARTMENT OF MATHEMATICS

SUBJECT NAME: DIFFERENTIAL EQUATIONS

SUBJECT CODE: SM23B

SEMESTER: III

PREPARED BY: PROF.T.N.REKHA

UNIVERSITY OF MADRAS
B.Sc. DEGREE COURSE IN MATHEMATICS
SYLLABUS WITH EFFECT FROM 2020-2021

BMA-CSC06

CORE-VI: DIFFERENTIAL EQUATIONS
(Common to B.Sc. Maths with Computer Applications)

Inst.Hrs : 4 **YEAR: II**
Credits : 4 **SEMESTER: III**

Learning outcomes:

Students will acquire knowledge

- About the methods of solving Ordinary and Partial Differential Equations.
- To introduce Differential Equation as a powerful tool in solving problems in Science.

UNIT I

Ordinary Differential Equations:Variable separable-Homogeneous Equation-Non-Homogeneous Equations of first degree in x and y-Linear Equation-Bernoulli's Equation-Exact differential equations.

Chapter 2: Section 1 to 6.

UNIT II

Equation of first order but not of higher degree: Equation solvable for dy/dx- Equation solvable for y-Equation solvable for x- Clairauts form-Linear Equations with constant coefficients-Particular integrals e^{ax}, sinax, cosax, x^m, Ve^{ax} where V is sinax or cosax or x^m.

Chapter 4: Section 1, 2.1, 2.2, 3.1.

Chapter 5: Section 4.

UNIT III

Simultaneous linear differential equations- Linear Equations of the Second Order -Complete solution in terms of a known integrals- Reduction to the Normal form- Change of the Independent Variable - Method of Variation of Parameters.

Chapter 6: Section- 6

Chapter 8:Section- 1,2,3,4.

UNIT IV

Partial differential equation: Formation of PDE by Eliminating arbitrary constants and arbitrary functions-complete integral-singular integral-General integral- Lagrange's Linear Equations Pp+Qq=R.

Chapter 12: Section- 1, 2, 3.1, 3.2, 4.

UNIT V

Special methods - Standard forms - Charpit's Methods - Related problems

Chapter 12: Section-5.1, 5.2, 5.3, 5.4, 6.

Contents and treatment as in

"Differential Equations and its applications", by S.Narayanan, T.K.Manikavachagam Pillay – S.Viswanathan (Printers and Publishers) Pvt. Ltd(2006).

Reference:

1.Mathematics for B.Sc-Branch-I Volume –III by P.Kandasamy ,K.Thilagavathy S.Chand Publications.

2.Differential equations with applications and historical notes by George F.Simmons, 2nd Ed,TataMcgraw Hill Publications .

3. Differential Equations by ShepleyL.Ross, 3 rdEd ,JohnWiely and sons 1984.

4 .Differential Equations by N.P.Bali,Laxmi Publications Ltd,New Delhi-2004.

5. Ordinary and Partial differential Equation by Dr.M.D.Raisinghania ,S.Chand.

e-Resources:

1.http://mathworld.wolfram.com
2. http://www.analyzemath.com/calculus/Differential_Equations/applications. html

UNIT- 1

Introduction to Differential Equations

The following topics are to be covered from differential equation of first order and first degree. Topics included here are from unit-3 of the syllabus according to choice base credit system effective from June-2010. The course code of the M-101 and title of the paper is *Geometry and calculus.*
Differential Equations of First Order and First Degree: Definition and method of solving of *homogeneous differential equations,* Definition and method of solving of *Linear differential equations of first order and first degree,* Definition and method of solving of *Bernoulli's differential equation* and Definition and methods of solving of *Exact differential equation.* **Differential Equations of First order and Higher Degree:** Differential equations of first order and first degree *solvable for x, solvable for y, solvable for p. Clairaut's form* of differential equation and *Lagrange's form* of differential equations.

Definition 1.1. Differential equation *is an equation which involves differentials or differential coefficients. For example,*

1. $\frac{dy}{dx} = x^2 + 2y.$

2. $r^2\frac{d^2\theta}{dr^2} = a.$ *Where a is constant.*

3. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E\sin\omega t.$

Definition 1.2. *A differential equation is said to be* linear in dependent variable *if,*

1. *dependent variable and all its derivatives present are in first degree.*

2. *dependent variable and its derivatives are not multiplies together.*

3. *dependent variable and its derivatives are not multiplied with itself.*

4. *no transcedental functions of dependent variable and/or its derivative occur.*

Remark 1.3. *A differential equation which is not linear is said to be* Non-linear. *It is nice exercise to find out some examples of linear and non linear differential equation. You can check from examples given in the exercises. (do it!)*

Definition 1.4. *An* ordinary differential equation (O. D. E.) *is a differential equation which involves only ordinary derivatives.*

Definition 1.5. *A* partial differential equation (P. D. E) *is a differential equation which involves only partial derivatives. For example,*

1. $\frac{\partial U}{\partial t} = c\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$.
2. $\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$.

Definition 1.6. *The* order of the differential equation *is defined to as the order of the highest derivative involved in the differential equation. Also, the* degree of the differential equation *is defined as the degree of the highest derivative involved in the differential equation, where all derivatives occurring therein are free from radicals and fraction.*

Examples 1.7. *(1) Decide the order and degree of the differential equation given by*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \int 3dx = \sin x.$$

Solution: The given differential equation is not free from integration sign. So, to decide order of a differential equation we have to differentiate with respect to x on both sides and make it free from integration.

$$\Longrightarrow x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3 = \cos x.$$

Here, order of the highest derivative involved is three. Therefore, order of differential equation is 3*, and degree of highest derivative is* 1*. Thus, order is* 3 *and degree is* 1.

(2) $\sqrt[4]{(y'')^5} = \sqrt{7+3(y')^2}$

Solution: To obtain degree of differential equation we have make differential equation free from radicals.

$$\therefore (\sqrt[4]{(y'')^5})^4 = (\sqrt{7+3(y')^2})^4.$$

$$(y'')^5 = (7+3(y')^2)^2.$$

$$\left(\frac{d^2y}{dx^2}\right)^5 = \left[7+3\left(\frac{dy}{dx}\right)\right]^2$$

Which shows that order of the given differential equation is 2 *and degree is* 5.

Definition 1.8. *1. A* solution *or* integral *or* primitive *of a differential equation is a relation between the variables which does not involve any derivatives and also satisfies given differential equation. For example, $y = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants, is a solution of the differential equation given by $\frac{d^2y}{dx^2} + y = 0$.*

2. *A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the* general solution *or* complete integral *or* complete primitive.

3. *The solution obtained from the general solution by giving particular values to the arbitrary constants is called* particular solution. *For example, $y = x^4 + 2$ is a particular solution of the differential equation $\frac{dy}{dx} = 4x^3$, where $c = 2$.*

4. *A solution which can not be obtained from a general solution is called* singular solution . *For example, $y = x\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2$. The general solution is given by $y = cx + 2c^2$, where c is an arbitrary constant. Also, $8y = x^2$ is a singular solution which can not be obtained by putting any value of c.*

Examples 1.9. *(1) Find the differential equation from $y = ax - a^2$, where a is an arbitrary constant.*
Solution: Differentiating $y = ax - a^2$ with respect to x we get $\frac{dy}{dx} = a$. Substituting we get desired differential equation $y = \left(\frac{dy}{dx}\right)x - \left(\frac{dy}{dx}\right)^2$.
(2)Form the differential equation from $y = Ae^{2x} + Be^{5x}$; where A and B are arbitrary constants.
Solution: Here, two arbitrary constants A and B are present, therefore to eliminate them we have to differentiate two times.

$$\therefore \frac{dy}{dx} = 2Ae^{2x} + 5Be^{5x}. \tag{1.1}$$

again by differentiating with respect to x we get.

$$\therefore \frac{d^2y}{dx^2} = 4Ae^{2x} + 25Be^{5x}. \tag{1.2}$$

Multiply equation $y = Ae^{2x} + Be^{5x}$ by -2 and adding in (4.2) we get

$$\frac{dy}{dx} - 2y = 3Be^{5x} \implies Be^{5x} = \frac{1}{3}\left[\frac{dy}{dx} - \frac{2}{3}y\right]. \tag{1.3}$$

Now multiply (4.1) by -5 and adding in (4.2) we get, $Ae^{2x} = \frac{5}{6}\frac{dy}{dx} - \frac{1}{6}\frac{d^2y}{dx^2}$. Thus by substituting values of constants we get

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0.$$

Which is required differential equation.

Exercise-1

Que-1. Find the differential equation from the following equations.

1. $xy = ce^{x} + be^{-x} + x^2$, where b and c are arbitrary constants.

2. $ax^2 + by^2 = 1$, where a and b are arbitrary constants.

3. $y = ax + bx^2$, where a and b are arbitrary constants.

4. $r^2 = a^2 cos2\theta$, where a is an arbitrary constant.

Que-2. Find out order and degree of the following differential equations.

1. $x^2 \frac{d^2y}{dx^2} - x(\frac{dy}{dx})^3 + y = cosx.$

2. $\frac{y'}{y} = \frac{d}{dx}[\frac{y''}{y'}].$

3. $(\frac{dy}{dx})^2 = \sqrt{1 + (\frac{dy}{dx})^2}.$

4. $\frac{d^2y}{dx^2} = 3\frac{dy}{dx} + \int xdx.$

Que-3. Show that $y = e^{2x}$ is a solution of a differential equation

$$3x^2\frac{d^2y}{dx^2} + 2(1-3x^2)\frac{dy}{dx} - 4y = 0.$$

Que-4. Prove that $y = 2x + 5e^{-x}$ is a particular solution of a differential equation

$$(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Que-5. Which curve is represented by a differential equation

$$2a\frac{d^2y}{dx^2} = 1?$$

Differential Equations of First Order and First Degree.

In order to solve the differential equation, we need to investigate, whether the solution exists. It is not always possible to find a real analytic solution of a given differential equation. For example, $\left(\frac{dy}{dx}\right)^2 = -5$ has no solution for any real value of y. In our case we shall discuss some of the special types of differential equations for which analytic solution exists. Only those differential equations which belong to or can be reduced to any one of the following type can be solved by standard procedure. These types are,

1. Differential equation in which variables are separable.
2. Homogeneous differential equations.
3. Nonhomogeneous differential equations which can be reduced to homogeneous differential equations.
4. Linear differential equations.
5. Bernoulli's differential equations. These are nonlinear types of differential equations which can be reduced to linear form.
6. Exact differential equations.

2.1 Differential equations in which variables are separable.

The general form of this type of equation is

$$M(x)dx + N(y)dy = 0, \tag{2.1}$$

which can be solved by direct integration as $\int M(x)dx + \int N(y)dy = c$, where c is an arbitrary constant. If the differential equation is given in the form

$$f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0, \tag{2.2}$$

then we can reduce it in the form of equation (2.1) by rewriting as

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0,$$

provided $f_2(x) \neq 0$, $g_1(y) \neq 0$. Also, if the given differential equation is in the form

$$\frac{dy}{dx} = f(ax + by + c), \tag{2.3}$$

then put $ax + by + c = u$, to convert it in general form. Let us see following examples to understand this method well.

Examples 2.1. *1.* $\frac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$.

Solution: The given differential equation is not in its general form. In order to solve the given differential equation first we will convert it into general form.

$$\frac{dy}{dx} = e^{-2y}(e^3x + x^2)$$
$$\Longrightarrow e^{2y}dy = (e^{3x} + x^2)dx$$
$$\Longrightarrow (e^{3x} + x^2)dx - e^{2y}dy = 0,$$

which is in the general form and hence the solution can be obtained by direct integration.

$$\Longrightarrow \int (e^{3x} + x^2)dx - \int e^{2y}dy = c$$
$$\Longrightarrow \frac{e^{3x}}{3} + \frac{x^3}{3} - \frac{e^{2y}}{2} = c$$
$$or\, 3e^{2y} = 2(e^{3x} + x^3) + c'.$$

Which is a general solution of the given differential equation and c' is an arbitrary constant.

2. *Obtain particular solution of* $\frac{dy}{dx} = (4x+y+1)^2$, *where* $y(0) = 1$
Solution: The given differential equation is not of the form of separable variable. Hence, to convert it into separable variable form we put $4x+y+1 = t$ *and* $\frac{dt}{dx} = 4 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{dt}{dx} - 4$. *Put these values in equation we get*

$$\frac{dt}{dx} - 4 = t^2$$

$$\therefore \frac{dt}{t^2+4} = dx.$$

$$\int \frac{dt}{t^2+4} = \int dx + c, \text{ where } c \text{ is an arbitrary constant.}$$

$$\therefore \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\therefore \frac{1}{2}\tan^{-1}\frac{4x+y+1}{2} = x + c$$

Put $x = 0$ *and* $y = 1$ *we get* $\tan(2c) = 1 \implies 2c = \frac{\pi}{4}$. *Thus, particular solution is given by*

$$4x+y+1 = 2\tan\left(2x + \frac{\pi}{4}\right).$$

2.2 Homogeneous differential equations

Definition 2.2. *Let* $E \subset \mathbb{R}^2$. *A function* $f : E \to \mathbb{R}$ *is said to be* homogeneous of degree n *if it can be written in the form* $f(x,y) = x^n \phi(\frac{y}{x})$.

Definition 2.3. *A differential equation is said to be* homogeneous differential equation *if it is of the form*

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \text{ or } \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.4}$$

Where $P(x,y)$ *and* $Q(x,y)$ *are homogeneous functions of equal degree in variables* x *and* y.

In order to solve homogeneous differential equations we need to follow mainly three following steps.

1. Put $y = vx$ in the given differential equation and evaluate $\frac{dy}{dx}$.
2. Substitute the values of y and $\frac{dy}{dx}$ in main equation and bring the equation in the form of separable variable.
3. Solve by the method of separable variable.

Examples 2.4. *1. Solve:* $(x^2+y^2)dx - 2xydy = 0$
Solution:

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{(1+\frac{y}{x})}{\frac{2y}{x}} \tag{2.5}$$

Put $y = vx$ *we get* $\frac{dy}{dx} = v + x\frac{dv}{dx}$. *Substitute these values in equation (2.5) we get,*

$$v + x\frac{dv}{dx} = \frac{1+v}{2v}$$
$$\therefore x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$
$$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$$
$$\therefore \frac{2v}{1-v^2}dv = \frac{1}{x}dx$$

Which is now in the separable variable form. So, solution can be obtain by direct integration. Integrating both side we get,

$$\therefore \int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$
$$\therefore -\log(1-v^2) = \log x + \log c \text{ where } c \text{ is an arbitrary constant.}$$
$$\therefore \log x + \log(1-v^2) = \log c', \text{ where } c' = c^{-1}$$
$$\therefore \log(x(1-v^2)) = \log c'$$

by taking exponential on both sides we get,

$$x(1-v^2) = c',$$

now substitute the value of v *in above equation, we get*

$$x^2 - y^2 = c'x$$

which is the general solution of the given differential equation.

2.3 Nonhomogeneous differential equations which can be reduced to homogeneous differential equations.

A differential equation of the form,

$$\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n} \tag{2.6}$$

is not homogeneous differential equation, but by making some change we can reduce it to the case of homogeneous differential equation.

Case-I $\frac{a}{l} \neq \frac{b}{m}$. In order to solve differential equation having this case, let $x = x' + h$ and $y = y' + k$, where h and k are constants.Also, $dx = dx'$ and $dy = dy'$. Then equation (2.6) reduces to

$$\frac{dy'}{dx'} = \frac{ax' + by' + ah + bk + c}{lx' + my' + lh + mk + n} \tag{2.7}$$

In this equation we select h and k by solving $ah+bk+c=0$ and $lh+mk+n=0$ such that equation (2.7) will turn out to homogeneous differential equation $\frac{dy'}{dx'}=\frac{ax'+by'}{lx'+my'}$, where $al-bm\neq 0$. Which is homogeneous in the variables x' and y'. So solve it by putting $y'=vx'$.
Case-II $\frac{a}{l}=\frac{b}{m}$. In this case $al-bm=0$,and hence h and k will be indetermined or infinity. Hence put $\frac{a}{l}=\frac{b}{m}=t$, where t is constant in equation (2.6) we get

$$\frac{dy}{dx}=\frac{(lx+my)t+c}{(lx+my)+n}. \tag{2.8}$$

Now by substitute $lx+my=t$ in equation (2.8) we can solve the given differential equation. Let us see the following examples to understand this method well.

Examples 2.5. *(1)*$\frac{dy}{dx}=\frac{y+x-2}{y-x-4}$. *Solution: The differential equation is given by*

$$\frac{dy}{dx}=\frac{y+x-2}{y-x-4} \tag{2.9}$$

is not homogeneous differential equation. By comparing with (2.6) we get $a=1, b=1, l=-1, m=1$. *Here,* $\frac{a}{l}=-1\neq\frac{b}{m}=1$. *Hence substitute* $x=x'+h$ *and* $y=y'+k$ *in equation (2.9) we get,*

$$\frac{dy'}{dx'}=\frac{y'+x'+(k+h-2)}{y'-x'+(k-h-4)} \tag{2.10}$$

To convert equation (2.10) in homogeneous differential equation we take $k+h-2=0$ *and* $k-h-4=0$, *by solving we get* $h=-1, k=3$. *Hence with these values of h and k equation (2.10) reduces to,*

$$\frac{dy'}{dx'}=\frac{y'+x'}{y'-x'}, \text{ which is homogeneous differential equation.} \tag{2.11}$$

In order to solve put $y'=vx'$ *and* $\frac{dy}{dx}=v+x'\frac{dv}{dx}$ *in equation (2.11) we obtain,*

$$v+x'\frac{dv}{dx'}=\frac{vx'+x'}{vx'-x'}=\frac{v+1}{v-1}$$

$$\therefore x'\frac{dv}{dx'}=\frac{v+1}{v-1}-v-\frac{1+2v-v^2}{v-1}$$

$$\therefore \frac{v-1}{1+2v-v^2}dv=\frac{dx'}{x'}, \text{ which is separable variable form}$$

By integrating term by term we get,

$$\int\frac{v-1}{1+2v-v^2}dv=\int\frac{dx'}{x'}+c, \text{ where c is an arbitrary constant.}$$

$$\therefore -\frac{1}{2}\int\frac{2-2v}{1+2v-v^2dv}=\log x'+c$$

$$\therefore \log\left(1+2\frac{y'}{x'}-\frac{y'^2}{x'^2}\right)+\log x'^2=-2c$$

$$\therefore \log(x'^2 + 2x'y' - y'^2) - \log x'^2 + \log x'^2 = -2c$$

$$\therefore x'^2 + 2x'y' - y'^2 = e^{-2c} = c'$$

by substituting $x' = x+1$ and $y' = y-3$, we get $x^2+2xy-y^2-4x+8y-14 = c'$, which is general equation of given differential equation. (2)$(x-y+2)dx + (2x-2y-4)dy = 0$

Solution: The differential equation is given by,

$$\frac{dy}{dx} = -\frac{x-y+2}{2(x-y)-4} \tag{2.12}$$

is not homogeneous differential equation. By comparing with (2.6) we get $a=-1, b=l, l=2, m=-2$. Here, $\frac{a}{l} = -\frac{1}{2} = \frac{b}{m}$. Therefore h and k can not be determined. Put $x-y=z$ and $1-\frac{dy}{dx} = \frac{dz}{dx}$ in equation (2.12) we get,

$$1 - \frac{dz}{dx} + \frac{z+2}{z-4} = 0$$

$$\therefore \frac{dz}{dx} + \frac{3z-2}{2z-4} = 0$$

$$\therefore \frac{2z-4}{3z-2}dz = dx, \text{ which is separable variable form.}$$

In order to get solution integrate the terms separately we get

$$\int \frac{2z-4}{3z-2}dz = \int dx + c, \text{ where } c \text{ is an arbitrary constant}$$

$$\therefore \int \frac{2}{3}\frac{3z-2-4}{3z-2}dz = \int dx + c$$

$$\therefore \frac{2}{3}\int \left(1 - \frac{4}{3z-2}\right)dz = x + c$$

$$\therefore \frac{2}{3}\left[x - y - \frac{4}{3}\log[3(x-y)-2]\right] = 3x + c', \text{ where } c' = 3c$$

$$\therefore x + 2y + \frac{8}{3}\log[3(x-y)-2] + c', \text{ which is a general solution.}$$

Exercise-II

Identify type of the following differential equations and solve them.

1. $2y\frac{dz}{dx} = x^2 + \sin 3x$. (Ans: $3y^2 = x^3 - \cos 3x + c$.)
2. $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$. (Ans: $\tan y = c(1-e^x)^3$.)
3. $\frac{y}{x}\frac{dy}{dx} + \frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$. (Ans: $2x^2 + y^2 + 3\log(x^2+y^2-2) = c$.)
4. $x^4\frac{dy}{dx} + x^3 y + cosec(xy) = 0$. (Ans: $\cos xy + \frac{1}{2x^2} = c$.)

5. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$. (Ans: $(x+a)(1-ay) = cy$.)

6. $x\frac{dy}{dx} = y + \cos^2\left(\frac{y}{x}\right)$. (Ans: $\tan\left(\frac{y}{x}\right) = \log|cx|$

7. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$. (Ans: $y = x\log y + cx$).

8. $y - x\frac{dy}{dx} = \sqrt{y^2 - x^2}$. (Ans: $y + \sqrt{y^2 - x^2} = c$.)

9. $\frac{x+y+1}{x-y+1}$. (Ans: $\tan^{-1}\frac{y}{x+1} = \log\left(c\sqrt{(x+1)^2 + y^2}\right)$).

10. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$. (Ans: $(x+y-2)(x-y)^{-3} = c$).

11. $(3y+2x+4)dx - (4x+6y+5)dy = 0$. (Ans: $21x - 42y + 9\log(14x+21y+22) = c'$).

12. $(2x+9y-20)dx = (6x+2y-10)dy$. (Ans: $(y-2x)^2 = c(x+2y-5)$).

2.4 Linear differential equations.

Definition 2.6. *A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x is said to be* linear differential equation of first order. *For example, $\frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x$ is linear differential equation of first order.*

In order to solve the linear differential equation we use the method of separable variable. Linear differential equation of first order is given by

$$\frac{dy}{dx} + Py = Q, \text{ where P and Q are either constants or functions of } x. \tag{2.13}$$

First we solve $\frac{dy}{dx} + Py = 0$ by using separable variable method. For

$$\int \frac{dy}{y} = -\int Pdx + c, \text{ where } c \text{ is an arbitrary constant.}$$

$$\log y = -\int Pdx + c'.$$

$$\therefore y = e^{-\int Pdx}e^{-c'}.$$

$$\therefore y = e^{-\int Pdx}c.$$

Now differentiate on both sides with respect to x we get,

$$e^{\int Pdx}\frac{dy}{dx} + ye^{\int Pdx}P = 0.$$

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = 0.$$

$$\therefore \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = 0. \tag{2.14}$$

Since $e^{\int Pdx} \neq 0$ we multiply equation (2.13) by $e^{\int Pdx}$ on both sides we get

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx}.$$

$$\therefore \frac{d}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx}.$$

By integrating on both sides we have

$$\int \frac{d}{dx}(ye^{\int Pdx})dx = \int Qe^{\int Pdx}dx + c.$$

$\therefore ye^{\int Pdx}dx = \int Qe^{\int Pdx}dx + c$, where c is an arbitrary constant. Which is the general solution of the given differentia

Remark 2.7. *Here we can solve the equation by multiplying the given differential equation by* $e^{\int Pdx}$ *and hence we call* $e^{\int Pdx}$ *an* integrating factor *denoted by* I. F *then here* $I.F = \int e^{Pdx}$. *Therefore the general formula for finding the solution of linear differential equation is given by*

$$y(I.F.) = \int Q(I.F.)dx + c.$$

Examples 2.8. *(1) Solve:* $(x+1)\frac{dy}{dx} + 2y = 1$.
Solution: To convert the given differential equation in general form of the linear differential equation we divide both side by $(x+1)$.

$$\therefore \frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x+1}.$$

Compare this with equation (2.13) we get $P = \frac{2}{x+1}$ *and* $Q = \frac{1}{x+1}$.

$$\therefore e^{\int Pdx} = e^{\int \frac{2}{x+1}dx} = e^{2\log(x+1)} = (x+1)^2.$$

Now we know the general formula for finding the solution of differential equation is

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx.$$

By substitutes values we get

$$y(x+1)^2 = \int \frac{1}{1+x}(1+x)^2 dx + c.$$

$$y(x+1)^2 = \int (x+1)dx + c = \frac{x^2}{2} + x + c.$$

$$y(x+1)^2 = \frac{x^2}{2} + x + c. \text{ Which is a general solution.}$$

(2) Solve: $(1+y^2)dx = (\tan^{-1}y - x)dy$.

In the given differential equation the term containing x is 1 with degree 1. Therefore the equation can be converted to a differential equation which is linear in x given by $\frac{dx}{dy}+Px=Q$.

$$\therefore \frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}$$

Comparing this equation with general form we get, $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1}y}{1+y^2}$.

$$\therefore I.F=e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

Now put this value in general formula given by $xe^{\int P\,dy}=\int Qe^{\int P\,dy}dy$ we get

$$xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+c$$

where c is an arbitrary constant. Now for right hand side integration we take $\tan^{-1}y=t$, $\frac{dy}{1+y^2}=dt$ we get

$$\therefore xe^{\tan^{-1}y}=\int te^t\,dt+c.$$

By integrating by parts we get

$$xe^{\tan^{-1}y}=te^t-\int 1e^t\,dt+c.$$

$$\therefore xe^{\tan^{-1}y}=(\tan^{-1}y-1)e^{\tan^{-1}y}+c$$

which is a general solution.

2.5 Bernoulli's differential equations.

Definition 2.9. *A differential equation of the form $\frac{dy}{dx}+Py=Qy^n, n\in\mathbb{R}\setminus\{0\}$ is said to be* Bernoulli's differential equation

In order to solve Bernoulli's differential equation we will use the method of solving linear differential equation. Bernoulli's differential equation is given by

$$\frac{dy}{dx}+Py=Qy^n, n\in\mathbb{R}\setminus\{0\}. \tag{2.15}$$

Divide both sides by y^n we get $y^{-n}\frac{dy}{dx}+y^{1-n}P=Q$. Now multiply by $(1-n)$ both sides we get

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)y^{1-n}P=(1-n)Q. \tag{2.16}$$

Now put $v=y^{(1-n)}$ and $\frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx}$ in equation (2.16) we get

$$\frac{dv}{dx}+(1-n)Pv=(1-n)Q \tag{2.17}$$

Which is linear in variable v and can be solved by method of linear differential equation. Hence substitute

$$\therefore I.F. = e^{\int Pdx} = e^{\int (1-n)Pdx}$$

in equation $ve^{\int Pdx} = \int Qe^{\int Pdx} + c$

$$\therefore ve^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} dx + c$$

$$\therefore y^{1-n} e^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} dx + c.$$

where c is an arbitrary constant . Which is a general solution.

Examples 2.10. *(1)Solve:* $x\frac{dy}{dx} + y = x^3 y^6$

Solution: The given differential equation is not linear in x also not linear y. To convert it into Bernoulli's form we divide the equation by xy^6 *we get*

$$y^{-6}\frac{dy}{dx} + y^{-5}\frac{1}{x} = x^2. \tag{2.18}$$

$\therefore$ *put* $y^{-5} = v$ *and* $-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$ *in equation (2.18) we get* $\frac{dv}{dx} - \frac{5}{x}v = -5x^2$ *which is linear in v. Hence comparing with general form of linear differential equation we get* $P = -\frac{5}{x}$ *and* $Q = -5x^2$*. Now*

$$I.F. = e^{\int Pdx} = e^{\int \frac{-5}{x} dx} = x^{-5}.$$

Now formula for solution is given by

$$ve^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

where c is an arbitrary constant.

$$\therefore y^{-5}x^{-5} = \int -5x^2 x^{-5} dx + c$$

$$\therefore y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c, \text{where } c \text{ is an arbitrary constant. Which is a general solution.}$$

(2) Solve: $x\frac{dy}{dx} - y = y^2 \log x.$

Solution: To convert this equation in form of Bernoulli's differential equation we divide both sides by x we get

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}y^2.$$

Now comparing with the general form of Bernoulli's differential equation $\frac{dy}{dx} + Py = Qy^n$*, we get* $P = -\frac{1}{x}; Q = \frac{\log x}{x}$ *with* $n = 2$*. Therefore the solution is given by*

$$y^{1-n}e^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} + c.$$

$$\therefore y^{-1}x = \int -1\frac{\log x}{x}xdx + c.$$

$$\therefore -\int \log x dx + c \implies -[\log x x - \int \frac{1}{x}x dx] + c.$$

$\therefore x = y(c + x - x\log x)$. *Which is a general solution of the given differential equation.*

Remark 2.11. *The general form of Bernoulli's differential equation* $\frac{dy}{dx} + Py = Qy^n$; $n \in \mathbb{R} \setminus \{0\}$ *is given by*

$$f'(y)\frac{dy}{dx} + f(y)P = Q.$$

In order to solve this we put $u = f(y)$ *we get* $\frac{du}{dx} = f'(y)\frac{dy}{dx}$ *in general form we get* $\frac{du}{dx} + Pu = Q$, *which is linear differential equation. Let us see the following examples to understand.*

Examples 2.12. *(1) Solve:* $\sin y\frac{dy}{dx} + x\cos y = x$.
Solution: Here $u = \cos y$ *and* $\frac{du}{dx} = -\sin y\frac{dy}{dx}$. *Substitute these values in given differential equation we get*

$$\frac{du}{dx} - xu = -x.$$

Which is linear differential equation in variable v. *Therefore solution is given by*

$$u(I.F.) = \int Q(I.F.)dx + c.$$

$$ue^{\frac{-x^2}{2}} = \int (-x)e^{\frac{-x^2}{2}}dx + c.$$

$$\cos y = \frac{1}{2} + ce^{\frac{x^2}{2}}. \text{ Which is a general solution.}$$

(2) Solve: $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x}(\log y)^2$.
Solution: Divide both sides by y we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = \frac{1}{x}(\log y)^2.$$

Now put $u = \log y$, *we get* $\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}$. *Substitute these values in above equation we get*

$$\frac{du}{dx} + \frac{u}{x} = \frac{u^2}{x} \implies \frac{1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = \frac{1}{x}$$

. *Which is in the form of Bernoulli's differential equation. By putting* $\frac{1}{u} = t$ *and solving it we get* $(\log y)^{-1} = 1 + cx$ *which is general solution of given differential equation.*

Exercise-III

Identify type of the following differential equations and solve them.

1. $\frac{dy}{dx} + y\cos x = \sin x \cos x$ (Ans: $y = \sin x + ce^{-\sin x} - 1$.)

2. $\frac{dy}{dx} + 2xy = 2x$, also $y = 3$ when $x = 0$ obtain a particular solution. (Ans: $y = 1 + ce^{-x^2}$ and P.S. is $y = 1 + 2e^{-x^2}$.)

3. $\frac{dy}{dx} + y\tan x = \sec x$. (Ans: $y = \sin x + c\cos x$.)

4. $\cos^2 x \frac{dy}{dx} + y = \tan x$. (Ans: $y = \tan x - 1 + ce^{-\tan x}$.)

5. $(1+x^2)dy = (\tan^{-1} x - y)dx$. (Ans: $y = \tan^{-1} x - 1 + ce^{-\tan^{-1} x}$.)

6. $x\frac{dy}{dx} + 2y = x^2 \log x$. (Ans: $y = \frac{x^2}{4}\log x - \frac{x^2}{16} + cx^{-2}$.)

7. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$. (Ans: $y\sin x = -5e^{\cos x + c}$.)

8. $\frac{dy}{dx} + 2y\tan x = \sin x$, also obtain particular solution with $y = 0$ when $x = \frac{\pi}{3}$. (Ans: $y\sec^2 x = \sec x + c$; P.S $= y\sec^2 x = \sec x - 2$)

9. $(x + 2y^3)\frac{dy}{dx} = y$. (Ans: $x = y^3 + cy$.)

10. $x\log x \frac{dy}{dx} + y = 2\log x$. (Ans: $y\log x = (\log x)^2 + c$.)

11. $\frac{dy}{dx} + y\tan x = y^3 \sec x$. (Ans: $\cos^2 x = y^2(c + 2\sin x)$)

12. $xy(1 + xy^2)\frac{dy}{dx} = 1$. (Ans: $\frac{1}{x} = (2 - y^2) + ce^{\frac{-y^2}{2}}$.)

13. $\frac{dy}{dx} + y\tan x = \frac{\cos x}{y}$. (Ans: $y^2 = \cos^2 x\left[c + \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$.)

14. $\sec^2 y \frac{dy}{dx} + x\tan y = x^3$. (Ans: $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$.)

15. $(x^3y^3 + xy)dx = dy$. (Ans: $y^{-1} = 2 - x^2 + ce^{\frac{-x^2}{2}}$.)

16. $\frac{dy}{dx} + y\cos x = y^3 \sin 2x$. (Ans: $y^{-2} = 2\sin x + 1 + ce^{2\sin x}$.)

17. $x\frac{dy}{dx} = y - \sqrt{y}$. (Ans: $4c^2 x = (y - 1 - c^2 x)^2$.)

18. $x^3 \frac{dy}{dx} - x^2 y + y^4 = 0$. (Ans: $y^3(3x + c) = x^3$.)

19. $\frac{dy}{dx} + y\log y = xye^x$. (Ans: $x\log y = (x-1)e^x + c$.)

2.6 Exact differential equations.

Definition 2.13. *A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be* exact *if there exists a function $f(x,y)$ such that $d[f(x,y)] = Mdx + Ndy$. That is,*

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy.$$

In other words if a differential equation can be obtain by direct differentiation of its solution, then we call it an exact differential equation.

Necessary and Sufficient Condition for differential equation $M(x,y)dx + N(x,y)dy = 0$ to be exact:

Theorem 2.14. *The necessary and sufficient condition for the differential equation $M(x,y)dx+N(x,y)dy = 0$ to be exact is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Where $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ denotes the partial derivatives of M and N with respect to y and x respectively.

In order to solve an differential equation of the type $M(x,y)dx + N(x,y)dy = 0$, first check the condition of exactness, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If the condition satisfied, then the given differential equation is exact and solution is given by

$$\int_{y\ \text{constant}} Mdx + \int \left(\text{Terms in } N \text{ which are independent of x}\right) dy = c.$$

Where c is an arbitrary constant.

Examples 2.15. *(1)Solve:* $(x^2 - ay)dx + (y^2 - ax)dy = 0$.
Solution: Here $M(x,y) = x^2 - ay$ *and* $N(x,y) = y^2 - ax$

$$\therefore \frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a.$$

Therefore the given differential equation is an exact differential equation. The solution is given by

$$\int_{y\ \text{constant}} Mdx + \int \left(\textit{Terms in N which are independent of x}\right) dy = c.$$

$$\therefore \int_{y\ \text{constant}} (x^2 - ay)dx + \int y^2 dy = c$$

$$\therefore \frac{x^3}{3} - ayx + \frac{y^3}{3} = c.$$

$x^3 + y^3 - 3axy = 3c$. *Which is a general solution.*

(2) Solve: $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0.$
Solution: We write this equation in the form $M(x,y)dx+N(x,y)dy=0$, *we get* $(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0$. *and also* $M(x,y)=y\cos x+\sin y+y$, $N(x,y)=\sin x+x\cos y+x$.

$$\therefore \frac{\partial M}{\partial y}=\cos x+\cos y+1=\frac{\partial N}{\partial x}.$$

Therefore the given differential equation is an exact differential equation. The solution is given by

$$\int_{y\ constant} M dx+\int \left(\text{Terms in } N \text{ which are independent of } x\right) dy=c.$$

$$\therefore \int_{y\ constant} (y\cos x+\sin y+y)dx+\int 0dx=c.$$

$$\therefore y\sin x+x\sin y+yx=c. \text{ Which is a general solution .}$$

Remark 2.16. *If condition* $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, *then the given differential equation is not exact. In this case, if there exist some function* $f(x,y)$ *of two variables such that*

$$f(x,y)[M(x,y)dx+N(x,y)dy=0]$$

become exact, then $f(x,y)$ *is called an* integrating factor *denoted by I.F. For example, the differential equation* $x\frac{dy}{dx}+2y+3x=0$ *is not exact, but by multiplying with* x *we get* $x^2\frac{dy}{dx}+2yx+3x^2=0$ *which is an exact differential equation. Thus, here integrating factor is* x.

Rules for Integrating factor for $M(x,y)dx+N(x,y)dy=0$**:**

1. If $M(x,y)dx+N(x,y)dy=0$ is homogeneous differential equation with $Mx+Ny\neq 0$, then integrating factor will be $\frac{1}{Mx+Ny}$.
2. If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is only function of x say $f(x)$, then $e^{\int f(x)dx}$ will be an integrating factor.
3. If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}$ is only function of y say $g(y)$, then $e^{\int g(y)dy}$ will be an integrating factor.
4. If given differential equation is of the form $f_1(x,y)ydx+f_2(x,y)xdy=0$, then integrating factor will be $\frac{1}{Mx-Ny}$, where $Mx-Ny\neq 0$.

Examples 2.17. *(1) Solve:* $(x^2+y^2+2x)dx+2ydy=0$.
Solution: Comparing the given differential equation with $M(x,y)dx+N(x,y)dy=0$, *we get* $M(x,y)=x^2+y^2+2x$ *and* $N(x,y)=2y$. *Here* $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that,* $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=1$ *which is only function of x say* $f(x)$. *Hence* $I.F.=e^{\int f(x)dx}=e^x$.

$\therefore I.F[(x^2+y^2+2x)dx+2ydy=0]$ *which is now reduced to an exact differential equation.*

Now, $e^x[(x^2+y^2+2x)dx+2ydy] = d((x^2+y^2)e^x) = 0$
Thus, the solution is $\int e^x[(x^2+y^2+2x)dx+2ydy] = \int d((x^2+y^2)e^x) = c$, *where c is an arbitrary constant.* $\therefore (x^2+y^2)e^x = c$ *is a general solution.*
(2) Solve: $(xy\sin(xy)+\cos(xy))ydx+(xy\sin(xy)-\cos(xy))xdy = 0.$
Solution: Comparing the given differential equation with $M(x,y)dx+N(x,y)dy=0$, *we get* $M(x,y) = (xy\sin(xy)+\cos(xy))y$ *and* $N(x,y) = (xy\sin(xy)-\cos(xy))x$. *Here* $\frac{\partial M}{\partial y} = x^2y^2\cos(xy)+yx\sin(xy)-yx\sin(xy)+\cos(xy) \neq x^2y^2\cos(xy)+3yx\sin(xy)-\cos(xy) = \frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that, it is of the form* $f_1(x,y)ydx+f_2(x,y)xdy=0$, *therefore integrating factor will be* $\frac{1}{Mx-Ny} = \frac{1}{2xy\cos(xy)}$, *where* $Mx-Ny \neq 0$.

$$\therefore I.F.[M(x,y)dx+N(x,y)dy] = \frac{1}{2xy\cos(xy)}[(xy\sin(xy)+\cos(xy))ydx+(xy\sin(xy)-\cos(xy))xdy]$$

is now reduced to exact differential equation. Thus, solution is given by,

$$\int_{y\ constant} \frac{y}{2}\tan(xy)+\frac{1}{2x}dx-\int\frac{1}{2y}dy = \log c,$$

where c is an arbitrary constant.

$$\therefore \frac{y}{2}\frac{\log\sec(xy)}{y}+\frac{1}{2}\log x-\frac{1}{2}\log y = \log c.$$

$$\therefore \log\sec(xy)+\log\frac{x}{y} = 2\log c$$

$$x = c' y\cos(xy),\ \textit{which is a general solution.}$$

(3)Solve: $x^2ydx-(x^3+y^3)dy=0.$
Solution: Comparing the given differential equation with $M(x,y)dx+N(x,y)dy=0$, *we get* $M(x,y) = x^2y$ *and* $N(x,y) = -(x^3+y^3)$. *Here* $\frac{\partial M}{\partial y} = x^2 \neq -3x^2 = \frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that given differential equation is homogeneous differential equation. Hence, I.F* $= \frac{1}{Mx+Ny} = \frac{-1}{y^4}$.

$$\therefore I.F.[M(x,y)dx+N(x,y)dy] = \frac{-1}{y^4}[x^2ydx-(x^3+y^3)dy]$$

is now reduced to exact differential equation. The solution is given by

$$\int_{y\ constant} \frac{-x^2}{y}dx+\int\frac{1}{y}dy = \log c,$$

where c is an arbitrary constant.

$$\therefore \frac{-x^3}{3y^3}+\log y = \log c.$$

$$\therefore \log y = \log c+\frac{-x^3}{3y^3}$$

$$\therefore y = ce^{\frac{x^3}{3y^3}},\ \textit{which is a general solution.}$$

Exercise-IV

1. Check the exactness of the following differential equations and solve it.

1. $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy$. (Ans: $x^5 - 5x^2y^2 + 5y^4x + 5\cos y = c$.)
2. $(\sin x\cos y + e^x)dx + (\cos(xy)x^2 + e^y)dy = 0$. (Ans: $e^x - \cos x\cos y + \tan y = c$.)
3. $(xy\cos(xy) + \sin(xy))dx + (\cos x\sin y + \sec^2 y)dy = 0$. (Ans: $x\sin(xy) + e^y = c$.)
4. $(2xy + y + -\tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$. (Ans: $x^2y + xy - x\tan y + \tan y = c$.)
5. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$. (Ans: $e^{xy^2} + x^4 - y^3 = c$.)
6. $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)ydy = 0$. (Ans: $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$.)
7. $y\sin 2xdx = (1 + y^2 + \cos^2 x)dy$. (Ans: $3y\cos 2x + 6y + 2y^3 = c$.)
8. $\frac{2x}{y^3}dx + \frac{y^2-3x^2}{y^4}dy = 0$. (Ans: $x^2 - y^2 = cy^3$.)
9. $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy$. (Ans: $y(x + \log x) + x\cos y = c$.)
10. $(\sin x\sin y + \sec^2 x)dx + (\tan^2 y - \cos x\cos y)dy = 0$. (Ans: $\tan x - \cos x\sin y + \tan y - y = c$.)

2. Solve the following differential equations using integrating factor.

1. $(xy\sin(xy) + \cos(xy))ydx + (xy\sin(xy) - \cos(xy))xdy = 0$. (Ans: $x = cy\cos xy$.)
2. $x^2ydx - (x^3 + y^3)dy = 0$. (Ans: $y = ce^{\frac{x^3}{3y^3}}$.)
3. $(y + y^2 - y^3)dx - (x + xy^2 - y)dy = 0$. (Ans: $x + xy + y\log y - xy^2 = cy$.)
4. $ydx + (y - x)dy = 0$. (Ans: $ye^{\frac{x}{y}} = c$.)
5. $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$. (Ans: $x - 2y\log x + 3y\log y = cy$.)

UNIT-II

Differential Equation of First orde rand Higher degree.

The general form of differential equation of first order and higher degree is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots + P_{n-1}\frac{dy}{dx} + P_n = 0.$$

Where each P_i is a function of x and y. If $\frac{dy}{dx} = p$, then the general form reduces to

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0.$$

Hence it also can be written as $F(x, y, p) = 0$. In this chapter we study following methods of solving differential equation of first order and higher degree.

Method of solving differential equation of the form $F(x, y, p) = 0$.

1. Differential equations which are solvable for p.
2. Differential equations which are solvable for x.
3. Differential equations which are solvable for y.
4. Clairaut's differential equations.
5. Lagrange's differential equations.

3.1 Differential equations which are solvable for p.

Suppose we can write the differential equation $F(x,y,p)=0$ of degree n in the form

$$(p-f_1(x,y))(p-f_2(x,y))(p-f_3(x,y))\cdots(p-f_n(x,y))=0. \tag{3.1}$$

Now comparing each factor with zero we get $p-f_i(x,y)=0$, where $i=1,2,\ldots,n$. Which is linear differential equation. Suppose solution of $p-f_i(x,y)=0$ is given by $F_i(x,y,c_i)=0$. Where c_i is an arbitrary constant. Instead of taking different c_i's in the general solution of $p-f_i(x,y)=0$ if we take only one c in all, then it makes no difference in general solution. Therefore general solution $p-f_i(x,y)=0$ will be $F_i(x,y,c)=0$. Then general solution of equation (3.1) is given by $F_1(x,y,c)F_2(x,y,c)\cdots F_n(x,y,c)=0$. Thus, differential equation of n degree and first order having linear factor $p-f_i(x,y)=0$ are known as *solvable for p.*

Examples 3.1. *(1)Solve:* $xyp^3+(x^2-2y^2)p^2-2xyp=0$
Solution: The given differential equation is of degree 3 *and therefore it has three linear factor.*

$$p[xyp^2+(x^2-2y^2)p-2xy]=0.$$

$$\therefore p[xyp^2+x^2p-2y^2p-2xy]=0.$$

$$\therefore p(xp-2y)(yp+x)=0.$$

Comparing these three linear factor with zero we get

1. $p=0 \implies y-c=0.$
2. $xp-2y=0 \implies \frac{dy}{y}=2\frac{dx}{x} \implies y=cx^2.$
3. $yp+x=0 \implies ydy+xdx=0 \implies x^2+y^2-2c=0.$

Therefore, the general solution is given by multiplying these three solutions of linear factors of given equation. $\therefore (y-c)(y-cx^2)(x^2+y^2-2c)=0$. *Which is a general solution.*
(2)Solve: $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}.$
Solution: put $p=\frac{dy}{dx}$ *we get* $p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}.$

$$\therefore p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)-1=0.$$

$$\therefore \left(p+\frac{y}{x}\right)\left(p-\frac{x}{y}\right)=0.$$

Now comparing the linear factors with zero we get

1. $\frac{dy}{dx}+\frac{y}{x}=0 \implies xdy+ydx=0. \implies d(xy)=0 \implies xy=c$
2. $\frac{dy}{dx}-\frac{x}{y}=0 \implies xdy-ydx=0. \implies x^2-y^2=c$

Thus, the general solution can be obtained by multiplying the general solutions of the linear factors of given differential equation.

$$(xy-c)(x^2-y^2-c)=0.$$

Which is a general solution.

Exercise-V

Solve the following differential equations.

1. $p^2-(x+3y)p+2y(x+y)=0$. (Ans. $(y-ce^{-2x})(x+y-1-ce^x)=0$.)
2. $p^2-7p+10$. (Ans. $(y-5x-c)(y-2x-c)=0$.)
3. $p(p+y)=x(x+y)$. (Ans. $(2y-x^2+c)(y+x+ce^{-x}-1)=0$.)
4. $yp^2+(x-y)p-x$. (Ans. $(x-y+c)(x^2+y^2+c)=0$.)
5. $p^3+2xp^2-y^2p^2-2xy^2p=0$. (Ans. $(y-c)(y+x^2-c)(xy+cy+1)=0$.)
6. $p^2+2py\cot x-y^2=0$. (Ans. $y(1\pm\cos x)=c$.)
7. $x^2p^2+xyp-6y^2=0$. (Ans. $(y-cx^2)(x^3y-c)=0$.)
8. $y^2p^2-x^2=0$. (Ans. $(x^2+y^2+c)(x^2-y^2+c)=0$.)
9. $p^2+2p\cos 2x-\sin^2 x=0$. (Ans. $(2y+2x+\sin 2x+c)=0$.)

3.2 Differential equations which are solvable for y.

If the differential equation of the form $F(x,y,p)=0$ can be written as $y=f(x,p)=0$, then it is said to be *solvable for y*. In order to solve these types of differential equation we differentiate with respect to x we get

$$\frac{dy}{dx}=p=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{dp}{dx}=F\left(x,p,\frac{dp}{dx}\right). \tag{3.2}$$

Which is in variable p and x. Hence its solution is given by $g(x,p,c)=0$. By eliminate p from equation (3.2) and $g(x,p,c)$ we get function $\phi(x,y,c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate p, then general solution can be obtained by taking $x=F_1(p,c)$ and $y=F_2(p,c)$. Where c is an arbitrary constant. Let us see following examples to understand this method.

Examples 3.2. *(1).Solve:* $xp^2-2yp+ax=0$
Solution: Here, $y=\frac{1}{2}xp+\frac{1}{2}\frac{ax}{p}$*; by differentiating with respect to x we get*

$$\frac{dy}{dx}=\frac{1}{2}p+\frac{1}{2}x\frac{dp}{dx}+\frac{a}{2p}-\frac{ax}{2p^2}\frac{dp}{dx}.$$

$$\therefore p = \frac{1}{2}p + \left(\frac{1}{2}x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{1}{2}\frac{a}{p}.$$

$$p = \left(x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{a}{p} \implies p^3 - p^2 x\frac{dp}{dx} + ax\frac{dp}{dx} - ap = 0.$$

$$\therefore (p^3 - a)\left(p - x\frac{dp}{dx}\right) = 0.$$

$$\therefore p - x\frac{dp}{dx} = 0 \text{ or } p^3 - a = 0.$$

$$\therefore \frac{dp}{p} = \frac{dx}{x} \implies \log p = \log x + \log c.$$

$$\therefore p = cx.$$

Now, substitute $p = cx$ *in* $y = \frac{1}{2}xp + \frac{1}{2}\frac{ax}{p}$ *we get,* $y = \frac{1}{2}cx^2 + \frac{1}{2}\frac{a}{c}$. *Which is a general solution.*

(2) $xp - y + x^{\frac{3}{2}} = 0.$

Solution:The given equation can be express in the form $y = f(x,p)$. *Therefore it is solvable for* y. $y = xp + x^{\frac{3}{2}}$. *Differentiate with respect to x we get,*

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}}.$$

$$\therefore p = p + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}} \implies \frac{dp}{dx} + \frac{3}{\sqrt{x}} = 0.$$

$$\therefore \int dp + \frac{3}{2}\int \frac{dx}{\sqrt{x}} = c \implies p + 3\sqrt{x} = c.$$

$$\therefore p = c - 3\sqrt{x}.$$

Now to eliminate p, substitute its value in equation $y = xp + x^{\frac{3}{2}}$ *we get,*

$$y = cx - 2x^{\frac{3}{2}}. \text{ Which is general solution.}$$

(3) Solve: $x + 2(xp - y) + p^2 = 0.$

Solution: The given equation can be express in the form $y = f(x,p)$. *Therefore it is solvable for* y. $y = \frac{1}{2}x + xp + \frac{1}{2}p^2$. *Differentiate with respect to x we get,*

$$\frac{dy}{dx} = p = \frac{1}{2} + p + x\frac{dp}{dx} + p\frac{dp}{dx}.$$

$$\therefore (x+p)\frac{dp}{dx} + \frac{1}{2} = 0.$$

Now put $x + p = u$ *we get* $1 + \frac{dp}{dx} = \frac{du}{dx}$.

$$\therefore u\left(\frac{du}{dx} - 1\right) + \frac{1}{2} = 0.$$

$$\therefore \frac{du}{dx} = \frac{2u-1}{2u} \implies \frac{2u}{2u-1}du = dx.$$

$$\therefore \int \left(1+\frac{1}{2u-1}\right) + \int dx + c.$$

$$\therefore u + \frac{1}{2}\log(2u-1) = x + c.$$

$$\therefore x + p + \frac{1}{2}\log(2x+2p-1) = x + c.$$

$$\therefore 2p + \frac{1}{2}\log(2x+2p-1) = c.$$

$$\therefore 2x + 2p - 1 = e^{2p-c}.$$

$$\therefore x = \frac{1}{2}e^{2p-c} + 1 - p.$$

Here we can not eliminate p from above equation. Hence, the general solution can be obtained from $y = \frac{1}{2}x + xp + \frac{1}{2}p^2$ *and* $x = \frac{1}{2}e^{2p-c} + 1 - p$.

3.3 Differential equations which are solvable for x.

If the differential equation of the form $F(x,y,p) = 0$ can be written as $x = f(y,p) = 0$, then it is said to be *solvable for* x. In order to solve these types of differential equation we differentiate with respect to y we get

$$\frac{dx}{dy} = p = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F\left(x,p,\frac{dp}{dy}\right). \tag{3.3}$$

Which is in variable p and y. Hence its solution is given by $g(y,p,c) = 0$. By eliminate p from equation (3.3) and $g(y,p,c)$ we get function $\phi(x,y,c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate p, then general solution can be obtained by taking $x = F_1(p,c)$ and $y = F_2(p,c)$. Where c is an arbitrary constant. Let us see following examples to understand this method.

Examples 3.3. *(1)Solve:* $y^2p^2 - 3xp + y = 0$.
Solution: The given differential equation is of the form $x = f(y,p)$, *where* $f(y,p) = \frac{1}{3}\left(\frac{y}{p} + y^2p\right)$. *Now differentiate with respect to y we get*

$$\therefore 3\frac{dx}{dy} = 3\frac{1}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2yp + y^2\frac{dp}{dy}.$$

$$\therefore 2yp - \frac{2}{p} + \left(y^2 - \frac{y}{p^2}\right)\frac{dp}{dy} = 0.$$

$$\therefore 2p(yp^2-1) + y(yp^2-1)\frac{dp}{dy} = 0.$$

$$\therefore (yp^2-1)\left(2p + y\frac{dp}{dy}\right) = 0.$$

We ignore $yp^2-1=0$ *we get and consider* $2p+y\frac{dp}{dy}=0$.

$$\therefore \frac{dp}{p}+2\frac{dy}{y}=0.$$

$$\therefore \log p+2\log y=\log c.$$

$$\therefore py^2=c \implies p=\frac{c}{y^2}.$$

Hence, substitute value of p *we get* $y^3-2cx+c^2=0$. *Which is a general solution. (2)Solve:* $x=p+\frac{1}{p}$. *Solution: It is easy too see that this differential equation is solvable for x. By differentiating with respect to y we get*

$$\therefore \frac{dx}{dy}=\frac{1}{p}=\frac{dp}{dy}-\frac{1}{p^2}\frac{dp}{dy}.$$

$$\therefore \frac{1}{p}=\left(1-\frac{1}{p^2}\right)\frac{dp}{dy} \implies \left(\frac{p^2-1}{p}\right)dp=dy.$$

$$\therefore \int\left(p-\frac{1}{p}\right)dp=\int dy+c.$$

$$\therefore y=\frac{p^2}{2}-\log p+c.$$

Where c is an arbitrary constant. Here, it is difficult to eliminate p. *Therefore, general solution can be obtained by taking* $x=p+\frac{1}{p}; y=\frac{p^2}{2}-\log p+c$.

Exercise-VI

1. $y=(1+p)x+p^2$. (Ans:$x=-2p+2+ce^{-p}; y=2-p^2+c(1+p)e^{-p}$.)
2. $xp-y+\sqrt{x}$. (Ans:$y=cx+2\sqrt{x}$.)
3. $y=2p+3p^2$. (Ans:$x=2p+3p^2; y=2\log p+3p+c$.)
4. $y+px=p^2x^4$. (Ans:$xy=c^2x-c$.)
5. $y^2p^2-3xp+y=0$. (Ans:$y^3-3cx+c^2=0$.)
6. $y=2px-p^2$. (Ans:$x=\frac{2}{3}p+cp^{-2}; y=\frac{1}{3}p^2+\frac{2c}{p}$.)
7. $y^2+p^2=0$. (Ans:$y=\pm\sin(x+c)$.)
8. $p^2y+2px=y$. (Ans:$y^2=2cx+c^2$.)
9. $y-2px=\tan^{-1}p$. (Ans:$2\sqrt{cx}+\tan^{-1}c$.)
10. $xp^2-yp-y=0$. (Ans:$c(1+p)e^p; y=cp^2c^p$.)
11. $y=x+a\tan^{-1}p$. (Ans:$x+c=\frac{a}{2}\left[\log(p-1)-\frac{1}{2}\log(1+p^2)-\tan^{-1}p\right]; y=x+a\tan^{-1}p$.)

12. $x^2 = a^2(1+p^2)$. (Ans:$x = a\sqrt{1+p^2}; y = \frac{a}{2}\left|p\sqrt{1+p^2} - \log(p+\sqrt{p+\sqrt{p^2+1}})\right| + c$.)

13. $p^2 = (p-1)y$. (Ans:$x = \log(p-1) + \frac{1}{p-1} + c; y = \frac{p^2}{p-1}$.)

14. $x = \frac{p}{1+p^2} + \tan^{-1} p$. (Ans:$x = x = \frac{p}{1+p^2} + \tan^{-1} p; y = c - \frac{1}{1+p}$.)

15. $p^2 - 4xyp + 8p^2 = 0$. (Ans:$c(c-4x)^2 = 64y$.)

3.4 Clairaut's differential equations.

Definition 3.4. *A differential equation of the form $y = px + f(p)$ is known as* Clairaut's differential equation.

It is easy to see that Clairaut's differential equation $y = px + f(p)$ is solvable for y. Hence, in order to solve we differentiate with respect to x on both sides we get,

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\implies (f'(p)+x)\frac{dp}{dx} = 0.$$

$$\implies \frac{dp}{dx} = 0 \text{ or } x + f'(p) = 0.$$

By taking the case $\frac{dp}{dx} = 0$ we get $p = \frac{dy}{dx} = c$. Where c is an arbitrary constant. Thus, by eliminating p from Clairaut's equation we have the family of straight lines given by $y = cx + f(c)$, as the general solution of Clairaut's differential equation. The later case $x + f'(p) = 0$ defines only one solution $y(x)$, so-called singular solution, whose graph is the envelope of the graphs of the general solutions. The singular solution is usually represented using parametric notation, as $(x(p), y(p))$, where p represents $\frac{dy}{dx}$.

Examples 3.5. *(1) Solve: $x^2(y - px) = yp^2$.*
Solution: The given differential equation is not Clairaut's differential equation, but by taking $x^2 = u$ and $y^2 = v$ we can convert it into the Clairaut's form. $x^2 = u \implies 2xdx = du$, and $y^2 = v \implies 2ydy = dv$. $\therefore \frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \implies p = \frac{x}{y}\frac{dv}{du}$. Now given equation reduces to

$$x^2\left(y - \frac{x^2}{y}\frac{dv}{du}\right) = y\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2.$$

$$y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2.$$

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2.$$

Which is Clairaut's differential equation. Hence, the general solution can be obtained by taking $\frac{dv}{du} = c$. *Hence* $v = cu + c^2$ *and* $y^2 = cx^2 + c^2$ *is the general solution.*
(2) Solve: $\sin px \cos y = \cos px \sin y + p$.
Solution: The given differential equation is not of the Clairaut's form. Notice that,

$$\sin px \cos y - \cos px \sin y = p \implies \sin(px - y) = p$$

.

$$px - y = \sin^{-1} p.$$

$$y = px + \sin^{-1} p, \textit{which is in Clairaut's form.}$$

$$p = c \implies y = cx + \sin^{-1} c, \textit{ which is a general solution.}$$

(3)Solve: $e^{4x}(p-1) + e^{2y}p^2 = 0$.
Solution: The given differential equation is not of the Clairaut's form, but by taking $e^{2x} = u$ *and* $e^{2y} = v$ *we can convert it into Clairaut's form.*

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2. \textit{ Which is in Clairaut's form.}$$

$$\frac{dv}{du} = c \implies v = uc + c^2 \implies e^{2y} = ce^{2x} + c^2. \textit{ Which is a general solution.}$$

3.5 Lagrange's differential equation.

Definition 3.6. *A differential equation of the form* $y = xf(p) + F(p)$ *is known as* Lagrange's differential equation.

It is easy to see that Lagrange's differential equation is solvable for y. Hence, in order to solve this differential equation we differentiate with respect to x on both sides we get

$$\frac{dy}{dx} = p = f(p) + xf'(p)\frac{dp}{dx} + F'(p)\frac{dp}{dx}.$$

$$\therefore p - f(p) = [xf'(p) + F'(p)]\frac{dp}{dx}.$$

$$\therefore \frac{dx}{dp} = \frac{xf'(p) + F'(p)}{p - f(p)}.$$

$$\therefore \frac{dx}{dp} = \frac{f'(p)}{p - f(p)}x + \frac{F'(p)}{p - f(p)}.$$

Which is linear in x and p. So it can be solved by method of linear differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only.

Remark 3.7. *1. An equation of the form* $x = yf(q) + F(q)$, *where* $q = \frac{dx}{dy}$ *is also known as Lagrange's differential equation and also can be solved by using method to solve differential equation which are solvable for* x.

Linear Differential Equations of Second and Higher Order

11.1 Introduction

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)=0$ in which the dependent variable $y(x)$ and its derivatives viz. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

11.2 Linear Differential Equations (LDE) with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$$k_0 \frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + - - - - - + k_{n-1} \frac{dy}{dx} + k_n y = F(x)$$

where $k's$ are constant and $F(x)$ is a function of x alone or constant.
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ..., $D \equiv \frac{d}{dx}$ are called differential operators.

11.3 Solving Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is particular integral.

When $F(x) = 0$, then solution of equation $f(D)y = 0$ is given by $y =$ C.F

11.3.1 Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + - - - - - + k_{n-1} m + k_n = 0$ ......③

Step 2: Solve the auxiliary equation given by ③

I. If the n roots of A.E. are real and distinct say $m_1, m_2, \dots m_n$

C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II. If two or more roots are equal i.e. $m_1 = m_2 = \dots = m_k,\ k \le n$

C.F. = $(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x} + \cdots + c_n e^{m_n x}$

III. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$

C.F. = $e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

IV. If 2 pairs of imaginary roots are equal i.e. $m_1 = m_2 = \alpha + i\beta$,

$m_3 = m_4 = \alpha - i\beta$

C.F. = $e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + \cdots + c_n e^{m_n x}$

Example 1 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

C.F. = $c_1 e^{3x} + c_2 e^{5x}$

Since $F(x) = 0$, solution is given by $y = $ C.F

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$

Example 2 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ ....①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. $= c_1e^x + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Example 3 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ ....①

By hit and trial $(m - 1)$ is a factor of ①

∴① May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m - 1) - 9m^2(m - 1) + 26m(m - 1) - 24(m - 1) = 0$

$\Rightarrow (m - 1)(m^3 - 9m^2 + 26m - 24) = 0$ ......②

By hit and trial $(m - 2)$ is a factor of ②

∴② May be rewritten as

$(m - 1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m - 1)[m^2(m - 2) - 7m(m - 2) + 12(m - 2)] = 0$

$\Rightarrow (m - 1)(m^2 - 7m + 12)(m - 2) = 0$

$\Rightarrow (m - 1)(m - 3)(m - 4)(m - 2) = 0$

$\Rightarrow m = 1,2,3,4$

C.F. $= c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example 4 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow \ m = 0, -1, -1$

C.F. = $c_1 + \ (c_2 + c_3x)e^{-x}$

Since $F(x) = 0$, solution is given by $\ y = $ C.F

$\Rightarrow y = \ \ c_1 + \ (c_2 + c_3x)e^{-x}$

Example 5 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + \ y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m + 1)^2(m - 1)^2 = 0$

$\Rightarrow \ m = -1, -1, 1, 1$

C.F. = $(c_1 + c_2x)e^{-x} + \ (c_3 + c_4x)e^{x}$

Since $F(x) = 0$, solution is given by $\ y = $ C.F

$\Rightarrow y = \ \ (c_1 + c_2x)e^{-x} + \ (c_3 + c_4x)e^{x}$

Example 6 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ ....①

By hit and trial $(m + 2)$ is a factor of ①

∴① May be rewritten as

$m^3 + 2m^2 - 2m^2 - \ 4m + 2m + \ 4 = 0$

$\Rightarrow m^2(m + 2) - 2m(m + 2) + 2(m + 2) = 0$

$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$

$\Rightarrow \ m = -2, 1 \pm i$

C.F. = $c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

Example 7 Solve the differential equation: $(D^2 - 2D + 5)^2 y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$........①

Solving ①, we get

$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$

C.F. = $e^{x}[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = e^{x}[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$

Example 8 Solve the differential equation: $(D^2 + 4)^3 y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3$........①

Solving ①, we get

$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i$

C.F. = $(c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

11.3.2 Shortcut Rules for Finding Particular Integral (P.I.)

Consider the equation $(D)y = F(x)$, $F(x) \neq 0$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Then P.I = $\frac{1}{f(D)} F(x)$. Clearly P.I. = 0 if $F(x) = 0$

Case I: When $F(x) = e^{ax}$

Use the rule P.I $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, $f(a) \neq 0$

In case of failure i.e. if $f(a) = 0$

P.I = $x\frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$, P.I. = $x^2\frac{1}{f''(a)}e^{ax}$, $f''(a) \neq 0$ and so on

Example 9 Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Solution: $\Rightarrow (D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. = $e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. = $\frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^{2x} = \frac{1}{f(2)}e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2) + 10}e^{2x} = \frac{1}{10}e^{2x}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10}e^{2x}$

Example 10 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. = $\frac{1}{f(D)}F(x) = \frac{1}{f(D)}e^x$, putting $D = 1$, $f(1) = 0$

$\therefore$P.I = $x\frac{1}{f'(D)}e^x$ $\quad\quad$ $\because$ P.I. = $x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. = $x\frac{1}{2D+1}e^x = \frac{1}{f'(1)}e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. = $\frac{xe^x}{3}$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{x} + \frac{xe^x}{3}$

Example 11 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x)$

$= \frac{1}{f(D)} (\sinh(2x+1) + 4^x)$

$= \frac{1}{D^2-4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2-4}\left(e^{x\log 4}\right)$

$\because \sinh x = \frac{e^x - e^{-x}}{2}$ and $4^x = e^{x \log 4}$

$= \frac{e}{2}\frac{1}{D^2-4} e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2-4} e^{-2x} + \frac{1}{D^2-4} e^{x\log 4}$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$\therefore$ P.I. $= \frac{e}{2} x \frac{1}{2D} e^{2x} - \frac{e^{-1}}{2} x \frac{1}{2D} e^{-2x} + \frac{1}{(log4)^2 - 4} e^{x \log 4}$

$\because$ P.I. $= x\frac{1}{f'(a)} e^{ax}$ if $f(a) = 0$

Now putting $D = 2, -2$ in first two terms respectively

$\Rightarrow$P.I. $= \frac{ex}{8} e^{2x} + \frac{e^{-1}x}{8} e^{-2x} + \frac{4^x}{(log4)^2 - 4}$ $\quad \because e^{x\log 4} = 4^x$

$\Rightarrow$P.I. $= \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log4)^2 - 4}$

$\Rightarrow$P.I. $= \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2 - 4}$ $\quad \because \cosh x = \frac{e^x + e^{-x}}{2}$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(\log 4)^2 - 4}$$

Case II: When $F(x) = Sin\,(ax+b) or\ Cos\,(ax+b)$

If $F(x) = Sin\,(ax+b)\ or\ Cos\,(ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D$, $D^4 = (D^2)^2 = a^4$,.......

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e. if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b)$, $f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.$= x^2\frac{1}{f''(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b)$, $f''(-a^2) \neq 0$

Example 12 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2{,}1$

C.F. $= c_1e^{-2x} + c_2e^x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\sin x = \frac{1}{D^2+D-2}\sin x$

putting $D^2 = -1^2 = -1$

P.I $= \frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{-1}{10}(D\sin x + 3\sin x)$

$= \frac{-1}{10}(\cos x + 3\sin x)$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1e^{-2x} + c_2e^{x} - \frac{1}{10}(\cos x + 3\sin x)$

Example 13 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2 x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. $= e^{-x}(c_1 + c_2x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}cos^2 x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1}e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

P.I $= \frac{1}{2} + \frac{1}{2}\frac{1}{2D-3}\cos 2x$

$= \frac{1}{2} + \frac{1}{2}\frac{2D+3}{4D^2-3^2}\cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$, Putting $D^2 = -4$

$\therefore$ P.I. $= \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Now $y = $ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2x) + \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Example 14 Solve the differential equation: $(D^2 + 9)y = \sin 2x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

C.F. $= c_1\cos 3x + c_2\sin 3x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\sin 2x\cos x = \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9\ ) = 0$ for the 1st term

$\therefore$ P.I. $= \frac{1}{2}\, x \frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$

$\because$P.I. $= x\frac{1}{f'(-a^2)}\ Sin\ (ax+b)\ .\ f'(-a^2) \neq 0$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n is a positive integer

P.I $= \frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 \pm \phi(D)]^{-1}$
2. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to n^{th} degree as $(n+1)^{th}$ derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(D)]^{-1}$in ascending powers of D

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Example 15 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2 - 1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. $= c_1e^{x} + c_2e^{-x}$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}(5x-2)$

$= \frac{1}{-(1-D^2)}(5x-2)$

$= -(1-D^2)^{-1}(5x-2)$

$= -[1+D^2+\cdots](5x-2)$

$= -(5x-2)$

$\therefore$P.I $= -5x+2$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{x} + c_2e^{-x} - 5x + 2$

Example 16 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow\ m^2(m^2+4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. $= (c_1 + c_2x) + (c_3\cos 2x + c_4\sin 2x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2+1)$

$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+1)$

$= \frac{1}{4D^2}\left[1-\frac{D^2}{4}+\cdots\right](x^2+1)$

$= \frac{1}{4D^2}\left(x^2+1-\frac{1}{2}\right)$

$= \frac{1}{4D^2}\left(x^2 + \frac{1}{2}\right)$

$= \frac{1}{4D}\int\left(x^2 + \frac{1}{2}\right)dx$

$= \frac{1}{4D}\left(\frac{x^3}{3} + \frac{x}{2}\right)$

$= \frac{1}{4}\int\left(\frac{x^3}{3} + \frac{x}{2}\right)dx$

$\therefore$ P.I $= \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x) + (c_3\cos 2x + c_4 \sin 2x) + \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Example 17 Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. $= e^{3x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-6D+9}(1 + x + x^2)$

$= \frac{1}{9\left(1-\frac{2D}{3}+\frac{D^2}{9}\right)}(1 + x + x^2)$

$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1}(1 + x + x^2)$

$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left(1 + x + x^2 + \frac{2}{3} + \frac{4x}{3} + \frac{2}{3}\right)$

$\therefore$ P.I $= \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = e^{3x}(c_1 + c_2 x) + \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Case IV: When $F(x) = e^{ax}g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)} e^{ax} g(x) = e^{ax}\left(\frac{1}{f(D+a)}\, g(x)\right)$

Example 18 Solve the differential equation: $(D^2 + 2)y = x^2 e^{3x}$

Solution: Auxiliary equation is: $m^2 + 2 = 0$

$\Rightarrow m^2 = -2$

$\Rightarrow m = \pm\sqrt{2}i$

C.F. $= (c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x))$

P.I. $= \frac{1}{f(D)}\, F(x) = \frac{1}{D^2+2} x^2 e^{3x}$

$$= e^{3x} \frac{1}{(D+3)^2+2} x^2$$

$$= e^{3x} \frac{1}{D^2+6D+11} x^2$$

$$= \frac{e^{3x}}{11} \frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)} x^2$$

$$= \frac{e^{3x}}{11}\left(1 + \left(\frac{6D}{11} + \frac{D^2}{11}\right)\right)^{-1} x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \left(\frac{6D}{11} + \frac{D^2}{11}\right) + \left(\frac{6D}{11} + \frac{D^2}{11}\right)^2 + \cdots\right] x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} + \cdots\right] x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121} + \cdots\right] x^2$$

$$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$$

$\therefore\ P.I = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Example 19 Solve the differential equation: $(D^3 + 1)y = e^{2x}\sin x$

Solution: Auxiliary equation is: $m^3 + 1 = 0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$

C.F. $= c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)}\, F(x) = \frac{1}{D^3+1} e^{2x}\sin x$

$= e^{2x}\frac{1}{(D+2)^3+1}\sin x$

$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$

$= e^{2x}\frac{1}{-D-6+12D+9}\sin x$, Putting $D^2 = -1$

$= e^{2x}\frac{1}{11D+3}\sin x$

$= e^{2x}\frac{11D-3}{121D^2-9}\sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D - 3)\sin x$, Putting $D^2 = -1$

$\therefore$ P.I $= -\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$-\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Example 20 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{f(D)} F(x)$$

$$= \frac{1}{f(D)}(x\sinh x)$$

$$= \frac{1}{D^2-4}\left(x\frac{e^x - e^{-x}}{2}\right) \qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{D^2-4}\left(x\frac{e^x}{2} - x\frac{e^{-x}}{2}\right)$$

$$= \frac{e^x}{2}\frac{1}{(D+1)^2-4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2-4}x$$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D-3)}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x$$

$$= \frac{e^x}{2}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}x - \frac{e^{-x}}{2}\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}x$$

$$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x$$

$$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$$

$$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$\therefore \text{P.I.} = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Example 21 Solve the differential equation: $(D^2+1)y = x^2 \sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

C.F. $= c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} x^2 \sin 2x$

$= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x}$

Now $\frac{1}{D^2+1} x^2 e^{i2x} = e^{i2x} \frac{1}{(D+2i)^2+1} x^2$

$$= e^{i2x} \frac{1}{D^2+4i^2+4iD+1} x^2$$

$$= e^{i2x} \frac{1}{D^2+4iD-3} x^2$$

$$= e^{i2x} \frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$= -\frac{1}{3}(\cos 2x + i \sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. $= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x} = -\frac{1}{3}\left(\frac{8x}{3}\cos 2x + \left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$$= -\frac{8x}{9}\cos 2x + \frac{1}{27}(26-9x^2)\sin 2x$$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{8x}{9} \cos 2x + \frac{1}{27}(26 - 9x^2) \sin 2x$$

Example 22 Solve the differential equation: $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2$

$\Rightarrow m = 2,2$

C.F. $= (c_1 + c_2 x)e^{2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-4D+4} x^2 e^{2x} \sin 2x$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= e^{2x} \frac{1}{D}\left[(x^2)\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right)\right]$$

$$= e^{2x} \frac{1}{D}\left[-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x\right]$$

$$= e^{2x}\left[-\frac{1}{2}\int x^2 \cos 2x + \frac{1}{2}\int x \sin 2x \, dx + \frac{1}{4}\int \cos 2x \, dx\right]$$

$$= e^{2x}\left[-\frac{1}{2}\left[(x^2)\left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(\frac{-\sin 2x}{8}\right)\right] + \right.$$

12x−cos2x2−1−sin2x4+14sin2x2

$$\therefore \text{P.I.} = e^{2x}\left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x\right]$$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2 x)e^{2x} + e^{2x}\left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x\right]$$

Case V: When $F(x) = x\, g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}(x\, g(x)) = x \frac{1}{f(D)} g(x) + \left(\frac{d}{dD} \frac{1}{f(D)}\right) g(x)$

Example 23 Solve the differential equation: $(D^2+9)y = x\cos x$

Solution: Auxiliary equation is: $m^2+9=0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. = $(c_1\cos 3x + c_2\sin 3x)$

P.I. $= \frac{1}{f(D)}\, F(x) \;= \frac{1}{D^2+9}\, x\cos x$

$= x\frac{1}{D^2+9}\cos x + \frac{-2D}{(D^2+9)^2}\cos x$

$= x\frac{1}{-1+9}\cos x + \frac{-2D}{(-1+9)^2}\cos x$, Putting $D^2=-1$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$\therefore$ P.I. $= \frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x + \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example 24 Solve the differential equation:

$$(D^2-1)y = x\sin x + (1+x^2)e^x$$

Solution: Auxiliary equation is: $m^2-1=0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^x + c_2e^x$

P.I. $= \frac{1}{f(D)}\, F(x) \;= \frac{1}{D^2-1}\,(x\sin x + (1+x^2)e^x)$

$= \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$

Now $\frac{1}{D^2-1}x\sin x == x\frac{1}{D^2-1}\sin x + \frac{-2D}{(D^2-1)^2}\sin x$

$$= x\frac{1}{-1-1}\sin x + \frac{-2D}{(-1-1)^2}\sin x. \qquad \text{Putting } D^2 = -1$$

$$= -\frac{1}{2}(x\sin x + \cos x)$$

Also $\frac{1}{D^2-1}(1+x^2)e^x = e^x\frac{1}{(D+1)^2-1}(1+x^2)$

$$= e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= e^x\frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right](1+x^2)$$

$$= e^x\frac{1}{2D}\left[1+x^2-x+\frac{1}{2}\right]$$

$$= e^x\frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$$

$\therefore$ P.I $= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^x + c_2e^x - \frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$$\frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$$

Example 25 Solve the differential equation: $(D^2+3D+2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\Rightarrow m = -1, -2$

C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$

$= e^{-x} \int D e^{e^x} dx - e^{-2x} \int e^x D e^{e^x} dx$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2[nd] term by parts

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int D e^{e^x} dx\right]$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$

$\therefore$ P.I. $= e^{-2x} e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

Example 26 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$= \frac{1}{(D-2i)(D+2i)} \tan 2x$

$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right)$ ……①

Now $\frac{1}{D-2i}\tan 2x = e^{2ix}\int e^{-2ix}\tan 2x\,dx$

$$= e^{2ix}\int(\cos 2x - i\sin 2x)\tan 2x\,dx$$

$$= e^{2ix}\int(\sin 2x - i\frac{sin^2 2x}{cos2x})\,dx$$

$$= e^{2ix}\int\left(\sin 2x - i\frac{1-cos^2 2x}{cos2x}\right)dx$$

$$= e^{2ix}\int(\sin 2x - i\sec 2x + i\cos 2x\,)\,dx$$

$$= e^{2ix}\left(-\frac{1}{2}\cos 2x - \frac{i}{2}log|\sec 2x + \tan 2x| + \frac{i}{2}\sin 2x\right)$$

$$\therefore \frac{1}{D-2i}\tan 2x = e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots ②$$

Replacing i by $-i$

$$\frac{1}{D+2i}\tan 2x = e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots ③$$

Using ②and ③ in ①

$$\text{P.I.} = \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$-\frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$=\frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix}log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix}log|\sec 2x + \tan 2x|\right]$$

$$= \frac{1}{4i}\left[-i\frac{e^{2ix}+e^{-2ix}}{2}log|\sec 2x + \tan 2x|\right]$$

$$\therefore \text{P.I} = -\frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Complete solution is: $y =$ C.F. + P.I.

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Exercise 11A

Solve the following differential equations:

1. $(D^3 + D^2 - 5D + 3)y = 0$ Ans. $y = (c_1x + c_2)e^x + c_3e^{-3x}$

2. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{3x}$ Ans. $y=c_1e^{2x}+c_2e^{3x}+e^{3x}(x-1)$

3. $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=e^{x}\cosh 2x$

Ans. $y=c_1e^{-3x}+c_2e^{2x}+\frac{1}{12}e^{3x}-\frac{1}{12}e^{-x}$

4. $(D-1)^2(D^2+1)^2y=e^{x}$

Ans. $y=(c_1x+c_2)e^{x}+(c_3x+c_4)\cos x+(c_5x+c_6)\sin x+\frac{x^2}{8}e^{x}$

5. $(D^2-6D+9)y=x^2+2e^{2x}$

Ans. $y=(c_1x+c_2)e^{3x}+\frac{1}{9}\left(x^2+\frac{4x}{8}+\frac{2}{3}\right)+2e^{2x}$

6. $(D^2+D-2)y=x+\sin x$

Ans. $y=c_1e^{-2x}+c_2e^{x}-\frac{1}{4}(2x+1)-\frac{1}{10}(\cos x+3\sin x)$

7. $(D^2+D)y=(1+e^{x})^{-1}$

Ans. $y=c_1+c_2e^{-x}+x-(1+e^{-x})\log(1+e^{x})$

8. $(D^2+5D+6)y=e^{-2x}sec^2x\,(1+2\tan x)$

Ans. $y=c_1e^{-2x}+c_2e^{-3x}+e^{-2x}(\tan x-1)$

9. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}-12y=(x-1)e^{2x}$

Ans. $y=c_1e^{2x}+c_2e^{-6x}+\frac{e^{2x}}{8}\left(\frac{x^2}{2}-\frac{9x}{8}\right)$

10. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=4x+e^{3x}$, given $y=1,\frac{dy}{dx}=-1$ when $x=0$

Ans. $y=-\frac{1}{2}e^{x}-2e^{2x}+2x+3+\frac{e^{3x}}{2}$

11.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

11.4.1 Cauchy's Linear Differential Equation

The differential equation of the form:

$$k_0\,x^n\frac{d^ny}{dx^n}+k_1\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+----+k_{n-1}\,x\frac{dy}{dx}+k_n\,y=F(x)$$

is called Cauchy's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \quad \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example 27 Solve the differential equation:

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\cos(\log x),\ x > 0 \qquad \text{.......①}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$ and $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

∴① May be rewritten as

$(D(D-1)(D-2) + 3D(D-1) + D + 8)y = 13\cos t$

$\Rightarrow (D^3 + 8)y = 13\cos t \ ,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

$\text{C.F.} = c_1e^{-2t} + e^t\left(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right)$

$\text{P.I.} = \frac{1}{f(D)}F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t \qquad$ Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{1}{5}(8\cos t + D\cos t)$

$= \frac{1}{5}(8\cos t - \sin t)$

$= \frac{1}{5}(8\cos(\log x) - \sin(\log x))$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right) +$$

$$\frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Example 28 Solve the differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1+x^2} \qquad \text{.......①}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ ① May be rewritten as

$(D(D-1) + D - 1)y = \frac{e^{3t}}{1+e^{2t}}$

$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}},\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^{-t} + c_2e^t$

$= \frac{c_1}{x} + c_2x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{(D-1)(D+1)}\frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$

$= \frac{1}{2}\left(e^{t}\int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^{t}\frac{e^{3t}}{1+e^{2t}}dt\right) \because \frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$

$= \frac{1}{2}\left(e^{t}\int \frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int \frac{e^{4t}}{1+e^{2t}}dt\right)$

Put $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$\therefore$ P.I $= \frac{1}{4}\left(e^{t}\int \frac{1}{1+u}du - e^{-t}\int \frac{u}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int \frac{1+u-1}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int \left(1-\frac{1}{1+u}\right)du\right)$

$= \frac{1}{4}(e^{t}\log(1+u) - e^{-t}(u - \log(1+u))$

$= \frac{1}{4}(e^{t}\log(1+e^{2t}) - e^{-t}(e^{2t} - \log(1+e^{2t}))$

$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2 - \log(1+x^2)\right)$

$= \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = \frac{c_1}{x} + c_2x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

$\Rightarrow y = \frac{c_1}{x} + c_3x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2)$, $c_3 = c_2 - \frac{1}{4}$

Example 29 Solve the differential equation:

$x^2D^2 - 2xD - 4y = x^2 + 2\log x$, $x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^{t}$ $\therefore \log x = t$

$\Rightarrow xD = \delta y, x^2D^2 = \delta(\delta-1)y$, $\delta \equiv \frac{d}{dt}$

$\therefore$① May be rewritten as

$(\delta(\delta-1)-2\delta-4)y = e^{2t}+2t$

$\Rightarrow (\delta^2-3\delta-4)y = e^{2t}+2t$

Auxiliary equation is: $m^2-3m-4=0$

$\Rightarrow (m+1)(m-4)=0$

$\Rightarrow m=-1,4$

C.F. $= c_1e^{-t}+c_2e^{4t}$

$= \frac{c_1}{x}+\frac{c_2}{x^4}$

P.I. $= \frac{1}{f(\delta)}F(x) = \frac{1}{\delta^2-3\delta-4}(e^{2t}+2t)$

$= \frac{1}{\delta^2-3\delta-4}e^{2t}+\frac{1}{\delta^2-3\delta-4}2t$

$= \frac{1}{-6}e^{2t}+2\frac{1}{-4\left(1-\frac{\delta^2}{4}+\frac{3\delta}{4}\right)}t$ Putting $\delta=2$ in the 1st term

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left(1-\left(\frac{\delta^2}{4}-\frac{3\delta}{4}\right)\right)^{-1}t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[1+\frac{\delta^2}{4}-\frac{3\delta}{4}+\cdots\right]t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[t-\frac{3}{4}\right]$

$\therefore$ P.I. $= \frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = \frac{c_1}{x}+\frac{c_2}{x^4}\frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

11.4.2 Legendre's Linear Differential Equation

The differential equation of the form: $k_0(ax+b)^n\frac{d^ny}{dx^n}+$

$$k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_{n-1}(ax+b)\frac{dy}{dx}+k_ny = F(x)$$

is called Legendre's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$(ax+b) = e^t \quad \Rightarrow t = \log(ax+b)$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{a}{ax+b}$

$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$, where $D \equiv \frac{d}{dt}$

Similarly $(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

Example 30 Solve the differential equation:

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1 \quad \text{.......①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(3x+2) = e^t \quad \therefore t = \log(3x+2)$

$\Rightarrow (3x+2)\frac{dy}{dx} = 3Dy$, $(3x+2)^2\frac{d^2y}{dx^2} = 3^2D(D-1)y$

Also $3x^2+4x+1 = \frac{1}{3}(9x^2+12x+3)$

$= \frac{1}{3}((3x)^2 + 2.3.2x + 4 - 4 + 3)$

$= \frac{1}{3}((3x+2)^2 - 1)$

$= \frac{1}{3}(e^{2t} - 1)$

$\therefore$ ① May be rewritten as

$(9D(D-1) + 9D - 36)y = \frac{1}{3}(e^{2t}-1)$

$\Rightarrow 9(D^2-4)y = \frac{1}{3}(e^{2t}-1)$

Auxiliary equation is: $9(m^2-4) = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{-2t} + c_2e^{2t}$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{9(D^2-4)} \frac{1}{3}(e^{2t} - 1)$$

$$= \frac{1}{27}\left(\frac{1}{(D^2-4)} e^{2t} - \frac{1}{(D^2-4)} e^{0t}\right)$$

$$= \frac{1}{27}\left(\frac{t}{2.2} e^{2t} - \frac{1}{(0-4)} e^{0t}\right)$$, Putting $D = 2$ in 1st term, it is a case of failure $\therefore \frac{1}{(D^2-4)} e^{2t} = t \frac{1}{f'(2)} e^{2x}$, also $D = 0$ in the 2nd term.

$$\therefore \text{P.I.} = \frac{1}{27}\left(\frac{t}{4} e^{2t} + \frac{1}{4}\right)$$

$$= \frac{1}{27}\left(\frac{\log(3x+2)}{4} (3x+2)^2 + \frac{1}{4}\right)$$

$$= \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Example 31 Solve the differential equation:

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1)), \quad x > -1 \text{ ....①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(x+1) = e^t \quad \therefore t = \log(x+1)$

$$\Rightarrow (x+1)\frac{dy}{dx} = Dy, \ (x+1)^2 \frac{d^2y}{dx^2} = 1^2 D(D-1)y$$

∴① May be rewritten as

$$(D(D-1) + D + 1)y = 2\sin t$$

$$\Rightarrow (D^2+1)y = 2\sin t$$

Auxiliary equation is: $(m^2+1) = 0$

$$\Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

$$= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} 2\sin t$

$= 2t \frac{1}{2D} \sin t$, Putting $D^2 = -1$, case of failure

$\therefore \frac{1}{(D^2+1)} \sin t = t \frac{1}{f'(D)} \sin t$

$= t \int \sin t \, dt = -t \cos t$

$\therefore$ P.I$= -\log(x+1) \cos(\log(x+1))$

Complete solution is: y = C.F. + P.I

$y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) - \log(x+1) \cos(\log(x+1))$

11.5 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2^{nd} and higher order differential equations with constant coefficients as well as variable coefficients.

Working rule

Consider a 2^{nd} order linear differential equation:

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$ ....①

1. Find complimentary function given as: C.F. = $c_1y_1 + c_2y_2$.

   where y_1 and y_2 are two linearly independent solutions of ①

2. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, W is called Wronskian of y_1 and y_2

3. Compute $u_1 = -\int \frac{y_2 F(x)}{W} dx$, $u_2 = \int \frac{y_1 F(x)}{W} dx$
4. Find P.I. = $u_1y_1 + u_2y_2$
5. Complete solution is given by: y = C.F. + P.I

Note: Method is commonly used to solve 2^{nd} order differential but it can be extended to solve differential equations of higher orders.

Example 32 Solve the differential equation: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

Auxiliary equation is: $(m^2+1) = 0$

$\Rightarrow\ m = \pm i$

C.F. = $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \sin x\ \operatorname{cosec} x\ dx = -\int 1 dx = -x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \cos x\ \operatorname{cosec} x\ dx = \int \cot x\, dx = log|\sin x|$

$\therefore$ P.I= $u_1 y_1 + u_2 y_2$

$= -x \cos x + \sin x\, log|\sin x|$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x\, log|\sin x|$

Example 33 Solve the differential equation: $(D^2 - 2D + 1)y = e^x$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - 2m + 1) = 0$

$\Rightarrow\ m = 1,1$

C.F. = $(c_1 + c_2 x)e^x = c_1 e^x + c_2 x\, e^x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^x$ and $y_2 = x\, e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x\, e^x \\ e^x & x\, e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x e^x e^x}{e^{2x}}\, dx = -\int x dx = -\frac{x^2}{2}$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x}{e^{2x}}\, dx = \int 1 dx = x$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$$= -\frac{x^2}{2}e^x + x^2e^x = \frac{x^2}{2}e^x$$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2\, x)e^x + \frac{x^2}{2}e^x$$

Example 34 Solve the differential equation: $\frac{d^2y}{dx^2} + 4y = x\sin 2x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2 + 4)y = x\sin 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1\cos 2x + c_2\sin 2x = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2F(x)}{W}dx = -\frac{1}{2}\int x\sin^2 2x\ dx = -\frac{1}{4}\int x(1-\cos 4x)\,dx$$

$$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right]$$

$$= \left[-\frac{x^2}{8} + \frac{x\sin 4x}{16} + \frac{\cos 4x}{64}\right]$$

$$u_2 = \int \frac{y_1F(x)}{W}dx = \frac{1}{2}\int x\sin 2x\cos 2x\ dx = \frac{1}{4}\int x\sin 4x\ dx$$

$$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right]$$

$$= \left[-\frac{x\cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$$= \cos 2x\left[-\frac{x^2}{8} + \frac{x\sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x\left[-\frac{x\cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x)$

$- \frac{x^2}{8}\cos 2x \ = \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: $y \ = \text{C.F.} + \text{P.I}$

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example 35 Solve the differential equation: $(D^2 - D - 2)y = e^{(e^x + 3x)}$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0$

$\Rightarrow \ m = -1,2$

C.F. = $c_1 e^{-x} + c_2 \, e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = \ e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x+3x)}}{3e^x} dx = \ -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = t dt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x+3x)}}{3e^x} dx = \ \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx \ \frac{1}{3}\int e^x e^{e^x} dx \ = \frac{e^{e^x}}{3}$

$\therefore$ P.I= $u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y \ = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$$

Example 36 Given that $\therefore y_1 = x$ and $y_2 = \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$

Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$

Given that $\therefore y_1 = x$ and $y_2 = \frac{1}{x}$

$\therefore$C.F. $= c_1y_1 + c_2y_2 = c_1x + \frac{c_2}{x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$u_1 = -\int \frac{y_2F(x)}{W}dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\,dx = \frac{1}{2}\int\frac{1}{x}dx = \frac{1}{2}\log x$

$u_2 = \int \frac{y_1F(x)}{W}dx = -\int x.\frac{1}{x}.\frac{x}{2}\,dx = -\frac{x^2}{4}$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$= \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1x + \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example 37 Solve the differential equation: $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2\log x$

using method of variation of parameters.

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given differential equation may be rewritten as

$(D(D-1)-4D+6)y = te^{2t}$

$\Rightarrow (D^2-5D+6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3)=0$

$\Rightarrow m = 2,3$

C.F. $= c_1e^{2t}+c_2e^{3t} = c_1y_1+c_2y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$u_1 = -\int \frac{y_2F(t)}{W}dt = -\int \frac{e^{3t}te^{2t}}{e^{5t}}\,dt = -\int t\,dt = -\frac{t^2}{2}$

$u_2 = \int \frac{y_1F(t)}{W}dt = \int \frac{e^{2t}te^{2t}}{e^{5t}}\,dt = \int te^{-t}dt = [(t)(-e^{-t})-(1)(e^{-t})]$

$= -te^{-t}-e^{-t}$

$\therefore$ P.I$= u_1y_1+u_2y_2$

$= -\frac{t^2}{2}e^{2t}-(te^{-t}+e^{-t})e^{3t}$

$= -\frac{t^2}{2}e^{2t}-te^{2t}-e^{2t} = -e^{2t}\left(\frac{t^2}{2}+t+1\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^{2t}+c_2e^{3t}-e^{2t}\left(\frac{t^2}{2}+t+1\right)$

or $y = c_1x^2+c_2\,x^3-x^2\left(\frac{(\log x)^2}{2}+\log x+1\right)$

$\Rightarrow y = c_3x^2+c_2\,x^3-\frac{x^2}{2}(\log x)^2-x^2\log x,\ c_3 = c_1-1$

UNIT-III

11.6 Solving Simultaneous Linear Differential Equations

Linear differential equations having two or more dependent variables with single independent variable are called simultaneous differential equations and can be of two types:

Type 1: $\boldsymbol{f_1(D)x + f_2(D)y = F(t)\,,\; g_1(D)x + g(D)y = G(t),\, D \equiv \frac{d}{dt}}$

Consider a system of ordinary differential equations in two dependent variables x and y and an independent variable t:

$$f_1(D)x + f_2(D)y = F(t)\,,\; g_1(D)x + g(D)y = G(t),\, D \equiv \frac{d}{dt}$$

Given system can be solved as follows:

1. Eliminate y from the given system of equations resulting a differential equation exclusively in x.
2. Solve the differential equation in x by usual methods to obtain x as a function of t.
3. Substitute value of x and its derivatives in one of the simultaneous equations to get an equation in y.
4. Solve for y by usual methods to obtain its value as a function of t.

Example 38 Solve the system of equations: $\frac{dx}{dt} + y = e^t,\; \frac{dy}{dt} - x = e^{-t}$

Solution: Rewriting given system of differential equations as:

$Dx + y = e^t$ ......①
$Dy - x = e^{-t}$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by D
$\Rightarrow D^2x + Dy = e^t$ .....③
Subtracting ② from ③, we get
$(D^2 + 1)x = e^t - e^{-t}$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x. Auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos t + c_2 \sin t$

P.I. $= \frac{1}{f(D)} F(t) = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1} e^t - \frac{1}{D^2+1} e^{-t}$

$= \frac{1}{2} e^t - \frac{1}{2} e^{-t}$, Putting $D = 1$ and $D = -1$ in 1st and 2nd terms respectively

$\therefore x = c_1 \cos t + c_2 \sin t + \frac{1}{2} e^t - \frac{1}{2} e^{-t}$ ....⑤

Using ⑤in ① $\Rightarrow D\left[c_1 \cos t + c_2 \sin t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right] + y = e^{t}$

$\Rightarrow \left[-c_1 \sin t + c_2 \cos t + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right] + y = e^{t}$

$\Rightarrow y = c_1 \sin t - c_2 \cos t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$ ......⑥

⑤ and ⑥ give the required solution.

Example 39 Solve the system of equations: $t\frac{dx}{dt} + y = 0$, $\frac{dy}{dt} + x = 0$

given that $x(1) = 1$, $y(-1) = 0$

Solution: Given system of equations is:

$t\frac{dx}{dt} + y = 0$......①
$t\frac{dy}{dt} + x = 0$ ......②,
Multiplying ① by $t\frac{d}{dt}$
$t\frac{d}{dt}(t\frac{dx}{dt} + y) = 0$
$\Rightarrow t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$ ......③
Subtracting ② from ③, we get
$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$.......④

which is Cauchy's linear differential equation in x with variable coefficients.

.Putting $t = e^{k}$ $\therefore \log t = k$

$\Rightarrow t\frac{dx}{dt} = Dx, t^2\frac{d^2x}{dt^2} = D(D-1)x$, $D \equiv \frac{d}{dk}$

$\therefore$ ④ may be rewritten as

$(D(D-1) + D - 1)x = 0$ ......⑤

$\Rightarrow (D^2 - 1)x = 0$

To solve ⑤ for x. Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^{k} + c_2e^{-k} = c_1t + \frac{c_2}{t}$

$\therefore x = c_1 t + \frac{c_2}{t}$ ……⑥

Using ⑥ in ① $\Rightarrow t\frac{d}{dt}\left(c_1 t + \frac{c_2}{t}\right) + y = 0$

$\Rightarrow c_1 t - \frac{c_2}{t} + y = 0$

$\Rightarrow y = -c_1 t + \frac{c_2}{t}$ ……..⑦

Also given that at $t = 1, x = 1$ and at $t = -1, y = 0$

Using in ⑥and ⑦ $c_1 + c_2 = 1,\ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$

Using $c_1 = c_2 = \frac{1}{2}$ in ⑥and ⑦, we get

$x = \frac{1}{2}\left(t + \frac{1}{t}\right),\ y = \frac{1}{2}\left(\frac{1}{t} - t\right)$

Example 40 Solve the system of equations:

$\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t$

Solution: Rewriting given system of differential equations as:

$D^2x + y = \sin t$……①
$D^2y + x = \cos t$ …..②, $D \equiv \frac{d}{dt}$
Multiplying ① by D^2
$D^2(D^2x + y) = D^2 \sin t$
$\Rightarrow D^4x + D^2y = -\sin t$ …..③
Subtracting ② from ③, we get
$(D^4 - 1)x = -\sin t - \cos t$ …….④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow m = \pm 1, \pm i$

C.F. $= c_1e^t + c_2e^{-t} + (c_3 \cos t + c_4 \sin t)$

P.I. $= \frac{1}{f(D)} F(t) = \frac{1}{D^4 - 1}(-\sin t - \cos t) = -\frac{1}{D^4 - 1}\sin t - \frac{1}{D^4 - 1}\cos t$

Putting $D^2 = -1$ i.e. $D^4 = 1$ in 1st and 2nd terms, it is a case of failure

$\therefore$ P.I. $= -t\frac{1}{4D^3}\sin t - t\frac{1}{4D^3}\cos t$

$= \frac{t}{4}\frac{1}{D}\sin t + \frac{t}{4}\frac{1}{D}\cos t$ putting $D^2 = -1$

$= -\frac{t}{4}\cos t + \frac{t}{4}\sin t$

$\therefore x = (c_1 e^t + c_2 e^{-t}) + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)$........⑤

Using ⑤in ①

$\Rightarrow D^2\left[c_1 e^t + c_2 e^{-t} + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)\right] + y = \sin t$

$\Rightarrow D\left[c_1 e^t - c_2 e^{-t} - c_3\sin t + c_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right]$
$+ y = \sin t$

$\Rightarrow \left[c_1 e^t + c_2 e^{-t} - c_3\cos t - c_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t)\right.$
$\left. + \frac{1}{4}(\cos t + \sin t)\right] + y = \sin t$

$\Rightarrow y = -(c_1 e^t + c_2 e^{-t}) + (c_3\cos t + c_4\sin t) + \left(\frac{t}{4} + \frac{1}{2}\right)(\sin t - \cos t)$ ...⑥

⑤ and ⑥ give the required solution.

Type II: Symmetric simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Simultaneous differential equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved by the method of grouping or the method of multipliers or both to get two independent solutions: $u = c_1, v = c_2$; where c_1 and c_2 are arbitrary constants.

Method of grouping: In this method, we consider a pair of fractions at a time which can be solved for an independent solution.

Method of multipliers: In this method, we multiply each fraction by suitable multipliers (not necessarily constants) such that denominator becomes zero.

If a, b, c are multipliers, then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+bdy+cdz}{aP+bQ+cR}$

Example 41 Solve the set of simultaneous equations:

$$\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$$

Taking x, y, z as multipliers, each fraction equals

$$\frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_1$

1st independent solution is: $u = x^2 + y^2 + z^2 = c_1$......①

Now for 2nd independent solution, taking last two members of the set of equations: $\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

$\Rightarrow (y-z)dy = (y+z)dz$

$\Rightarrow\ ydy - (zdy + ydz) - zdz = 0$

$\Rightarrow\ ydy - d(yz) - zdz = 0$

Integrating, we get

$\frac{y^2}{2} - yz - \frac{z^2}{2} = c'_2$

$\Rightarrow\ v = y^2 - 2yz - z^2 = c_2$ ....②

① and ② give the required solution.

Exercise 11B

Q1. Solve the following differential equations:

i. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Ans. $\langle y = (c_1 + c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2\rangle$

ii. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Ans. $\langle y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}\rangle$

iii. $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Ans. $\langle y = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4}\rangle$

iv. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos(\log(x+1))$

Ans.$\langle y = c_1\cos(\log(x+1)) + c_2 \sin(\log(x+1)) + 2\log(x+$
$1.sin\log x+1$

Q2. Solve the following differential equations using method of variation of parameters

i. $\frac{d^2y}{dx^2} + y = x\sin x$

Ans. $\langle y = c_1\cos x + c_2\sin x + \frac{1}{8}\cos x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x\rangle$

ii. $(D^2-1)y = e^{-2x}\sin e^{-x}$

Ans. $\langle y = c_1e^x + c_2e^{-x} - \sin e^{-x} - e^x\cos e^{-x}\rangle$

iii. $(D^2-2D)y = e^x\sin x$

Ans. $\langle y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x\rangle$

iv. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x\log x$

Ans: $\langle y = c_1 + c_2e^{2x} + \frac{x^2}{4}e^x(2\log x - 3)\rangle$

Q2. Solve the following set of simultaneous differential equations

i. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$

Ans:$\langle x = e^{6t}(c_1\cos t + c_2\sin t), y = e^{6t}(c_1-c_2)\cos t + (c_1+c_2)\sin t)\ \rangle$

ii. $(D+1)x + (2D+1)y = e^t\ ,\ (D-1)x + (D+1)y = 1$

Ans: $\langle x = c_1e^t + c_2e^{-2t} + 2e^{-t}, y = 3c_1e^t + 2c_2e^{-2t} + 3e^{-t}\rangle$

iii. $\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$

Ans: $\langle xy - z = c_1 , x^2 - y^2 + z^2 = c_2\rangle$

11.7 Previous Years Solved Questions

Q1. Solve $(D^2 + D + 1)^2(D - 2)y = 0$

$\langle \boldsymbol{Q1(h), GGSIPU, December\ 2012}\rangle$

Solution: Auxiliary equation is: $(m^2 + m + 1)^2(m - 2)y = 0$…….①

Solving ①, we get

$\Rightarrow m = 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

C.F. = $c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Q2. Solve $(D^2 - 1)y = \cosh x \cos x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2012}\rangle$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^x + c_2e^{-x}$

P.I. $= \frac{1}{f(D)} F(x)$

$= \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\cos x\right)$ $\qquad \because \cosh x = \frac{e^x+e^{-x}}{2}$

$= \frac{1}{D^2-1}\left(\frac{e^x}{2}\cos x + \frac{e^{-x}}{2}\cos x\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$

$$= \frac{e^x}{2} \frac{1}{(D^2+2D)} \cos x + \frac{e^{-x}}{2} \frac{1}{D^2-2D} \cos x$$

$$= \frac{e^x}{2} \frac{1}{2D-1} \cos x + \frac{e^{-x}}{2} \frac{1}{-2D-1} \cos x \qquad \text{Putting } D^2 = -1$$

$$= \frac{e^x}{2} \frac{2D+1}{4D^2-1} \cos x - \frac{e^{-x}}{2} \frac{2D-1}{4D^2-1} \cos x$$

$$= -\frac{e^x}{10}(2D+1) \cos x + \frac{e^{-x}}{10}(2D-1) \cos x \quad \text{Putting } D^2 = -1$$

$$= -\frac{e^x}{10}(-2 \sin x + \cos x) + \frac{e^{-x}}{10}(-2 \sin x - \cos x)$$

$$\therefore \text{P.I.} = \frac{e^x}{10}(2 \sin x - \cos x) - \frac{e^{-x}}{10}(2 \sin x + \cos x)$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{10}(2 \sin x - \cos x) - \frac{e^{-x}}{10}(2 \sin x + \cos x)$$

Q3. Solve $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$ by the method of variation of parameters.

$(Q9(a), \ \boldsymbol{GGSIPU}, \ \boldsymbol{December\ 2012})$

Solution: $\Rightarrow (D^2 + 4)y = 4 \tan 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2 \cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\frac{4}{2} \int \sin 2x \tan 2x \ dx = -2 \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$-2 \int \frac{1-\cos^2 2x}{\cos 2x} dx = -2 \int (\sec 2x - \cos 2x) \, dx$$

$$= -2 \left[\frac{1}{2} \log|\sec 2x + \tan 2x| - \frac{1}{2} \sin 2x \right]$$

$= [\sin 2x - \log|\sec 2x + \tan 2x|]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int 4 \tan 2x \cos 2x \; dx = \; 2\int \sin 2x \; dx$

$= -\cos 2x$

$\therefore$ P.I= $u_1 y_1 + u_2 y_2$

$= \cos 2x\, [\sin 2x - \log|\sec 2x + \tan 2x|] - \sin 2x \cos 2x$

$= -\cos 2x \log|\sec 2x + \tan 2x|$

Complete solution is: $y \; = \;$ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$

Q4. Solve the system of equations: $\frac{dx}{dt} + x \; = y + e^t$, $\frac{dy}{dt} + y \; = x + e^t$

$\langle \boldsymbol{Q9(b),\; GGSIPU,\;\; December\; 2012} \rangle$

Solution: Rewriting given system of differential equations as:

$(D+1)x - y = e^t$ ......①
$(D+1)y - x = e^t$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by $(D+1)$
$\Rightarrow (D+1)^2 x - (D+1)y = (D+1)e^t$
$(D^2 + 2D + 1)x - (D+1)y = 2e^t$ .....③
Adding ② and ③, we get
$(D^2 + 2D)x = 3e^t$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 2m = 0$

$\Rightarrow m = 0, -2$

C.F. = $c_1 + c_2 e^{-2t}$

P.I. $= \frac{1}{f(D)} F(t) \; = 3 \frac{1}{D^2 + 2D} e^t$

$= e^t$ Putting $D = 1$

$\therefore x = c_1 + c_2 e^{-2t} + e^t$ ....⑤

Using ⑤ in ① $\Rightarrow D[c_1 + c_2e^{-2t} + e^t] + c_1 + c_2e^{-2t} + e^t - y = e^t$

$\Rightarrow -2c_2e^{-2t} + e^t + c_1 + c_2e^{-2t} - y = 0$

$\Rightarrow y = c_1 - c_2e^{-2t} + e^t$ ......⑥

⑤ and ⑥ give the required solution.

Q5. Solve by method of variation of parameters $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2013} \rangle, \langle \boldsymbol{Q3(b), GGSIPU, 2^{nd} term\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$(m - 3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. $= (c_1 + c_2 x)e^{3x} = c_1e^{3x} + c_2x\, e^{3x} = c_1y_1 + c_2y_2$

$\therefore y_1 = e^{3x}$ and $y_2 = xe^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$u_1 = -\int \frac{y_2F(x)}{W}dx = -\int \frac{xe^{3x}e^{3x}}{x^2e^{6x}}dx = -\int \frac{1}{x}dx = -\log x$

$u_2 = \int \frac{y_1F(x)}{W}dx = \int \frac{e^{3x}e^{3x}}{x^2e^{6x}}dx = \int \frac{1}{x^2}dx = -\frac{1}{x}$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$= -e^{3x}\log x - e^{3x} = -e^{3x}(1 + \log x)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2x)e^{3x} - e^{3x}(1 + \log x)$

Q6. Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2013} \rangle$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2+2m+1)=0$

$\Rightarrow m(m+1)^2=0$

$\Rightarrow\ m=0,-1,-1$

C.F. $= c_1 +\ e^{-x}(c_2+c_3x)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^3+2D^2+D}(e^{2x}+\sin 2x)$

$= \frac{1}{D^3+2D^2+D}e^{2x} + \frac{1}{D^3+2D^2+D}\sin 2x$

$= \frac{1}{18}e^{2x} + \frac{1}{-4D-8+D}\sin 2x$, putting $D=2$ in 1st term, $D^2=-4$ in the 2nd term

$= \frac{1}{18}e^{2x} - \frac{3D-8}{(3D+8)(3D-8)}\sin 2x = \frac{1}{18}e^{2x} - \frac{3D-8}{(9D^2-64)}\sin 2x$

$= \frac{1}{18}e^{2x} + \frac{1}{100}(3\mathrm{D}-8)\sin 2x$

$= \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Complete solution is: $y\ =\ $ C.F. + P.I

$\Rightarrow y\ = c_1 +\ e^{-x}(c_2+c_3x) + \ \frac{1}{18}e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Q7. Solve $(D^2-2D+1)y = xe^x\cos x$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2014}\rangle$

Solution: Auxiliary equation is: $m^2-2m+1=0$

$\Rightarrow (m-1)^2$

$\Rightarrow m=1{,}1$

C.F. $= (c_1+c_2x)e^x$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-2D+1}xe^x\cos x$

$= e^x\frac{1}{(D+1)^2-2(D+1)+1}x\cos x$

$= e^x\frac{1}{D^2}x\cos x$

$= e^x \frac{1}{D} \int x \cos x \, dx$

$= e^x \frac{1}{D} [(x)(\sin x) - (1)(-\cos x)]$

$= e^x \frac{1}{D} [x \sin x + \cos x]$

$= e^x [\int x \sin x \, dx + \int \cos x \, dx]$

$= e^x \left[[(x)(-\cos x) - (1)(-\sin x)] + \sin x \right]$

$\therefore$ P.I. $= e^x [-x \cos x + 2 \sin x]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + e^x[-x \cos x + 2 \sin x]$

Q8. Solve by M.O.V.P. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$

(*Q8(b), GGSIPU, December* 2014)

Solution: Given differential equation may be rewritten as

$(D^2 - 2D + 1)y = e^x \log x$

$\therefore$ Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2$

$\Rightarrow m = 1,1$

C.F. $= (c_1 + c_2 x)e^x = c_1 y_1 + c_2 y_2$
$\therefore y_1 = e^x$ and $y_2 = xe^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x\,e^x\, e^x \log x}{e^{2x}} dx = -\int x \log x \, dx$

$\int x \log x \, dx = I = \left[(x)(x \log x - x) - (1)\left(I - \frac{x^2}{2} \right) \right]$

$\because \int \log x \, dx = x \log x - x$

$\Rightarrow 2I = x^2 \log x - x^2 + \frac{x^2}{2}$

$$\Rightarrow I = \int x \log x \, dx \; = \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$$\therefore u_1 = \frac{x^2}{4} - \frac{x^2}{2}\log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x \log x}{e^{2x}} \, dx = \int \log x \, dx = \; x\log x - x$$

$$\therefore \text{P.I} = \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right) e^x + (x\log x - x)xe^x$$

$$= e^x \left(\frac{x^2}{4} - \frac{x^2}{2}\log x + x^2\log x - x^2\right)$$

$$= \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Complete solution is: $y \; = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Q9. Solve $(D-1)^2(D+1)^2 = \sin^2\frac{x}{2} + e^x + x$

$$\langle \boldsymbol{Q1(a), GGSIPU, December\ 2015} \rangle$$

Solution: Auxiliary equation is: $(m-1)^2(m+1)^2 = 0$

$$\Rightarrow \; m = 1,1,-1,-1$$

$$\text{C.F.} = \; (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x}$$

$$\text{P.I} = \frac{1}{f(D)}\, F(x) \; = \frac{1}{((D-1)(D+1))^2}\left(\sin^2\frac{x}{2} + e^x + x\right)$$

$$= \frac{1}{2}\frac{1}{D^4-2D^2+1}(1-\cos x) + \frac{1}{D^4-2D^2+1}\; e^x + \frac{1}{D^4-2D^2+1}x$$

$$= \frac{1}{2}\frac{1}{D^4-2D^2+1}\; e^{0x} - \frac{1}{2}\frac{1}{D^4-2D^2+1}\cos x + \frac{1}{D^4-2D^2+1}\; e^x + \frac{1}{D^4-2D^2+1}x$$

Now $\frac{1}{2}\frac{1}{D^4-2D^2+1}\; e^{0x} = \frac{1}{2}$, putting $D = 0$

Also $\frac{1}{2}\frac{1}{D^4-2D^2+1}\cos x = \frac{1}{8}\cos x$ putting $D^2 = -1$

Again $\frac{1}{D^4-2D^2+1}\; e^x = \; x\frac{1}{4D^3-4D}\; e^x$ as $f(1) = 0$, a case of failure 2 times

$= x^2 \frac{1}{12D^2-4} e^x = \frac{x^2}{8} e^x$, putting $D = 1$

And $\frac{1}{D^4-2D^2+1} x = \frac{1}{1+(D^4-2D^2)} x = [1 + (D^4 - 2D^2)]^{-1} x = x$

$\therefore$ P.I.$= \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{8} e^x + x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} - \frac{1}{8}\cos x + \frac{x^2}{8} e^x + x + \frac{1}{2}$

Q.10 Solve $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^4 \sin x$

$\langle Q3(b), GGSIPU, December\ 2015 \rangle$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t$ $\therefore \log x = t$

$\Rightarrow x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Equation may be rewritten as

$(D(D-1) - 4D + 6)y = e^{4t} \sin e^t$

$\Rightarrow (D^2 - 5D + 6)y = e^{4t} \sin e^t$; $D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2,3$

C.F. $= c_1 e^{2t} + c_2 e^{3t}$

$= c_1 x^2 + c_2 x^3$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-5D+6} e^{4t} \sin e^t$

$= e^{4t} \frac{1}{(D+4)^2-5(D+4)+6} \sin e^t$

$= e^{4t} \frac{1}{D^2+3D+2} \sin e^t = e^{4t} \frac{1}{(D+1)(D+2)} \sin e^t$

$=e^{4t}\left[\frac{1}{(D+1)}-\frac{1}{(D+2)}\right]\sin e^t = e^{4t}\left[\frac{1}{(D+1)}\sin e^t-\frac{1}{(D+2)}\sin e^t\right]$

$=e^{4t}[e^{-t}\int e^t \sin e^t\, dt - e^{-2t}\int e^{2t}\sin e^t\, dt]$

$\because \frac{1}{(D+a)}F(t)=e^{-at}\int e^{at}F(t)dt$

$=e^{4t}[e^{-t}(-cose^t)-e^{-2t}(-e^t\cos e^t+\sin e^t]$

Solving the two integrals by putting $e^t=u, \therefore e^t dt=du$

$\therefore$ P.I $=-e^{2t}\sin e^t=-x^2\sin x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1x^2+c_2x^3-x^2\sin x$

UNIT-IV

IIIII

9.1. PARTIAL DIFFERENTIAL EQUATIONS are those equations which contain partial differential coefficients, independent variables and dependent variables.

The independent variables will be denoted by x and y and the dependent variable by z. The partial differential coefficients are denoted as follows:

$$\frac{\partial z}{\partial x} = p, \qquad \frac{\partial z}{\partial y} = q.$$

$$\frac{\partial^2 z}{\partial x^2} = r, \qquad \frac{\partial^2 z}{\partial x \partial y} = s, \qquad \frac{\partial^2 z}{\partial y^2} = t$$

9.2. ORDER of a partial differential equation is the same as that of the order of the highest differential coefficient in it.

9.3 CLASSIFICATION

Consider the equation. $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0$ ... (1)

Where A, B, C may be constants or functions of x and y. Now the equation (1) is

1. Parabolic; if $B^2 - 4AC = 0$

2. Elliptic; if $B^2 - 4AC < 0$

3. Hyperbolic; if $B^2 - 4AC > 0$

9.4 METHOD OF FORMING PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is formed by two methods.

(i) By eliminating arbitrary constants.

(ii) By eliminating arbitrary functions.

(*i*) **Method of elimination of arbitrary constants**

Example 1. *Form a partial differential equation from*

$$x^2 + y^2 + (z - c)^2 = a^2.$$

Solution. $x^2 + y^2 + (z - c)^2 = a^2$ ...(1)

(1) contains two arbitrary constants a and c.

Differentiating (1) partially w.r.t. x we get

$$2x + 2(z - c)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \qquad x + (z - c)\,p = 0 \qquad ...(2)$$

Differentiating (1) partially w.r.t. y we get

$$2y + 2(z - c)\frac{\partial z}{\partial y} = 0$$

$$y + (z - c)\, q = 0 \qquad ...(3)$$

Let us eliminate c from (2) and (3)

From (2) $\qquad (z - c) = -\dfrac{x}{p}$

Putting this value of $z - c$ in (3), we get $y - \dfrac{x}{p} q = 0$

or $\qquad yp - xq = 0$ **Ans.**

(*ii*) **Method of elimination of arbitrary functions**

Example 2. *Form the partial differential equation from*

$$z = f(x^2 - y^2)$$

Solution. $\qquad z = f(x^2 - y^2) \qquad ...(1)$

Differentiating (1) w.r.t x and y

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x \qquad ...(2)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)\,(-2y) \qquad ...(3)$$

Dividing (2) by (3) we get $\dfrac{p}{q} = \dfrac{-x}{y}$ or $py = -qx$

or $\qquad yp + xq = 0$ **Ans.**

EXERCISE 9.1

Form the partial differential equation

1. $z = (x + a)(y + b)$ **Ans.** $pq = z$
2. $(x - h)^2 + (y - k)^2 + z^2 = a^2$ **Ans.** $z^2(p^2 + q^2 + 1) = a^2$
3. $2z = (a\,x + y)^2 + b$ **Ans.** $p\,x + q\,y = q^2$
4. $ax^2 + by^2 + z^2 = 1$ **Ans.** $z(px + qy) = z^2 - 1$
5. $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ **Ans.** $yp - xq = 0$
6. $z = f(x^2 + y^2)$ **Ans.** $yp - xq = 0$
7. $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ *(A.M.I.E., Winter 2001)* **Ans.** $2z = xp + yq$
8. $f(x + y + z,\ x^2 + y^2 + z^2) = 0$ **Ans.** $(y - z)p + (z - x)q = x - y$

9.5 SOLUTION OF EQUATION BY DIRECT INTEGRATION

Example 3. *Solve* $\qquad \dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Solution. $\qquad \dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Integrating w.r.t. 'x', we get $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{2}\sin(2x + 3y) + f(y)$

Integrating w.r.t. x, we get $\dfrac{\partial z}{\partial y} = -\dfrac{1}{4}\cos(2x + 3y) + x \int f(y)\, dy + g(y)$

$$= -\frac{1}{4}\cos(2x+3y) + x\phi(y) + g(y)$$

Integrating w.r.t. 'y' we get

$$z = \frac{1}{12}\sin(2x+3y) + x\int\phi(y)\,dy + \int g(y)\,dy$$

$$z = -\frac{1}{12}\sin(2x+3y) + x\phi_1(y) + \phi_2(y) \qquad \textbf{Ans.}$$

Example 4. *Solve* $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

subject to the condition $z(x, 0) = x^2$ *and* $z(1, y) = \cos y$.

Solution $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$

On integrating w.r.t. x, we obtain $\frac{\partial z}{\partial y} = \frac{x^3}{3}y + f(y)$

Integrating w.r.t. y, we obtain $z = \frac{x^3}{3}\cdot\frac{y^2}{2} + \int f(y)\,dy + g(x)$

$$\left[F(y) = \int f(y)\,dy\right]$$

or $$z = \frac{x^3y^2}{6} + F(y) + g(x) \qquad \ldots(1)$$

Condition 1: Putting $z = x^2$ and $y = 0$ in (1), we get

$$x^2 = 0 + F(0) + g(x)$$

Putting the value of $g(x)$ in (1), we get $z = \frac{x^3y^2}{6} + F(y) + x^2 - F(0) \qquad \ldots(2)$

Condition 2: $z(1, y) = \cos y$

Putting $x = 1$ and $z = \cos y$ in (2), we get

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

Putting the value of $F(y)$ in (2), we obtain

$$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + F(0) + x^2 - F(0)$$

or $$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2 \qquad \textbf{Ans.}$$

Example 5. *Solve* $\frac{\partial^2 z}{\partial y^2} = z$, *if* $y = 0$, $z = e^x$ *and* $\frac{\partial z}{\partial y} = e^{-x}$

Solution. If z is a function of y alone, then

$$z = \sinh y \cdot f(x) + \cosh y \cdot \phi(x) \quad \ldots(1)$$

$\frac{\partial^2 z}{\partial y^2} = z \Rightarrow (D^2 - 1)z = 0 \Rightarrow m = \pm 1$

$\Rightarrow z = Ae^y + Be^{-y} = A\sinh y + B\cosh y$

$= f(x)\sinh y + \phi(x) \cdot \cosh y$

On putting $y = 0$ and $z = e^x$ in (1), we obtain

$$e^x = \phi(x)$$

(1) becomes $\qquad z = \sinh y \,.\, f(x) + \cosh y \,.\, e^x \qquad ...(2)$

On differentiating (2) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = \cosh y \cdot f(x) + \sinh y \cdot e^x \qquad ...(3)$$

On putting $y = 0$ and $\frac{\partial z}{\partial y} = e^{-x}$ in (3), we obtain

$$e^{-x} = f(x)$$

(2) becomes, $\qquad z = e^{-x} \sinh y + e^x \cosh y$ **Ans.**

EXERCISE 9.2

Solve the following:

1. $\frac{\partial^2 z}{\partial x \partial y} = xy^2$ **Ans.** $z = \frac{x^2 y^3}{6} + f(y) + \phi(x)$
2. $\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$ **Ans.** $z = e^y \sin x + f(y) + \phi(x)$
3. $\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$ **Ans.** $z = \frac{y^2}{2} \log x + 2xy + f(y) + \phi(x)$
4. $\frac{\partial^2 z}{\partial x^2} = a^2 z$, when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$ **Ans.** $z = \sin x + e^y \cos x$
5. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ if $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$, and $z = 0$ when y is an odd multiple of $\frac{\pi}{2}$.

   Ans. $z = \cos x \cos y + \cos y$
6. The partial differential equation $y \frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y} = 0$ is elliptic if

   (*a*) $x^2 = y^2$ (*b*) $x^2 < y^2$ (*c*) $x^2 + y^2 > 1$ (*d*) $x^2 + y^2 = 1$

   (*A.M.I.E.T.E., Dec. 2004*) **Ans.** (*b*)

9.6 LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE

$$Pp + Qq = R$$

where P, Q, R are the functions of x, y, z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

Solution. $\qquad Pp + Qq = R \qquad ...(1)$

This form of the equation is obtained by eliminating an arbitrary function f from

$$f(u, v) = 0 \qquad ...(2)$$

where u, v are functions of x, y, z.

Differentiating (2) partially w.r.t. to x and y.

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad ...(3) \quad \text{and} \quad \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0 \quad ...(4)$$

Let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4).

From (3), $\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right]=-\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right]$ ...(5)

From (4), $\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right]=-\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right]$ ...(6)

Dividing (5) by (6), we get $\dfrac{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.p}{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}.q}=\dfrac{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.p}{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}.q}$

or $\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.p\right]\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right]=\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}.q\right]\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}.p\right]$

$$\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}.q+\frac{\partial u}{\partial z}\times p\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}.pq$$

or $$=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}.p+\frac{\partial u}{\partial z}.q\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}.pq$$

$$\left[\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}\right]p+\left[\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\right]q=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x} \quad ...(7)$$

If (1) and (7) are the same, then the coefficients of p, q are equal.

$$P=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}$$
$$Q=\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z} \quad ...(8)$$
$$R=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}$$

Now suppose $u=c_1$ and $v=c_2$ are two solutions, where a, b are constants.

Differentiating $u=c_1$ and $v=c_2$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \quad ...(9)$$

and $$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \quad ...(10)$$

Solving (9) and (10), we get

$$\frac{dx}{\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}} \quad ...(11)$$

From (8) and (11) $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Solutions of these equations are $u=c_1$ *and* $v=C_2$

$\therefore$ *$f(u, v)=0$ is the required solution of (1).*

9.7 WORKING RULE

First step. Write down the auxiliary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Second step. Solve the above auxiliary equations.

Let the two solutions be $u = c_1$ and $v = c_2$

Third step. Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$$Pp + Qq = R.$$

Example 6. *Solve the following partial differential equation*

$$yq - xp = z, \qquad where\ p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

Solution. $yq - xp = z$

Here the auxiliary equations are

$$\Rightarrow \quad \frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

$\Rightarrow$ $-\log x = \log y - \log a$ (From first two equations)

$\Rightarrow$ $xy = a$ ...(1)

$\Rightarrow$ $\log y = \log z + \log b$ (From last two equations)

$$\frac{y}{z} = b \quad ...(2)$$

From (1) and (2)

Hence the solution is $f\left(xy, \frac{y}{z}\right) = 0$ **Ans.**

Example 7. *Solve* $y^2p - xyq = x(z - 2y)$ *(A.M.I.E. Summer 2001)*

Solution. $y^2p - xyq = x(z - 2y)$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad ...(1)$$

Considering first two members of the equations

$$\frac{dx}{y} = \frac{dy}{-x} \quad \Rightarrow \quad x\,dx = -y\,dy$$

Integrating $\frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2} \quad \Rightarrow \quad x^2 + y^2 = C_1$ ...(2)

From last two equations of (1)

$$-\frac{dy}{y} = \frac{dz}{z - 2y}$$

$\Rightarrow$ $-z\,dy + 2y\,dy = y\,dz \quad \Rightarrow \quad 2y\,dy = y\,dz + z\,dy$

On integration, we get

$$y^2 = yz + C_2$$

$$y^2 - yz = C_2 \quad ...(3)$$

From (2) and (3)

$$x^2 + y^2 = f(y^2 - yz)$$ **Ans.**

Example 8. *Solve* $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$ *(A.M.I.E., Summer 2001)*

Solution. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ ...(1)

The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

or

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

$$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(x+y+z)(y-z)} = \frac{dz - dx}{(x+y+z)(z-x)}$$

$$\frac{dx - dy}{(x-y)} = \frac{dy - dz}{(y-z)} = \frac{dz - dx}{(z-x)} \quad ...(2)$$

Intergrating first members of (2), we have

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log\frac{x-y}{y-z} = \log c_1 \quad \text{or} \quad \frac{x-y}{y-z} = c_1$$

Similarly from last two members of (2), we have

$$\frac{y-z}{z-x} = c_2$$

The required solution is

$$f\left[\frac{x-y}{y-z}, \frac{y-z}{z-x}\right] = 0$$ **Ans.**

9.8 METHOD OF MULTIPLIERS

Let the auxiliary equations be

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$$

l, m, n may be constants or functions of x, y, z then we have

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lp + mQ + nR}$$

l, m, n are chosen in such a way that

$$lP + mQ + nR = 0$$

Thus $$l\,dx + m\,dy + n\,dz = 0$$

Solve this differential equation, if the solution is $u = c_1$

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is $v = C_2$.

$\therefore$ Required solution is $f(u, v) = 0$.

Example 9. *Solve*

$$(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$$ *(A.M.I.E. Winter 2001)*

Solution. $(mz - ny)\dfrac{\partial z}{\partial x} + (nx - lz)\dfrac{\partial z}{\partial y} = ly - mx$

Here, the auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using multipliers x, y, z we get

$$\text{Each fraction} = \frac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$

which on integration gives $x^2 + y^2 + z^2 = c_1$ ...(1)

Again using multipliers, l, m, n, we get

$$\text{each fraction} = \frac{l\,dx + m\,dy + n\,dz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\therefore \Rightarrow \qquad l dx + m\,dy + n\,dz = 0$

which, on integration gives

$$lx + m\,y + n\,z = c_2 \qquad ...(2)$$

Hence from (1) and (2), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$ **Ans.**

Example 10. *Find the general solution of*

$$x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2)$$

Solution. $x(z^2 - y^2)\dfrac{\partial z}{\partial x} + y(x^2 - z^2)\dfrac{\partial z}{\partial y} = z(y^2 - x^2)$

The auxiliary simultaneous equations are

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \qquad ...(1)$$

Using multipliers x, y, z we get

Each term of (1) is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$

On integration $\qquad x^2 + y^2 + z^2 = C_1$ ...(2)

Again (1) can be written as

$$\frac{\frac{dx}{x}}{z^2 - y^2} = \frac{\frac{dy}{y}}{x^2 - z^2} = \frac{\frac{dz}{z}}{y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$\Rightarrow \quad \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0$

$\Rightarrow \quad \log x + \log y + \log z = \log C_2$

$\Rightarrow \quad \log xyz = \log C_2 \qquad \Rightarrow \qquad xyz = C_2$ ...(3)

From (2) and (3), the general solution is $xyz = f(x^2 + y^2 + z^2)$ **Ans.**

Example 11. *Solve the partial differential equation*

$$\frac{y-z}{yz}p = \frac{z-x}{zx}q = \frac{x-y}{xy}$$ (*A.M.I.E., Winter 2001*)

Solution. $$\frac{y-z}{yz}p = \frac{z-x}{zx}q = \frac{x-y}{xy}$$

Multiplying by xyz, we get

$$x(y-z)p + y(z-x)q = z(x-y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} \quad \dots (1)$$

$$= \frac{dx+dy+dz}{0}$$

$\therefore$ $$dx + dy + dz = 0$$

Which on integration gives

$$x + y + z = a \quad \dots (2)$$

Again (1) can be written

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{(y-z)+(z-x)+(x-y)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

or $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integration we get

$$\log x + \log y + \log z = \log b \Rightarrow \log xyz = \log b \Rightarrow xyz = b \quad \dots (3)$$

From (2) and (3) the general solution is

$$xyz = f(x+y+z)$$ **Ans.**

Example 12. *Solve* $(x^2-y^2-z^2)p + 2xy\,q = 2xz$. (*A.M.I.E., Summer, 2004, 2000*)

Solution. $(x^2-y^2-z^2)p + 2xyq = 2xz$

Here the auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \dots (1)$$

From the last two members of (1) we have dz

$$\frac{dy}{y} = \frac{dz}{z}$$

which on integration gives

$$\log y = \log z + \log a \quad \text{or} \quad \log \frac{y}{z} = \log a$$

or $$\frac{y}{z} = a \quad \dots (2)$$

Using multipliers x, y, z we have

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{2x\,dx+2y\,dy+2z\,dz}{(x^2+y^2+z^2)}=\frac{dz}{z}$$

which on integration gives

$$\log(x^2+y^2+z^2)=\log z+\log b$$

$$\frac{x^2+y^2+z^2}{z}=b \qquad ...(3)$$

Hence from (2) and (3), the required solution is

$$x^2+y^2+z^2=z\,f\left(\frac{y}{z}\right) \qquad \textbf{Ans.}$$

Example 13. *Solve the differential equation*

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z.$$

Solution. $$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z. \qquad ...(1)$$

The auxiliary equations of (1) are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \qquad ...(2)$$

Take first two members of (2) and integrate them

$$-\frac{1}{x}=-\frac{1}{y}+c$$

$$\frac{1}{x}-\frac{1}{y}=c_1 \qquad ...(3)$$

(2) can be written as $\dfrac{\frac{dx}{x}}{x}=\dfrac{\frac{dy}{y}}{y}+\dfrac{\frac{dz}{z}}{x+y}=\dfrac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{(x+y)-(x+y)}$

or $$\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0$$

On integration we get

or $$\log x+\log y-\log z=\log c_2$$

or $$\log\frac{xy}{z}=\log c_2 \qquad \text{or} \qquad \frac{xy}{z}=c_2 \qquad ...(4)$$

From (3) and (4) we have

$$f\left[\frac{1}{x}-\frac{1}{y},\frac{xy}{z}\right]=0 \qquad \textbf{Ans.}$$

Example 14. *Find the general solution of*

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=xyt$$

Solution. The auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dt}{t}=\dfrac{dz}{xyt}$ ...(1)

Taking the first two members and integrating, we get

$$\log x=\log y+\log a$$

$\Rightarrow \quad \log x = \log ay \quad \Rightarrow \quad x = ay \quad \Rightarrow \quad y/x = a$ ...(2)

Similarly, from the 2nd and 3rd members

$$\frac{t}{y} = b \qquad ...(3)$$

Multiplying the equations (1) by xyt, we get

$$dz = \frac{ty\,dx}{1} = \frac{tx\,dy}{1} = \frac{xy\,dt}{1} = \frac{ty\,dx + tx\,dy + xy\,dt}{3}$$

Integrating,

$$z = \frac{1}{3}xyt + c \quad or \quad z - \frac{1}{3}xyt = c \qquad ...(4)$$

From (2), (3) and (4) the solution is

$$z - \frac{1}{3}xyt = f\left(\frac{y}{x}\right) + \phi\left(\frac{t}{y}\right) \qquad \textbf{Ans.}$$

Example 15. *Solve* $(y + z)p - (x + z)\,q = x - y$

Solution. $(y+z)p - (x+z)q = x - y$ ... (1)

$\therefore$ The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} \qquad ...(2)$$

$$\Rightarrow \quad \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y+z-(x+z)+x-y}$$

$$\Rightarrow \quad \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

Thus, we have $dx + dy + dz = 0$

which on integration gives $x + y + z = c_1$. ...(3)

Let us use multipliers $(x, y, -z)$ for (2)

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy + z\,dz}{x(y+z) - y(x+z) - z(x-y)}$$

or

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy - z\,dz}{0}$$

Integrating $x\,dx + y\,dy - z\,dz = 0$, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

or

$$x^2 + y^2 - z^2 = 2c_2 \qquad ...(4)$$

From (3) and (4), we get the required solution

$$f(x + y + z,\ x^2 + y^2 - z^2) = 0 \qquad \textbf{Ans.}$$

Example 16. *Solve* $zp + yq = x$

Solution. $zp + yq = x$ ...(1)

The auxiliary equations are $\dfrac{dx}{z} = \dfrac{dy}{y} = \dfrac{dz}{x}$

(*i*) (*ii*) (*iii*)

From (*i*) and (*ii*) $\quad \frac{dx}{z} = \frac{dz}{x} \quad$ or $\quad x\,dx = z\,dz$

$\Rightarrow \quad \frac{x^2}{2} = \frac{z^2}{2} - \frac{c_1}{2}$ or $\quad x^2 = z^2 - c_1 \qquad$ ...(2)

$\Rightarrow \quad z = \sqrt{x^2 + c_1}$

Putting the value of z in (1)

$$\frac{dx}{\sqrt{x^2 + c_1}} = \frac{dy}{y}$$

$$\sinh^{-1}\frac{x}{\sqrt{c_1}} = \log y + c_2 \quad \text{or} \quad \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y = c_2 \qquad \text{...(3)}$$

From (2) and (3), the required solution is

$$f(z^2 - x^2) = \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y$$

Ans.

Example 17. *Solve* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$. *(A.M.I.E., Summer 2000)*

Solution. $\quad px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3) \qquad$ ...(1)

$\Rightarrow \quad px(z - 2y^2) + qy(z - y^2 - 2x^3) = z(z - y^2 - 2x^3)$

Here the auxiliary equations are

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)} \qquad \text{...(2)}$$

From the last two members of (2) we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which gives on integration

$\log y = \log z + \log a \quad$ or $\quad y = az \qquad$ ...(3)

From the first and third members of (2) we have

$$\frac{dx}{x(z - 2y^2)} = \frac{dz}{z(z - y^2 - 2x^3)} \qquad \text{Put } y = az$$

$$\Rightarrow \quad \frac{dx}{x(z - 2a^2z^2)} = \frac{dz}{z(z - a^2z^2 - 2x^3)}$$

$$\frac{dx}{x(1 - 2a^2z)} = \frac{dz}{z - a^2z^2 - 2x^3}$$

$\Rightarrow \quad z\,dx - a^2z^2dx - 2x^3dx = x\,dz - 2a^2xz\,dz$

$\Rightarrow \quad (x\,dz - z\,dx) - a^2(2xz\,dz - z^2\,dx) + 2x^3dx = 0$

On integrating, we have

$$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = b \qquad \text{...(4)}$$

From (3) and (4), we have

$$\frac{y}{z} = f\left(\frac{z}{x} - \frac{a^2z^2}{x} + x^2\right)$$

Ans.

EXERCISE 9.3

Solve the following partial differential equations :

1. $p\tan x + q\tan y = \tan z$ **Ans.** $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

2. $y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + 1$ *(AMIE Winter 2002)* **Ans.** $f(x - y) = \log y - \tan^{-1} z$

3. $(y - z)p + (x - y)q = z - x$ **Ans.** $f(x + y + z, x^2 + 2yz) = 0$

4. $(y + zx)p - (x + yz)q = x^2 - y^2$ **Ans.** $f(x^2 + y^2 - z^2) = (x - y)^2 - (z + 1)^2$

5. $zx\frac{\partial z}{\partial x} - zy\frac{\partial z}{\partial y} = y^2 - x^2$ **Ans.** $f(x^2 + y^2 + z^2, xy) = 0$

6. $pz - qz = z^2 + (x + y)^2$ **Ans.** $[z^2 + (x + y)^2]\, e^{-2x} = f(x + y)$

7. $p + q + 2xz = 0$ **Ans.** $f(x - y) = x^2 + \log z$

8. $x^2p + y^2q + z^2 = 0$ **Ans.** $f\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{y} + \frac{1}{z}\right) = 0$

9. $(x^2 + y^2)p + 2xyq = (x + y)z$ **Ans.** $f\left(\frac{x + y}{z}, \frac{2y}{x^2 - y^2}\right) = 0$

10. $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 2x - e^y + 1$ **Ans.** $f(2x + y) = z - \frac{(2x + 1)^2}{4} - \frac{e^y}{2}$

11. $p + 3q = 5z + \tan(y - 3x)$ **Ans.** $f(y - 3x) = \frac{e^{5x}}{5z + \tan(y - 3x)}$

12. $xp - yq + x^2 - y^2 = 0$ **Ans.** $f(xy) = \frac{x^2}{2} + \frac{y^2}{2} + z$

13. $(x + y)\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = z - 1$ **Ans.** $f(x - y) = \frac{x + y}{(z - 1)^2}$

14. $(x^3 + 3xy^2)\frac{\partial z}{\partial x} + (y^3 + 3x^2y)\frac{\partial z}{\partial y} = 2(x^2 + y^2)z$ **Ans.** $f\left(\frac{xy}{z^2}, (x - y)^{-2} - (x + y)^{-2}\right) = 0$

15. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$ **Ans.** $(x^2 + y^2 + z^2) = f(y^2 - 2yz - z^2)$

16. Find the solution of the equation $\frac{x\partial z}{\partial y} - \frac{y\partial z}{\partial x} = 0$, which passes through the curve $z = 1$, $x^2 + y^2 = 4$ **Ans.** $f(x^2 + y^2 - 4, z - 1) = 0$

17. $2x(y + z^2)p + y(2y + z^2)q = z^3$ *(AMIE Winter 2005)*

18. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$ **Ans.** $u = 4e^{-x + \frac{3y}{2}}$

19. $4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u$, when $t = 0$, $u = 3e^{-x} - e^{-5x}$ **Ans.** $u = 3e^{-x+t} - 3e^{-5x+2t}$

9.9 PARTIAL DIFFERENTIAL EQUATIONS NON-LINEAR IN p AND q.

We give below the methods of solving non-linear partial differential equations in certain standard form only.

Type I. *Equation of the Type* $f(p, q) = 0$ *i.e.*, equations containing p and q only.

Method. Let the required solution be

$$z = ax + by + c \qquad \ldots(1)$$

$$\therefore \quad \frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b.$$

On putting these values in $f(p, q) = 0$
we get $f(a, b) = 0,$
From this, find the value of b in terms of a and substitute the value of b in (1), that will be the required solution.

Example 18. *Solve* $p^2 + q^2 = 1$ ...(1)

Solution. Let $z = ax + by + c$ ...(2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

On substituting the values of p and q in (1), we have

$$\therefore \quad a^2 + b^2 = 1 \text{ or } b = \sqrt{1-a^2}$$

Putting the value of b in (2), we get $z = ax + \sqrt{1-a^2}\, y + c$

This is the required solution. **Ans.**

Example 19. *Solve* $x^2p^2 + y^2q^2 = z^2$. *(RGPV, Bhopal, Feb. 2008)*

Solution. This equation can be transformed in the above type.

$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\Rightarrow \quad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{Z}\frac{\partial z}{\partial y}\right)^2 = 1 \Rightarrow \left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2 = 1 \quad ...(1)$$

Let $\frac{\partial z}{z} = \partial Z, \quad \frac{\partial x}{x} = \partial X, \quad \frac{\partial y}{y} = \partial Y,$

$\therefore \quad \log z = Z, \quad \log x = X, \quad \log y = Y$

$\therefore$ (1) can be written as

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad ...(2)$$

$$\Rightarrow \quad P^2 + Q^2 = 1$$

Let the required solution be

$$Z = aX + bY + c$$

$$P = \frac{\partial Z}{\partial X} = a, \; Q = \frac{\partial Z}{\partial Y} = b$$

From (2) we have

$$a^2 + b^2 = 1 \text{ or } b = \sqrt{1-a^2}$$

$$Z = aX + \sqrt{1-a^2}\; Y + c$$

$$\log z = a \log x + \sqrt{1-a^2}\, \log y + c$$ **Ans.**

EXERCISE 9.4

Solve the following partial differential equations

1. $pq = 1$ **Ans.** $z = ax + \frac{1}{a}y + c$

2. $\sqrt{p} + \sqrt{q} = 1$ **Ans.** $z = ax + (1 - \sqrt{a})^2 y + c$

3. $p^2 - q^2 = 1$ **Ans.** $z = ax - \sqrt{(a^2 - 1)}\; y + c$

4. $pq + p + q = 0$ **Ans.** $z = ax - \frac{a}{1+a}y + c$

Type II. *Equation of the type*

$$z = px + qy + f(p, q)$$

Its solution is $z = ax + by + f(a, b)$

Example 20. *Solve* $z = px + qy + p^2 + q^2$

Solution. $z = px + qy + p^2 + q^2$ $\qquad p = a, q = b$

Its solution is $z = ax + by + a^2 + b^2$ **Ans.**

Example 21. *Solve* $z = px + qy + 2\sqrt{pq}$

Solution. $z = px + qy + 2\sqrt{pq}$

Its solution is $z = ax + by + 2\sqrt{ab}$ **Ans.**

Type III. Equation of the type f(z, p, q) = 0 equations not containing x and y.

Let z be a function of u where

$$u = x + ay.$$

$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial u}{\partial y} = a$$

Then
$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = \frac{dz}{du}(a)$$

On putting the values of p and q in the given equation $f(z, p, q) = 0$, it becomes

$f\left(z, \frac{dz}{dy}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the first order.

Rule. Assume $u = x + ay$; replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ in the given equation and then solve the ordinary differential equation obtained.

Example 22. *Solve*

$$p(1 + q) = qz$$

Solution. $p(1 + q) = qz$ ... (1)

Let $u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

(1) becomes

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z \qquad \text{or} \qquad 1 + a\frac{\partial z}{\partial u} = az$$

$$\Rightarrow \qquad a\frac{dz}{du} = az - 1 \Rightarrow du = \frac{a\,dz}{az - 1}$$

Integrating, we get

$$u = \log(az - 1) + \log c$$

$$x + ay = \log c\,(az - 1)$$ **Ans.**

Example 23. *Solve* $p(1+q^2) = q(z-a)$.

Solution. Let $u = x + by$

So that $$p = \frac{dz}{du} \quad \text{and} \quad q = b\frac{dz}{du}$$

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left[1+b^2\left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a)$$

$$1+b^2\left(\frac{dz}{du}\right)^2 = b(z-a) \text{ or } b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

$$\int\frac{b\,dz}{\sqrt{bz-ab-1}} = \int du + c$$

$$2\sqrt{bz-ab-1} = u+c$$

$$4(bz-ab-1) = (u+c)^2$$

$$4(bz-ab-1) = (x+by+c)^2$$ **Ans.**

Example 24. *Solve* $z^2(p^2x^2+q^2) = 1$ ...(1)

Solution. $z^2(p^2x^2+q^2) = 1$

$$\Rightarrow \quad z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \quad \Rightarrow \quad z^2\left[\left(\frac{\partial z}{\frac{\partial x}{x}}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1$$

$$\Rightarrow \quad z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \quad ...(2)$$

where $\frac{\partial x}{x} = \partial X$ or $\log x = X$

Let $$u = X + ay$$

$$\frac{\partial z}{\partial X} = \frac{dz}{du} \quad \text{and} \quad \frac{\partial z}{\partial y} = a\frac{dz}{du}$$

Then (2) becomes

$$z^2\left[\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2\right] = 1 \Rightarrow \left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2}$$

$$\Rightarrow \quad \left(\frac{dz}{du}\right)^2 = \frac{1}{z^2(1+a^2)} \Rightarrow \frac{dz}{du} = \frac{1}{z\sqrt{1+a^2}} \Rightarrow z\,dz = \frac{du}{\sqrt{1+a^2}}$$

$$\Rightarrow \quad \int z\,dz = \int\frac{du}{\sqrt{1+a^2}}+c \text{ or } \frac{z^2}{2} = \frac{u}{\sqrt{1+a^2}}+c$$

$$\sqrt{1+a^2}\,\frac{z^2}{2} = u + c\sqrt{1+a^2}$$

$$= X+ay+c\sqrt{1+a^2}$$

$$= \log x+ay+c\sqrt{1+a^2}$$ **Ans.**

EXERCISE 9.5

Solve

1. $z^2(p^2+q^2+1)=1$ **Ans.** $(1-z^2)^{\frac{1}{2}}=-\dfrac{x+ay}{\sqrt{1+a^2}}+c$

2. $1+q^2=q(z-a)$ **Ans.** $\dfrac{x+by}{b}+\dfrac{1}{4}(z-a)^2=\dfrac{1}{4}(z-a)\sqrt{(z-a)^2-2^2}+4\cosh^{-1}\left(\dfrac{z-a}{2}\right)$

3. $x^2p^2+y^2q^2=z$ **Ans.** $2\sqrt{z}=\dfrac{\log x+a\log y}{\sqrt{1+a^2}}+c$

Type IV. *Equation of the type* $f_1(x, p)=f_2(y, q)$

In these equations, z is absent and the terms containing x and p can be written on one side and the terms containing y and q can be written on the other side.

Method. Let $f_1(x, p)=f_2(y, q)=a$

$f_1(x, p)=a$, solve it for p. Let $p=F_1(x)$

$f_2(y, q)=a$, solve it for q. Let $q=F_2(y)$

Since $$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \Rightarrow dz=p\,dx+q\,dy$$

$$\Rightarrow dz=F_1(x)\,dx+F_2(y)\,dy \Rightarrow z=\int F_1(x)\,dx+\int F_2(y)\,dy+c$$

Example 25. *Solve* $p-x^2=q+y^2$.

Solution. $p-x^2=q+y^2=c$ (say)

i.e. $p=x^2+c$ and $q=c-y^2$

Putting these values of p and q in

$$dz=pdx+qdy = (x^2+c)\,dx+(c-y^2)\,dy$$

$$z=\left(\frac{x^3}{x}+cx\right)+\left(cy-\frac{y^3}{3}\right)+c_1$$ **Ans.**

Example 26. *Solve* $p^2+q^2=z^2(x+y)$.

Solution. $p^2+q^2=z^2(x+y) \Rightarrow \left(\dfrac{p}{z}\right)^2+\left(\dfrac{q}{z}\right)^2=(x+y)$

$$\Rightarrow \left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2=x+y \Rightarrow \left(\frac{\frac{\partial z}{z}}{\partial x}\right)^2+\left(\frac{\frac{\partial z}{z}}{\partial y}\right)^2=x+y$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=x+y \quad \text{where } \frac{\partial z}{z}=\partial Z \text{ or } \log z=Z$$

$$\Rightarrow p^2+Q^2=x+y \Rightarrow p^2-x=y-Q^2=a$$

$$P^2-x=a \Rightarrow P=\sqrt{a+x}$$

$$y-Q^2=a \Rightarrow Q=\sqrt{y-a}$$

Therefore, the equation $dZ=\dfrac{\partial Z}{\partial x}dx+\dfrac{\partial Z}{\partial y}dy$

$dZ=Pdx+Qdy$ gives

$$dZ=\sqrt{a+x}\,dx+\sqrt{y-a}\,dy$$

$$z = \int\sqrt{a+x}\,dx + \int\sqrt{y-a}\,dy + c$$

$\Rightarrow$ $$\log z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$$ **Ans.**

EXERCISE 9.6

Solve

1. $q - p + x - y = 0$ **Ans.** $2z = (x+a)^2 + (y+a)^2 + b$
2. $\sqrt{p} + \sqrt{q} = 2x$ **Ans.** $z = \frac{1}{6}(2x-a)^3 + a^2y + b$
3. $q = xp + p^2$ **Ans.** $z = -\frac{x^2}{4} + \left\{\frac{x\sqrt{x^2+4a}}{4} + a\log\left(x+\sqrt{x^2+4a}\right)\right\} + ay + b$
4. $z^2(p^2+q^2) = x^2+y^2$

   Ans. $z^2 = x\sqrt{x^2+a} + a\log(x+\sqrt{x^2+a}) + y\sqrt{y^2-a} - a\log(y+\sqrt{y^2-a}) + 2b$
5. $z(p^2+q^2) = x + y$ **Ans.** $z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + b$
6. $p^2 - q^2 = x - y$ **Ans.** $z = \frac{2}{3}(x+c)^{3/2} + \frac{2}{3}(y+c)^{3/2} + c_1$
7. $(p^2+q^2)y = qz$ **Ans.** $z^2 = (cx+a)^2 + c^2y^2$
8. Tick √ the correct answer.

   (*a*) The partial differential equation from $z = (a+x)^2 + y$ is

   (*i*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (*ii*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$ (*iii*) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (*iv*) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$

   (*b*) The solution of $xp + yq = z$ is

   (*i*) $f(x, y) = 0$ (*ii*) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ (*iii*) $f(xy, yz) = 0$ (*iv*) $f(x^2, y^2) = 0$

   (*c*) The solution of $p + q = z$ is

   (*i*) $f(x+y, y+\log z) = 0$ (*ii*) $f(xy, y\log z) = 0$

   (*iii*) $f(x-y, y-\log z) = 0$ (*iv*) None of these

   (*d*) The solution of $(y-z)p + (z-x)q = x - y$ is

   (*i*) $f(x+y+z) = xyz$ (*ii*) $f(x^2+y^2+z^2) = xyz$

   (*iii*) $f(x^2+y^2+z^2, x^2y^2z^2) = 0$ (*iv*) $f(x+y+z) = x^2+y^2+z^2$

   Ans. (*a*) (*i*), (*b*) (*ii*), (*c*) (*iii*), (*d*), (*iv*)

9.10 CHARPIT'S METHOD

General method for solving partial differential equation with two independent variables.

Solution. Let the general partial differential equation be

$$f(x, y, z, p, q) = 0 \qquad ...(1)$$

Since z depends on x, y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy \qquad ...(2)$$

The main aim in Charpits method is to find another relation between the variables x, y, z and p, q. Let the relation be

$$\phi(x, y, z, p, q) = 0 \qquad ...(3)$$

On solving (1) and (3), we get the values of p and q.

UNIT-V

Charpit's and Jacobi's Methods for Solving Non-linear Partial Differential Equations of Order One

4.1 Introduction

We have already seen that a linear partial differential equation of order one with two or more independent variables is solved by Lagrange's method. But, in many problems of science and engineering when we arrive at a non-linear partial differential equation of order one with two or more independent variables then we require new methods of solution. Thus, we describe Charpit's and Jacobi's methods for solving non-linear partial differential equations of order one. These methods are also applicable for solving linear partial differential equations of order one.

4.2 Charpit's Method for Solving Non-linear Partial Differential Equation of Order One

This method is used for solving non-linear partial differential equations of order one involving two independent variables.

A general method for solving the most general non-linear partial differential equation of order one i.e., the method for solving

$$f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

involving two independent variables x and y is given by Charpit and is known as **Charpit's method**. This method is applied to solve only those partial differential equations which cannot be solved by already known methods. The basic idea in Charpit's method is the introduction of another partial differential equation of order one of the form

$$g(x, y, z, p, q) = 0 \qquad \text{...(2)}$$

involving the two independent variables x, y and the dependent variable z, alongwith p and q. After introducing (2), we solve equations (1) and (2) for p and q and then substitute these values in the equation

$$dz = pdx + qdy \qquad \text{...(3)}$$

Now, if the solution of (3) exists, then this solution is the complete solution of equation (1).

To determine g, we differentiate (1) and (2) partially w.r.t. x and y. Thus, we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0, \frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial g}{\partial q}\frac{\partial q}{\partial x}=0,$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0 \ , \ \frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}q+\frac{\partial g}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial g}{\partial q}\frac{\partial q}{\partial y}=0$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair of the above, we have

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial g}{\partial p}-\left(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p}=0$$

or

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \qquad \text{...(4)}$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from the second pair, we have

$$\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0$$
...(5)

Since $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}$, therefore, the last term in (4) is the same as the last term in (5) except for a minus sign and hence they cancel on adding equations (4) and (5).

Adding (4) and (5) and re-arranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}$$

$$+\left(\frac{-\partial f}{\partial q}\right)\frac{\partial g}{\partial x}+\left(\frac{-\partial f}{\partial q}\right)\frac{\partial g}{\partial y}=0 \qquad ...(6)$$

This is a linear partial differential equation of order one with x, y, *z*, *p* and *q* as independent variables and *g* as the dependent variable. Thus, it can be solved by Lagrange's method.

Therefore, we have the following system of auxiliary equations:

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{\frac{-\partial f}{\partial p}}=\frac{dy}{\frac{-\partial f}{\partial q}}=\frac{dg}{0}$$
...(7)

Any two fractions of (7) which involve *p* or *q* or both can be taken to give a relation between p and q, which is solved for p and q with equation (1). Substituting these values of p and q in (3) and then integrating it, we get the complete solution of the given partial differential equation.

The auxiliary equations given by (7) are called as **Charpit's auxiliary equations** or simply **Charpit's equations**.

4.3 Working Rules of Charpit's Method for Solving Non-Linear Partial Differential Equations of Order One with Two Independent Variables

The following steps are required while using Charpit's method for solving non-linear partial differential equation of order one:

Step 1. Transfer all the terms of given PDE to L.H.S. and denote the entire expression in L.H.S. by $f(x, y, z, p, q)$.

Step 2. Write down the Charpit's auxiliary equations.

Step 3. Find the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, etc. occurring in Charpit's auxiliary equations. Put them in Charpit auxiliary equations and simplify.

Step 4. Choose two proper fractions from Charpit's auxiliary equations such that the resulting integral may come out as simplest relation involving at least one of p or q or both.

Step 5. The simplest relation of **step 4** is solved along with given partial differential equation to find p and q. Put these values of p and q in $dz = pdx + qdy$ which on integration gives the complete integral of the given partial differential equation.

The singular and general integrals may be obtained in the usual manner.

SOLVED EXAMPLES

Example 1. Find a complete integral of the partial differential equation $3p^2 = q$ by Charpit's method.

Solution : The given partial differential equation may be written as

$$f(x, y, z, p, q) \equiv 3p^2 - q = 0 \qquad ...(1)$$

$\therefore$ Charpit's auxiliary equations for (1) are :

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{\frac{-\partial f}{\partial p}}=\frac{dy}{\frac{-\partial f}{\partial q}}$$

or
$$\frac{dp}{0+p.0}=\frac{dq}{0+q.0}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1}$$
...(2)

Taking the first fraction of (2), we obtain $dp = 0$

Integrating it, we get $p = a$ ...(3)

Substituting the value $p = a$ in (1), we get $q = 3a^2$ ...(4)

Now, putting the values of p and q respectively from (3) and (4) in $dz = pdx + qdy$, we obtain

$$dz = adx + 3a^2 dy$$

Integrating it, we obtain $z = ax + 3a^2y + b$ ...(5)

Thus, the required complete integral is given by (5).

Example 2. Find a complete integral of $p^2 - y^2q = y^2 - x^2$ by Charpit's method.

Solution : Here, given partial differential equation may be written as

$$f(x, y, z, p, q) \equiv p^2 - y^2q - y^2 + x^2 = 0 \qquad ...(1)$$

$\therefore$ Charpit's auxiliary equations for (1) are :

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{\frac{-\partial f}{\partial p}}=\frac{dy}{\frac{-\partial f}{\partial q}}$$

or $$\frac{dp}{2x}=\frac{dq}{-2yq-2y}=\frac{dz}{-2p^2+y^2q}=\frac{dx}{-2p}=\frac{dy}{y^2}$$
...(2)

Taking the first and fourth fractions of (2), we obtain

$$pdp + xdx = 0$$

Integrating it, we get $$p^2+x^2=a^2 \qquad ...(3)$$

Solving (1) and (3) for p and q, we get

$$p=\sqrt{a^2-x^2} \quad \text{and} \quad q=a^2y^{-2}-1$$

Putting these values of p and q in $dz = pdx + qdy$, we have

$$dz=\sqrt{a^2-x^2}\,dx+(a^2y^{-2}-1)\,dy$$

Integrating it, we obtain

$$z=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)-\frac{a^2}{y}-y \qquad ...(4)$$

Thus, the required complete integral is given by (4).

EXERCISE 4 (A)

Find complete integral in each problem by Charpit's method:

1. $z = px + qy + pq$ **2.** $p^2 + px + q = z$

3. $(p + q)(z - px - qy) = 1$ **4.** $p + q = 3pq$

5. Find a complete integral of $z^2 = pqxy$ by Charpit's method.

6. Using Charpit's method, find a complete integral of $p^2x + q^2y = z$

7. Find a complete integral of $2z + p^2 + qy + 2y^2 = 0$ by Charpit's method.

8. Find a complete integral of $(p^2 + q^2)\, x = pz$ by Charpit's method.

ANSWERS

1. $z = ax + by + ab$

2. $z = ax + a^2 + be^y$

3. $(a+b)(z - ax - by) = 1$

4. $az = b - \frac{1}{2}(y + ax)^2$

5. $\log z = a \log x + \frac{1}{a} \log y + \log b$ or $z = b x^a y^{1/a}$